Spectra of noncommutative spaces.

Alexander Rosenberg

Introduction

Noncommutative spectral theory was started by P. Gabriel [Gab] with the definition of the injective spectrum of a locally noetherian abelian category and application of this notion to the reconstruction of a noetherian scheme from its category of quasi-coherent sheaves. Elements of the Gabriel's spectrum of an abelian category are isomorphism classes of its indecomposable injectives. The Gabriel's spectrum of the category of modules over a commutative noetherian ring is naturally isomorphic to the prime spectrum of this ring.

Trying to find a spectrum of an associative ring which does not rely on the noetherian hypothesis and is more closely related with representation theory, I run into a notion of the *left spectrum* of a ring (see [R1], [R, Ch.1]) which is a certain set of left ideals endowed with a *specialization* preorder, and containing left maximal ideals as closed points. Reformulated in terms of (quotient) modules, the left spectrum gave rise to a spectrum of an abelian category (see [R2], or [R, Ch.3]). This spectrum can be realized as a certain family of Serre subcategories. In particular, to each point of the spectrum, there corresponds a localization at this point with a remarkable property: the spectrum of the quotient category has only one closed point. Such categories are called *local.* If a local category has simple objects, then all of them are isomorphic to each other. Local categories suggested the notion of *S-spectrum* (in [R, Ch.6] it is called *flat spectrum*). Its points are Serre subcategories such that the corresponding quotient category is local. If the category is locally noetherian, then the S-spectrum is naturally isomorphic to the Gabriel's spectrum. But, even for locally noetherian categories, the S-spectrum is, usually, much bigger than the spectrum of [R2].

In [R4], several other spectra appeared. Those are related with exact localizations of arbitrary (not necessarily abelian) categories and triangulated categories. An abundance of new spectra made me suspect that the previously found spectra are not unique, as they made an impression at the beginning, and there exist pattern producing these spectra and the spectra of [R4], and, probably, some other spectra which make sense. Such pattern do exist, indeed, and explaining them is the main purpose of this text.

For the readers' convenience, original definitions of the left spectrum of a ring, and the spectrum and S-spectrum of an abelian category are gathered in Appendix, together with a couple of other notions from [R] which reappear in the main text in a different disguise.

The main construction is presented in Section 1. Roughly, it runs as follows. With any category, $\mathfrak{H}$, we associate two spectra, $\mathfrak{Spec}^0(\mathfrak{H})$ and $\mathfrak{Spec}^1(\mathfrak{H})$. These spectra are subcategories of $\mathfrak{H}$. And, by construction, there is a natural functor $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{H}$ (different from the inclusion functor). Given a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$, we define (in 1.6) two spectra, $\mathfrak{Spec}^0(\mathfrak{G}, F)$ and $\mathfrak{Spec}^1(\mathfrak{G}, F)$, of the pair $(\mathfrak{G}, F)$ as pullbacks of $\mathfrak{Spec}^0(\mathfrak{H})$ and $\mathfrak{Spec}^1(\mathfrak{H})$ along F. If $\mathfrak{H}$ is a preorder (which is the case of our main examples), then there exists a canonical morphism $\mathfrak{Spec}^0(\mathfrak{G}, F) \longrightarrow \mathfrak{Spec}^1(\mathfrak{G}, F)$. In particular, there is a canonical morphism $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\mathfrak{H})$.

Taking as F the inclusion map of the preorder of Serre subcategories to the preorder of topologizing subcategories of an abelian category, we recover the spectrum of [R2] mentioned above. If $\mathfrak{H}$ is the preorder of saturated multiplicative systems of a category (resp. triangulated category) and F is the identical functor, we recover the basic spectra of an arbitrary category (resp. triangulated category) introduced in [R4]. These and some other applications of the general construction are sketched in Section 2.

Spectra considered in Section 2 are related with saturated (left and right) multiplicative systems, or what is the same, with exact (i.e. preserving finite limits and colimits) localizations. In the case of an abelian or triangulated category, they correspond to thick subcategories. There are categories with only trivial saturated multiplicative systems. A fundamental example is the category $Sets$ of sets which belong to a given universe. It has no non-trivial right multiplicative systems, but has plenty of saturated left multiplicative systems. The latter are in bijective correspondence with right exact (– preserving colimits) localizations. In Section 3 we apply the pattern of Section 1 to the preorder of saturated left multiplicative systems of a category and obtain, as a result, left versions of the spectra discussed in Section 2 (and in [R4]).

In Section 4, we look at injective objects and related localizations and spectra, in particular, the Gabriel's spectrum. Injective objects play important role not only in abelian (Grothendieck) categories, but also in a large class of non-additive categories which includes toposes. Therefore, the exposition here is not restricted to abelian or even additive categories. We define a *left exact* multiplicative system as a saturated left multiplicative systems such that the corresponding localization functor preserves strict monomorphisms. In the case of abelian categories, left exact multiplicative systems are precisely saturated (left and right) multiplicative systems. On the other hand, in the case of the category $Sets$, every saturated left system is left exact, but, as it is mentioned above, there are no non-trivial right saturated multiplicative systems.

To any injective object E of a category C, there corresponds a left exact multiplicative system Σ_E which consists of all arrows s such that $Hom_C(s, E)$ is an isomorphism. Injective spectra, in particular (the non-additive version of) the Gabriel's spectrum, are obtained by applying to this correspondence the general formalism of Section 1.

It is worth to mention that left exact multiplicative systems are usually more important, at least from spectral prospective, than injective objects. For instance, if C is the category $Sets^1$ of non-empty sets, then there are only trivial left multiplicative systems of the form Σ_E. In particular, injective spectra are trivial. On the other hand, the preorder of left exact multiplicative systems is isomorphic to the order of infinite cardinals. Both spectra, $\mathfrak{Spec}^0$ and $\mathfrak{Spec}^1$, of this preorder are naturally isomorphic to the order of non-limit infinite cardinals.

The purpose of this work is to explain what stands behind the known constructions of spectra and give a couple of curious examples. There is no attempt to make the list of applications and examples complete (i.e. include all applications which seem to be important ones) and, with more reason, no attempt to impose choices. The reader might make a different choice of the functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ and use the 'spectral cuisine' of Section 1 to produce other spectra which could be appropriate for something.

Main constructions of this work were guessed in the process of writing [R4] and are partly motivated by [R4], [R2], and [R]. Section 1.5, where 'local' properties of spectra are discussed, is related to [R3]. In 'Complementary facts', we discuss associated points following the lead of the corresponding part of [R4]. A considerable part of the paper might be regarded as comments to [R4] and [R].

The work was partially supported by the NSF grant DMS-0070921.

1. General pattern.

Fix a category $\mathfrak{H}$. Let $\mathfrak{H}_0$ denote the full subcategory of $\mathfrak{H}$ whose objects are initial objects of $\mathfrak{H}$. Thus, $\mathfrak{H}_0$ is either empty, or a groupoid. Let $\mathfrak{H}^1$ denote the full subcategory of $\mathfrak{H}$ defined by $Ob\mathfrak{H}^1 = Ob\mathfrak{H} - Ob\mathfrak{H}_0$.

1.1. Definition. We call $\mathfrak{H}$ *local* if the category $\mathfrak{H}^1$ has an initial object.

1.1.1. Note. It follows that if $\mathfrak{H}$ is local, than $\mathfrak{H}$ has initial objects, i.e. $\mathfrak{H}_0 \neq \emptyset$.

1.1.2. Example. The preorder $\{x \to y\}$ is local, since $\mathfrak{H}^1$ has only one object, y, and one morphism, id_y.

1.1.3. Example. Let R be an associative commutative unital ring, and let IR denote the set of its ideals. The preorder $(IR, \supseteq)$ is local iff the ring R is local, i.e. there exists a maximal ideal in R which contains all other proper ideals.

1.2. The spectrum $\mathfrak{Spec}^1(\mathfrak{H})$. We denote by $\mathfrak{Spec}^1(\mathfrak{H})$ the full subcategory of the category $\mathfrak{H}$ generated by all $x \in Ob\mathfrak{H}$ such that the category $x\backslash\mathfrak{H}$ is local. We call $\mathfrak{Spec}^1(\mathfrak{H})$ the *local spectrum* of $\mathfrak{H}$.

In other words, an object x of $\mathfrak{H}$ belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ iff there exists an object $x^\star$ of $\mathfrak{H}$ and an arrow $x \xrightarrow{\gamma_x} x^\star$ such that γ_x is not an isomorphism and if $x \xrightarrow{f} y$ is not an isomorphism, then there exists a unique arrow $x^\star \xrightarrow{\bar{f}} y$ such that $f = \bar{f} \circ \gamma_x$. The morphism $x \xrightarrow{\gamma_x} x^\star$ (in particular, the object $x^\star$) is determined by these conditions uniquely up to isomorphism.

1.2.1. Note. It follows from this definition and 1.1.1 that $\mathfrak{H}$ is local iff it has initial objects and they belong to $\mathfrak{Spec}^1(\mathfrak{H})$.

1.2.1.1. Example. Let $\mathfrak{H} = Sets$. The category $Sets$ has one initial object – $\emptyset$. It is local: every one element set is an initial object of the category $Sets^1$. Notice that the spectrum $\mathfrak{Spec}^1(Sets^1)$ is empty. Therefore $\mathfrak{Spec}^1(Sets)$ consists of one point which is the initial object $\emptyset$.

1.2.2. Maximal proper objects and $\mathfrak{Spec}^1(\mathfrak{H})$. We call an object x of the category $\mathfrak{H}$ *proper* if there exists an arrow $x \longrightarrow y$ which is not an isomorphism. We call a proper object, x, *maximal* if any two proper morphisms, $y_1 \xleftarrow{s} x \xrightarrow{t} y_2$, are isomorphic; that is there exists an isomorphism $y_1 \xrightarrow{u} y_2$ such that $t = u \circ s$.

We denote by $\mathcal{M}ax(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by maximal proper objects. It follows that $\mathcal{M}ax(\mathfrak{H})$ is a groupoid which is connected iff all maximal proper objects are isomorphic to each other.

If for every proper object, y, there is an arrow from y to a maximal proper object, then the groupoid $\mathcal{M}ax(\mathfrak{H})$ is connected iff $\mathfrak{H}^{op}$ is a local category.

1.2.2.1. Proposition. $\mathcal{M}ax(\mathfrak{H}) \subseteq \mathfrak{Spec}^1(\mathfrak{H})$.

Proof. In fact, if x is an object of $\mathcal{M}ax(\mathfrak{H})$, then the category $x\backslash\mathfrak{H}$ is equivalent to the preorder $\{x \to y\}$, hence it is local (cf. 1.1.2). ■

1.2.2.2. Example. If $\mathfrak{H}$ is the preorder $(I_\ell R, \subseteq)$ of left ideals of an associative unital ring R, then $\mathcal{M}ax(\mathfrak{H})$ coincides with the set $Max_\ell R$ of left maximal ideals of R regarded as a discrete category. The category $Max_\ell R$ is connected iff R has only one left maximal ideal, μ. Notice that in this case the left ideal μ is two-sided, because for every $r \in R - \mu$, the ideal $(\mu : r) = \{a \in R \mid ar \in \mu\}$ is a maximal left ideal, hence it coincides with μ.

1.2.3. Minimal proper objects. We call $x \in Ob\mathfrak{H}$ a *minimal proper* object of the category $\mathfrak{H}$ if x is a maximal proper object of $\mathfrak{H}^{op}$. We denote by $\mathcal{M}in(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by minimal proper objects. By definition, $\mathcal{M}in(\mathfrak{H})$ is isomorphic to $\mathcal{M}ax(\mathfrak{H}^{op})$. In particular, $\mathcal{M}in(\mathfrak{H})$ is a groupoid which is connected iff $\mathfrak{H}$ is a local category. By 1.2.2.1, $\mathcal{M}in(\mathfrak{H}) \subseteq \mathfrak{Spec}^1(\mathfrak{H}^{op})$.

1.2.3.1. Example. Let C_X be a category with an initial object, and let $C_{\mathfrak{M}(X)}$ be the subcategory of C_X formed by all monoarrows of C_X. Then $\mathcal{M}in(C_{\mathfrak{M}(X)})$ is the groupoid of all simple objects of the category C_X. Isomorphism classes of simple objects can be regarded as a *naive spectrum* of C_X.

The groupoid $\mathcal{M}in(C_{\mathfrak{M}(X)})$ is connected (that is the category $C_{\mathfrak{M}(X)}$ is local) iff the category C_X has a unique, up to isomorphism, simple object.

1.2.3.2. Note. A useful version of 1.2.3.1 is obtained by taking instead of $C_{\mathfrak{M}(X)}$ the subcategory $C_{\mathfrak{M}_\mathfrak{s}(X)}$ generated by all strict monomorphisms of the category C_X.

Recall that a monomorphism $L \xrightarrow{h} M$ is called *strict* if every arrow $L' \longrightarrow M$ which equalizes all pairs of arrows $M \underset{g_2}{\overset{g_1}{\rightrightarrows}} N$ equalized by h is represented as the composition of h and an arrow $L' \longrightarrow L$ uniquely determined by this property. If an arrow $L \xrightarrow{h} M$ is such that there exists a fibred coproduct $M \coprod_L M$, then h is a strict monomorphism iff the canonical diagram $L \xrightarrow{h} M \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} M \coprod_L M$ is exact.

The groupoid $\mathcal{M}in(C_{\mathfrak{M}_\mathfrak{s}(X)})$ is generated by objects which are simple in a "strict" sense. For instance, if C_X is the category of continuous representaions of a topological algebra in topological vector spaces, objects of $\mathcal{M}in(C_{\mathfrak{M}_\mathfrak{s}(X)})$ are topologically irreducible representations of this algebra.

1.2.3.3. Example: injectives and the Gabriel's spectrum. Let C_X be a category with finite limits. An object E of the category C_X is called *injective* if the functor $C_X(-, E) : C_X^{op} \longrightarrow Sets$ preserves strict epimorphisms (in other words, $C_X(\mathfrak{j}, E)$ is a surjective map for any strict monomorphism $\mathfrak{j}$). We denote by $C_{\mathfrak{I}(X)}$ the subcategory of C_X formed by injective objects and strict monomorphisms (see 1.2.3.2) If follows that if E is an injective object, than any strict monomorphism $E \xrightarrow{g} M$ is a split monomorphism; i.e. $h \circ g = id_E$ for some $M \xrightarrow{h} E$.

We call an arrow in C_X a *zero morphism* if it factors through an initial object (if any).

We call an injective object E of the category C_X *indecomposable* if the only nonzero idempotent $E \longrightarrow E$ is the identical morphism. Equivalently, any strict monomorphism $E_1 \longrightarrow E$ with E_1 injective and non-initial, is an isomorphism.

Objects of the groupoid $\mathcal{M}in(C_{\mathfrak{I}(X)})$ are precisely indecomposable injectives of the full subcategory C_X^1 of the category C_X formed by non-initial objects. Isomorphism classes of indecomposable injectives are points of the Gabriel's spectrum.

1.2.4. Functorial properties. Let $\mathfrak{H} \xrightarrow{F} \widetilde{\mathfrak{H}}$ be a functor. For any $x \in Ob\mathfrak{H}$, the functor F induces a functor $x\backslash\mathfrak{H} \xrightarrow{F_x} F(x)\backslash\widetilde{\mathfrak{H}}$. Suppose that the functor F is such that F_x is an equivalence of categories. Then F induces a functor $\mathfrak{Spec}^1(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\widetilde{\mathfrak{H}})$.

A typical example is the functor

$$y\backslash\mathfrak{H} \xrightarrow{f_\star} z\backslash\mathfrak{H}, \quad (y, y \xrightarrow{g} v) \longmapsto (z, z \xrightarrow{gf} v),$$

corresponding to a morphism $z \xrightarrow{f} y$, or the canonical functor $y\backslash\mathfrak{H} \longrightarrow \mathfrak{H}$.

1.3. Supports. For any $x \in Ob\mathfrak{H}$, we denote by $\mathfrak{Supp}_{\mathfrak{H}}(x)$ the full subcategory of $\mathfrak{H}$ generated by all $y \in Ob\mathfrak{H}$ such that $\mathfrak{H}(x,y) = \emptyset$. We call $\mathfrak{Supp}_{\mathfrak{H}}(x)$ the *support of x in $\mathfrak{H}$*.

1.3.1. Proposition. *(a) For any two objects, x and y, of the category $\mathfrak{H}$, there exists an arrow $x \to y$ iff $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.*

(b) Let $\{x_i \mid i \in J\}$ be a set of objects of $\mathfrak{H}$ such that there exists a coproduct, $\coprod_{i\in J} x_i$. Then

$$\mathfrak{Supp}_{\mathfrak{H}}\Big(\coprod_{i\in J} x_i\Big) = \bigcup_{i\in J} \mathfrak{Supp}_{\mathfrak{H}}(x_i). \tag{1}$$

Proof. (a) If there exists a morphism $x \longrightarrow y$ and $\mathfrak{H}(x,z) = \emptyset$, then, obviously, $\mathfrak{H}(y,z) = \emptyset$, hence $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$.

If $\mathfrak{H}(x,y) = \emptyset$, i.e. $y \in \mathfrak{Supp}_{\mathfrak{H}}(x)$, then, since $y \notin Ob\mathfrak{Supp}_{\mathfrak{H}}(y)$, the inclusion $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$ does not hold.

(b) Since $\mathfrak{H}(\coprod_{i\in J} x_i, z) \simeq \prod_{i\in J} \mathfrak{H}(x_i, z)$, $\mathfrak{H}(\coprod_{i\in J} x_i, z) = \emptyset$ iff $\mathfrak{H}(x_i, z) = \emptyset$ for some $i \in J$, whence the equality (1). ∎

1.3.2. Support in $\mathfrak{Spec}^1(\mathfrak{H})$. For any $x \in Ob\mathfrak{H}$, we denote the intersection $\mathfrak{Supp}_{\mathfrak{H}}(x) \bigcap \mathfrak{Spec}^1(\mathfrak{H})$ by $\mathfrak{Supp}^1_{\mathfrak{H}}(x)$ and call it the *support of x in* $\mathfrak{Spec}^1(\mathfrak{H})$. Evidently, 1.3.1(b) is still true if $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is replaced by $\mathfrak{Supp}^1_{\mathfrak{H}}(x)$, as well as a half of 1.3.1(a): if $\mathfrak{H}(x,y)$ is not empty, then $\mathfrak{Supp}^1_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}^1_{\mathfrak{H}}(y)$.

1.4. The spectrum $\mathfrak{Spec}^0(\mathfrak{H})$. We denote by $\mathfrak{Spec}^0(\mathfrak{H})$ the full subcategory of $\mathfrak{H}$ generated by $x \in Ob\mathfrak{H}$ such that $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is not empty and has a final object, $\widehat{x}$.

1.4.1. Proposition. *Let $\mathfrak{H}$ be local. Then initial objects of $\mathfrak{H}^1$ belong to $\mathfrak{Spec}^0(\mathfrak{H})$.*

Proof. Let $\mathfrak{H}_0$ be the full subcategory (groupoid) of $\mathfrak{H}$ generated by all initial objects of $\mathfrak{H}$. If x is an initial object of the category $\mathfrak{H}^1$, then $\mathfrak{Supp}_{\mathfrak{H}}(x)$ coincides with $\mathfrak{H}_0$.

In fact, suppose that there is an arrow, $x \xrightarrow{f} y$, for some $y \in Ob\mathfrak{H}_0$. Since y is an initial object of the category $\mathfrak{H}$, there exists a unique morphism $y \xrightarrow{g} x$. By the universal property of y, the composition $y \xrightarrow{fg} y$ is the identical morphism. Since x is an initial object of the category $\mathfrak{H}^1$, the composition $x \xrightarrow{gf} x$ is the identical morphism too. This means that the morphism $x \xrightarrow{f} y$ is an isomorphism which contradicts to the fact that x is not an initial object of the category $\mathfrak{H}$.

Thus, $\mathfrak{H}_0$ is a subcategory of $\mathfrak{Supp}_{\mathfrak{H}}(x)$. Since for every $z \in Ob\mathfrak{H}^1 = Ob\mathfrak{H} - Ob\mathfrak{H}_0$ there is a (unique) morphism $x \longrightarrow z$, the subcategory $\mathfrak{Supp}_{\mathfrak{H}}(x)$ is contained in $\mathfrak{H}_0$; i.e. $\mathfrak{Supp}_{\mathfrak{H}}(x) = \mathfrak{H}_0$.

Since $\mathfrak{H}_0$ is a connected groupoid, every object of $\mathfrak{H}_0$ is final. ■

1.4.2.1. Example. The spectrum $\mathfrak{Spec}^0(Sets)$ coincides with the subcategory $Sets^1$ of all non-empty sets, because the support of any non-empty set consists of only $\emptyset$.

1.4.2. Proposition. *The full subcategory, $\mathfrak{Spec}^0(\mathfrak{H})_0$, generated by initial objects of $\mathfrak{Spec}^0(\mathfrak{H})$ coincides with be the full subcategory $\mathfrak{H}_1 = (\mathfrak{H}^1)_0$ of the category $\mathfrak{H}$ generated by initial objects of the subcategory $\mathfrak{H}^1$.*

Proof. By definition, the subcategory $\mathfrak{Supp}_{\mathfrak{H}}(x)$ (cf. 1.4) is not empty for every object x of $\mathfrak{Spec}^0(\mathfrak{H})$. Therefore initial objects of the category $\mathfrak{H}$ do not belong to $\mathfrak{Spec}^0(\mathfrak{H})$, i.e. $\mathfrak{Spec}^0(\mathfrak{H})$ is contained in the subcategory $\mathfrak{H}^1$. In particular, the subcategory $\mathfrak{Spec}^0(\mathfrak{H})_0$ of initial objects of $\mathfrak{Spec}^0(\mathfrak{H})$ is contained in $\mathfrak{H}_1 = (\mathfrak{H}^1)_0$. The converse inclusion is a consequence of 1.4.1. ■

1.4.3. Corollary. *Let $|\mathfrak{Spec}^1(\mathfrak{H})|$ denote the set of isomorphism classes of objects of $\mathfrak{Spec}^1(\mathfrak{H})$. Then*

$$Ob\mathfrak{Spec}^1(\mathfrak{H}) = \bigcup_{x \in |\mathfrak{Spec}^1(\mathfrak{H})|} \big\{ y \mid (y, x \to y) = \widehat{z},\ z \in Ob\mathfrak{Spec}^0(x\backslash\mathfrak{H})_0 \big\}. \tag{1}$$

In particular,

$$Ob\mathfrak{Spec}^1(\mathfrak{H}) \subseteq \bigcup_{x \in |\mathfrak{Spec}^1(\mathfrak{H})|} \big\{ y \mid (y, x \to y) = \widehat{z},\ z \in Ob\mathfrak{Spec}^0(x\backslash\mathfrak{H}) \big\}. \tag{2}$$

Here $\widehat{z}$ is a final object of the category $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(z)$ (cf. 1.4).

Proof. The formula (1) follows from 1.4.2 applied to the category $x\backslash\mathfrak{H}$. ■

1.4.4. Lemma. *A choice for every $x \in Ob\mathfrak{Spec}^0(\mathfrak{H})$ of a final object, $\widehat{x}$, of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$ extends to a functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$.*

Proof. In fact, if $x, y \in Ob\mathfrak{Spec}^0(\mathfrak{H})$, and there is a morphism $x \longrightarrow y$, then $\mathfrak{Supp}_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(y)$. Therefore there exists a unique morphism $\widehat{x} \longrightarrow \widehat{y}$. ■

1.4.5. Remark. Notice that the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ is faithful iff $\mathfrak{H}$ is a preorder, i.e. for any pair of objects, x, y, of $\mathfrak{H}$, there is at most one morphism $x \longrightarrow y$.

1.4.6. Proposition. *Suppose the category $\mathfrak{H}$ is a preorder with finite coproducts (i.e. supremums of pairs of objects). Then the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ takes values in $\mathfrak{Spec}^1(\mathfrak{H})$, i.e. it induces a functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\theta_{\mathfrak{H}}} \mathfrak{Spec}^1(\mathfrak{H})$.*

Proof. For any $x \in Ob\mathfrak{Spec}^0(\mathfrak{H})$, the final object, $\widehat{x}$, of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$ belongs to $\mathfrak{Spec}^1(\mathfrak{H})$. More expicitly, we claim that the canonical coprojection, $\widehat{x} \longrightarrow x \sqcup \widehat{x}$, is an initial object of the category $(\widehat{x}\backslash\mathfrak{H})^1$.

In fact, let $\widehat{x} \xrightarrow{g} y$ be a morphism. Then one of two things happens: either $y \in Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, or not. If $y \in Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, then, since $\widehat{x}$ is a final object of the category $\mathfrak{Supp}_{\mathfrak{H}}(x)$, there is a unique morphism $y \xrightarrow{h} \widehat{x}$. It follows from the universal property of $\widehat{x}$ that $h \circ g = id_{\widehat{x}}$. By hypothesis, $\mathfrak{H}$ is a preorder, in particular, h is a monomorphism. Therefore, h is an isomorphism inverse to g.

If $y \notin Ob\mathfrak{Supp}_{\mathfrak{H}}(x)$, then there exists an arrow $x \longrightarrow y$ which, together with $\widehat{x} \xrightarrow{g} y$, determines (and is determined by) a morphism $(x \sqcup \widehat{x}, \widehat{x} \to x \sqcup \widehat{x}) \longrightarrow (y, \widehat{x} \xrightarrow{g} y)$. Since $\mathfrak{H}$ is a preorder, this is all we need. ■

1.4.6.1. Example. Let $\mathfrak{H}$ be a category with initial objects and such that $\mathfrak{H}(x, y) \neq \emptyset$ for every $x \in Ob\mathfrak{H}$ and $\mathfrak{H}(y, z) = \emptyset$ if y is not an initial object and z is an initial object. Then $\mathfrak{Supp}_{\mathfrak{H}}(x) = \mathfrak{H}_0$ for any $x \in Ob\mathfrak{H}^1$. Therefore, $\mathfrak{Spec}^0(\mathfrak{H})$ coincides with $\mathfrak{H}^1$. If $\mathfrak{H}$ is a preorder, then, under conditions, $\mathfrak{H}^1$ is a connected groupoid, hence $\mathfrak{Spec}^1(\mathfrak{H}) = \mathfrak{H}_0$ and the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\theta_{\mathfrak{H}}} \mathfrak{Spec}^1(\mathfrak{H})$ is a category equivalence.

What might happen if $\mathfrak{H}$ is not a preorder is illustrated by the following.

1.4.6.2. Example. If $\mathfrak{H} = Sets$, then, by 1.2.1.1, $\mathfrak{Spec}^1(\mathfrak{H}) = \{\emptyset, id_{\emptyset}\}$ and by 14.2.1, $\mathfrak{Spec}^0(\mathfrak{H}) = Sets^1$ – the category of all non-empty sets. There is only one functor $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(\mathfrak{H})$.

1.4.7. Proposition. *Suppose that the category $\mathfrak{H}$ is a preorder with coproducts. Then*

$$Ob\mathfrak{Spec}^1(\mathfrak{H}) = \bigcup_{x \in |\mathfrak{Spec}^1(\mathfrak{H})|} \big\{y \mid (y, x \to y) = \widehat{z},\ z \in Ob\mathfrak{Spec}^0(x\backslash\mathfrak{H})\big\}. \tag{3}$$

Proof. The assertion follows from 1.4.6 and 1.4.3(2). ■

1.4.8. Support in $\mathfrak{Spec}^0(\mathfrak{H})$. For any object x of the category $\mathfrak{H}$, we denote by $\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ the preimage of $\mathfrak{Supp}_{\mathfrak{H}}(x)$ by the functor $\mathfrak{Spec}^0(\mathfrak{H}) \longrightarrow \mathfrak{H}$, $z \longmapsto \widehat{z}$, (cf. 1.4.4) and call it the *support of x in* $\mathfrak{Spec}^0(\mathfrak{H})$. This means that $\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ is a full subcategory of $\mathfrak{H}$ whose objects are all objects z of $\mathfrak{Spec}^0(\mathfrak{H})$ such that $\mathfrak{H}(x, \widehat{z}) = \emptyset$, or, equivalently, $\mathfrak{Supp}_{\mathfrak{H}}(z) \subseteq \mathfrak{Supp}_{\mathfrak{H}}(x)$. By 1.3.1, the latter means precisely that there exists a morphism $z \to x$. Thus, $\mathfrak{Supp}^0_{\mathfrak{H}}(x)$ is a full subcategory of $\mathfrak{H}$ generated by all objects z of $\mathfrak{Spec}^0(\mathfrak{H})$ such that $z \to x$.

1.4.9. Proposition. *(a) The map $x \longmapsto \mathfrak{Supp}^0_{\mathfrak{H}}(x)$ is functorial: if there exists an arrow $x \to y$, then $\mathfrak{Supp}^0_{\mathfrak{H}}(x) \subseteq \mathfrak{Supp}^0_{\mathfrak{H}}(y)$.*

(b) Let $\{x_i \mid i \in J\}$ be a set of objects of $\mathfrak{H}$ such that there exists a coproduct, $\coprod_{i \in J} x_i$. Then

$$\mathfrak{Supp}^0_{\mathfrak{H}}\big(\coprod_{i \in J} x_i\big) = \bigcup_{i \in J} \mathfrak{Supp}^0_{\mathfrak{H}}(x_i). \qquad (1)$$

Proof. (a) The assertion follows from the fact that

$$Ob\mathfrak{Supp}^0_{\mathfrak{H}}(x) = \{z \in Ob\mathfrak{Spec}^0(\mathfrak{H}) \mid \mathfrak{H}(z, x) \neq \emptyset\}$$

(see the discussion in 1.4.8).

(b) An object z of $\mathfrak{Spec}^0(\mathfrak{H})$ belongs to $\mathfrak{Supp}^0_{\mathfrak{H}}\big(\coprod_{i \in J} x_i\big)$ iff $\mathfrak{H}\big(\coprod_{i \in J} x_i, \widehat{z}\big) = \emptyset$. Since $\mathfrak{H}\big(\coprod_{i \in J} x_i, \widehat{z}\big) = \prod_{i \in J} \mathfrak{H}(x_i, \widehat{z})$, this occurs iff $\mathfrak{H}(x_i, \widehat{z}) = \emptyset$ for some $i \in J$. ■

1.5. Topologies and spectra.

1.5.1. Generalities on topologies. Let τ be a topology on $\mathfrak{H}^{op}$, i.e. τ is a function which assigns to every object x of $\mathfrak{H}$ a set, $\tau(x)$, of subfunctors of the functor $\mathfrak{H}(x, -)$ (called the *refinements* of x) satisfying the following conditions:

(a) for every arrow $x \xrightarrow{f} y$ of $\mathfrak{H}$ and every $R \in \tau(x)$, the fibre product, R^f, of $R \longrightarrow \mathfrak{H}(x, -) \xleftarrow{\mathfrak{H}(f,-)} \mathfrak{H}(y, -)$ belongs to $\tau(y)$;

(b) If $R \in \tau(x)$ and E is a subfunctor of $\mathfrak{H}(x, -)$ such that $E^f \in \tau(y)$ for any $f \in R(y)$ and any y, then $E \in \tau(x)$.

1.5.1.1. Cocovers. A family of arrows $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ generates a subfunctor, $R_{\widetilde{x}}$, of $\mathfrak{H}(x, -)$ defined as follows: $R_{\widetilde{x}}(y)$ consists of all arrows $x \longrightarrow y$ which factor through $x \xrightarrow{u_i} x_i$ for some $i \in J$. The family $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ is called a *cocover* (or a cover in $\mathfrak{H}^{op}$) for the topology τ if $R_{\widetilde{x}} \in \tau(x)$.

1.5.1.2. Sheaves. Subcanonical topologies. A functor $\mathfrak{H} \xrightarrow{F} Sets$ (viewed as a presheaf of sets on $\mathfrak{H}^{op}$) is called a sheaf (on $(\mathfrak{H}^{op}, \tau)$) if for every $x \in Ob\mathfrak{H}$ and any refinement R of x, the map $F(x) \longrightarrow Hom(R, F)$ induced by the embedding $R \hookrightarrow \mathfrak{H}(x, -)$ and the Yoneda isomorphism $F(x) \simeq Hom(\mathfrak{H}(x, -), F)$, is a bijection.

The topology τ on $\mathfrak{H}^{op}$ is called *subcanonical* if every representable presheaf, i.e. a functor of the form $\mathfrak{H}(x, -)$, $x \in Ob\mathfrak{H}$, is a sheaf.

1.5.1.3. Cosieves. It is convenient sometimes to describe topologies on $\mathfrak{H}^{op}$ in terms of cosieves. Recall that a *cosieve* in a category $\mathcal{A}$ is a full subcategory, $\mathcal{B}$, of $\mathcal{A}$ such that for every $x \in Ob\mathcal{B}$, all arrows $x \longrightarrow y$ belong to $\mathcal{B}$.

Let x be an object of the category $\mathfrak{H}$. There is a one-to-one correspondence between cosieves of the category $x\backslash\mathfrak{H}$ and subfunctors of the functor $\mathfrak{H}(x, -)$. Namely, an object $(y, x \xrightarrow{\xi} y)$ of $x\backslash\mathfrak{H}$ belongs to the cosieve $\widetilde{R}$ corresoponding to a subfunctor R of $\mathfrak{H}(x, -)$ iff the morphism $x \xrightarrow{\xi} y$ is an element of $R(y)$.

Thus, a topology, τ, on $\mathfrak{H}^{op}$ can be described as a function which assigns to each object of $\mathfrak{H}$ a non-empty family of cosieves of the category $x\backslash\mathfrak{H}$, which are also called *refinements of* x, satisfying the conditions reflecting properties (a) and (b) of 1.5.1.

One can see that a topology τ on $\mathfrak{H}^{op}$ is subcanonical iff for every $x \in Ob\mathfrak{H}$ each refinement of x is a terminal cone. In other words, for every refinement $\widetilde{R}$ of x, the limit of the canonical functor $R \longrightarrow \mathfrak{H}$, $(y, x \xrightarrow{\xi} y) \longmapsto y$, is isomorphic to x.

1.5.1.4. Pretopologies on $\mathfrak{H}^{op}$. A *pretopology* on $\mathfrak{H}^{op}$ is a function, τ, which assigns to each object x of $\mathfrak{H}$ a family, τ_x, of sets of arrows $\{x \to x_i \mid i \in J\}$ (in $\mathfrak{H}$) having the following properties:

(a) for every $x \in Ob\mathfrak{H}$, $\{id_x\} \in \tau_x$;

(b) if $\{x \to x_i \mid i \in J\} \in \tau_x$ and $\{x_i \to x_{ij} \mid j \in J_i\} \in \tau_{x_i}$ for every $i \in J$, then $\{x \to x_{ij} \mid i \in J,\ j \in J_i\} \in \tau_x$;

(c) for any $\widetilde{x} = \{x \to x_i \mid i \in J\} \in \tau_x$ and any morphism $x \xrightarrow{\phi} y$, there exists $\widetilde{y} = \{y \xrightarrow{v_j} y_j \mid j \in I\} \in \tau_y$ such that the morphism ϕ can be lifted to a morphism $\widetilde{x} \xrightarrow{\widetilde{\phi}} \widetilde{y}$. The latter means that for every $j \in I$, the composition of $x \xrightarrow{\phi} y$ and $y \longrightarrow y_j$ factors through $x \longrightarrow x_i$ for some $i \in J$.

Elements of τ_x are called *cocovers* of x. Arrows which belong to cocovers are interpreted as closed subsets. The corresponding arrows in $\mathfrak{H}^{op}$ are viewed as open subsets.

Every pretopology determines a topology obtained by taking cosieves (or subfunctors of representable functors) associated with cocovers.

1.5.2. Proposition. *Suppose that τ is a subcanonical topology on $\mathfrak{H}^{op}$; and let $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ be a cocover for τ. Then*

$$\mathfrak{Spec}^1(x\backslash\mathfrak{H}) = \bigcup_{i \in J} \mathfrak{Spec}^1(x_i\backslash\mathfrak{H}).$$

Here $\mathfrak{Spec}^1(x_i\backslash\mathfrak{H})$ is identified with its image in $x\backslash\mathfrak{H}$ via the morphism $x \xrightarrow{u_i} x_i$.

Proof. Let an object $\widetilde{z} = (z, x \xrightarrow{\mathfrak{f}} z)$ belong to $\mathfrak{Spec}^1(x\backslash\mathfrak{H})$. By definition, this means that the subcategory $(\widetilde{z}\backslash(x\backslash\mathfrak{H}))^1$ has an initial object. Notice that the category $\widetilde{z}\backslash(x\backslash\mathfrak{H})$ is isomorphic to the category $z\backslash\mathfrak{H}$. Thus, the category $(z\backslash\mathfrak{H})^1$ has an initial object, $(z^\star, z \xrightarrow{\xi} z^\star)$. By condition, ξ is not an isomorphism.

Let $R_{\widetilde{x}}$ be a refinement of x associated with the cocover $\widetilde{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$; and let $R^{\xi}_{\widetilde{x}}$ be the corresponding refinement of the object z. Notice that there exists $y \in Ob\mathfrak{H}$ and an element $z \xrightarrow{g} y$ of $R^{\xi}_{\widetilde{x}}(y)$ such that $(y, z \xrightarrow{g} y) \notin Ob(z\backslash\mathfrak{H})^1$.

In fact, if such element would not exist, then, since $(z^\star, z \xrightarrow{\xi} z^\star)$ is an initial object of $(z\backslash\mathfrak{H})^1$, every element $z \xrightarrow{g} y$ of $R^{\xi}_{\widetilde{x}}(y)$ factors through $z \xrightarrow{\xi} z^\star$, and this factorization is unique. Since the topology τ is subcanonical, (z, id_z) is an initial object of the sieve $\widehat{R}^{\xi}_{\widetilde{x}}$ associated with $R^{\xi}_{\widetilde{x}}$. Therefore, there exists a unique morphism $(z^\star, \xi) \longrightarrow (z, id_z)$. But, this cannot happen (see the argument of 1.4.6).

Thus, there exists an element $z \xrightarrow{g} y$ of $R^{\xi}_{\widetilde{x}}(y)$ such that $(y, z \xrightarrow{g} y) \notin Ob(z\backslash\mathfrak{H})^1$, or, what is the same, the arrow $z \xrightarrow{g} y$ is an isomorhism. By the definition of $\widehat{R}^{\xi}_{\widetilde{x}}$, there exists a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{\mathfrak{f}} & z \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle g} \\ x_i & \xrightarrow{\mathfrak{f}_i} & y \end{array} \tag{4}$$

for some $i \in J$. Since the arrow g in (4) is an isomorphism, it follows from (4) that $x \xrightarrow{\mathfrak{f}} z$ factors through the element $x \xrightarrow{u_i} x_i$ of the cocover $\widetilde{x}$. Therefore, the object $(z, \mathfrak{f})$ of $\mathfrak{Spec}^1(x\backslash\mathfrak{H})$ is the image of an object $(z, g\mathfrak{f}_i)$ of $\mathfrak{Spec}^1(x_i\backslash\mathfrak{H})$; hence the assertion. ■

For any $x \in Ob\mathfrak{H}$, let $\mathcal{U}_{\mathfrak{H}}(x)$ denote the full subcategory of $\mathfrak{H}$ generated by all $y \in Ob\mathfrak{H}$ such that $\mathfrak{H}(y, x) = \emptyset$. Thus, $\mathcal{U}_{\mathfrak{H}}(x)$ coincides with $\big(\mathfrak{Supp}_{\mathfrak{H}^{op}}(x)\big)^{op}$.

1.5.3. Proposition. *Let $x \in Ob\mathfrak{H}$ be such that for any $z \in Ob\mathfrak{H}$, there exists a coproduct, $x \sqcup z$. Then the map $z \longmapsto (x \sqcup z, x \to x \sqcup z)$ defines a functor*

$$\mathfrak{Spec}^0(\mathfrak{H}) \bigcap \mathcal{U}_{\mathfrak{H}}(x) \longrightarrow \mathfrak{Spec}^0(x\backslash\mathfrak{H}).$$

Proof. For any $y \in Ob\mathfrak{H}$, we set $f^*(y) = (y \sqcup x, x \to y \sqcup x)$. The map $y \mapsto f^*(y)$ extends to a functor, $\mathfrak{H} \xrightarrow{f^*} x\backslash\mathfrak{H}$, which is left adjoint to the functor $x\backslash\mathfrak{H} \xrightarrow{f_*} \mathfrak{H}$, $(v, x \to v) \mapsto v$.

Let z be an object of the subcategory $\mathfrak{Spec}^0(\mathfrak{H}) \bigcap \mathcal{U}_{\mathfrak{H}}(x)$, and let $\widehat{z}$ be a final object of the category $\mathfrak{Supp}_{\mathfrak{H}}(z)$. Since $z \in Ob\mathcal{U}_{\mathfrak{H}}(x)$, i.e. $\mathfrak{H}(z, x) = \emptyset$, there exists a unique morphism $x \to \widehat{z}$. We claim that the $(\widehat{z}, x \to \widehat{z})$ is a final object of the category $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(f^*(z))$.

In fact, $x\backslash\mathfrak{H}(f^*(z), (y, x \to y)) \simeq \mathfrak{H}(z, y)$ which shows that $(y, x \to y)$ is an object of $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(f^*(z))$ iff y is an object of $\mathfrak{Supp}_{\mathfrak{H}}(z)$. Therefore, $(\widehat{z}, x \to \widehat{z})$ belongs to $\mathfrak{Supp}_{x\backslash\mathfrak{H}}(f^*(z))$ and, moreover, is a final object of this category. ■

1.5.4. Corollary. *Let $x \xrightarrow{u} y$ be a morphism of $\mathfrak{H}$ such that for any other morphism, $x \xrightarrow{v} z$, there exists a fibred coproduct, $y \sqcup_x z$. Then the functor*

$$y\backslash\mathfrak{H} \xrightarrow{u_*} x\backslash\mathfrak{H}, \quad (z, y \xrightarrow{g} z) \longmapsto (z, x \xrightarrow{gu} z),$$

has a left adjoint, u^*; and u^* induces a functor

$$\mathfrak{Spec}^0(x\backslash\mathfrak{H}) \bigcap \mathcal{U}_{x\backslash\mathfrak{H}}(y, u) \longrightarrow \mathfrak{Spec}^0(y\backslash\mathfrak{H}).$$

Proof. The fact is a consequence of 1.5.3 applied to the category $x\backslash\mathfrak{H}$. ■

1.5.5. Proposition. *Let $\mathbf{x} = \{x \xrightarrow{u_i} x_i \mid i \in J\}$ be a set of arrows such that the cone $x \longrightarrow \widetilde{R}_{\mathbf{x}}$ is terminal (i.e. $x = \lim \widetilde{R}_{\mathbf{x}}$) and for any arrow $x \longrightarrow y$, there exist fibred coproducts $x_i \sqcup_x y$. If $\mathfrak{H}$ is a preorder, then the image of the canonical map*

$\mathfrak{Spec}^0(x\backslash\mathfrak{H}) \longrightarrow \mathfrak{Spec}^1(x\backslash\mathfrak{H})$ *is contained in the union of images of* $\mathfrak{Spec}^0(x_i\backslash\mathfrak{H})$, $i \in J$, *in* $\mathfrak{Spec}^1(x\backslash\mathfrak{H})$.

Proof. By 1.5.4, there are natural functors

$$\mathfrak{Spec}^0(x\backslash\mathfrak{H})\bigcap\mathcal{U}_{x\backslash\mathfrak{H}}(x_i,u_i) \longrightarrow \mathfrak{Spec}^0(x_i\backslash\mathfrak{H})$$

Therefore, it suffices to show that for any object (z,ξ) of $\mathfrak{Spec}^0(x\backslash\mathfrak{H})$, there exists $i \in J$ such that there are no morphisms from (z,ξ) to (x_i,u_i).

Suppose that for each $i \in J$, there is an arrow $(z,\xi) \longrightarrow (x_i,u_i)$. Since $\mathfrak{H}$ is a preorder, these arrows determine a cone $z \longrightarrow \widetilde{R}_{\mathbf{x}}$. By hypothesis, $x = \lim \widetilde{R}_{\mathbf{x}}$, hence there exists a morphism $(z,\xi) \longrightarrow (x, id_x)$. Since (x,id_x) is an initial object of the category $x\backslash\mathfrak{H}$, this means that for any object of $x\backslash\mathfrak{H}$ there is an arrow from (z,ξ) to this object, which cannot happen, because (z,ξ) belongs to $\mathfrak{Spec}^0(x\backslash\mathfrak{H})$. Thus, there exists $i \in J$ such that there are no morphisms from (z,ξ) to (x_i,u_i). ∎

1.5.6. The spectrum of a precosite. Let τ be a pretopology on $\mathfrak{H}^{op}$. We call the pair $(\mathfrak{H},\tau)$ a *precosite*. Let $\mathfrak{H}_\tau$ denote the subcategory of $\mathfrak{H}$ formed by arrows which belong to some cocovers. For every $x \in Ob\mathfrak{H}$, we denote by τ_x the induced pretopology on $\mathfrak{H}^{op}/x = (x\backslash\mathfrak{H})^{op}$. We denote by $\mathfrak{Spec}^0(x\backslash\mathfrak{H},\tau_x)$ the full subcategory of $x\backslash\mathfrak{H}$ generated by the images of $\mathfrak{Spec}^0(y\backslash\mathfrak{H})$ for all arrows $x \to y$ of the subcategory $\mathfrak{H}_\tau$. We call $\mathfrak{Spec}^0(x\backslash\mathfrak{H},\tau_x)$ the *spectrum* of the precosite $(x\backslash\mathfrak{H},\tau_x)$.

Thus, if the category $\mathfrak{H}$ has an initial object, we obtain the spectrum, $\mathfrak{Spec}^0(\mathfrak{H},\tau)$, of the precosite $(\mathfrak{H},\tau)$. If, in addition, all arrows of the subcategory $\mathfrak{H}_\tau$ are isomorphisms (for instance, the pretopology τ is discrete), then $\mathfrak{Spec}^0(\mathfrak{H},\tau)$ coincides with $\mathfrak{Spec}^0(\mathfrak{H})$.

1.5.6.1. Proposition. *Suppose that* $\mathfrak{H}_\tau = \mathfrak{H}$ *and* $\mathfrak{H}$ *is a preorder with finite coproducts and an initial object. Then* $\mathfrak{Spec}^0(\mathfrak{H},\tau)$ *is isomorphic to* $\mathfrak{Spec}^1(\mathfrak{H})$.

Proof. The assertion is a consequence of 1.4.6. ∎

1.5.6.2. Proposition. *To any morphism,* $x \longrightarrow y$, *of the subcategory* $\mathfrak{H}_\tau$, *there corresponds an inclusion* $\mathfrak{Spec}^0(y\backslash\mathfrak{H},\tau_y) \subseteq \mathfrak{Spec}^0(x\backslash\mathfrak{H},\tau_x)$, *i.e. the map* $x \longmapsto \mathfrak{Spec}^0(x\backslash\mathfrak{H},\tau_x)$ *extends to a functor* $\mathfrak{H}_\tau^{op} \longrightarrow Cat$.

Proof. The fact follows from definitions. ∎

1.6. Relative spectra. Let $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ be a functor. We define the relative spectra, $\mathfrak{Spec}^1(\mathfrak{G},F)$ and $\mathfrak{Spec}^0(\mathfrak{G},F)$, via cartesian squares

$$\begin{array}{ccc} \mathfrak{Spec}^1(\mathfrak{G},F) & \xrightarrow{\theta_F^1} & \mathfrak{G} \\ {\scriptstyle \pi_1^F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathfrak{Spec}^1(\mathfrak{H}) & \xrightarrow{\theta_{\mathfrak{H}}^1} & \mathfrak{H} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathfrak{Spec}^0(\mathfrak{G},F) & \xrightarrow{\vartheta_F} & \mathfrak{G} \\ {\scriptstyle \pi_0^F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathfrak{Spec}^0(\mathfrak{H}) & \xrightarrow{\vartheta_{\mathfrak{H}}} & \mathfrak{H} \end{array} \qquad (1)$$

(in the bicategorical sense, i.e. the squares quasi-commute), where $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ is the canonical functor of 1.4.4.

Explicitly, objects of the category $\mathfrak{Spec}^1(\mathfrak{G}, F)$ are triples $(z, x; \phi)$, where z is an object of $\mathfrak{Spec}^1(\mathfrak{H})$, $x \in Ob\mathfrak{G}$, and ϕ is an isomorphism $z \xrightarrow{\sim} F(x)$. Morphisms from $(z, x; \phi)$ to $(z', x'; \phi')$ are given by pairs of arrows, $z \xrightarrow{g} z'$ and $x \xrightarrow{h} x'$ such that the diagram

$$\begin{array}{ccc} z & \xrightarrow{g} & z' \\ \phi \big\downarrow \wr & & \wr \big\downarrow \phi' \\ F(x) & \xrightarrow{F(h)} & F(x') \end{array}$$

commutes. The projections $\mathfrak{Spec}^1(\mathfrak{H}) \xleftarrow{\pi_1^F} \mathfrak{Spec}^1(\mathfrak{G}, F) \xrightarrow{\theta_F^1} \mathfrak{G}$ in the left diagram (1) are defined by $\pi_1^F(z, x; \phi) = z$ and $\theta_F^1(z, x; \phi) = x$.

Similarly, objects of the category $\mathfrak{Spec}^0(\mathfrak{G}, F)$ are triples $(z, x; \psi)$, where z is an object of $\mathfrak{Spec}^0(\mathfrak{H})$, $x \in Ob\mathfrak{G}$, and ψ is an isomorphism $\vartheta_{\mathfrak{H}}(z) \xrightarrow{\sim} F(x)$.

1.6.1. Proposition. *Let i be 0 or 1. The map $(\mathfrak{G}, F) \longmapsto \mathfrak{Spec}^i(\mathfrak{G}, F)$ extends to a pseudo-functor $\mathfrak{Spec}^i : Cat/\mathfrak{H} \longrightarrow Cat$.*

Proof. The assertion follows from the universal property of cartesian squares. ■

1.6.2. Proposition. *Suppose $\mathfrak{H}$ is a preorder with finite coproducts. Then for every functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$, there is a canonical functor*

$$\mathfrak{Spec}^0(\mathfrak{G}, F) \xrightarrow{\vartheta_{(\mathfrak{G},F)}} \mathfrak{Spec}^1(\mathfrak{G}, F). \tag{2}$$

The family $\{\vartheta_{(\mathfrak{G},F)} \mid (\mathfrak{G}, F) \in ObCat/\mathfrak{H}\}$ is a morphism of pseudo-functors,

$$\mathfrak{Spec}^0 \xrightarrow{\vartheta} \mathfrak{Spec}^1. \tag{3}$$

Proof. Since $\mathfrak{H}$ is a preorder with finite coproducts, the functor $\mathfrak{Spec}^0(\mathfrak{H}) \xrightarrow{\vartheta_{\mathfrak{H}}} \mathfrak{H}$ takes values in $\mathfrak{Spec}^1(\mathfrak{H})$, hence it factors through the embedding $\mathfrak{Spec}^1(\mathfrak{H}) \longrightarrow \mathfrak{H}$ (see 1.4.6). By the universal property of cartesian squares, there exists a unique functor (2) such that $\theta_F^1 \circ \vartheta_{(\mathfrak{G},F)} = \vartheta_F$ and $\pi_1^F \circ \vartheta_{(\mathfrak{G},F)} = \pi_0^F$ (see the diagram (2)).

It is useful to have an explicit description of the functor (2) in terms of the descripitons of $\mathfrak{Spec}^0(\mathfrak{G}, F)$ and $\mathfrak{Spec}^1(\mathfrak{G}, F)$ given above. The functor $\vartheta_{(\mathfrak{G},F)}$ maps an object $(z, x; \psi)$ of $\mathfrak{Spec}^0(\mathfrak{G}, F)$ to the object $(\vartheta_{\mathfrak{H}}(z), x; \psi)$ of $\mathfrak{Spec}^1(\mathfrak{G}, F)$.

It follows from this description that $\vartheta = \{\vartheta_{(\mathfrak{G},F)} \mid (\mathfrak{G}, F) \in ObCat/\mathfrak{H}\}$ is a morphism of pseudo-functors. ■

2. Applications: spectra of 'spaces'.

2.0. Preliminaries on 'spaces' and localizations.

'Spaces' here are spaces of noncommutative algebraic geometry. In the simplest (or the most abstract) setting, they are represented by categories. Morphisms of 'spaces' are functors regarded as inverse image functors. We denote by C_X the category representing a 'space' X and by f^* a functor $C_Y \longrightarrow C_X$ representing a morphism $X \xrightarrow{f} Y$. Formally, 'spaces' are objects of the category Cat^{op} opposite to the category Cat. The *bicategory of 'spaces'* is the bicategory Cat^{op}. The *category of 'spaces'* is the category $|Cat|^o$ having same objects as Cat^{op}. Morphisms from X to Y are isomorphism classes of (inverse image) functors $C_Y \longrightarrow C_X$.

2.0.1. Localizations and conservative morphisms. Let X be a 'space' and Σ a family of arrows of the category C_X. We denote by $\Sigma^{-1}X$ the object of $|Cat|^o$ such that the corresponding category coincides with (the standard realization of) the category of fractions ot C_X for Σ (cf. [GZ], 1.1): $C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$. We call $\Sigma^{-1}X$ the *'space' of fractions of* X *for* Σ. The canonical *localization functor* $C_X \xrightarrow{q_\Sigma^*} \Sigma^{-1}C_X$ is regarded as an inverse image functor of a morphism, $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$.

For any morphism $f : X \longrightarrow Y$ in $|Cat|^o$, we denote by Σ_f the family of all morphisms s of the category C_Y such that $f^*(s)$ is invertible (notice that Σ_f does not depend on the choice of an inverse image functor f^*). Thanks to the universal property of localizations, f^* is represented as the composition of the localization functor $p_f^* = p_{\Sigma_f}^* : C_Y \longrightarrow \Sigma_f^{-1}C_Y$ and a uniquely determined functor $f_\mathfrak{c}^* : \Sigma_f^{-1}C_Y \longrightarrow C_X$. In other words, $f = p_f \circ f_\mathfrak{c}$ for a uniquely determined morphism $f_\mathfrak{c} : X \longrightarrow \Sigma_f^{-1}Y$.

A morphism $f : X \longrightarrow Y$ is called *conservative* if Σ_f consists of isomorphisms only, or, equivalently, p_f is an isomorphism.

A morphism $f : X \longrightarrow Y$ is called a *localization* if $f_\mathfrak{c}$ is an isomorphism, i.e. the functor $f_\mathfrak{c}^*$ is an equivalence of categories.

Thus, $f = p_f \circ f_\mathfrak{c}$ is a decomposition of a morphism f into a localization and a conservative morphism.

2.0.2. Multiplicative systems. A family of arrows Σ of a category C_X is called a *left multiplicative system* if it has the following properties:

(S1) Σ is closed under composition and contains all identical arrows of C_X.

(SL2) Every diagram $M' \xleftarrow{s} M \xrightarrow{f} L$, where $s \in \Sigma$, can be completed to a commutative square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "s"'] & L \arrow[d, "s'"] \\ M' \arrow[r, "f'"] & L' \end{tikzcd}$$

where $s' \in \Sigma$.

(SL3) If $M \underset{g}{\overset{f}{\rightrightarrows}} N$ is a pair of arrows such that $f \circ s = g \circ s$ for some $s \in \Sigma$, then there exists a morphism $N \xrightarrow{t} N'$ of Σ such that $t \circ f = t \circ g$.

A family $\Sigma \subseteq HomC_X$ is a *right multiplicative system* if it has dual properties. Finally, Σ is called *multiplicative system* if it is both right and left multiplicative.

We denote by $\mathcal{SM}_\ell(X)$ (resp. by $\mathcal{SM}_\mathfrak{r}(X)$) the family of all left (resp. right) multiplicative systems in C_X. We denote by $\mathcal{SM}(X)$ the family $\mathcal{SM}_\ell(X)\bigcap\mathcal{SM}_\mathfrak{r}(X)$ of all multiplicative systems in C_X.

We regard $\mathcal{SM}_\ell(X)$, $\mathcal{SM}_\mathfrak{r}(X)$, and $\mathcal{SM}(X)$ as preorders with respect to $\subseteq$.

2.0.3. Saturated multiplicative systems. Let Σ be a family of morphisms of the category C_X. Let q_{Σ} be the localization morphism $\Sigma^{-1}X \longrightarrow X$ and $C_X \xrightarrow{q_{\Sigma}^*} C_{\Sigma^{-1}X} = \Sigma^{-1}C_X$ its canonical inverse image functor.

The family $\Sigma^\mathfrak{s} = \Sigma_{q_{\Sigma}}$ of all arrows of C_X which q_{Σ}^* transfers into isomorphisms (cf. 1.2) is called the *saturation* of Σ. A family of morphisms Σ is called *saturated* if Σ coincides with its saturation.

We denote by $\mathcal{S}^\mathfrak{s}\mathcal{M}_\ell(X)$ (resp. by $\mathcal{S}^\mathfrak{s}\mathcal{M}_\mathfrak{r}(X)$) the family of all saturated left (resp. right) multiplicative systems in C_X. We denote by $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ the family of all saturated (left and right) multiplicative systems in C_X; that is $\mathcal{S}^\mathfrak{s}\mathcal{M}(X) = \mathcal{S}^\mathfrak{s}\mathcal{M}_\ell(X)\bigcap\mathcal{S}^\mathfrak{s}\mathcal{M}_\mathfrak{r}(X)$.

We regard $\mathcal{S}^\mathfrak{s}\mathcal{M}_\ell(X)$, $\mathcal{S}^\mathfrak{s}\mathcal{M}_\mathfrak{r}(X)$, and $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ as preorders with respect to $\subseteq$.

2.0.3.1. Saturated multiplicative systems and thick subcategories. If C_X is an abelian category, then there is an isomorphism between the preorder $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ of saturated multiplicative systems of C_X and the preorder $\mathfrak{Th}(X)$ of thick subcategories of C_X. Recall that a full subcategory $\mathbb{T}$ of C_X is *thick* if it is closed under taking subquotients and extensions. This isomorphism is given by the map which assigns to each thick subcategory $\mathbb{T}$ of C_X the family $\Sigma_\mathbb{T}$ of all arrows s of C_X such that both kernel and cokernel of s belong to $\mathbb{T}$. Inverse isomorphism assigns to a multiplicative system Σ the kernel of the localization at Σ.

2.0.4. Right exact, left exact, and exact morphisms. A morphism $X \xrightarrow{f} Y$ is called *right exact* (resp. *left exact*, resp *exact*), if its inverse image functor preserves colimits (resp. limits, resp. both limits and colimits) of arbitrary finite diagrams.

The following assertion is a reformulation of Propositions 1.3.1 and 1.3.4 in [GZ].

2.0.4.1. Proposition. *(a) Let Σ be a left multiplicative system in C_X. Then the canonical morphism $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$ is right exact.*

(b) Let $f = p_f \circ f_\mathfrak{c}$ be the canonical decomposition of a morphism $f : X \longrightarrow Y$ into a conservative morphism $X \xrightarrow{f_\mathfrak{c}} \Sigma_f^{-1}Y$ and a localization $\Sigma_f^{-1}Y \xrightarrow{p_f} Y$. Suppose C_Y has finite limits (resp. finite colimits). Then f is left exact (resp. right exact) iff the family of arrows Σ_f is a left (resp. right) multiplicative system. In this case both the localization p_f and the conservative morphism $f_\mathfrak{c}$ are left (resp. right) exact.

In particular, if the category C_Y has limits and colimits of finite diagrams, then f is exact iff both the localization p_f and the conservative component $f_\mathfrak{c}$ are exact. The exactness of p_f is equivalent to that $\Sigma_f \in \mathcal{S}^\mathfrak{s}\mathcal{M}(X)$.

Thus, if a category C_X admits finite colimits, then saturated left multiplicative systems of C_X classify right exact localizations of C_X.

2.0.5. Closed multiplicative systems. Let $\Sigma \subseteq HomC_X$. We say that an object, M, of the category C_X is Σ-*torsion free* if every morphism $M \xrightarrow{t} N$ which belongs to Σ is a strict monomorphism.

We say that Σ is *closed*, or *right closed*, if for every $M \in ObC_X$, there exist a Σ-torsion free object $\widetilde{M}$ and an arrow $M \longrightarrow \widetilde{M}$ of Σ which belongs to Σ.

We denote by $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ the preorder of all closed saturated multiplicative systems of the category C_X.

2.0.6. Flat multiplicative systems. We call a saturated multiplicative system Σ *flat* if the localization functor $C_X \xrightarrow{q^*_\Sigma} \Sigma^{-1}C_X$ has a right adjoint.

We denote by $\mathfrak{Lc}(X)$ the preorder of all flat multiplicative systems of C_X. By [R4, 5.2.1], every flat multiplicative system is closed, i.e. $\mathfrak{Lc}(X) \subseteq \mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$.

2.1. The spectra of exact localizations.

For a 'space' X, we take as $\mathfrak{H}$ the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems of C_X and set

$$\mathbf{Spec}^1_{\mathfrak{L}}(X) = \mathfrak{Spec}^1(\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)) \quad \text{and} \quad \mathbf{Spec}^0_{\mathfrak{L}}(X) = \mathfrak{Spec}^0(\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)).$$

Since $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ is a preorder, there exists a canonical injective morphism

$$\mathbf{Spec}^0_{\mathfrak{L}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L}}(X), \quad \Sigma \longmapsto \widehat{\Sigma}$$

(cf. 1.4.6). Notice that the support, $\mathfrak{Supp}_{\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)}(\Sigma)$, of Σ consists of all saturated multiplicative systems of C_X which do not contain Σ, and its final object, $\widehat{\Sigma}$, is the union of all multiplicative systems which belong to the support of Σ.

2.2. Closed spectra and flat spectra.

The *closed* spectra of a 'space' X are relative spectra corresponding to the inclusion functor, $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X) \xrightarrow{\mathfrak{J}_{\mathfrak{C}}(X)} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$, where $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ is the preorder of all closed saturated multiplicative systems of C_X (cf. 2.0.5). Thus,

$$\mathbf{Spec}^i_{\mathfrak{C}}(X) = \mathfrak{Spec}^i(\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X), \mathfrak{J}_{\mathfrak{C}}(X)), \quad i = 0, 1.$$

Similarly, the *flat* spectra of X are relative spectra of $(\mathfrak{Lc}(X), \mathfrak{J}_{\mathfrak{L}}(X))$, where $\mathfrak{Lc}(X)$ is the preorder of all flat saturated multiplicative systems of C_X (cf. 2.0.6) and $\mathfrak{J}_{\mathfrak{L}}(X)$ is the inclusion functor $\mathfrak{Lc}(X) \hookrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$. Thus,

$$\mathbf{Spec}^i_{\mathfrak{f}\mathfrak{L}}(X) = \mathfrak{Spec}^i(\mathfrak{Lc}(X), \mathfrak{J}_{\mathfrak{L}}(X)), \quad i = 0, 1.$$

2.3. Spectra of 'spaces' represented by abelian categories.

Fix a 'space' X such that C_X is an abelian category.

2.3.1. Thick spectra. We take as $\mathfrak{H}$ the preorder $\mathfrak{Th}(X)$ of thick subcategories of C_X and set

$$\mathbf{Spec}^i_{\mathfrak{Th}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}(X)), \quad i = 0, 1.$$

Since the preorder $\mathfrak{Th}(X)$ is naturally isomorphic to the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems, the isomorphism $\mathfrak{Th}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ induces isomorphisms

$$\mathbf{Spec}^i_{\mathfrak{Th}}(X) \xrightarrow{\sim} \mathbf{Spec}^i_{\mathfrak{L}}(X), \quad i = 0, 1$$

such that the diagram

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{Th}}(X) & \xrightarrow{\sim} & \mathbf{Spec}^0_{\mathfrak{L}}(X) \\ \downarrow & & \downarrow \\ \mathbf{Spec}^1_{\mathfrak{Th}}(X) & \xrightarrow{\sim} & \mathbf{Spec}^1_{\mathfrak{L}}(X) \end{array}$$

commutes. Here the vertical arrows are canonical embeddings of 1.4.6.

2.3.2. Representatives of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$. Let $C_{\mathfrak{M}(X)}$ denote the subcategory of C_X formed by all monomorphisms of C_X. The map which assigns to each object, M, of the category C_X the smallest thick subcategory, $[M]_\bullet$, containing M defines a functor $C_{\mathfrak{M}(X)} \xrightarrow{\mathfrak{F}_X} \mathfrak{Th}(X)$. We denote by $Spec^0_{\mathfrak{Th}}(X)$ the preimage, $\mathfrak{F}_X^{-1}(\mathbf{Spec}^0_{\mathfrak{Th}}(X))$, of the spectrum $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$. An object M of $Spec^0_{\mathfrak{Th}}(X)$ is regarded as a representative of the object $[M]_\bullet$ of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$.

It follows from [R4, 7.1.1] that the functor

$$Spec^0_{\mathfrak{Th}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{Th}}(X), \quad M \longmapsto [M]_\bullet, \tag{1}$$

is surjective. Namely, if $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$, then $\mathcal{P} = [M]_\bullet$ for any $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$. Here $\widehat{\mathcal{P}}$ is the union of all thick subcategories of C_X which do not contain $\mathcal{P}$.

2.3.3. Closed and flat spectra. Let $\mathfrak{CTh}(X)$ be the preorder of coreflective thick subcategories of the category C_X, and $\mathfrak{J}_{\mathfrak{CT}}(X)$ the inclusion functor $\mathfrak{CTh}(X) \hookrightarrow \mathfrak{Th}(X)$. Recall that a full subcategory $\mathbb{T}$ of C_X is *coreflective* if the inclusion functor $\mathbb{T} \longrightarrow C_X$ has a right adjoint. In other words, every object of C_X has the biggest subobject which belongs to $\mathbb{T}$.

The coreflective spectra of X are defined by

$$\mathbf{Spec}^i_{\mathfrak{CTh}}(X) = \mathfrak{Spec}^i(\mathfrak{CTh}(X), \mathfrak{J}_{\mathfrak{CT}}(X)), \quad i = 0, 1.$$

By [R4, 7.2.1], the isomorphism $\mathfrak{Th}(X) \xrightarrow{\sim} \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ induces an isomorphism of the preorder $\mathfrak{CTh}(X)$ and the preorder $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of closed saturated multiplicative systems. Therefore, $\mathbf{Spec}^i_{\mathfrak{CTh}}(X)$ is isomorphic to the closed spectrum, $\mathbf{Spec}^i_{\mathfrak{C}}(X)$ defined in 2.2.

Let $\mathfrak{Th}_{\mathfrak{c}}(X)$ denote the preorder of all thick subcategories $\mathbb{T}$ such that the localization functor $C_X \longrightarrow C_X/\mathbb{T}$ has a right adjoint. And let $\mathfrak{J}_{\mathfrak{c}}(X)$ denote the inclusion functor $\mathfrak{Th}_{\mathfrak{c}}(X) \hookrightarrow \mathfrak{Th}(X)$. Since the preorder $\mathfrak{Th}_{\mathfrak{c}}(X)$ is isomorphic to the preorder $\mathfrak{Lc}(X)$ of flat saturated systems of C_X (cf. 2.2), the spectrum

$$\mathbf{Spec}^i_{\mathfrak{Th}_{\mathfrak{c}}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}_{\mathfrak{c}}(X), \mathfrak{J}_{\mathfrak{c}}(X))$$

is isomorphic to the corresponding flat spectrum $\mathbf{Spec}^i_{\mathfrak{fL}}(X)$ defined in 2.2. Here $i = 0, 1$.

2.3.4. Relative Serre spectrum. Fix an abelian category C_X. Let $\mathbb{T}$ be a subcategory of C_X. We denote by $\mathbb{T}^-$ the full subcategory of C_X generated by all objects L of C_X such that any nonzero subquotient of L has a nonzero subobject which belongs to $\mathbb{T}$. By [R, III.2.3.2], for any subcategory $\mathbb{T}$, the category $\mathbb{T}^-$ is thick, and $(\mathbb{T}^-)^- = \mathbb{T}^-$.

A subcategory $\mathbb{T}$ of C_X is called a *Serre subcategory* if $\mathbb{T} = \mathbb{T}^-$.

Recall that X (or the category C_X) has the property (sup) if for any ascending chain, Ω, of subobjects of an object M, the supremum of Ω exists, and for any subobject L of M, the natural morphism

$$sup(N \cap L \mid N \in \Omega) \longrightarrow (sup\Omega) \cap L$$

is an isomorphism.

2.3.4.1. Lemma. *(a) Any coreflective thick subcategory of an abelian category C_X is a Serre subcategory.*

(b) If X has the property (sup), then any Serre subcategory of C_X is coreflective.

Proof. See [R, III.2.4.4]. ■

Let $\mathfrak{Se}(X)$ be the preorder of Serre subcategories of C_X and $\mathfrak{J}_{\mathfrak{Se}}(X)$ the inclusion functor $\mathfrak{Se}(X) \hookrightarrow \mathfrak{Th}(X)$. The *Serre spectra*, or, shortly, S-spectra of X are defined by

$$\mathbf{Spec}^i_{\mathfrak{Se}}(X) = \mathfrak{Spec}^i(\mathfrak{Se}(X), \mathfrak{J}_{\mathfrak{Se}}(X)), \quad i = 0, 1.$$

Since $\mathfrak{Th}_{\mathfrak{c}}(X) \subseteq \mathfrak{CTh}(X) \subseteq \mathfrak{Se}(X)$, there are inclusions of spectra,

$$\mathbf{Spec}^i_{\mathfrak{Th}_{\mathfrak{c}}}(X) \subseteq \mathbf{Spec}^i_{\mathfrak{CTh}}(X) \subseteq \mathbf{Spec}^i_{\mathfrak{Se}}(X), \quad i = 0, 1.$$

By 2.3.4.1, if C_X is a category with the property (sup), then $\mathfrak{CTh}(X) = \mathfrak{Se}(X)$, in particular the spectra $\mathbf{Spec}^i_{\mathfrak{Se}}(X)$ and $\mathbf{Spec}^i_{\mathfrak{CTh}}(X)$ coincide. If C_X is a Grothendieck category, then $\mathfrak{Th}_{\mathfrak{c}}(X) = \mathfrak{CTh}(X) = \mathfrak{Se}(X)$, hence in this case,

$$\mathbf{Spec}^i_{\mathfrak{Th}_{\mathfrak{c}}}(X) = \mathbf{Spec}^i_{\mathfrak{CTh}}(X) = \mathbf{Spec}^i_{\mathfrak{Se}}(X), \quad i = 0, 1.$$

2.3.4.3. The category $Spec^0_{\mathfrak{s}}(X)$. We denote by $Spec^0_{\mathfrak{s}}(X)$ the full subcategory of the category $C_{\mathfrak{M}(X)}$ generated by nonzero objects M such that $M \in Ob[N]_\bullet$ if there exists a nonzero arrow $N \longrightarrow M$ (see [R4, 7.3.4]). By [R4, 7.3.5], $Spec^0_{\mathfrak{s}}(X) \subseteq Spec^0_{\mathfrak{Se}}(X)$.

It is easy to see that a nonzero object M belongs to $Spec^0_{\mathfrak{s}}(X)$ iff $[L]_\bullet = [M]_\bullet$ for any nonzero subobject L of M.

2.3.4.4. Proposition. *The image, $\mathbf{Spec}^0_{\mathfrak{s}}(X)$, of the map*

$$Spec^0_{\mathfrak{s}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{Th}}(X), \quad M \longmapsto [M]_\bullet, \qquad (2)$$

contains $\mathbf{Spec}^0_{\mathfrak{CTh}}(X)$. If the category C_X has the property (sup), then the image of (2) coincides with $\mathbf{Spec}^0_{\mathfrak{CTh}}(X)$.

Proof. Let $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{C}\mathfrak{T}\mathfrak{h}}(X)$; i.e. $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)$ such that the thick subcategory $\widehat{\mathcal{P}}$ is coreflective. Let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}$. Since $\widehat{\mathcal{P}}$ is coreflective, M has a $\widehat{\mathcal{P}}$-torsion, $\mathfrak{t}_{\widehat{\mathcal{P}}}M$. Replacing M by the quotient $M/\mathfrak{t}_{\widehat{\mathcal{P}}}M$, we can assume that M is $\widehat{\mathcal{P}}$-torsion free. Since $\widehat{\mathcal{P}} = \langle M\rangle_\bullet$, it follows from [R4, 7.3.6] that M is an object of $Spec^0_{\mathfrak{s}}(X)$ such that $[M]_\bullet = \mathcal{P}$.

By [R4, 7.3.5], the subcategory $\langle M\rangle_\bullet$ is a Serre subcategory for every object M of $Spec^0_{\mathfrak{s}}(X)$. If the category C_X has the property (sup), then, by [R4, 7.3.8], every Serre subcategory of C_X is coreflective. ∎

2.3.5. Spectra related to topologizing subcategories. Let $\mathfrak{T}(X)$ denote the preorder of all topologizing subcategories of the category C_X. Let $\mathfrak{J}^{\mathfrak{t}}_X$ denote the inclusion functor $\mathfrak{T}\mathfrak{h}(X) \longrightarrow \mathfrak{T}(X)$.

Thus, we have two spectra associated to this functor,

$$\mathbf{Spec}^i_{\mathfrak{t}}(X) = \mathfrak{Spec}^i(\mathfrak{T}\mathfrak{h}(X), \mathfrak{J}^{\mathfrak{t}}_X), \quad i = 0, 1.$$

and a canonical morphism from one to another, $\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{t}}(X)$.

2.3.5.1. Proposition. *(a) There is a natural map* $\mathbf{Spec}^1_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$.

(b) The functor $\mathfrak{T}(X) \longrightarrow \mathfrak{T}\mathfrak{h}(X)$ *which assigns to every topologizing subcategory* $\mathbb{T}$ *the smallest thick subcategory,* $\mathbb{T}_\bullet$*, containing* $\mathbb{T}$*, induces a functor* $\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{T}\mathfrak{h}}(X)$.

Proof. (a) The category $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ is defined by the cartesian square

$$\begin{array}{ccc} \mathbf{Spec}^1_{\mathfrak{t}}(X) & \longrightarrow & \mathfrak{T}\mathfrak{h}(X) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{Spec}^1(\mathfrak{T}(X)) & \longrightarrow & \mathfrak{T}(X) \end{array}$$

in which the right vertical arrow and the lower horizontal arrow are inclusions (see 1.6). It follows from this description (or from the explicit description of relative spectra in 1.6) that objects of $\mathbf{Spec}^1_{\mathfrak{t}}(X)$ are naturally identified with thick subcategories, $\mathcal{P}$, which are objects of $\mathfrak{Spec}^1(\mathfrak{T}(X))$. The latter means that that there exists the smallest topologizing subcategory, $\mathcal{P}^{\mathfrak{t}}$, properly containing $\mathcal{P}$. Therefore, the smallest thick subcategory, $[\mathcal{P}^{\mathfrak{t}}]_\bullet$, containing $\mathcal{P}^{\mathfrak{t}}$ is the smallest thick subcategory *properly* containing $\mathcal{P}$, hence $\mathcal{P}$ belongs to $\mathbf{Spec}^1_{\mathfrak{T}\mathfrak{h}}(X)$.

(b) The spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ is defined by the cartesian square

$$\begin{array}{ccc} \mathbf{Spec}^0_{\mathfrak{t}}(X) & \longrightarrow & \mathfrak{T}\mathfrak{h}(X) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{Spec}^0(\mathfrak{T}(X)) & \longrightarrow & \mathfrak{T}(X) \end{array} \qquad (1)$$

(see 1.6). By definition, objects of $\mathfrak{Spec}^0(\mathfrak{T}(X))$ are topologizing subcategories, $\mathcal{P}$, such that $\mathfrak{Supp}_{\mathfrak{T}(X)}(\mathcal{P})$ has a final object. This means, precisely, that the union, $\widehat{\mathcal{P}}^{\mathfrak{t}}$, of all topologizing subcategories which do not contain $\mathcal{P}$ is a topologizing subcategory. The lower horizontal arrow of the diagram (1) maps an element $\mathcal{P}$ to $\widehat{\mathcal{P}}^{\mathfrak{t}}$.

It follows that objects of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ can be identified with topologizing subcategories, $\mathcal{P}$, such that $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is a thick subcategory. Therefore, $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is a final object of the support $\mathfrak{Supp}_{\mathfrak{Th}(X)}(\mathcal{P}_\bullet)$, where $\mathcal{P}_\bullet$ is the smallest thick subcategory containing $\mathcal{P}$. This implies that the map

$$\mathbf{Spec}^0_{\mathfrak{t}}(X) \longrightarrow \mathfrak{Th}(X), \quad \mathcal{P} \longmapsto \mathcal{P}_\bullet,$$

takes values in $\mathbf{Spec}^0_{\mathfrak{Th}}(X)$, hence the assertion. ∎

2.3.5.2. Representatives of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. Let $C_{\mathfrak{M}(X)}$ denote the subcategory of C_X formed by all monomorphisms of C_X. The map which assigns to each object, M, of the category C_X the smallest topologizing subcategory, $[M]$, containing M defines a functor $C_{\mathfrak{M}(X)} \xrightarrow{\mathfrak{F}_X} \mathfrak{T}(X)$. We denote by $Spec^0_{\mathfrak{t}}(X)$ the preimage, $\mathfrak{F}_X^{-1}(\mathbf{Spec}^0_{\mathfrak{t}}(X))$, of the spectrum $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. An object M of $Spec^0_{\mathfrak{t}}(X)$ is regarded as a representative of the object $[M]$ of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$.

2.3.5.3. Proposition. *The functor*

$$Spec^0_{\mathfrak{t}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{t}}(X), \quad M \longmapsto [M],$$

is surjective.

Proof. Let $\mathcal{P}$ be an object of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$. For any $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}^{\mathfrak{t}}$, the union, $\langle M\rangle$, of all topologizing subcategories of C_X which do not contain the object M coincides with $\widehat{\mathcal{P}}^{\mathfrak{t}}$. In fact, $\langle M\rangle \subseteq \widehat{\mathcal{P}}^{\mathfrak{t}}$, because $M \in Ob\mathcal{P}$; and $\widehat{\mathcal{P}}^{\mathfrak{t}} \subseteq \langle M\rangle$, because $M \notin Ob\widehat{\mathcal{P}}^{\mathfrak{t}}$.

It remains to notice that $[M] = \mathcal{P}$. Clearly $[M] \subseteq \mathcal{P}$. The inverse inclusion, $\mathcal{P} \subseteq [M]$, holds because if $\mathcal{P} \nsubseteq [M]$, then $[M] \subseteq \langle M\rangle$ which is impossible by the definition of $\langle M\rangle$. ∎

2.3.6. Closed spectra defined by topologizing subcategories. Let $\mathfrak{J}^{\mathfrak{s}}_X$ be the inclusion functor $\mathfrak{CTh}(X) \longrightarrow \mathfrak{T}(X)$. This functor creates two spectra,

$$\mathbf{Spec}^i_{\mathfrak{Ct}}(X) = \mathfrak{Spec}^i(\mathfrak{CTh}(X), \mathfrak{J}^{\mathfrak{s}}_X), \quad i = 0, 1.$$

and a canonical morphism from one to another, $\mathbf{Spec}^0_{\mathfrak{Ct}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Ct}}(X)$.

2.3.6.1. The spectrum $Spec(X)$. We denote by $Spec(X)$ the full subcategory of the category $C_{\mathfrak{M}(X)}$ generated by nonzero objects M such that $M \in Ob[N]$ if there exists a nonzero morphism $N \longrightarrow M$.

Since $[N] \subseteq [N]_\bullet$ for every object N, it follows that $Spec(X) \subseteq Spec^0_{\mathfrak{s}}(X)$. In particular, $Spec(X) \subseteq Spec^0_{\mathfrak{Se}}(X)$.

It is easy to see that a nonzero object M belongs to $Spec(X)$ iff $[L] = [M]$ for every nonzero subobject L of M. In other words, the functor

$$C_{\mathfrak{M}(X)} \longrightarrow \mathfrak{T}(X), \quad M \longmapsto [M],$$

maps every nonzero (mono)morphism $L \longrightarrow M$ to the (identical) isomorphism.

2.3.6.2. Proposition. *The image* $\mathbf{Spec}(X)$ *of the map*

$$Spec(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{t}}(X), \quad M \longmapsto [M], \tag{1}$$

contains $\mathbf{Spec}^0_{\mathfrak{Ct}}(X)$*. If the category* C_X *has the property (sup), then the image of (1) coincides with* $\mathbf{Spec}^0_{\mathfrak{Ct}}(X)$*.*

Proof. Let $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{Ct}}(X)$; i.e. $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{t}}(X)$ such that $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is a coreflective thick subcategory. Let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}^{\mathfrak{t}}$. Since $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is coreflective, M has a $\widehat{\mathcal{P}}^{\mathfrak{t}}$-torsion, $\mathfrak{t}_{\widehat{\mathcal{P}}^{\mathfrak{t}}}M$. Replacing M by the quotient $M/\mathfrak{t}_{\widehat{\mathcal{P}}^{\mathfrak{t}}}M$, we assume that M is $\widehat{\mathcal{P}}^{\mathfrak{t}}$-torsion free. Since $\widehat{\mathcal{P}}^{\mathfrak{t}}$ is the union, $\langle M\rangle$, of all topologizing subcategories of C_X which do not contain M, it follows that M is an object of $Spec(X)$ such that $[M] = \mathcal{P}$.

Since $\langle M\rangle$ is thick, it coincides with $\langle M\rangle_\bullet$. In particular, it is (by [R4, 7.3.5]) a Serre subcategory. If the category C_X has the property (sup), then, by [R4, 7.3.8], every Serre subcategory of C_X is coreflective. ■

2.4. Spectra defined by Serre subcategories.

Let $\mathfrak{H}$ be the preorder $\mathfrak{Se}(X)$ of all Serre subcategories of the category C_X. Thus, we have two spectra and an embedding:

$$\mathfrak{Spec}^0(\mathfrak{Se}(X)) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X)).$$

2.4.1. Proposition. *There are natural functors*

$$\mathbf{Spec}^i_{\mathfrak{Se}}(X) \longrightarrow \mathfrak{Spec}^i(\mathfrak{Se}(X)), \quad i = 0, 1,$$

such that the diagram

$$\begin{array}{ccc}
\mathbf{Spec}^0_{\mathfrak{Se}}(X) & \longrightarrow & \mathfrak{Spec}^0(\mathfrak{Se}(X)) \\
\downarrow & & \downarrow \\
\mathbf{Spec}^1_{\mathfrak{Se}}(X) & \longrightarrow & \mathfrak{Spec}^1(\mathfrak{Se}(X))
\end{array}$$

commutes.

Proof. The functor $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X))$ is the inclusion. The functor $\mathbf{Spec}^0_{\mathfrak{Se}}(X) \longrightarrow \mathfrak{Spec}^0(\mathfrak{Se}(X))$ assigns to each object $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{Se}}(X)$ the Serre subcategory $\mathcal{P}^-$. ■

2.5. Spectra defined by closed cosubspaces.

Let $\mathfrak{CT}(X)$ denote the preorder of all coreflective topologizing subcategories of the category C_X. Let $\mathfrak{J}^{\mathfrak{ct}}_X$ denote the inclusion functor $\mathfrak{CTh}(X) \longrightarrow \mathfrak{CT}(X)$.

Thus, we have two spectra associated to this functor,

$$\mathbf{Spec}^i_{\mathfrak{c}}(X) = \mathfrak{Spec}^i(\mathfrak{Th}(X), \mathfrak{J}^{\mathfrak{ct}}_X), \quad i = 0, 1.$$

and a canonical morphism from one to another, $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{c}}(X)$.

If the category C_X has the property (sup), then the preorder $\mathfrak{CTh}(X)$ of coreflective thick subcategories coincides with the preorder $\mathfrak{Se}(X)$ of Serre subcategories.

2.5.1. Proposition. *Suppose that the category C_X has the property (sup).*
(a) There is a natural map $\mathbf{Spec}^1_{\mathfrak{c}}(X) \longrightarrow \mathfrak{Spec}^1(\mathfrak{Se}(X)$.
(b) The functor

$$\mathfrak{CT}(X) \longrightarrow \mathfrak{Se}(X), \ \mathbb{T} \longmapsto \mathbb{T}^-,$$

(see 2.4) induces a functor $\mathbf{Spec}^0_{\mathfrak{c}}(X) \longrightarrow \mathfrak{Spec}^0(\mathfrak{Se}(X)$.

Proof. The argument is similar to that of 2.3.5.1. Details are left to the reader. ■

2.5.2. The spectrum $Spec^0_{\mathfrak{c}}(X)$. We denote by $Spec^0_{\mathfrak{c}}(X)$ the full subcategory of the category $C_{\mathfrak{M}(X)}$ generated by nonzero objects M such that $M \in Ob[N]_{\mathfrak{c}}$ if there exists a nonzero morphism $N \longrightarrow M$. Here $[N]_{\mathfrak{c}}$ denotes the smallest coreflective topologizing subcategory of C_X containing the object N.

Since $[N]_{\mathfrak{c}} \subseteq [N]^-$ for every object N, it follows that $Spec^0_{\mathfrak{c}}(X) \subseteq Spec^0_{\mathfrak{s}}(X)$. In particular, $Spec(X) \subseteq Spec^0_{\mathfrak{Se}}(X)$.

2.5.2.1. Remarks. (a) It is easy to show that a nonzero object M belongs to $Spec^0_{\mathfrak{c}}(X)$ iff $[L]_{\mathfrak{c}} = [M]_{\mathfrak{c}}$ for every nonzero subobject L of M. In other words, the functor

$$C_{\mathfrak{M}(X)} \longrightarrow \mathfrak{CT}(X), \quad M \longmapsto [M]_{\mathfrak{c}},$$

maps every nonzero (mono)morphism $L \longrightarrow M$ to the (identical) isomorphism.

(b) Suppose the category C_X has infinite coproducts. Then one can show that, for any object $M \in ObC_X$, objects of the subcategory $[M]_{\mathfrak{c}}$ are subquotients of a coproduct of a set of copies of M, while objects of the subcategory $[M]$ are subquotients of a coproduct of a *finite* set of copies of M.

Thus, a nonzero object M belongs to $Spec^0_{\mathfrak{c}}(X)$ iff M is a subquotient of a coproduct of a set of copies of any of its nonzero subobjects. And a nonzero object M' belongs to $Spec(X)$ iff M' is a subquotient of a coproduct of a finite set of copies of any of its nonzero subobjects.

2.5.3. Proposition. *Suppose that the category C_X has the property (sup). Then the map*

$$Spec^0_{\mathfrak{c}}(X) \longrightarrow \mathbf{Spec}^0_{\mathfrak{c}}(X), \quad M \longmapsto [M]_{\mathfrak{c}}, \tag{1}$$

is surjective.

Proof. Let $\mathcal{P}$ is an object of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$; i.e. $\mathcal{P}$ is a coreflective topologizing subcategory such that the union, $\widehat{\mathcal{P}}^{\mathfrak{ct}}$, of all coreflective topologizing subcategories of C_X which do not contain $\mathcal{P}$ is a Serre subcategory. Let $M \in Ob\mathcal{P} - Ob\widehat{\mathcal{P}}^{\mathfrak{ct}}$. Since $\widehat{\mathcal{P}}^{\mathfrak{ct}}$ is coreflective, we can and will assume M that M is $\widehat{\mathcal{P}}^{\mathfrak{ct}}$-torsion free. Since $\widehat{\mathcal{P}}^{\mathfrak{ct}}$ is the union, $\langle M\rangle_{\mathfrak{c}}$, of all topologizing subcategories of C_X which do not contain M, it follows that M is an object of $Spec^0_{\mathfrak{c}}(X)$ such that $[M]_{\mathfrak{c}} = \mathcal{P}$. ■

Recall that an object M of a category C_X is *of finite type* if the functor $C_X(M,-)$ preserves colimits of filtered systems of monomorphisms. If the category C_X has the property (sup), then M is of finite type iff the following condition holds: if M is the supremum of a family, $\mathfrak{F}$, of its subobjects, then M is the supremum of a finite subfamily of $\mathfrak{F}$. If C_X is the category of modules over some associative ring, then its objects of finite type are finitely generated modules.

2.5.4. Proposition. *Suppose that the category C_X has the property (sup) and every nonzero object of C_X has a nonzero subobject of finite type. Then* $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$.

Proof. The inclusion $\mathbf{Spec}(X) \subseteq \mathbf{Spec}^0_{\mathfrak{c}}(X)$ holds without any additional hypothesis. The inverse inclusion is a consequence of the following observations.

(a) Thanks to the property (sup), the smallest coreflective subcategory spanned by a topologizing subcategory, $\mathbb{T}$, is generated by objects which are supremums of objects of $\mathbb{T}$. In particular, for any object N of the category C_X, objects of the subcategory $[N]_{\mathfrak{c}}$ are supremums (of a filtered family) of their subobjects which belong to $[N]$. This implies that every object of finite type of the category $[N]_{\mathfrak{c}}$ belongs to $[N]$.

(b) Let $\mathcal{P}$ be an object of $\mathbf{Spec}^0_{\mathfrak{c}}(X)$. By 2.5.3, $\mathcal{P} = [M]_{\mathfrak{c}}$ for some object M of $Spec^0_{\mathfrak{c}}(X)$. Suppose that M is of finite type. Then M belongs to $Spec(X)$.

In fact, M belongs to the subcategory $[N]_{\mathfrak{c}}$ for any nonzero subobject N of M. By (a), since M is of finite type, it belongs to $[N]$. This means that M is an object of $Spec(X)$.

(c) Since $[M]_{\mathfrak{c}} = [L]_{\mathfrak{c}}$ for any nonzero subobject, L, of M, and, by hypothesis, M has a nonzero subobject of finite type, we can choose M to be of finite type. ■

2.5.4.1. Corollary. *If C_X is the category of left (or right) modules over an associative unital ring, then* $\mathbf{Spec}^0_{\mathfrak{c}}(X) = \mathbf{Spec}(X)$.

2.6. Spectra of 'spaces' represented by triangulated categories.

The reader is referred to [Ve1] and [Ve2] for background on triangulated categories. Here we only fix notations. Recall that a *$\mathbb{Z}$-category* is a pair (C_X, γ), where C_X is a category and γ is an action of $\mathbb{Z}$ on C_X; i.e. γ is a monoidal functor from $\mathbb{Z}$ to the monoidal category $\mathfrak{End}(C_X)$ of functors $C_X \longrightarrow C_X$. Here $\mathbb{Z}$ is regarded as a discrete category with the monoidal structure given by $m \odot n = m + n$.

A triangulated category is a triple $(C_X, \gamma; \mathfrak{D})$, where (C_X, γ) is an additive $\mathbb{Z}$-category, and $\mathfrak{D}$ a class of 'distinguished' triangles having certain properties (Tr1) – (Tr4) formulated by Verdier (cf. [Ve1] or [Ve2, Ch.2]). We shall denote a triangulated category $(C_X, \gamma; \mathfrak{D})$ by $\mathcal{CT}_{\mathfrak{X}}$ and regard it as a triangulated category representing a *triangulated 'space'* $\mathfrak{X}$.

2.6.1. The spectra of exact localizations. Let $\mathcal{CT}_{\mathfrak{X}} = (C_X, \gamma; \mathfrak{D})$ be a triangulated category. Let $\mathfrak{Th}(\mathfrak{X})$ denote the preorder of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$. Following the standard procedure, we associate with the preorder $\mathfrak{Th}(\mathfrak{X})$ two spectra of the 'space' $\mathfrak{X}$ represented by the triangulated category $\mathcal{CT}_{\mathfrak{X}}$

$$\mathbf{Spec}^i_{\mathfrak{Th}}(\mathfrak{X}) = \mathfrak{Spec}^i(\mathfrak{Th}(\mathfrak{X})), \quad i = 0, 1.$$

and a canonical morphism from one to another,

$$\mathbf{Spec}^0_{\mathfrak{Th}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Th}}(\mathfrak{X}) \tag{1}$$

which assigns to every object $\mathcal{P}$ of $\mathbf{Spec}^0_{\mathfrak{Th}}(\mathfrak{X})$ the union, $\widehat{\mathcal{P}}^{\mathfrak{tr}}$, of all thick triangulated subcategories of $\mathcal{CT}_{\mathfrak{X}}$ which do not contain the subcategory $\mathcal{P}$.

2.6.2. Flat spectra. Let $\mathfrak{Se}(\mathfrak{X})$ denote the family of all thick triangulated categories $\mathbb{T}$ of C_X such that the localization functor $C_X \xrightarrow{q^*_{\mathbb{T}}} C_X/\mathbb{T}$ has a right adjoint, $q_{\mathbb{T}*}$.

The *flat* spectra of $\mathfrak{X}$ are relative spectra

$$\mathbf{Spec}^i_{\mathfrak{fL}}(\mathfrak{X}) = \mathfrak{Spec}^i(\mathfrak{Se}(\mathfrak{X}), \mathfrak{J}_{\mathfrak{S}}), \quad i = 0, 1,$$

corresponding to the inclusion functor $\mathfrak{Se}(\mathfrak{X}) \xrightarrow{\mathfrak{J}_{\mathfrak{S}}} \mathfrak{Tht}(\mathfrak{X})$. The morphism (1) induces a canonical morphism

$$\mathbf{Spec}^i_{\mathfrak{fL}}(\mathfrak{X}) \longrightarrow \mathbf{Spec}^i_{\mathfrak{fL}}(\mathfrak{X}). \tag{2}$$

Let $\mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$ denote the full subpreorder of $\mathfrak{Tht}(\mathfrak{X})$ whose objects are thick triangulated subcategories $\mathcal{Q}$ such that ${}^\perp\mathcal{Q}$ belongs to $\mathbf{Spec}^1_{\mathfrak{fL}}(\mathfrak{X})$ and every thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp\mathcal{Q}$ contains $\mathcal{Q}$; i.e. ${}^\perp\mathcal{Q} \vee \mathcal{Q}$ is the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp\mathcal{Q}$. Here ${}^\perp\mathcal{Q}$ is the *left orthogonal to* $\mathcal{Q}$, i.e. the full subcategory of $\mathcal{CT}_{\mathfrak{X}}$ generated by all objects L such that $\mathcal{CT}_{\mathfrak{X}}(L, M) = 0$ for every $M \in Ob\mathcal{Q}$.

2.6.2.1. Proposition. *(a) The map*

$$\mathfrak{Tht}(\mathfrak{X}) \longrightarrow \mathfrak{Tht}(\mathfrak{X}), \quad \mathcal{Q} \longmapsto {}^\perp\mathcal{Q},$$

induces an isomorphism

$$\mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X}) \xrightarrow{\sim} \mathbf{Spec}^1_{\mathfrak{fL}}(\mathfrak{X}). \tag{3}$$

(b) $\mathbf{Spec}^0_{\mathfrak{fL}}(\mathfrak{X}) = \mathbf{Spec}^0_{\mathfrak{Tht}}(\mathfrak{X}) \bigcap \mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$. *The canonical morphism (2) is the composition of the inclusion* $\mathbf{Spec}^0_{\mathfrak{fL}}(\mathfrak{X}) \hookrightarrow \mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$ *and the isomorphism (3).*

Proof. See [R4, 12.7.1]. ∎

For any object M of the category $\mathcal{CT}_{\mathfrak{X}}$, let $[M]_{\mathfrak{t}}$ denote the smallest thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ containing M, and let ${}^\perp M$ be the left orthogonal to $[M]_{\mathfrak{t}}$.

2.6.2.2. Proposition. *(a) If $\mathcal{Q}$ is an element of* $\mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$, *then $\mathcal{Q} = [M]_{\mathfrak{t}}$ (hence ${}^\perp\mathcal{Q} = {}^\perp M$) for any nonzero object M of $\mathcal{Q}$.*

(b) The following properties of a nonzero object M of the triangulated category $\mathcal{CT}_{\mathfrak{X}}$ are equivalent:

(i) The thick envelope, $[M]_{\mathfrak{t}}$, of M belongs to $\mathbf{Spec}^{1/2}_{\mathfrak{fL}}(\mathfrak{X})$.

(ii) The inclusion functor ${}^\perp M \longrightarrow \mathcal{CT}_{\mathfrak{X}}$ has a right adjoint and M belongs to any thick triangulated subcategory of $\mathcal{CT}_{\mathfrak{X}}$ properly containing ${}^\perp M$.

(iii) $[M]_{\mathfrak{t}}$ is a minimal nonzero thick subcategory, and ${}^\perp M$ belongs to $\mathbf{Spec}^1_{\mathfrak{fL}}(\mathfrak{X})$.

(iv) M belongs to every nonzero thick triangulated subcategory of $({}^\perp M)^\perp$ and such that the inclusion functor ${}^\perp M \longrightarrow \mathcal{CT}_{\mathfrak{X}}$ has a right adjoint.

Proof. See [R4, 12.7.2]. ∎

3. The left spectra.

There are 'spaces' with only trivial saturated multiplicative systems. They might be called *simple* in the same sense as a ring with only trivial two-sided ideals is called simple. If X is such a 'space', then $\mathbf{Spec}^1_{\mathfrak{L}}(X) = \{Iso(C_X)\}$ and $\mathbf{Spec}^0_{\mathfrak{L}}(X) = \{HomC_X\}$. It follows that all other spectra introduced above (closed, flat, etc.) are one-element sets too. Some of simple 'spaces' have quite meaningful left spectra. The latter are associated with the preorder of saturated left multiplicative systems.

A fundamental example of a simple 'space' is the 'space' represented by the category $Sets = Sets_{\mathfrak{U}}$ of sets which belong to a given universe $\mathfrak{U}$.

3.1. Basic left spectra. Let X be a 'space'. We take as $\mathfrak{H}$ the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ of saturated left multiplicative systems on C_X and set

$$\mathbf{Spec}^i_{\mathfrak{L},\ell}(X) = \mathfrak{Spec}^i(\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)), \quad i = 0, 1.$$

By 1.4.6, there exists a canonical injective map

$$\mathbf{Spec}^0_{\mathfrak{L},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{L},\ell}(X), \quad \Sigma \longmapsto \widehat{\Sigma}.$$

3.1.1. The spectrum $\mathbf{Spec}^1_{\mathfrak{L},\ell}(X)$ and left local quotient 'spaces'. We call a 'space' Y *left local* if the preorder $(\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(Y), \subseteq)$ is local, i.e. there is the smallest non-trivial saturated left multiplicative system on C_Y. It follows that $\mathbf{Spec}^1_{\mathfrak{L},\ell}(X)$ is formed by all $\Sigma \in \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ such that the quotient 'space' $\Sigma^{-1}X$ is left local.

3.2. Closed left spectra. Let $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$ denote the preorder of all closed saturated left multiplicative systems on C_X (cf. 2.0.5). They give rise to the spectra

$$\mathbf{Spec}^i_{\mathfrak{f},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)), \quad i = 0, 1.$$

and the relative spectra

$$\mathbf{Spec}^i_{\mathfrak{C},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X), \mathfrak{J}_{\mathfrak{C},\ell}), \quad i = 0, 1,$$

where $\mathfrak{J}_{\mathfrak{C},\ell}$ denotes the embedding $\mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X) \hookrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$.

They are related by canonical injective maps

$$\mathbf{Spec}^0_{\mathfrak{f},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{f},\ell}(X),$$

$$\mathbf{Spec}^0_{\mathfrak{C},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{C},\ell}(X)$$

(see 1.4.6 and 1.6.2).

3.3. Continuous left multiplicative systems and continuous left spectra. A left multiplicative system Σ in C_X is called *continuous* if the corresponding localization functor $C_X \longrightarrow \Sigma^{-1}C_X$ has a right adjoint.

Let $\mathfrak{Lc}_\ell(X)$ denote the preorder of continuous saturated left multiplicative systems. By [R4, 5.2.1, 5.2.2], every continuous saturated left multiplicative system is closed, i.e. $\mathfrak{Lc}_\ell(X) \subseteq \mathfrak{C}\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$.

Let $\mathfrak{J}_{\mathfrak{c},\ell}$ denote the embedding $\mathfrak{Lc}_\ell(X) \hookrightarrow \mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(X)$. This data provides us with the *continuous left spectra*

$$\mathbf{Spec}^i_{\mathfrak{Fl},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{Lc}_\ell(X)), \quad i = 0, 1.$$

and the *relative continuous left spectra*

$$\mathbf{Spec}^i_{\mathfrak{fL},\ell}(X) = \mathfrak{Spec}^i(\mathfrak{Lc}_\ell(X), \mathfrak{J}_{\mathfrak{c},\ell}), \quad i = 0, 1,$$

together with the canonical injective maps

$$\mathbf{Spec}^0_{\mathfrak{Fl},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{Fl},\ell}(X),$$

$$\mathbf{Spec}^0_{\mathfrak{fL},\ell}(X) \longrightarrow \mathbf{Spec}^1_{\mathfrak{fL},\ell}(X).$$

3.3.1. Another realization of continuous localizations and continuous left spectra. Fix a 'space' X. Consider the preorder $\mathfrak{fL}_\ell(X)$ of all strictly full subcategories C_Y of C_X such that the inclusion functor $C_Y \xrightarrow{\iota_{Y*}} C_X$ has a left adjoint $C_X \xrightarrow{\iota_Y^*} C_Y$. These functors are regarded as resp. direct and inverse image functors of a *continuous strictly full embedding* $Y \overset{\iota_Y}{\hookrightarrow} X$. The map which assigns to every such subcategory the family of arrows $\Sigma_{\iota_Y^*} = {\iota_Y^*}^{-1}(Iso(C_Y))$ is an isomorphism of the preorder $(\mathfrak{fL}_\ell(X), \supseteq)$ onto the preorder $(\mathfrak{Lc}_\ell(X), \subseteq)$ of continuous saturated left multiplicative systems.

Thus, the continuous left spectrum $\mathbf{Spec}^1_{\mathfrak{fL}}(X)$ can be identified with the preorder of all continuous strictly full embeddings $Y \overset{\iota_Y}{\hookrightarrow} X$ such that Y is a left local 'space'.

Let $\mathfrak{fL}^\star_\ell(X)$ denote the set $\mathfrak{fL}(X) - \{id_X\}$ of all proper continuous strictly full embeddings. Thanks to the isomorphism $(\mathfrak{Lc}(X), \subseteq) \xrightarrow{\sim} (\mathfrak{fL}(X), \supseteq)$ elements of $\mathbf{Spec}^1_{\mathfrak{Fl},\ell}(X)$ can be identified with continuous strictly full embeddings $Y \overset{\iota_Y}{\hookrightarrow} X$ such that the preorder $(\mathfrak{fL}^\star(Y), \subseteq)$ of proper strictly full continuous embeddings into Y has the biggest element.

3.4. The left spectra of Sets. Let $\mathcal{E}$ denote the 'space' represented by the category $Sets = Sets_{\mathfrak{U}}$ of sets which belong to a fixed universe $\mathfrak{U}$, i.e. $C_{\mathcal{E}} = Sets$.

The preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(\mathcal{E})$ of right saturated multiplicative systems on *Sets* is trivial: it consists only of $Iso(C_{\mathcal{E}})$ and $HomC_{\mathcal{E}}$. In particular, the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(\mathcal{E})$ of saturated multiplicative systems on *Sets* is trivial.

For an infinite cardinal number α, let Σ_α denote the family of all maps $M \xrightarrow{f} N$ (– morphisms of $C_{\mathcal{E}}$) such that

(a) $M \neq \emptyset$ if $N \neq \emptyset$,

(b) $Card(N - f(M)) < \alpha$,

(c) There exists a subset M' of M such that $Card(M - M') < \alpha$ and the restriction of the map f to M' is injective.

Let $\Sigma_{\alpha *}$ denote the family of all maps $M \xrightarrow{f} N$ satisfying (b) and (c) only. So that $\Sigma_\alpha \subset \Sigma_{\alpha *}$. Explicitly, $\Sigma_{\alpha *} = \Sigma_\alpha \bigcup \{\emptyset \longrightarrow N \mid Card(N) < \alpha\}$.

Both Σ_α and $\Sigma_{\alpha *}$ are saturated left multiplicative systems on $Sets$. Moreover, every saturated left multiplicative system on $Sets$ is either Σ_α or $\Sigma_{\alpha *}$ for a suitable infinite cardinal α (see [GZ], I.2.5f) and I.3.5).

Let $\mathfrak{Sp}(\mathfrak{U})$ denote the order of non-limit cardinals which belong to the universe $\mathfrak{U}$.

3.4.1. Proposition. *(a) Let α be an infinite cardinal number. Then the following conditions are equivalent:*

(i) α is a non-limit cardinal,

(ii) Σ_α belongs to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(b) The map $\alpha \longmapsto \Sigma_\alpha$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\varrho^0} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(c) If α is a non-limit infinite cardinal, then $\Sigma_{\alpha-1*}$ belongs to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$. The map $\alpha \longmapsto \Sigma_{\alpha-1*}$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\varrho^1} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ such that the diagram

$$\begin{array}{ccc}
\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}) & \xrightarrow{\theta_\ell(\mathcal{E})} & \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}) \\
\varrho^0 \nwarrow & & \nearrow \varrho^1 \\
& \mathfrak{Sp}(\mathfrak{U}) &
\end{array} \qquad (1)$$

commutes. In particular, the canonical preorder morphism $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}) \xrightarrow{\theta_\ell(\mathcal{E})} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ is an isomorphism.

Proof. (a) Fix an infinite cardinal number α.

(i)⇒(ii). Let α be a non-limit cardinal. Then the union, $\widehat{\Sigma}_\alpha$ of all elements of the support of Σ_α (that is the union of all saturated left multiplicative system on $C_{\mathcal{E}}$ which do not contain Σ_α) coincides with $\Sigma_{\alpha-1*}$. This shows that Σ_α is an element of $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(ii)⇒(i). If α is a limit cardinal, then $\widehat{\Sigma}_\alpha = \bigcup\limits_{\beta<\alpha} \Sigma_{\beta *} \supseteq \bigcup\limits_{\beta<\alpha} \Sigma_\beta = \Sigma_\alpha$. This shows that the support, $\mathfrak{Supp}(\Sigma_\alpha)$, of Σ_α does not have the final object, i.e. the left multiplicative system Σ_α does not belong to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(b) Let α be any cardinal of a set from $\mathfrak{U}$. The support of $\Sigma_{\alpha *}$ consists of all Σ_β and all $\Sigma_{\gamma *}$ with $\gamma < \alpha$. Thus $\widehat{\Sigma}_{\alpha *} = HomC_{\mathcal{E}^1} \bigcup \{\emptyset \longrightarrow N \mid Card(N) < \gamma,\ \gamma < \alpha\}$, where $C_{\mathcal{E}^1}$ is the full subcategory of $C_{\mathcal{E}}$ formed by all non-empty sets. It follows from this description that $\widehat{\Sigma}_{\alpha *}$ is not closed under composition, hence the support of $\Sigma_{\alpha *}$ does not have the final object, i.e. the left multiplicative system $\Sigma_{\alpha *}$ does not belong to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$. Since every saturated left multiplicative system on $Sets$ is either Σ_α, or $\Sigma_{\alpha *}$, this proves that the map $\alpha \longmapsto \Sigma_\alpha$ is an isomorphism $\mathfrak{Sp}(\mathfrak{U}) \longrightarrow \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E})$.

(c) Let α be an infinite non-limit cardinal number. That $\Sigma_{\alpha-1*}$ is an element of $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ follows, together with the commutativity of the diagram (1), from the fact that $\Sigma_{\alpha-1*}$ is the final object of the support of Σ_α (see the argument (i)⇒(ii) above) and 1.4.6. Clearly the map

$$\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\varrho^1} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}), \quad \alpha \longmapsto \Sigma_{\alpha-1*}, \qquad (2)$$

is a morphism of preorders. It remains to show that this map is bijective.

In fact, for any pair of infinite cardinal numbers β and γ such that $\beta < \gamma$, the system Σ_β is contained properly in Σ_γ and in $\Sigma_{\beta*}$. On the other hand, $\Sigma_\beta = \Sigma_\gamma \bigcap \Sigma_{\beta*}$. Therefore Σ_β does not belong to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ for any infinite cardinal number β. Therefore elements of $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$ are systems $\Sigma_{\beta*}$ for some β. Suppose $\Sigma_{\beta*}$ belongs to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E})$, i.e. there exists the smallest system $\Sigma^\star_{\beta*}$ in $\mathcal{S}^\ell\mathcal{M}_{\mathfrak{r}}(\mathcal{E})$ properly containing $\Sigma_{\beta*}$. Since $\Sigma_{\beta*}$ is not contained in Σ_σ for any σ, the system $\Sigma^\star_{\beta*}$ should coincide with $\Sigma_{\alpha*}$ for some α. But, $\Sigma_{\beta*} \subsetneq \Sigma_{\gamma*}$ iff $\beta \lneq \gamma$. Therefore, $\beta \lneq \alpha$ and there are no intermediate cardinal numbers, i.e. α is a non-limit cardinal number and $\beta = \alpha - 1$. ∎

3.4.2. Other spectra. Let $\Sigma \subseteq HomC_{\mathcal{E}}$. By definition, and object M of $C_{\mathcal{E}}$ is Σ-torsion free if every morphism $M \longrightarrow M'$ which belongs to Σ is a monomorphism. Suppose Σ is Σ_α or $\Sigma_{\alpha*}$ for some infinite cardinal number α. Then a set M is Σ-torsion free iff $Card(M) \leq 1$ (that is either $M = \emptyset$, or M is a one-element set). It follows from the definitions of Σ_α and $\Sigma_{\alpha*}$ that objects of $C_{\mathcal{E}}$ having a morphism to a Σ-torsion free object are precisely sets N such that $Card(N) < \alpha$. This shows that the only closed saturated left multiplicative systems on $C_{\mathcal{E}}$ are $Iso(C_{\mathcal{E}})$ and $HomC_{\mathcal{E}}$. Since, by [R4, 5.2.2], every continuous saturated left multiplicative system is closed, there are no non-trivial continuous saturated left multiplicative systems either. Therefore, $\mathbf{Spec}^0_{\mathfrak{f},\ell} = \mathbf{Spec}^0_{\mathfrak{Fl},\ell} = \{HomC_{\mathcal{E}}\}$ and $\mathbf{Spec}^1_{\mathfrak{f},\ell} = \mathbf{Spec}^1_{\mathfrak{Fl},\ell} = \{Iso(C_{\mathcal{E}})\}$ (see notations in 3.2 and 3.3).

The relative continuous and closed left spectra (cf. 3.2, 3.3) are empty.

3.4.3. Sets without emptyset. Let $C_{\mathcal{E}^1} = Sets^1_{\mathfrak{U}}$, where (in accordance with notations in Section 1), $Sets^1_{\mathfrak{U}}$ is the full subcategory of $Sets_{\mathfrak{U}}$ formed by non-empty sets which belong to the universe $\mathfrak{U}$. It follows that the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\mathfrak{r}}(\mathcal{E}^1)$ of saturated right multiplicative systems on $Sets^1_{\mathfrak{U}}$ is trivial and the set $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_\ell(\mathcal{E}^1)$ of saturated left multiplicative systems consists of all systems $\Sigma^1_\alpha = \Sigma_\alpha \bigcap HomC_{\mathcal{E}^1}$, where α runs through infinite cardinal numbers (notice that $\Sigma_{\alpha*} \bigcap HomC_{\mathcal{E}^1} = \Sigma^1_\alpha$ for any α). There is a following analogue of 3.4.1:

3.4.4. Proposition. *(a) Let α be an infinite cardinal number. Then the following conditions are equivalent:*

(i) α is a non-limit cardinal,

(ii) Σ^1_α belongs to $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1)$.

(b) The map $\alpha \longmapsto \Sigma^1_\alpha$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\nu^0} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1)$.

(c) If α is a non-limit infinite cardinal, then $\Sigma^1_{\alpha-1}$ belongs to $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$. The map $\alpha \longmapsto \Sigma^1_{\alpha-1}$ defines an isomorphism of preorders $\mathfrak{Sp}(\mathfrak{U}) \xrightarrow{\nu^1} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ such that the diagram

$$\begin{tikzcd} \mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1) \arrow[rr, "\theta_\ell(\mathcal{E}^1)"] & & \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1) \\ & \mathfrak{Sp}(\mathfrak{U}) \arrow[ul, "\nu^0"] \arrow[ur, "\nu^1"'] & \end{tikzcd} \qquad (1)$$

commutes. In particular, the canonical morphism $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1) \xrightarrow{\theta_\ell(\mathcal{E}^1)} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ is an isomorphism.

Proof. The assertion follows from 3.4.1. Details are left to the reader. ■

The canonical map $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1) \xrightarrow{\theta_\ell(\mathcal{E}^1)} \mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ (of 1.4.6) assigns to each element Σ^1_α of $\mathbf{Spec}^0_{\mathfrak{L},\ell}(\mathcal{E}^1)$ the system $\Sigma^1_{\alpha-1}$. Notice that the inverse map assigns to each element Σ of $\mathbf{Spec}^1_{\mathfrak{L},\ell}(\mathcal{E}^1)$ the smallest saturated left multiplicative system properly containing Σ.

3.5. Example: finite sets. Let $C_{\mathcal{E}_\mathfrak{f}}$ be the category $Set_\mathfrak{f}$ of finite sets. There are no non-trivial left or right multiplicative systems on $Sets_\mathfrak{f}$; so that the 'space' $\mathcal{E}_\mathfrak{f}$ can be viewed as an analog of a 'point' – the spectrum of a field.

4. Left exact multiplicative systems and injective spectra.

4.1. Left exact multiplicative systems. Fix a 'space' X. We call a saturated left multiplicative system Σ *left exact* if the localization functor $C_X \xrightarrow{q^*_\Sigma} \Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ maps strict monomorphisms to strict monomorphisms. Let $\mathcal{S}^\mathfrak{s}_{\ell s}\mathcal{M}(X)$ denote the preorder of all left exact multiplicative systems.

If C_X is an abelian category, then every left exact multiplicative system is a right multiplicative system, i.e. $\mathcal{S}^\mathfrak{s}_{\ell s}\mathcal{M}(X) = \mathcal{S}^\mathfrak{s}\mathcal{M}(X)$.

4.2. Proposition. *Suppose that the category C_X has finite colimits and kernels of pairs of arrows. Let $C_X \xrightarrow{f^*} C_Y$ be a right exact functor (i.e. it preserves finite colimits) which maps strict monomorphisms to strict monomorphisms, and let $\Sigma = \Sigma_{f^*} = \{s \in HomC_X \mid f^*(s) \in Iso(C_X)\}$. Then both functors, p^*_f and q^*_f, in the canonical decomposition $f^* = p^*_f q^*_f$ (here q^*_f is the localization functor $C_X \longrightarrow \Sigma^{-1}C_X$ and p^*_f is a conservative functor) are right exact and map strict monomorphisms to strict monomorphisms. In particular, $\Sigma \in \mathcal{S}^\mathfrak{s}_{\ell s}\mathcal{M}(X)$.*

Proof. (a) By [GZ, I.3.4], the functors q^*_f and p^*_f are right exact and $\Sigma = \Sigma_{f^*}$ belongs to $\mathcal{S}^\mathfrak{s}_\ell\mathcal{M}(X)$. It remains to show that q^*_f and p^*_f map strict monomorphisms to strict monomorphisms.

(b) Let $L \xrightarrow{\mathfrak{j}} M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M_1 = M \coprod_L M$ be an exact diagram. We claim that its image by the localization functor q^*_f is exact too.

In fact, let a morphism $q^*_f(N) \xrightarrow{g'} q^*_f(M)$ equalize the pair $q^*_f\big(M \underset{p_2}{\overset{p_1}{\rightrightarrows}} M_1\big)$. Since Σ is a left multiplicative system, there exist arrows $N \xrightarrow{g} M' \xleftarrow{s} M$ such that $s \in \Sigma$ and $g' = q^*_f(s)^{-1}q^*_f(g)$, and there exist commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{p_i} & M_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s_i} \\ M' & \xrightarrow{p'_i} & N_i \end{array} \qquad i = 1, 2,$$

with $s_1, s_2 \in \Sigma$. By the same reason, there exists a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{s_1} & N_1 \\ {\scriptstyle s_2}\big\downarrow & & \big\downarrow{\scriptstyle s_2'} \\ N_2 & \xrightarrow{s_1'} & N_1' \end{array}$$

with $s_2' \in \Sigma$. Since the system Σ is saturated and the arrows s_1, s_2, s_2' belong to Σ, the remaining arrow, s_1', belongs to Σ too. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\mathfrak{j}} & M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M_1 \\ & & {\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ N & \xrightarrow{g} & M' & \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} & N_1' \end{array} \tag{1}$$

where s and $t = s_2' s_1$ are arrows from Σ and $\phi_i = s_i' p_i'$. It follows from the definition of $N \xrightarrow{g} M'$ and the commutativity of the diagram (1) that $q_f^*(\phi_1 g) = q_f^*(\phi_2 g)$. Since Σ is saturated, this means precisely that there exists an arrow $N_1' \xrightarrow{u} M_1''$ in Σ such that $(u\phi_1)g = (u\phi_2)g$. Set $\phi_i' = u\phi_i$, $i = 1, 2$, and let $L' \xrightarrow{\mathfrak{j}'} M'$ denote the kernel of the pair of arrows $M' \overset{\phi_1'}{\underset{\phi_2'}{\rightrightarrows}} M_1''$. Then we have the diagram $L' \xrightarrow{\mathfrak{j}'} M' \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} M_1' \xrightarrow{\sigma} M_1''$, where $M_1' = M' \coprod_{L'} M'$, π_1, π_2 are coprojections, and σ is a morphism uniquely determined by the equalities $\sigma\pi_i = \phi_i'$, $i = 1, 2$. Combining these decompositions with (1), we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & L & \xrightarrow{\mathfrak{j}} & M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M_1 & \xrightarrow{t} & N_1' \\ & & {\scriptstyle s'}\big\downarrow & & {\scriptstyle s}\big\downarrow & & & & \big\downarrow{\scriptstyle u} \\ N & \xrightarrow{g'} & L' & \xrightarrow{\mathfrak{j}'} & M' & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & M_1' & \xrightarrow{\sigma} & M_1'' \end{array} \tag{2}$$

Here $N \xrightarrow{g'} L' \xrightarrow{\mathfrak{j}'} M'$ is the unique decomposition of the morphism $N \xrightarrow{g} M'$ and the arrow $L \xrightarrow{s'} L'$ is uniquely determined by the commutativity of the diagram (2) and the fact that $L' \xrightarrow{\mathfrak{j}'} M'$ is the kernel of the pair of arrows $M' \overset{\sigma\pi_1}{\underset{\sigma\pi_2}{\rightrightarrows}} M_1''$. Applying the localization functor q_f^* to (2) and using that arrows s, t, u, σ belong to Σ, we obtain a commutative diagram

$$\begin{array}{ccccccc} & & q_f^*(L) & \xrightarrow{q_f^*(\mathfrak{j})} & q_f^*(M) & \rightrightarrows & q_f^*(M_1) \\ & & {\scriptstyle q_f^*(s')}\big\downarrow & & \wr\big\downarrow & & \big\downarrow\wr \\ q_f^*(N) & \xrightarrow{q_f^*(g')} & q_f^*(L') & \xrightarrow{q_f^*(\mathfrak{j}')} & q_f^*(M') & \rightrightarrows & q_f^*(M_1') \end{array} \tag{3}$$

Since two of the three vertical arrows in (3) are isomorphisms and the diagrams

$$q_f^*(L) \xrightarrow{q_f^*(\mathfrak{j})} q_f^*(M) \rightrightarrows q_f^*(M_1)$$

and

$$q_f^*(L') \xrightarrow{q_f^*(\mathfrak{j}')} q_f^*(M') \rightrightarrows q_f^*(M_1')$$

are exact, the third one, $q_f^*(s')$, is an isomorphism, or, equivalently, $s' \in \Sigma$. This shows that any morphism $q_f^*(N) \xrightarrow{g'} q_f^*(M)$ which equalizes the pair $q_f^*\big(M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1\big)$ factors uniquely through the kernel of this pair. Notice that our argument shows the existence of this kernel.

(c) Let $q_f^*(L) \xrightarrow{\xi'} q_f^*(M')$ be a morphism in $\Sigma^{-1}C_X$ and $L \xrightarrow{\xi} M \xleftarrow{s} M'$ morphisms such that $s \in \Sigma$ and $\xi' = q_f^*(s)^{-1}q_f^*(\xi)$ (cf. (b) above). Consider the cokernel diagram

$$L \xrightarrow{\ \xi\ } M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1 = M \coprod_L M$$

of the morphism $L \xrightarrow{\xi} M$. Since the localization functor q_f^* preserves finite colimits, the diagram

$$q_f^*\big(L \xrightarrow{\ \xi\ } M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1 = M \coprod_L M\big)$$

is isomorphic to the cokernel diagram of the morphism $q_f^*(L) \xrightarrow{\xi'} q_f^*(M')$. Let $L' \xrightarrow{\mathfrak{j}} M$ denote the kernel of the pair of arrows $M \overset{p_1}{\underset{p_2}{\rightrightarrows}} M_1$, and let $L \xrightarrow{t} L' \xrightarrow{\mathfrak{j}} M$ be the canonical decomposition of $L \xrightarrow{\xi} M$.

It follows from the construction that $q_f^*(L) \xrightarrow{\xi'} q_f^*(M')$ is a strict monomorphism iff $t \in \Sigma$, hence the morphism ξ' is isomorphic to $q_f^*(L') \xrightarrow{q_f^*(\mathfrak{j})} q_f^*(M)$. Therefore $p_f^*(\xi')$ is isomorphic to $p_f^*\big(q_f^*(L') \xrightarrow{q_f^*(\mathfrak{j})} q_f^*(M)\big) = f^*(L') \xrightarrow{f^*(\mathfrak{j})} f^*(M)$. By hypothesis, the functor f^* preserves strict monomorphisms, hence $p_f^*(\xi')$ is a strict monomorphism. ∎

The following assertion is suggested by (the part (b) of) the argument of 4.2.

4.2.1. Proposition. *Suppose the category C_X has kernels of pairs of arrows and for any arrow $L \longrightarrow M$, there exists a pushforward $M \coprod_L M$. Then the following conditions on a left saturated multiplicative system Σ are equivalent:*

(a) Σ is left exact.

(b) If in the commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\ \mathfrak{j}\ } & M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M_1 = M \coprod_L M \\ {\scriptstyle s'}\downarrow & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ L' & \xrightarrow{\ \mathfrak{j}'\ } & M' & \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} & M'' \end{array} \qquad (4)$$

the rows are exact and the vertical arrows s and t belong to Σ, then the left vertical arrow belongs to Σ too.

Proof. (a)$\Rightarrow$(b). The diagram (4) gives rise to the commutative diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\mathfrak{j}} & M & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & M_1 & \xrightarrow{t} & M'' \\ s'\big\downarrow & & s\big\downarrow & & \big\downarrow u & & \big\downarrow id_{M''} \\ L' & \xrightarrow{\mathfrak{j}'} & M' & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & M_1' & \xrightarrow{\sigma} & M'' \end{array} \tag{5}$$

where $M_1' = M'\coprod_{L'} M'$, $M_1 \xrightarrow{\sigma} M''$ is uniquely determined by $\sigma\pi_i = \phi_i,\ i = 1, 2$; and $M_1 \xrightarrow{u} M_1'$ is uniquely defined by the left square in (5) (due to the functoriality of coproducts). Applying the localization functor $C_X \xrightarrow{q^*} \Sigma^{-1}C_X$ to the diagram (5), we obtain the diagram

$$\begin{array}{ccccc} q^*(L) & \xrightarrow{\widetilde{\mathfrak{j}}} & q^*(M) & \overset{p_1'}{\underset{p_2'}{\rightrightarrows}} & q^*(M_1) \\ q^*(s')\big\downarrow & & q^*(s)\big\downarrow & & u'\big\downarrow\big\uparrow\sigma' \\ q^*(L') & \xrightarrow{\widetilde{\mathfrak{j}'}} & q^*(M') & \overset{\pi_1'}{\underset{\pi_2'}{\rightrightarrows}} & q^*(M_1') \end{array} \tag{6}$$

where $\sigma' = q^*(t)^{-1}q^*(\sigma)$. Since the system Σ is left exact, the functor q^* maps the diagrams $L \longrightarrow M \rightrightarrows M_1$ and $L' \longrightarrow M' \rightrightarrows M_1'$ to exact diagrams. Therefore the commutative square

$$\begin{array}{ccc} q^*(M) & \rightrightarrows & q^*(M_1) \\ q^*(s)^{-1}\big\uparrow & & \big\uparrow\sigma' \\ q^*(M) & \rightrightarrows & q^*(M_1) \end{array}$$

and (6) yield a commutative square

$$\begin{array}{ccc} q^*(L) & \xrightarrow{\widetilde{\mathfrak{j}}} & q^*(M) \\ \widetilde{\sigma}\big\uparrow & & \big\uparrow q^*(s)^{-1} \\ q^*(L') & \xrightarrow{\widetilde{\mathfrak{j}'}} & q^*(M') \end{array}$$

It follows from the universal property of kernels that $\widetilde{\sigma}q^*(s') = id_{q^*(L)}$, hence $\widetilde{\sigma}$ is a strict epimorphism. On the other hand, the equality $\widetilde{\mathfrak{j}}\widetilde{\sigma} = q^*(s)^{-1}\widetilde{\mathfrak{j}'}$ (together with the fact that $\widetilde{\mathfrak{j}'}$ is a monomorphism) implies that $\widetilde{\sigma}$ is a monomorphism; hence it is an isomorphism.

(b)⇒(a). The argument is a repetition of the part (b) of the argument of 4.2. Details are left to the reader. ■

4.2.2. Example: left multiplicative systems in Sets. Let $C_{\mathcal{E}} = Sets$. Then $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(\mathcal{E}) = \mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell s}(\mathcal{E})$, i.e. every saturated left multiplicative system on *Sets* is left exact.

In fact, $\mathcal{S}^{\mathfrak{s}}\mathcal{M}_{\ell}(\mathcal{E})$ consists of systems Σ_α and $\Sigma_{\alpha *}$, where α is an infinite cardinal (which belongs to a given universe). Recall that $\Sigma_{\alpha *} = \Sigma_\alpha \bigcup \{\emptyset \longrightarrow N \mid Card(N) < \alpha\}$ and Σ_α is the family of all maps $M \xrightarrow{f} N$ (– morphisms of $C_{\mathcal{E}}$) such that

(a) $M \neq \emptyset$ if $N \neq \emptyset$,

(b) $Card(N - f(M)) < \alpha$,

(c) There exists a subset M' of M such that $Card(M - M') < \alpha$ and the restriction of the map f to M' is injective (see 3.4).

Notice that the conditions (b) and (c) (defining $\Sigma_{\alpha *}$) are equivalent to the following condition which explains the meaning of $\Sigma_{\alpha *}$ and is more convenient for our purposes:

(b') There exists a subset M' of M such that the restriction of the map f to M' is injective and $Card(M - M') < \alpha > Card(N - f(M'))$.

Let Σ be Σ_α or $\Sigma_{\alpha *}$. Consider a commutative square

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{j}} & M \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ L' & \xrightarrow{\mathrm{j}'} & M' \end{array}$$

in $C_{\mathcal{E}} = Sets$ whose horizontal arrows are monomorphisms and the right vertical arrow, $M \xrightarrow{t} M'$, belongs to Σ. Then $L \xrightarrow{s} L'$ belongs to Σ too.

Suppose first that $M = \emptyset$. Then $L = \emptyset$. If $\Sigma = \Sigma_\alpha$, then $L' = M' = \emptyset$. In particular, $L \xrightarrow{s} L'$ belongs to Σ.

If $M = \emptyset$ and $\Sigma = \Sigma_{\alpha *}$, then $Card(M') < \alpha$. Since $L' \xrightarrow{\mathrm{j}'} M'$ is an injective map, and $Card(L') \leq Card(M') < \alpha$. Therefore $L \xrightarrow{s} L'$ belongs to $\Sigma_{\alpha *}$.

Suppose now that $M \neq \emptyset$ and L and L' are subsets of resp. M and M'. The map $M \xrightarrow{t} M'$ belongs to Σ iff there exists a subset M'' of M such that the restriction of t to M'' is injective and $Card(M - M'') < \alpha > Card(M' - t(M''))$ (see (b') above). Then the restriction of the map $L \xrightarrow{s} L'$ to $L \cap M''$ is injective and both $Card(L - L \cap M'')$ and $Card(L' - s(L \cap M''))$ are smaller than α. The latter means that $s \in \Sigma$.

Now it follows from 4.2.1 that Σ is a left exact multiplicative system.

4.3. Injective objects. Fix a 'space' X. Let C_{X_s} denote the subcategory of C_X formed by all objects of C_X and split monomorphisms.

An object E of the category C_X is called *injective* (or *strictly injective*) if the functor $C_X(-, E) : C^{op} \longrightarrow Sets$ maps strict epimorphisms (of C^{op}, i.e. strict monomoprhisms of C_X) to epimorphisms. We denote by $C_{\mathfrak{I}(X)}$ the subcategory of C_X formed by injective objects and split monomorphisms (or, what is the same, strict monomorphisms, see 1.2.3.3) between them. For any object E of the category C_X, let Σ_E denote the family of all arrows of C_X which the functor $C_X(-, E)$ transforms into invertible morphisms.

4.3.1. Proposition. *(a) Suppose that the category C_X has finite colimits. Then the map $E \longmapsto \Sigma_E$ extends to a functor $C^{op}_{X_s} \longrightarrow \mathcal{S}^{\mathfrak{s}}_{\ell}\mathcal{M}(X)$.*

(b) If, in addition, C_X has kernels of pairs of arrows. Then the map $E \longmapsto \Sigma_E$ defines a functor $C^{op}_{\mathfrak{I}(X)} \longrightarrow \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$.

Proof. (a) Since the category C_X has finite colimits, it follows from [GZ, 1.3.4] that Σ_E belongs to $\mathcal{S}^{\mathfrak{s}}_{\ell}\mathcal{M}(X)$ for every object E.

Let $E_1 \xrightarrow{u} E$ be a split monomorphism; and let $L \xrightarrow{s} M$ be a morphism of C_X such that $C_X(s,E)$ is an isomorphism. Then $C_X(s,E_1)$ is an isomorphism. In fact, there exists a morphism $E \xrightarrow{v} E_1$ such that $v \circ u = id_{E_1}$. Thus, there are two commutative diagrams

$$\begin{array}{ccc} C_X(M,E) & \xrightarrow{C_X(s,E)} & C_X(L,E) \\ {\scriptstyle C_X(M,v)}\downarrow & & \downarrow{\scriptstyle C_X(L,v)} \\ C_X(M,E_1) & \xrightarrow{C_X(s,E_1)} & C_X(L,E_1) \end{array} \qquad \begin{array}{ccc} C_X(M,E_1) & \xrightarrow{C_X(s,E_1)} & C_X(L,E_1) \\ {\scriptstyle C_X(M,u)}\downarrow & & \downarrow{\scriptstyle C_X(L,u)} \\ C_X(M,E) & \xrightarrow{C_X(s,E)} & C_X(L,E) \end{array}$$

such that the vertical arrows and the upper horizontal arrow of the first diagram are surjective (hence the remaining arrow, $C_X(s,E_1)$ is surjective) and the vertical arrows and the lower horizontal arrow of the second diagram are injective, hence $C_X(s,E_1)$ is injective. Therefore, $C_X(s,E_1)$ is bijective, i.e. $s \in \Sigma_{E_1}$.

(b) If E is an injective object, then, by 4.2, the localization at Σ_E preserves strict monomorphisms, i.e. Σ_E belongs to $\mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$. ■

4.4. Proposition. *Let $X \xrightarrow{f} Y$ be a continuous morphism with an inverse image functor f^* and a direct image functor f_*. Suppose that the functor f^* maps strict monomorphisms to strict monomorphisms. Then f_* maps injective objects to injective objects.*

Proof. If E is an injective object in C_Y, then the functor $C_X(f^*(-),E)$ maps strict monomorphisms of the category C_Y to (strict) epimorphisms, because f^* preserves strict monomorphisms and, since E is injective, $C_X(-,E)$ maps strict monomorphisms to epimorphisms. But, $C_Y(-,f_*(E)) \simeq C_X(f^*(-),E)$, hence the assertion. ■

4.5. Proposition. *(a) Let Σ be a left multiplicative system in C_X. Then $\Sigma \subseteq \Sigma_E$ for any Σ-torsion free injective object E.*

(b) Suppose that for every morphism $L \longrightarrow M$ in C_X, there exists a fibred coproduct $M \coprod_L M$ and the pair of coprojections $M \rightrightarrows M \coprod_L M$ has a kernel. Then for any $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$, the image of any injective Σ-torsion free object E in the quotient category $\Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ is an injective object.

(c) Let $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ be a continuous multiplicative system such that $\Sigma^{-1}C_X$ has a conservative family of injective objects. Then $\Sigma = \bigcap_{E \in \mathfrak{F}} \Sigma_E$, for some family $\mathfrak{F}$ of Σ-torsion free injective objects.

Proof. (a) Let Σ be a left multiplicative system and E a Σ-torsion free injective object. The claim is that for any arrow $L \xrightarrow{s} M$ in Σ, the map

$$C_X(M,E) \xrightarrow{C_X(s,E)} C_X(L,E), \quad f \longmapsto f \circ s, \tag{1}$$

is bijective. In fact, let $L \xrightarrow{f} E$ be an arbitrary morphism. Since $s \in \Sigma$ and Σ is a left multiplicative system, there exists a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f'} & M' \\ s\uparrow & & \uparrow t \\ L & \xrightarrow{f} & E \end{array}$$

with $t \in \Sigma$. Since E is Σ-torsion free, the arrow $E \xrightarrow{t} M'$ is a strict monomorphism. Every strict monomorphism from an injective object splits, i.e. there exists a morphism $M' \xrightarrow{g} E$ such that $g \circ t = id_E$. Therefore $(g \circ f') \circ s = g \circ t \circ f = f$. This proves the surjectivity of the map (1). Suppose now that $M \overset{p_1}{\underset{p_2}{\rightrightarrows}} E$ is a pair of arrows such that $p_1 \circ s = p_2 \circ s$. Since Σ is a left multiplicative system and $s \in \Sigma$, there exists an arrow $E \xrightarrow{u} N$ in Σ such that $u \circ p_1 = u \circ p_2$. The arrow u is a (strict) monomorphism, because E is Σ-torsion free, hence $p_1 = p_2$. This shows that the map (1) is injective.

(b) Let q^* be the localization functor $C_X \longrightarrow \Sigma^{-1}C_X$. Consider a diagram

$$q^*(E) \xleftarrow{f} q^*(L) \xrightarrow{\mathfrak{j}'} q^*(M) \tag{1}$$

such that $\mathfrak{j}'$ is a strict monomorphism. Since Σ is a left multiplicative system, the diagram (1) corresponds to the diagram

$$E \xrightarrow{s} K \xleftarrow{f'} L \xrightarrow{\mathfrak{j}''} M' \xleftarrow{t} M. \tag{2}$$

Here $s, t \in \Sigma$ and $f = q^*(s)^{-1}q^*(f')$ and $q^*(\mathfrak{j}') = q^*(t)^{-1}q^*(\mathfrak{j}'')$. Since E is Σ-torsion free, the arrow s is a strict monomorphism. It is a split monomorphism, because E is injective; i.e. there exists an arrow $K \xrightarrow{h} E$ such that $hs = id_E$. In particular, $g^*(h) = g^*(s)^{-1}$. Thus, $f = q^*(\mathfrak{v})$, where $\mathfrak{v} = hf' : L \longrightarrow E$. Consider the decomposition of the arrow $L \xrightarrow{\mathfrak{j}''} M'$ into $L \xrightarrow{u'} L' \xrightarrow{\mathfrak{j}} M'$, where $L' \xrightarrow{\mathfrak{j}} M'$ is the kernel of the pair $M' \overset{p_1}{\underset{p_2}{\rightrightarrows}} M' \coprod_L M'$. Since $q^*(\mathfrak{j}') = q^*(t)^{-1}q^*(\mathfrak{j}'')$ is a strict monomorphism, $q^*(\mathfrak{j}'')$ is a strict monomorphism. The localization functor q^* is right exact [GZ, I.3.1]; in particular, the diagram

$$q^*\big(L \xrightarrow{\mathfrak{j}''} M' \overset{p_1}{\underset{p_2}{\rightrightarrows}} M' \coprod_L M'\big)$$

is isomorphic to

$$q^*(L) \xrightarrow{q^*(\mathfrak{j}'')} q^*(M') \rightrightarrows q^*(M') \coprod_{q^*(L)} q^*(M'). \tag{3}$$

Since $q^*(\mathfrak{j}'')$ is a strict monomorphism, the diagram (3) is exact. By hypothesis, q^* maps strict monomorphisms to strict monomorphisms, hence the diagram

$$q^*(L') \xrightarrow{q^*(\mathfrak{j})} q^*(M') \overset{q^*(p_1)}{\underset{q^*(p_2)}{\rightrightarrows}} q^*(M') \coprod_{q^*(L)} q^*(M'). \tag{3'}$$

is exact too. By the universal property of kernels, this implies that $q^*(L) \xrightarrow{q^*(u')} q^*(L')$ is an isomorphism. Since Σ is saturated, $u' \in \Sigma$. Therefore, there exists a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\mathfrak{v}} & E \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u''} \\ L' & \xrightarrow{\mathfrak{v}} & E' \end{array}$$

with $u'' \in \Sigma$. Since E is Σ-torsion free and injective, there exists $E' \xrightarrow{w} E$ such that $wu'' = id_E$. Thus, we obtain a diagram $E \xleftarrow{\mathfrak{v}'} L' \xrightarrow{\mathfrak{j}} M'$ in which $\mathfrak{j}$ is a strict monomorphism. Therefore, there exists a morphism $E \xrightarrow{g} M'$ such that $g\mathfrak{j} = \mathfrak{v}'$. But, then $\gamma f = \mathfrak{j}'$, where $\gamma = q^*(t)^{-1}q^*(g)$ (see notations above).

(c) Fix $\Sigma \in \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ such that $\Sigma^{-1}C_X = C_{\Sigma^{-1}X}$ has a conservative family, $\mathfrak{F}$, of injective objects, i.e. $\{C_{\Sigma^{-1}X}(-,E) \mid E \in \mathfrak{F}\}$ is a conservative family of functors. Fix a direct image functor $C_{\Sigma^{-1}X} \xrightarrow{q_{\Sigma *}} C_X$ of the localization $\Sigma^{-1}X \xrightarrow{q_\Sigma} X$. Since the localization functor $C_X \xrightarrow{q^*_\Sigma} C_{\Sigma^{-1}X}$ maps strict monomorphisms to strict monomorphisms, the functor $q_{\Sigma *}$ maps injectives to Σ-torsion free injectives (see 4.4). By the (a), we have the inclusion $\Sigma \subseteq \bigcap\limits_{E\in\mathfrak{F}} \Sigma_{q_{\Sigma *}(E)}$. The inverse inclusion follows from the conservativity of the family $\{C_{\Sigma^{-1}X}(-,E) \mid E \in \mathfrak{F}\}$. ∎

4.6. Example. Suppose that C_X is an abelian category with injective hulls (e.g. C_X is a Grothendieck category). Then the conditions of 4.5 hold. Moreover, every closed multiplicative system Σ is the intersection of the systems Σ_E, where E runs through a family of Σ-torsion free injectives.

In fact, closed multiplicative systems are in bijective correspondence with coreflective thick subcagories, $\Sigma \longmapsto \mathbb{T}_\Sigma$, where $Ob\mathbb{T}_\Sigma = \{M \in ObC_X \mid (0 \longrightarrow M) \in \Sigma\}$. If E is an injective object, then $\mathbb{T}_{\Sigma_E} = {}^\perp E$, i.e. $Ob\mathbb{T}_{\Sigma_E} = \{M \in ObC_X \mid C_X(M,E) = 0\}$. Notice that $\Sigma_\mathbb{T}$-torsion free objects are precisely $\mathbb{T}$-torsion free objects, i.e. objects which have no nonzero subobjects from $\mathbb{T}$.

Fix a coreflective thick subcategory $\mathbb{T}$ and denote by $q^*_\mathbb{T}$ the localization functor $C_X \longrightarrow C_X/\mathbb{T}$, Let M be a nonzero $\mathbb{T}$-torsion free object. If $M \longrightarrow E$ is an essential monomorphism, then E is $\mathbb{T}$-torsion free object too. Let $\mathfrak{F}$ be a family of $\mathbb{T}$-torsion free objects such that $\{q^*_\mathbb{T}(M) \mid M \in \mathfrak{F}\}$ generates the quotient category $C_X/\mathbb{T} = C_{X/\mathbb{T}}$, i.e. $\{q^*_\mathbb{T}(M) \mid M \in \mathfrak{F}\}^\perp = 0$. For each $M \in \mathfrak{F}$ we chose an injective hull $E(M)$ of M. It follows from 4.5(a) that $\Sigma_\mathbb{T} \subseteq \bigcap\limits_{M\in\mathfrak{F}} \Sigma_{E(M)}$, or, equivalently, $\mathbb{T} \subseteq \bigcap\limits_{M\in\mathfrak{F}} {}^\perp E(M)$. We leave verifying the inverse inclusion to the reader.

Suppose that C_X is a Grothendieck category. Then every closed multiplicative system, Σ, is flat and the corresponding quotient category $C_{\Sigma^{-1}X}$, is a Grothendieck category too. In particular, it has a set of generators, $\mathfrak{F}$. By the argument above, $\Sigma = \bigcap\limits_{M\in\mathfrak{F}} \Sigma_{E(M)} = \Sigma_{E_\mathfrak{F}}$, where $E_\mathfrak{F} = \prod\limits_{M\in\mathfrak{F}} E(M)$. Here we use the fact that injective hulls and small products exist in a Grothendieck category [BD, 6.3.1, 6.3.2]. Thus, we have recovered a well known assertion: every Serre subcategory of a Grothendieck category is of the form ${}^{\perp}E$ for some injective object E.

4.7. Example. The conditions of 4.5 hold if C_X is an elementary (Lawvere-Tierney) topos, in particular if C_X is a Grothendieck topos. In fact, by a Lawvere-Tierney theorem [J, 1.26], *in a topos, all partial maps are representable.* This means that for any object M, there exists a monomorphism $M \xrightarrow{\eta_M} \widetilde{M}$ such that any diagram $L \xleftarrow{\mathfrak{j}} L' \xrightarrow{f} M$ with a monomorphic left arrow is uniquely completed to a commutative square

$$\begin{array}{ccc} L' & \xrightarrow{f} & M \\ {\scriptstyle \mathfrak{j}}\downarrow & & \downarrow{\scriptstyle \eta_M} \\ L & \xrightarrow{\widetilde{f}} & \widetilde{M} \end{array} \qquad (1)$$

It follows from the uniqueness of $\widetilde{f}$ in (1) that the map $M \longmapsto \widetilde{M}$ defines a functor $C_X \xrightarrow{\mathfrak{I}_X} C_X$ and $\eta = \{\eta_M \mid M \in ObC_X\}$ is a functor morphism $Id_{C_X} \longrightarrow \mathfrak{I}_X$. Moreover, for every object M, the object $\mathfrak{I}_X(M) = \widetilde{M}$ is injective [J, 1.27].

Note that $\mathfrak{I}_X(\prod_{i\in I} M_i) \simeq \prod_{i\in I} \mathfrak{I}_X(M_i)$ provided the product $\prod_{i\in I} M_i$ exists, and if $Y \xrightarrow{f} X$ is a *geometric morphism* (that is f is continuous and f^* is (left) exact), then there is a functor isomorphism $f_*\mathfrak{I}_Y \simeq \mathfrak{I}_X f_*$.

Let Σ be a flat multiplicative system in C_X. Set $Y = \Sigma^{-1}X$ and denote by q the canonical morphism $Y \longrightarrow X$. The quotient category $C_Y = \Sigma^{-1}C_X$ is a topos too. Let $\mathfrak{F}$ be a family of generators in C_Y. Then $\bigcap_{M\in\mathfrak{F}} \Sigma_{\mathfrak{I}_Y(M)} = Iso(C_Y)$. Therefore $\Sigma = \bigcap_{M\in\mathfrak{F}} \Sigma_{q_*\mathfrak{I}_Y(M)} = \bigcap_{M\in\mathfrak{F}} \Sigma_{\mathfrak{I}_X q_*(M)}$.

Suppose now that C_X is a Grothendieck topos. Then $C_Y = \Sigma^{-1}C_X$ is a Grothendieck topos. In particular, C_Y has small products and a *set* of generators, $\mathfrak{F}$. Therefore $\Sigma = \bigcap_{M\in\mathfrak{F}} \Sigma_{\mathfrak{I}_X q_*(M)} = \Sigma_{\mathfrak{I}_X(M_{\mathfrak{F}})}$, where $M_{\mathfrak{F}} = \prod_{M\in\mathfrak{F}} q_*(M) \simeq q_*(\prod_{M\in\mathfrak{F}} M)$.

4.8. Note. The examples 4.6 and 4.7 suggest that abelian categories with injective hulls might be regarded as abelian versions of elementary toposes, and Grothendieck categories are abelian analogs of Grothendieck toposes.

4.9. Example. Let $C_{\mathcal{E}} = Sets$ (like in 4.2.2). Then all objects of $C_{\mathcal{E}}$ are injective (and projective). Fix an object E of $C_{\mathcal{E}}$. If E is a one-element set, or the empty set, then $\Sigma_E = HomC_{\mathcal{E}}$. If $Card(E) \geq 2$, then $\Sigma_E = Iso(C_{\mathcal{E}})$.

Evidently, the empty set and one-element sets are the only indecomposable injectives.

Let $\Sigma \subseteq HomC_{\mathcal{E}}$. By definition, an object M of $C_{\mathcal{E}}$ is Σ-torsion free if every morphism $M \longrightarrow M'$ which belongs to Σ is a monomorphism. Suppose Σ is a saturated left multiplicative system, i.e. it coincides either with Σ_α or with $\Sigma_{\alpha*}$ for some infinite cardinal number α (cf. 4.2.2). Then a set M is Σ-torsion free iff $Card(M) \leq 1$; that is, again, either $M = \emptyset$, or M is a one-element set.

It follows from the definitions of Σ_α and $\Sigma_{\alpha*}$ that objects of $C_{\mathcal{E}}$ having a morphism to a Σ-torsion free object are precisely sets N such that $Card(N) < \alpha$. This shows that the only closed saturated left multiplicative systems on $C_{\mathcal{E}}$ are $Iso(C_{\mathcal{E}})$ and $HomC_{\mathcal{E}}$. Since, by [R4, 5.2.2], every continuous saturated left multiplicative system is closed, there are no non-trivial continuous saturated left multiplicative systems either.

4.10. The injective spectrum and the Gabriel spectrum.

4.10.1. Relatively maximal objects. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. We call an object x of $\mathfrak{G}$ *relatively maximal*, or *F-maximal*, if F transforms any arrow $x \longrightarrow y$ into an isomorphism. We denote by $\mathfrak{Max}(\mathfrak{G}, F)$ the full subcategory of $\mathfrak{G}$ generated by relatively maximal objects.

4.10.2. Injective spectrum. Suppose that the category C_X has colimits of finite diagrams. Let F be the functor

$$C^{op}_{\mathfrak{I}(X)} \longrightarrow \mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X), \quad E \longmapsto \Sigma_E$$

(see 4.3.1). We denote $\mathfrak{Max}(C^{op}_{\mathfrak{I}(X)}, F)$ by $ISpec(X)$. It follows that the objects of $ISpec(X)$ are injective objects E such that $\Sigma_E = \Sigma_{E_1}$ for every nontrivial split monomorphism $E_1 \longrightarrow E$. We denote by $\mathbf{ISpec}(X)$ the full subcategory of the $\mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ generated by the image of $ISpec(X)$ and call it the *injective spectrum of X*.

4.10.2.1. Note. The injective spectrum is introduced in [R, 6.5], in a slightly different way, in the case when C_X is an abelian category.

4.10.3. The Gabriel's spectrum.

Recall that an object, E, of the category C_X is *indecomposable* if every nontrivial idempotent $E \xrightarrow{\mathfrak{p}} E$ is id_E (see 1.2.3.3).

We denote by $\widehat{ISpec}(X)$ the groupoid $\mathcal{M}in(C_{\mathfrak{I}(X)})$ formed by indecomposable injectives and their isomorphisms. It follows that $\widehat{ISpec}(X) \subseteq ISpec(X)$.

The Gabriel's spectrum is the full subpreorder, $\widehat{\mathbf{ISpec}}(X)$, of the preorder $\mathcal{S}^{\mathfrak{s}}_{\ell s}\mathcal{M}(X)$ spanned by multiplicative systems Σ_E, where E runs through indecomposable injectives of C_X. In particular, the Gabriel's spectrum is contained in the injective spectrum: $\widehat{\mathbf{ISpec}}(X) \subseteq \mathbf{ISpec}(X)$.

4.10.3.1. Note. The Gabriel's spectrum is introduced in [Gab] for a (locally noetherian) abelian category. Its elements are defined as isomorphism classes of indecomposable injectives. The preorder inherited from $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ is opposite to the *specialization* preorder.

4.11. Injective spectrum of an abelian category. Fix an abelian category C_X. Let $E \in ObC_X$, and let ${}^\perp E$ be the full subcategory of the category C_X generated by all objects M which are left orthogonal to E, i.e. $C_X(M, E) = 0$. If E_1 is a subobject of the object E, then ${}^\perp E \subseteq {}^\perp E_1$.

If E is an injective object, then ${}^\perp E$ is a Serre subcategory of the category C_X and the map $E \longmapsto {}^\perp E$ is a functor $\mathfrak{J}(X)^{op} \longrightarrow \mathfrak{Se}(X)$. In particular, the spectrum $\mathbf{ISpec}(X)$ can be identified with the subpreorder of the preorder $\mathfrak{Se}(X)$ of Serre subcategories of C_X generated by the image of the map $ISpec(X) \longrightarrow \mathfrak{Se}(X),\ E \longmapsto {}^\perp E$.

4.11.1. Proposition. *Let C_X be an abelian category with the property (sup).*
(a) If every object of C_X has an injective hull, then $\mathfrak{Spec}^1(\mathfrak{Se}(X)) \subseteq \mathbf{ISpec}(X)$. In particular, $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathbf{ISpec}(X)$.
(b) If C_X has a Gabriel-Krull dimension, then

$$\mathbf{Spec}^1_{\mathfrak{Se}}(X) = \mathfrak{Spec}^1(\mathfrak{Se}(X)) = \mathbf{ISpec}(X) = \widehat{\mathbf{I}}\mathbf{Spec}(X).$$

Proof. (a) Let $\mathcal{P}$ be an object of $\mathfrak{Spec}^1(\mathfrak{Se}(X))$, i.e. there exists the smallest Serre subcategory, $\mathcal{P}^{\mathfrak{s}}$, properly containing $\mathcal{P}$. Let $M \in Ob\mathcal{P}^{\mathfrak{s}} - Ob\mathcal{P}$. Since, thanks to the property (sup), every Serre subcategory of C_X, in particular $\mathcal{P}$, is coreflective, we can and will assume that M is $\mathcal{P}$-torsion free. Let $E(M)$ be an injective hull of the object M. Then $E(M)$ is $\mathcal{P}$-torsion free, because M is $\mathcal{P}$-torsion free, hence $\mathcal{P} \subseteq {}^\perp E(M)$. Notice that ${}^\perp E(M)$ cannot contain $\mathcal{P}$ properly, because if $\mathcal{P} \neq {}^\perp E(M)$, then ${}^\perp E(M)$, being a Serre subcategory, would contain $\mathcal{P}^{\mathfrak{s}}$, in particular, it would contain the object M which is not the case. This verifies the inclusion $\mathfrak{Spec}^1(\mathfrak{Se}(X)) \subseteq \mathbf{ISpec}(X)$. By 2.4.1, $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathfrak{Spec}^1(\mathfrak{Se}(X))$, hence $\mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathbf{ISpec}(X)$.

(b) If C_X has a Gabriel-Krull dimension, then, by [R4, 7.9.1], $\mathbf{Spec}^1_{\mathfrak{s}}(X) = \mathbf{Spec}^-(X)$ and by [R, 6.6.1.1, 6.6.1.2], $\mathbf{Spec}^-(X) = \mathbf{ISpec}(X) = \widehat{\mathbf{I}}\mathbf{Spec}(X)$. The assertion follows now from the inclusions $\mathbf{Spec}^-(X) \subseteq \mathbf{Spec}^1_{\mathfrak{Se}}(X) \subseteq \mathfrak{Spec}^1(\mathfrak{Se}(X)) \subseteq \mathbf{ISpec}(X)$. ∎

C. Complementary facts: relative support and associated points.

C.1. Relative support. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. For an $x \in Ob\mathfrak{G}$, we call $\mathfrak{Supp}_{\mathfrak{H}}(F(x))$ the *support of* x *in* $\mathfrak{H}$, or the *relative support of* x.

C.2. Weakly associated points. For any $x \in Ob\mathfrak{G}$, we denote by $Ass^1_{(\mathfrak{G},F)}(x)$ the full subcategory of $\mathfrak{Spec}^1(\mathfrak{H})$ generated by all objects z for which there exists $\widetilde{x} \longrightarrow x$ such that $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$ and there is an arrow $F(\widetilde{x}) \longrightarrow z^\star$. As before, $(z, z \to z^\star)$ denotes an initial object of $(z\backslash\mathfrak{H})^1$.

It follows from this definition that $Ass^1_{(\mathfrak{G},F)}(x)$ is a subcategory of the relative support, $\mathfrak{Supp}_{\mathfrak{H}}(F(x))$, of the object x.

We call objects of $Ass^1_{(\mathfrak{G},F)}(x)$ *weakly associated points of* x *in* $(\mathfrak{G}, F)$.

If F is the identical functor $\mathfrak{H} \longrightarrow \mathfrak{H}$, we shall write $Ass^1_{\mathfrak{H}}(x)$ instead of $Ass^1_{(\mathfrak{H},Id_{\mathfrak{H}})}(x)$ and call objects of this category *weakly associated points of* x. It follows that objects of $Ass^1_{\mathfrak{H}}(x)$ are $z \in Ob\mathfrak{Spec}^1(\mathfrak{H})$ such that there exist arrows $z^\star \longleftarrow \widetilde{x} \longrightarrow x$ and z belongs to $\mathfrak{Supp}_{\mathfrak{H}}(\widetilde{x})$.

C.3. Associated points. Fix a functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$. For an object x of $\mathfrak{G}$, we denote by $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$ the full subcategory of $\mathfrak{Spec}^1(\mathfrak{H})$ generated by all objects z such that there exists $\widetilde{x} \in Ob\mathfrak{G}$ having the following properties:

(a) there exist arrows $\widetilde{x} \longrightarrow x$ and $F(\widetilde{x}) \longrightarrow z^\star$;

(b) if there exists an arrow $y \longrightarrow \widetilde{x}$, then $\mathfrak{H}(F(y), z) = \emptyset$.

We call objects of $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$ *associated points* of the object x in $(\mathfrak{G}, F)$.

It follows from the condition (b) that z belongs to $\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$. Together with the condition (a), this means that $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x) \subseteq Ass^1_{(\mathfrak{G},F)}(x)$.

If F is the identical functor $\mathfrak{H} \longrightarrow \mathfrak{H}$, we shall write $\mathfrak{Ass}^1_{\mathfrak{H}}(x)$ instead of $\mathfrak{Ass}^1_{(\mathfrak{H},Id_{\mathfrak{H}})}(x)$. It follows that objects of $\mathfrak{Ass}^1_{\mathfrak{H}}(x)$ are $z \in Ob\mathfrak{Spec}^1(\mathfrak{H})$ for which there exists a pair of arrows $z^\star \longleftarrow \widetilde{x} \longrightarrow x$ with $\widetilde{x}$ such that there is no diagram of the form $z \longleftarrow y \longrightarrow \widetilde{x}$.

C.4. Associated points and weakly associated points in $\mathfrak{Spec}^0(\mathfrak{H})$. We define $\mathfrak{Ass}^0_{(\mathfrak{G},F)}(x)$, resp. $Ass^0_{(\mathfrak{G},F)}(x)$, as full subcategories of $\mathfrak{H}$ generated by all $z \in Ob\mathfrak{Spec}^0(\mathfrak{H})$ such that $\widehat{z}$ is an object of $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$, resp. an object of $Ass^1_{(\mathfrak{G},F)}(x)$.

Consider the following two properties:

(sup1) If $x \in Ob\mathfrak{H}$ is the supremum of a filtered system, $\{x_i \mid i \in J\}$, of its subobjects, then for any morphism $\widetilde{x} \longrightarrow x$, there exists a cofinal subset $I \subseteq J$ such that for every $i \in I$, there exists a fibre product, $\widetilde{x}_i = \widetilde{x} \times_x x_i$, and the canonical arrow $colim(\widetilde{x}_i | i \in I) \longrightarrow \widetilde{x}$ is an isomorphism.

(sup2) If $x \in Ob\mathfrak{H}$ is the supremum of a filtered system, $\{x_i \mid i \in J\}$, of its subobjects, then for any morphism $\widetilde{x} \longrightarrow x$, there exists a diagram $\widetilde{x} \longleftarrow y \longrightarrow x_i$ for some $i \in J$.

C.5. Proposition. *(a) If $x = colim(x_i \mid i \in J)$, then*

$$\bigcup_{i \in J} Ass^1_{(\mathfrak{G},F)}(x_i) \subseteq Ass^1_{(\mathfrak{G},F)}(x) \quad \textit{and} \quad \bigcup_{i \in J} \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i) \subseteq \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x).$$

(b) Let $x \in Ob\mathfrak{H}$ be the supremum of a filtered system, $\{x_i \mid i \in J\}$, of its subobjects.

(i) If $\mathfrak{H}$ is a preorder with the property (sup1) and F preserves colimits of filtered systems, then

$$Ass^1_{(\mathfrak{G},F)}(x) = \bigcup_{i\in J} Ass^1_{(\mathfrak{G},F)}(x_i).$$

(ii) Suppose $\mathfrak{G}$ possesses the property (sup2). Then

$$\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x) = \bigcup_{i\in J} \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i).$$

Proof. (a) Let $z \in ObAss^1_{(\mathfrak{G},F)}(x_i)$, that is z belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ and there exists a pair of arrows, $z^\star \longleftarrow F(\widetilde{x}_i)$ and $\widetilde{x}_i \longrightarrow x_i$, such that $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}_i))$. Since there is an arrow $x_i \longrightarrow x$, same $\widetilde{x}_i$ serves for x. Therefore $Ass^1_{(\mathfrak{G},F)}(x_i) \subseteq Ass^1_{(\mathfrak{G},F)}(x)$ for all $i \in J$. Similar argument shows the inclusion $\bigcup_{i\in J} \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i) \subseteq \mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$.

(b)(i) Suppose that $(\mathfrak{G}, F)$ possesses the property (sup1). Let z be an object of $Ass^1_{(\mathfrak{G},F)}(x)$; i.e. z belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ and there exists a pair of arrows, $z^\star \longleftarrow F(\widetilde{x})$ and $\widetilde{x} \longrightarrow x$, such that $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$. By the property (sup1), a fibre product $\widetilde{x}_i = \widetilde{x} \times_x x_i$ exists for $i \in I$, where I is a cofinal subset of J, and the canonical arrow $colim(\widetilde{x}_i | i \in I) \longrightarrow \widetilde{x}$ is an isomorphism. If for every $i \in I$, there is an arrow $F(\widetilde{x}_i) \longrightarrow z$, then there is an arrow $F(\widetilde{x}) \longrightarrow z$, that is z does non belong to $\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}))$, which contradicts to the hypothesis. Thus, $z \in Ob\mathfrak{Supp}_{\mathfrak{H}}(F(\widetilde{x}_i))$ for some $i \in I$, which means that z is an object of $Ass^1_{(\mathfrak{G},F)}(x_i)$.

(ii) Let now $(\mathfrak{G}, F)$ have the property (sup2). Let z be an object of $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x)$; i.e. z belongs to $\mathfrak{Spec}^1(\mathfrak{H})$ and there exists a pair of arrows, $z^\star \longleftarrow F(\widetilde{x})$ and $\widetilde{x} \longrightarrow x$, such that if $\mathfrak{H}(F(y), z) \neq \emptyset$, then $\mathfrak{G}(y, \widetilde{x}) = \emptyset$. By the property (sup2), for some $i \in J$, there exists a pair of arrows $\widetilde{x} \longleftarrow \widetilde{x}_i \longrightarrow x_i$. It follows that if $\mathfrak{H}(F(y), z) \neq \emptyset$, then $\mathfrak{G}(y, \widetilde{x}_i) = \emptyset$, hence z belongs to $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(x_i)$. ■

C.6. Example: supports and associated points of a family of arrows. Fix a category C_X. Let $\mathfrak{H}$ be the preorder $\mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ of saturated multiplicative systems of C_X. Let $\mathfrak{G}$ be the preorder (with respect to inclusion) of non-empty families of arrows of the category C_X, and let $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ be the functor which assigns to each family S the intersection, $[S]_\bullet$, of all saturated multiplicative systems containing S.

The support, $\mathfrak{Supp}(F(S))$, of a family of arrows S in $\mathfrak{H} = \mathcal{S}^{\mathfrak{s}}\mathcal{M}(X)$ consists of all saturated multiplicative systems Σ which do not contain S.

Weakly associated points of S are saturated multiplicative systems Σ such that there exists $\widetilde{S} \subseteq S$ which is not contained in Σ, but is contained in $\Sigma^\star$. Here $\Sigma^\star$ is the intersection of all saturated multiplicative systems of C_X properly containing Σ.

Thus, the preorder $Ass_{(\mathfrak{G},F)}(S)$ of weakly associated points of a family of arrows S coincides with the preorder $Ass_{\mathfrak{L}}(S)$ of weakly associated points of S defined in [R4, 9.4.2].

Notice that in this case, *associated points* and weakly associated points of S coincide.

In fact, by definition C.3, *associated points* of S are saturated multiplicative systems Σ having the following property: there exists a non-empty subfamily, T, of S such that $T \cap \Sigma = \emptyset$. Let Σ be a weakly associated point of S, i.e. there exists $\widetilde{S} \subseteq S$ such that

$\Sigma \not\supseteq \widetilde{S} \subseteq \Sigma^\star$. The family $T = \widetilde{S} - \Sigma$ is non-empty, and $T \cap \Sigma = \emptyset$. This means, precisely, that Σ is an associated point of S.

C.7. Supports and associated points of objects. Let $\mathfrak{G}$, $\mathfrak{H}$ and F be same as in C.6; i.e. $\mathfrak{G}$ is a preorder of families of arrows of a category C_X, $\mathfrak{H}$ is the preorder $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ of saturated multiplicative systems of C_X, and F maps every family S to the smallest saturated multiplicative system containing S. Fix a functor $C_Y \xrightarrow{\phi^*} C_X$. The functor ϕ^* determines, for every $M \in ObC_Y$, a functor $C_Y/M \xrightarrow{\phi^*_M} C_X/\phi^*(M)$. The map

$$ObC_Y \longrightarrow \mathfrak{G}, \quad M \longmapsto \phi^*_M(Ob(C_Y/M)),$$

defines a functor $C_Y \xrightarrow{\Phi} \mathfrak{G}$. Let $C_Y \xrightarrow{\Upsilon_\phi} \mathfrak{H}$ denote the composition of the functor Φ with the functor F. The functor Υ_ϕ provides the notions of the support, weakly associated points and associated points in $\mathfrak{H} = \mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ of any object M of the category C_Y.

The support of an object M consists of all saturated multiplicative systems Σ such that $\phi^*(\xi) \notin \Sigma$ for some arrow $L \xrightarrow{\xi} M$.

A saturated multiplicative system Σ of C_X is a *weakly associated point* of an object M of the category C_Y iff there exists a morphism $L \longrightarrow M$ such that $\phi^*(C_Y(N,L)) \subseteq \Sigma^\star$ for all $N \in ObC_Y$ and $\phi^*(\xi) \notin \Sigma$ for some arrow $M' \xrightarrow{\xi} M$.

Finally, Σ is an *associated point* of M iff there exists a morphism $L \longrightarrow M$ such that $\Sigma \not\ni \phi^*(\xi) \in \Sigma^\star$ for every arrow $N \xrightarrow{\xi} L$.

C.8. A canonical setting. Let C_Y denote the subcategory of C_X formed by all non-initial objects of C_X and strict monomorphisms; and let ϕ^* be the inclusion functor $C_Y \hookrightarrow C_X$. Applying the construction of C.7, we obtain non-trivial notions of the support and weakly associated and associated points of any non-initial object of the category C_X.

C.8.1. The case of an abelian category. If C_X is an abelian category, these notions are equivalent to those introduced in [R4, 9.4, 10.8]. The equivalence is given by the isomorphism between the preorder $\mathcal{S}^\mathfrak{s}\mathcal{M}(X)$ of saturated multiplicative systems and the preorder $\mathfrak{Th}(X)$ of thick subcategories of the category C_X. By definition, a saturated system $\Sigma_\mathbb{T}$ corresponding to a thick subcategory $\mathbb{T}$ belongs to the support of an object M of C_X iff there exists a monomorphism $N \xrightarrow{g} M$ which does not belong to $\Sigma_\mathbb{T}$. This means, precisely, that $Cok(g)$ does not belong to the subcategory $\mathbb{T}$. Thus, $\Sigma_\mathbb{T}$ belongs to the support of M iff M does not belong to $\mathbb{T}$.

A multiplicative system $\Sigma_\mathbb{T}$ is a *weakly associated point* of an object M if there exists a subobject N of M such that all monoarrows $L \longrightarrow N$ belong to $\Sigma^\star_\mathbb{T} = \Sigma_{\mathbb{T}^\star}$, but some of them do not belong $\Sigma_\mathbb{T}$. This means that N is an object of the subcategory $\mathbb{T}^\star$ which does not belong to $\mathbb{T}$. Here $\mathbb{T}^\star$ is the intersection of all thick subcategories of C_X properly containing $\mathbb{T}$.

A multiplicative system $\Sigma_\mathbb{T}$ is an *associated point* of an object M if there is nonzero subobject N of M such that every nonzero monoarrow $L \longrightarrow N$ belongs to $\Sigma_{\mathbb{T}^\star}$ and does not belong $\Sigma_\mathbb{T}$. This means that the subobject N belongs to $\mathbb{T}^\star$ and is $\mathbb{T}$-torsion free.

C.8.2. The direct description. Fix an abelian category C_X. Let $\mathfrak{G}$ be the subcategory $C_{\mathfrak{M}^\star(X)}$ of C_X formed by all nonzero monomorphisms and all nonzero objects of

C_X. Let $\mathfrak{H}$ be the preorder $\mathfrak{Th}(X)$ of thick subcategories of the category C_X. The functor $\mathfrak{G} \xrightarrow{F} \mathfrak{H}$ assigns to every object M of the category $C_{\mathfrak{M}^\star(X)}$ the smallest thick subcategory, $[M]_\bullet$, containing M.

The support, $\mathfrak{Supp}(F(M))$, of the object M in $\mathfrak{H} = \mathfrak{Th}(X)$ consists of all thick subcategories $\mathbb{T}$ such that $F(M) = [M]_\bullet \not\subseteq \mathbb{T}$, or, equvalently, $M \notin Ob\mathbb{T}$.

A thick subcategory $\mathbb{T}$ is a *weakly associated point* of a nonzero object M iff there exists a subobject $\widetilde{M}$ of M which belongs to $Ob\mathbb{T}^\star - Ob\mathbb{T}$.

A thick subcategory $\mathbb{T}$ is an *associated point* of a nonzero object M iff there exists a nonzero subobject $\widetilde{M}$ of M which belongs to $\mathbb{T}^\star$ and is $\mathbb{T}$-torsion free.

Thus the preorder $Ass^1_{(\mathfrak{G},F)}(M)$ coincides with the preorder $Ass^1_{\mathfrak{Th}}(M)$ of weakly associated points of M in the sense of [R4, 10.1]. The preorder $\mathfrak{Ass}^1_{(\mathfrak{G},F)}(M)$ coincides with the preorder $\mathfrak{Ass}^1_{\mathfrak{L}}(M)$ introduced in [R4, 10.8].

Appendix

The purpose of this appendix is to remind the definitions of the spectra introduced in [R1], [R2], and [R, Ch.6]. The main reference is [R].

A1. The left spectrum of a ring. Let $I_\ell R$ denote the set of left ideals of an associative unital ring R. For two left ideals, $\mathfrak{m}$, $\mathfrak{n}$, we write $\mathfrak{m} \le \mathfrak{n}$ if there exists a finite set, x, of elements of R such that $(\mathfrak{m} : x) = \{r \in R \mid rx \subset \mathfrak{m}\}$ is contained in $\mathfrak{n}$. This relation is a preorder. If $\mathfrak{m}$ is a two-sided ideal, then $\mathfrak{m} \le \mathfrak{n}$ iff $\mathfrak{m} \subseteq \mathfrak{n}$. In particular, if the ring R is commutative, then the preorder $\le$ coincides with the inclusion.

The *left spectrum* of the ring R is the preorder $(Spec_\ell R, \le)$, where $Spec_\ell R$ consists of all left ideals, $\mathfrak{p}$, in R such that $(\mathfrak{p} : r) \le \mathfrak{p}$ for any $r \in R - \mathfrak{p}$.

Since for a commutative ring, the relation $\le$ is the inclusion, the left spectrum of a commutative ring coincides with its prime spectrum endowed with the *specialization* preorder, $\subseteq$.

A2. The preorder $\succ$ and the spectrum $\mathbf{Spec}(X)$. Let C_X be an abelian category. Given two objects, M and N, of the category C_X, we write $M \succ N$ if N is a subquotient of a finite coproduct of copies of M. The relation $\succ$ is a preorder on ObC_X.

For any object $M \in ObC_X$, let $[M]$ (resp. $\langle M \rangle$) denote the full subcategory of C_X generated by all $N \in ObC_X$ such that $M \succ N$ (resp. $N \not\succ M$). It is easy to see that $M \succ L \Leftrightarrow [L] \subseteq [M] \Leftrightarrow \langle L \rangle \subseteq \langle M \rangle$.

A2.1. Note. A full subcategory, $\mathbb{T}$, of C_X is called *topologizing* if it contains all subquotients of its objects and is closed under finite coproducts. One can check $[M]$ is the smallest topologizing subcategory of C_X containing the object M, and $\langle M \rangle$ is the union of all topologizing subcategories of C_X which do not contain M.

A.2.2. $\mathbf{Spec}(X)$. We denote by $Spec(X)$ a subpreorder of $(ObC_X, \succ)$ formed by all nonzero objects M of C_X such that $L \succ M$ for any nonzero subobject L of M.

The spectrum $\mathbf{Spec}(X)$ consists of subcategories $[M]$, where $M \in Spec(X)$. The preorder $\succ$ induces on $\mathbf{Spec}(X)$ the partial order $\supseteq$.

Every simple object of C_X belongs to $Spec(X)$, and if $M,\ N$ are simple objects, then $M \succ N$ iff they are isomorphic. Moreover, if M is a simple object and N is a nonzero object, then $M \succ N$ iff N is a coproduct of finite number of copies of M.

This shows that simple objects are maximal objects of C_X for the preorder $\succ$, and isomorphism classes of simple objects are maximal elements of $(\mathbf{Spec}(X), \supseteq)$.

A2.3. Connections with the left spectrum of a ring. Let C_X be the category $R-mod$ of left modules over a ring R. Then for any pair of left ideals, $\mathfrak{m},\ \mathfrak{n}$ the relation $R/\mathfrak{m} \succ R/\mathfrak{n}$ is equivalent to the relation $\mathfrak{m} \leq \mathfrak{n}$ (cf. A1).

By [R, Ch.3, 4.3], the map $\mathfrak{m} \longmapsto R/\mathfrak{m}$ assigns to each element of $Spec_\ell R$ an object of $Spec(X)$ and this map is an equivalence of preorders $(Spec_\ell R, \leq) \longrightarrow (Spec(X), \succ)$, that is the map $\mathfrak{m} \longmapsto [R/\mathfrak{m}]$ induces an isomorphism of the partial order associated with the left spectrum $(Spec_\ell R, \leq)$ onto the spectrum $(\mathbf{Spec}(X), \supseteq)$ of X.

A3. Local categories. An abelian category C_X is called *local* if it has a nonzero object, P, such that $M \succ P$ for any nonzero object M. An objects P possessing this property is called (in [R]) *quasifinal*. All quasifinal objects belong to $Spec(X)$ and are equivalent one to another. They define a unique maximal element of the spectrum $\mathbf{Spec}(X)$.

It follows that if C_X is local and has simple objects, then all of them are isomorphic one to another. If C_X is the category of modules over a commutative ring R, then C_X is local iff the ring R is local.

A4. The complete spectrum. The *complete spectrum* of a 'space' X represented by an abelian category C_X is the partial order $(\mathbf{Spec}^1(X), \subseteq)$ of all thick subcategories $\mathcal{P}$ of C_X such that the quotient category $C_X/\mathcal{P}$ is local.

A4.1. Note. The notions of a local category and a thick subcategory are selfdual. Therefore the complete spectrum of a category C_X is naturally isomorphic to the complete spectrum of its opposite category, C_X^{op}. Setting $C_{X^o} = C_X^{op}$, we can write this as $\mathbf{Spec}^1(X) \simeq \mathbf{Spec}^1(X^o)$.

A5. S-spectrum. For any subcategory $\mathbb{T}$ of an abelian category C_X, let $\mathbb{T}^-$ denote the full subcategory of C_X generated by all $M \in ObC_X$ such that any nonzero subquotient of M has a nonzero subobject which is isomorphic to an object of $\mathbb{T}$. This construction has the following properties ([R, Ch.3, 2.3.2.1):

(a) The subcategory $\mathbb{T}^-$ is thick.

(b) $(\mathbb{T}^-)^- = \mathbb{T}^-$.

A subcategory $\mathbb{T}$ is called a *Serre* subcategory if $\mathbb{T}^- = \mathbb{T}$. In particular, every Serre subcategory is thick.

The *S-spectrum* (in [R] it is called the *flat spectrum*), $\mathbf{Spec}^-(X)$ is formed by all Serre subcategories, $\mathcal{P}$, of C_X such that the quotient category $C_X/\mathcal{P}$ is local.

A5.1. Proposition ([R, Ch.3, 3.3.2). *Let C_X be an abelian category. For any object M of $Spec(X)$, the full subcategory $\langle M\rangle$ defined by $Ob\langle M\rangle = \{N \in ObC_X \mid N \not\succ M\}$ belongs to* $\mathbf{Spec}^-(X)$.

Thus, $\mathbf{Spec}(X)$ is naturally embedded into $\mathbf{Spec}^-(X)$.

References.

[Gab] P.Gabriel, Des catégories abéliennes, Bull. Soc. Math. France, 90 (1962), 323-449

[GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Springer Verlag, Berlin-Heidelberg-New York, 1967

[J] P.T. Johnstone, Topos theory, Academic Press, London-New York-San Francisco, 1977

[R] A.L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer Academic Publishers, Mathematics and Its Applications, v.330 (1995), 328 pages.

[R1] Left spectrum, Levitzki radical, and noncommutative schemes, Proc. Natnl. Acad., v.87, 1990, pp. 8583-8586

[R2] A.L. Rosenberg, Noncommutative local algebra, Geometric and Functional Analysis (GAFA), v.4, no.5 (1994), 545-585

[R3] A.L. Rosenberg, Underlying spaces of noncommutative schemes, preprint MPIM, 2003, 43 pp

[R4] A.L. Rosenberg, Spectra related with localizations, preprint MPIM, 2003, 77 pp

[Ve1] J.–L. Verdier, Catégories dérivées, Séminaire de Géométrie Algébrique 4 1/2, Cohomologie Étale, LNM 569, pp. 262–311, Springer, 1977

[Ve2] J.–L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque, v.239, 1996