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COMPLETE LEFT-INVARIANT AFFINE STRUCTURES ON THE OSCILLATOR GROUP

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Abstract. *The goal of this paper is to provide a method, based on the theory of extensions of left-symmetric algebras, for classifying left-invariant affine structures on a given solvable Lie group of low dimension. To better illustrate our method, we shall apply it to classify complete left-invariant affine structures on the oscillator group.*

1 Introduction

It is a well known result (see [1], [17]) that a simply connected Lie group G which admits a complete left-invariant affine structure, or equivalently G acts simply transitively by affine transformations on $\mathbb{R}^n$, must be solvable. It is also well known that not every solvable (even nilpotent) Lie group can admit an affine structure [3]. On the other hand, given a simply connected solvable Lie group G which can admit a complete left-invariant structure, it is important to classify all such possible structures on G.

Our goal in the present paper is to provide a method for classifying left-invariant affine structures on a given solvable Lie group of low dimension. Since the classification has been completely achieved up to dimension four in the nilpotent case (see [10], [13], [15]), we shall illustrate our method by applying it to the classification of complete left-invariant affine structures on the remarkable solvable non-nilpotent 4-dimensional Lie group O_4, known as the *oscillator group*. Recall that O_4 can be viewed as a semidirect product of the real line with the Heisenberg group. Recall also that the Lie algebra $\mathcal{O}_4$

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of O_4 (that we shall call *oscillator algebra*) is the Lie algebra with generators e_1, e_2, e_3, e_4, and with nonzero brackets

$$[e_1, e_2] = e_3, \ [e_4, e_1] = e_2, \ [e_4, e_2] = e_1.$$

Since left-invariant affine structures on a Lie group G are in one-to-one correspondence with left-symmetric structures on its Lie algebra $\mathcal{G}$ [13], we shall carry out the classification of complete left-invariant affine structures on O_4 in terms of complete (in the sense of [20]) left-symmetric structures on $\mathcal{O}_4$.

The paper is organized as follows. In Section 2, we shall recall the notion of extensions of Lie algebras and its relationship to the notion of $\mathcal{G}$-kernels. In Section 3, we shall give some necessary definitions, notations, and basic results on left-symmetric algebras and their extensions. In Section 4, we shall consider complete non-simple real left-symmetric structures on the oscillator algebra $\mathcal{O}_4$. We shall show that, if A_4 is a complete non-simple left-symmetric algebra whose Lie algebra is $\mathcal{O}_4$, then A_4 contains a proper two-sided ideal whose associated Lie algebra is isomorphic to the center $Z(\mathcal{O}_4) \cong \mathbb{R}$ or the commutator ideal $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$ of $\mathcal{O}_4$. In the latter case, we shall show that the so-called center of A_4 is nontrivial, and therefore we can get A_4 as a central (in some sense that will be defined later) extension of a complete 3-dimensional left-symmetric algebra A_3 by the trivial left-symmetric algebra $\mathbb{R}$.

In Section 5, we shall show that, in both cases above, we have a short exact sequence (which turns out to be central) of left-symmetric algebras of the form

$$0 \to \mathbb{R} \xrightarrow{i} A_4 \xrightarrow{\pi} A_3 \to 0,$$

where A_3 is a complete left-symmetric algebra whose Lie algebra is isomorphic to the Lie algebra $\mathcal{E}(2)$ of the group of Euclidean motions of the plane. We shall then show that, up to left-symmetric isomorphism, there are only two non-isomorphic complete left-symmetric structures on $\mathcal{E}(2)$, and we shall use these to carry out all complete non-simple left-symmetric structures on $\mathcal{O}_4$. We shall see that one of these two left-symmetric structures on $\mathcal{E}(2)$ yields exactly one complete left-symmetric structure on $\mathcal{O}_4$. However, the second one yields a two-parameter family of complete left-symmetric algebras $A_4(s,t)$ whose associated Lie algebra is $\mathcal{O}_4$, and the conjugacy class of $A_4(s,t)$ is given as follows: $A_4(s',t')$ is isomorphic to $A_4(s,t)$ if and only if

$(s', t') = (\alpha s, \pm t)$ for some $\alpha \in \mathbb{R}^*$. By using the Lie group exponential maps, we shall deduce the classification of complete left-invariant affine structures on the oscillator group O_4 in terms of simply transitive actions of subgroups of the affine group $Aff(\mathbb{R}^4) = GL(\mathbb{R}^4) \ltimes \mathbb{R}^4$ (see Theorem 22).

Throughout this paper, all vector spaces, Lie algebras, and left-symmetric algebras are supposed to be over the filed $\mathbb{R}$, unless otherwise specified. We shall also suppose that all Lie groups are connected and simply connected.

2 Extensions of Lie algebras

Recall that a Lie algebra $\widetilde{\mathcal{G}}$ is an extension of the Lie algebra $\mathcal{G}$ by the Lie algebra $\mathcal{A}$ if there exists a short exact sequence of Lie algebras

$$0 \to \mathcal{A} \xrightarrow{i} \widetilde{\mathcal{G}} \xrightarrow{\pi} \mathcal{G} \to 0. \tag{1}$$

In other words, if we identify the elements of $\mathcal{A}$ with their images in $\widetilde{\mathcal{G}}$ via the injection i, then $\mathcal{A}$ is an ideal in $\widetilde{\mathcal{G}}$ such that $\widetilde{\mathcal{G}}/\mathcal{A} \cong \mathcal{G}$.

Two extensions $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ are called equivalent if there exists an isomorphism of Lie algebras φ such that the following diagram commutes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{A} & \xrightarrow{i_1} & \widetilde{\mathcal{G}}_1 & \xrightarrow{\pi_1} & \mathcal{G} & \longrightarrow 0 \\
& id_{\mathcal{A}} \downarrow & & \downarrow \varphi & & \downarrow id_{\mathcal{G}} & \\
0 \longrightarrow & \mathcal{A} & \xrightarrow{i_2} & \widetilde{\mathcal{G}}_2 & \xrightarrow{\pi_2} & \mathcal{G} & \longrightarrow 0
\end{array}$$

The notion of extensions of a Lie algebra $\mathcal{G}$ by an abelian Lie algebra $\mathcal{A}$ is well known (see for instance, the books [8] and [12]). In light of [19], we shall describe here the notion of extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by a Lie algebra $\mathcal{A}$ which is not necessarily abelian.

Suppose that a vector space extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by another Lie algebra $\mathcal{A}$ is known, and we want to define a Lie structure on $\widetilde{\mathcal{G}}$ in terms of the Lie structures of $\mathcal{G}$ and $\mathcal{A}$. Let $\sigma : \mathcal{G} \to \widetilde{\mathcal{G}}$ be a section, that is, a linear map such that $\pi \circ \sigma = id$. Then the linear map $\Psi : (a, x) \mapsto i(a) + \sigma(x)$ from $\mathcal{A} \oplus \mathcal{G}$ onto $\widetilde{\mathcal{G}}$ is an isomorphism of vector spaces.

For (a, x) and (b, y) in $\mathcal{A} \oplus \mathcal{G}$, a commutator on $\widetilde{\mathcal{G}}$ must satisfy

$$\begin{aligned}
[i(a) + \sigma(x), i(b) + \sigma(y)] &= i([a, b]) + [\sigma(x), i(b)] \\
&\quad + [i(a), \sigma(y)] + [\sigma(x), \sigma(y)]
\end{aligned} \tag{2}$$

Now we define a linear map $\phi : \mathcal{G} \rightarrow End\,(\mathcal{A})$ by

$$\phi\,(x)\,a = [\sigma\,(x)\,, i\,(a)] \tag{3}$$

On the other hand, since

$$\pi\,([\sigma\,(x)\,, \sigma\,(y)]) = \pi\,(\sigma\,([x, y]))\,,$$

it follows that there exists an alternating bilinear map $\omega : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{A}$ such that

$$[\sigma\,(x)\,, \sigma\,(y)] = \sigma\,[x, y] + \omega\,(x, y)\,.$$

In summary, by means of the isomorphism above, $\widetilde{\mathcal{G}} \cong \mathcal{A} \oplus \mathcal{G}$ and its elements may be denoted by (a, x) with $a \in \mathcal{A}$ and x is simply characterized by its coordinates in $\mathcal{G}$. The commutator defined by (2) is now given by

$$[(a, x)\,, (b, y)] = ([a, b] + \phi\,(x)\,b - \phi\,(y)\,a + \omega\,(x, y)\,, [x, y])\,, \tag{4}$$

for all $(a, x) \in \widetilde{\mathcal{G}} \cong \mathcal{A} \oplus \mathcal{G}$.

Now, it is easy to see that this is actually a Lie bracket (i.e, it verifies the Jacobi identity) if and only if the following three conditions are satisfied

1. $\phi\,(x)\,[b, c] = [\phi\,(x)\,b, c] + [b, \phi\,(x)\,c]\,,$
2. $[\phi\,(x)\,, \phi\,(y)] = \phi\,([x, y]) + ad_{\omega(x,y)},$
3. $\omega\,([x, y]\,, z) - \omega\,(x, [y, z]) + \omega\,(y, [x, z]) = \phi\,(x)\,\omega\,(y, z) + \phi\,(y)\,\omega\,(z, x) + \phi\,(z)\,\omega\,(x, y)\,.$

Remark 1 *We see that the condition (1) above is equivalent to say that $\phi\,(x)$ is a derivation of $\mathcal{A}$. In other words, $\mathcal{G}$ is actually acting by derivations, that is, $\phi : \mathcal{G} \rightarrow Der\,(\mathcal{A})$. The condition (2) indicates clearly that if $\mathcal{A}$ is supposed to be abelian, then $\mathcal{A}$ becomes a $\mathcal{G}$-module in a natural way, because in this case the linear map $\phi : \mathcal{G} \rightarrow Der\,(\mathcal{A})$ given by $\phi\,(x)\,a = [\sigma\,(x)\,, i\,(a)]$ is well defined. The condition (3) is equivalent to the fact that, if $\mathcal{A}$ is abelian, ω is a 2-cocycle (i.e., $\delta_\phi \omega = 0$, where δ_ϕ refers to the coboundary operator corresponding to the action ϕ).*

If now $\sigma' : \mathcal{G} \to \widetilde{\mathcal{G}}$ is another section, then $\sigma' - \sigma = \tau$ for some linear map $\tau : \mathcal{G} \to \mathcal{A}$, and it follows that the corresponding morphism and 2-cocycle are, respectively, $\phi' = \phi + ad \circ \tau$ and $\omega' = \omega + \delta_\phi \tau + \frac{1}{2}[\tau,\tau]$, where ad stands here and below (if there is no ambiguity) for the adjoint representation in $\mathcal{A}$, and where $[\tau,\tau]$ has the following meaning : Given two linear maps $\alpha,\beta : \mathcal{G} \to \mathcal{A}$, we define $[\alpha,\beta](x,y) = [\alpha(x),\beta(y)] - [\alpha(y),\beta(x)]$. In particular, we have $\frac{1}{2}[\tau,\tau](x,y) = [\tau(x),\tau(y)]$. Note here that the Lie algebra $\mathcal{A}$ is not necessarily abelian. Therefore, $\omega' - \omega$ is a 2-coboundary if and only if $[\tau(x),\tau(y)] = 0$ for all $x,y \in \mathcal{G}$. Equivalently, $\omega' - \omega$ is a 2-coboundary if and only if τ has its range in the center $Z(\mathcal{A})$ of $\mathcal{A}$. In that case, we get $\omega' - \omega = \delta_\phi \tau \in B^2_\phi(\mathcal{G}, Z(\mathcal{A}))$, the group of 2-coboundaries for $\mathcal{G}$ with values in $Z(\mathcal{A})$.

To overcome all these difficulties, we proceed as follows. Let $C^2(\mathcal{G},\mathcal{A})$ be the abelian group of all 2-cochains, i.e. alternating bilinear mappings $\mathcal{G} \times \mathcal{G} \to \mathcal{A}$. For a given $\phi : \mathcal{G} \to Der(\mathcal{A})$, let $T_\phi \in C^2(\mathcal{G},\mathcal{A})$ be defined by

$$T_\phi(x,y) = [\phi(x),\phi(y)] - \phi([x,y]), \quad \text{for all } x,y \in \mathcal{G}.$$

If there exists some $\omega \in C^2(\mathcal{G},\mathcal{A})$ such that $T_\phi = ad \circ \omega$ and $\delta_\phi \omega = 0$, then the pair (ϕ,ω) is called a *factor system* for $(\mathcal{G},\mathcal{A})$. Let $Z^2(\mathcal{G},\mathcal{A})$ be the set of all factor systems for $(\mathcal{G},\mathcal{A})$. It is well known that the equivalence classes of extensions of a Lie algebra $\mathcal{G}$ by a Lie algebra $\mathcal{A}$ are in one-to-one correspondence with the elements of the quotient space $Z^2(\mathcal{G},\mathcal{A})/C^1(\mathcal{G},\mathcal{A})$, where $C^1(\mathcal{G},\mathcal{A})$ is the space of linear maps from $\mathcal{G}$ into $\mathcal{A}$ (see for instance [19], Theorem II.7). Note that if we assume that $\mathcal{A}$ is abelian, then we meet the well known result (see for instance [7]) stating that for a given action $\phi : \mathcal{G} \to End(\mathcal{A})$, the equivalence classes of extensions of $\mathcal{G}$ by $\mathcal{A}$ are in one-to-one correspondence with the elements of the second cohomology group

$$H^2_\phi(\mathcal{G},\mathcal{A}) = Z^2_\phi(\mathcal{G},\mathcal{A})/B^2_\phi(\mathcal{G},\mathcal{A}).$$

In the present paper, we shall be concerned with the special case where $\mathcal{A}$ is non-abelian and $\mathcal{G}$ is the field $\mathbb{R}$, and henceforth the cocycle ω is identically zero.

Remark 2 *It is worth noticing that the construction above is closely related to the notion of $\mathcal{G}$-kernels (considered for Lie algebras firstly in [18]) . On $\{\phi : \mathcal{G} \to Der(\mathcal{A}) : T_\phi = ad \circ \omega,$ for some $\omega \in C^2(\mathcal{G},\mathcal{A})\}$, define an equivalence relation by $\phi \sim \phi'$ if and only if $\phi' = \phi + ad \circ \tau$, for some linear map*

$\tau : \mathcal{G} \to \mathcal{A}$. *The equivalence class* $[\phi]$ *of* ϕ *is called a* $\mathcal{G}$-kernel. *It turns out that if* $\mathcal{A}$ *is abelian, then a* $\mathcal{G}$*-kernel is nothing but a* $\mathcal{G}$*-module. By considering the quotient morphism* $\Pi : Der(\mathcal{A}) \to Out(\mathcal{A}) = Der(\mathcal{A})/ad_{\mathcal{A}}$*, and remarking that* $\Pi \circ ad \circ \tau = 0$ *for any linear map* $\tau : \mathcal{G} \to \mathcal{A}$*, we can naturally associate to each* $\mathcal{G}$*-kernel* $[\phi]$ *the morphim* $\phi = \Pi \circ [\phi] : \mathcal{G} \to Out(\mathcal{A})$.

3 Extensions of left-symmetric algebras

The notion of a *left-symmetric algebra* arises naturally in various areas of mathematics and physics. It originally appeared in the works of Vinberg [21] and Koszul [14] concerning convex homogeneous cones and bounded homogeneous domains, respectively. It also appears, for instance, in connection with Yang-Baxter equation and integrable hydrodynamic systems (cf. [4], [11], [16]).

A left-symmetric algebra $(A, .)$ is a finite-dimensional algebra A in which the products, for all $x, y, z \in A$, satisfy the identity

$$(xy)z - x(yz) = (yx)z - y(xz), \tag{5}$$

where here and frequently during this paper we simply write xy instead of $x \cdot y$.

It is clear that an associative algebra is a left-symmetric algebra. Actually, for a left-symmetric algebra A, if $(x, y, z) = (xy)z - x(yz)$ is the associator of x, y, z, then we see that (5) is equivalent to $(x, y, z) = (y, x, z)$ This means that left-symmetric algebras are natural generalizations of associative algebras.

Now if A is a left-symmetric algebra, then the commutator

$$[x, y] = xy - yx \tag{6}$$

defines a structure of Lie algebra on A, called the *associated Lie algebra*. On the other hand, if $\mathcal{G}$ is a Lie algebra with a left-symmetric product $\cdot$ satisfying

$$[x, y] = x \cdot y - y \cdot x,$$

then we say that the left-symmetric structure is *compatible* with the Lie structure on $\mathcal{G}$.

Suppose now we are given a Lie group G with a left-invariant flat affine connection ∇, and define a product $\cdot$ on the Lie algebra $\mathcal{G}$ of G by

$$x \cdot y = \nabla_x y, \tag{7}$$

for all $x, y \in \mathcal{G}$. Then, the conditions on the connection ∇ for being flat and torsion-free are now equivalent to the conditions (5) and (6), respectively.

Conversely, suppose that G is a simply connected Lie group with Lie algebra $\mathcal{G}$, and suppose that $\mathcal{G}$ is endowed with a left-symmetric product $\cdot$ which is compatible with the Lie bracket of $\mathcal{G}$. We define an operator ∇ on $\mathcal{G}$ according to identity (7), and then we extend it by left-translations to the whole Lie group G. This clearly defines a left-invariant flat affine structure on G. In summary, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the left-invariant flat affine structures on G are in one-to-one correspondence with the left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure.

Let A be a left-symmetric algebra, and let the left and right multiplications L_x and R_x by the element x be defined by $L_x y = x \cdot y$ and $R_x y = y \cdot x$. We say that A is *complete* if R_x is a nilpotent operator, for all $x \in A$. It turns out that, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the complete left-invariant flat affine structures on G are in one-to-one correspondence with the complete left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure (see for example [13]). It is also known that an n-dimensional simply connected Lie group admits a complete left-invariant flat affine structure if and only if it acts simply transitively on $\mathbb{R}^n$ by affine transformations (see [13]). A simply connected Lie group which is acting simply transitively on $\mathbb{R}^n$ by affine transformations must be solvable according to [1], but it is worth noticeable that there exist solvable (even nilpotent) Lie groups which do not admit affine structures (see [3]).

We close this section by fixing some notations which we will use in what follows. For a left-symmetric algebra A, we can easily check that the subset

$$T(A) = \{x \in A : L_x = 0\} \tag{8}$$

is a two-sided ideal in A. Geometrically, if G is a Lie group which acts simply transitively on $\mathbb{R}^n$ by affine transformations then $T(\mathcal{G})$ corresponds to the set of translational elements in G, where $\mathcal{G}$ is endowed with the complete left-symmetric product corresponding to the action of G on $\mathbb{R}^n$. It has been conjectured in [1] that every nilpotent Lie group G which acts simply transitively on $\mathbb{R}^n$ by affine transformations contains a translation which lies in the center of G, but this conjecture turned out to be false (see [9]).

We discussed in the last section the problem of extension of a Lie algebra by another Lie algebra. Similarly, we shall briefly discuss in this section the problem of extension of a left-symmetric algebra by another left-symmetric algebra. To our knowledge, the notion of extensions of left-symmetric algebras has been considered for the first time in [13], to which we refer for more details.

Suppose we are given a vector space A as an extension of a left-symmetric algebra K by another left-symmetric algebra E . We want to define a left-symmetric structure on A in terms of the left-symmetric structures given on K and E. In other words, we want to define a left-symmetric product on A for which E becomes a two-sided ideal in A such that $A/E \cong K$; or equivalently,

$$0 \to E \to A \to K \to 0$$

becomes a short exact sequence of left-symmetric algebras.

Theorem 3 ([13]) *There exists a left-symmetric structure on A extending a left-symmetric algebra K by a left-symmetric algebra E if and only if there exist two linear maps λ, $\rho : K \to End\,(E)$ and a bilinear map $g : K \times K \to E$ such that, for all $x, y, z \in K$ and $a, b \in E$, the following conditions are satisfied.*

(i) $\lambda_x (a \cdot b) = \lambda_x (a) \cdot b + a \cdot \lambda_x (b) - \rho_x (a) \cdot b,$

(ii) $\rho_x ([a, b]) = a \cdot \rho_x (b) - b \cdot \rho_x (a)\,,$

(iii) $[\lambda_x, \lambda_y] = \lambda_{[x,y]} + L_{g(x,y)-g(y,x)}$, *where $L_{g(x,y)-g(y,x)}$ denotes the left multiplication in E by $g\,(x, y) - g\,(y, x)$.*

(iv) $\left[\lambda_x, \rho_y\right] = \rho_{x\cdot y} - \rho_y \circ \rho_x + R_{g(x,y)}$, *where $R_{g(x,y)}$ denotes the right multiplication in E by $g\,(x, y)$.*

(v) $g\,(x, y \cdot z) - g\,(y, x \cdot z) + \lambda_x\,(g\,(y, z)) - \lambda_y\,(g\,(x, z)) - g\,([x, y]\,, z)$

$-\rho_z\,(g\,(x, y) - g\,(y, x)) = 0.$

If the conditions of Theorem 3 are fulfilled, then the extended left-symmetric product on $A \cong K \times E$ is given by

$$(x, a) \cdot (y, b) = \big(x \cdot y, a \cdot b + \lambda_x\,(b) + \rho_y\,(a) + g\,(x, y)\big)\,. \tag{9}$$

It is remarkable that if the left-symmetric product of E is trivial, then the conditions of Theorem 3 simplify to the following three conditions:

(i) $[\lambda_x, \lambda_y] = \lambda_{[x,y]}$, i.e. λ is a representation of Lie algebras,

(ii) $\left[\lambda_x, \rho_y\right] = \rho_{x\cdot y} - \rho_y \circ \rho_x$.

(iii) $g(x, y\cdot z) - g(y, x\cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) - g([x,y], z)$

$-\rho_z(g(x,y) - g(y,x)) = 0.$

In this case, E becomes a K-bimodule and the extended product given by (9) simplifies too.

Recall that if K is a left-symmetric algebra and V is a vector space, then we say tha V is a K-bimodule if there exist two linear maps $\lambda,\ \rho : K \to End(V)$ which satisfy the conditions (i) and (ii) stated above.

Let K be a left-symmetric algebra, and let V be a K-bimodule. Let $L^p(K,V)$ be the space of all p-linear maps from K to V, and define two coboundary operators $\delta_1 : L^1(K,V) \to L^2(K,V)$ and $\delta_2 : L^2(K,V) \to L^3(K,V)$ as follows : For a linear map $h \in L^1(K,V)$ we set

$$\delta_1 h(x,y) = \rho_y(h(x)) + \lambda_x(h(y)) - h(x\cdot y), \tag{10}$$

and for a bilinear map $g \in L^2(K,V)$ we set

$$\begin{aligned}\delta_2 g(x,y,z) &= g(x, y\cdot z) - g(y, x\cdot z) + \lambda_x(g(y,z)) - \lambda_y(g(x,z)) \\ &\quad -g([x,y], z) - \rho_z(g(x,y) - g(y,x)).\end{aligned} \tag{11}$$

It is straightforward to check that $\delta_2 \circ \delta_1 = 0$. Therefore, if we set $Z^2_{\lambda,\rho}(K,V) = \ker \delta_2$ and $B^2_{\lambda,\rho}(K,V) = \operatorname{Im}\delta_1$, we can define a notion of second cohomology for the actions λ and ρ by simply setting $H^2_{\lambda,\rho}(K,V) = Z^2_{\lambda,\rho}(K,V)/B^2_{\lambda,\rho}(K,V)$.

As in the case of extensions of Lie algebras, we can prove that for given linear maps $\lambda,\ \rho : K \to End(V)$, the equivalent classes of extensions $0 \to V \to A \to K \to 0$ of K by V are in one-to-one correspondence with the elements of the second cohomology group $H^2_{\lambda,\rho}(K,V)$.

3.1 Central extensions of left-symmetric algebras

The notion of central extensions known for Lie algebras may analogously be defined for left-symmetric algebras. Let A be a left-symmetric extension of a left-symmetric algebra K by another left-symmetric algebra E, and let $\mathcal{G}$ be the Lie algebra associated to A. Define *the center* $C(A)$ of A to be

$$C(A) = T(A) \cap Z(\mathcal{G}) = \{x \in A : x \cdot y = y \cdot x = 0, \quad \text{for all } y \in A\}, \quad (12)$$

where $Z(\mathcal{G})$ is the center of the Lie algebra $\mathcal{G}$ and $T(A)$ is the two-sided ideal of A defined by (8).

Definition 4 *The extension* $0 \to E \xrightarrow{i} A \xrightarrow{\pi} K \to 0$ *of left-symmetric algebras is said to be* central *(resp.* exact*) if* $i(E) \subseteq C(A)$ *(resp.* $i(E) = C(A)$*).*

Remark 5 *It is not difficult to show that if the extension* $0 \to E \xrightarrow{i} A \xrightarrow{\pi} K \to 0$ *is central, then both the left-symmetric product and the* K*-bimodule on* E *are trivial (i.e.,* $a \cdot b = 0$ *for all* $a, b \in E$*, and* $\lambda = \rho = 0$*). In this case, the left-symmetric given by (9) simplifies to* $(x, a) \cdot (y, b) = (x \cdot y, g(x, y))$.

We will require the following lemma, whose proof is immediate from the above remark.

Lemma 6 *Let* $0 \to E \xrightarrow{i} A \xrightarrow{\pi} K \to 0$ *be a central extension of left-symmetric algebras. Then,* A *is complete if and only if* E *and* K *are complete.*

Remark 7 *We should notice here that an announcement of Lemma 6 in the case of an arbitrary extension appeared in [6].*

Let now $[g]$ denote the cohomology class associated to the extension $0 \to E \xrightarrow{i} A \xrightarrow{\pi} K \to 0$, and let

$$I_{[g]} = \{x \in K : x \cdot y = y \cdot x = 0 \text{ and } g(x, y) = g(y, x) = 0, \text{ for all } y \in K\}.$$

The set $I_{[g]}$ is well defined because any other element in $[g]$ takes the form $g + \delta_1 h$, with $\delta_1 h(x, y) = -h(x \cdot y)$. The following lemma can be easily proved (see [13]).

Lemma 8 *The extension* $0 \to E \xrightarrow{i} A \xrightarrow{\pi} K \to 0$ *is exact if and only if* $I_{[g]} = 0$.

Let now K be a left-symmetric algebra, and E a trivial K-bimodule. Denote by $(A, [g])$ the central extension $0 \to E \to A \to K \to 0$ corresponding to the cohomology class $[g] \in H^2(K, E)$. Let $(A, [g])$ and $(A', [g'])$ be two central extensions of K by E, and let $\mu \in Aut(E) = GL(E)$ and $\eta \in Aut(K)$, where $Aut(E)$ and $Aut(K)$ are the groups of left-symmetric automorphisms of E and K, respectively. It is clear that, if $h \in L^1(K, E)$, then the linear mapping $\psi : A \to A'$ defined by

$$\psi(x, a) = (\eta(x), \mu(a) + h(x))$$

is an isomorphism provided $g'(\eta(x), \eta(y)) = \mu(g(x, y)) - \delta_1 h(x, y)$ for all $(x, y) \in K \times K$, i.e. $\eta^*[g'] = \mu_*[g]$.

This allows us to define an action of the group $G = Aut(E) \times Aut(K)$ on $H^2(K, E)$ by setting

$$(\mu, \eta) . [g] = \mu_* \eta^* [g], \tag{13}$$

or equivalently, $(\mu, \eta) .g(x, y) = \mu(g(\eta(x), \eta(y)))$ for all $x, y \in K$.

Denoting the set of all exact central extensions of K by E by

$$H^2_{ex}(K, E) = \left\{[g] \in H^2(K, E) : I_{[g]} = 0\right\},$$

and the orbit of $[g]$ by $G_{[g]}$, it turns out that the following result is valid (see [13]).

Proposition 9 *Let $[g]$ and $[g']$ be two classes in $H^2_{ex}(K, E)$. Then, the central extensions $(A, [g])$ and $(A', [g'])$ are isomorphic if and only if $G_{[g]} = G_{[g']}$. In other words, the classification of the exact central extensions of K by E is, up to left-symmetric isomorphism, the orbit space of $H^2_{ex}(K, E)$ under the natural action of $G = Aut(E) \times Aut(K)$.*

4 Non-simple real left-symmetric structures on the oscillator algebra

Recall that the Heisenberg group H_3 is the 3-dimensional Lie group diffeomorphic to $\mathbb{R} \times \mathbb{C}$ with the group law

$$(v_1, z_1) \cdot (v_2, z_2) = (v_1 + v_2 + \frac{1}{2} \operatorname{Im}(\overline{z_1} z_2), z_1 + z_2),$$

for all $v_1, v_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$.

Let $\lambda > 0$, and let $G = \mathbb{R} \ltimes H_3$ be equipped with the group law

$$(t_1, v_1, z_1) \cdot (t_2, v_2, z_2) = (t_1 + t_2, v_1 + v_2 + \frac{1}{2} \operatorname{Im}(\overline{z_1} z_2 e^{i\lambda t_1}), z_1 + z_2 e^{i\lambda t_1}),$$

for all $t_1, t_2 \in \mathbb{R}$ and $(v_1, z_1), (v_2, z_2) \in H_3$. This is a 4-dimensional Lie group with Lie algebra $\mathcal{G}$ having a basis $\{e_1, e_2, e_3, e_4\}$ such that

$$[e_1, e_2] = e_3, \ [e_4, e_1] = \lambda e_2, \ [e_4, e_2] = -\lambda e_1,$$

and all the other brackets are zero.

It follows that the derived series is given by

$$\mathcal{D}^1\mathcal{G} = [\mathcal{G}, \mathcal{G}] = span\{e_1, e_2, e_3\}, \ \mathcal{D}^2\mathcal{G} = span\{e_3\}, \ \mathcal{D}^3\mathcal{G} = \{0\},$$

and therefore $\mathcal{G}$ is a (non-nilpotent) 3-step solvable Lie algebra.

When $\lambda = 1$, G is known as the *oscillator group.* We shall denote it by O_4, and we shall denote its Lie algebra by $\mathcal{O}_4$ and call it the *oscillator algebra.*

Let A_4 be a complete non-simple real left-symmetric algebra whose associated Lie algebra is $\mathcal{O}_4$. To continue, we first need to state the following straightforward lemmas.

Lemma 10 *Let A be a left-symmetric algebra with Lie algebra $\mathcal{G}$, and R a two-sided ideal in A. Then, the Lie algebra $\mathcal{R}$ associated to R is an ideal in $\mathcal{G}$.*

Lemma 11 *The oscillator algebra $\mathcal{O}_4$ contains only two proper ideals which are $Z(\mathcal{O}_4) \cong \mathbb{R}$ and $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$.*

Since A_4 is not simple, let I be a proper two-sided ideal in A_4. It follows that we have a short exact sequence of left-symmetric algebras

$$0 \to I \xrightarrow{i} A_4 \xrightarrow{\pi} J \to 0. \tag{14}$$

If $\mathcal{I}$ is the Lie subalgebra associated to I then, by Lemma 10, $\mathcal{I}$ is an ideal in $\mathcal{O}_4$. From Lemma 11, it follows that there are two cases to consider according to whether $\mathcal{I}$ is isomorphic to $\mathcal{H}_3$ or $\mathbb{R}$.

Next, we shall focus on the case where $\mathcal{I}$ is isomorphic to $\mathcal{H}_3 \cong [\mathcal{O}_4, \mathcal{O}_4]$. In this case, the short exact sequence (14) becomes

$$0 \to I_3 \xrightarrow{i} A_4 \xrightarrow{\pi} \mathbb{R} \to 0, \tag{15}$$

where I_3 is a complete 3-dimensional left-symmetric algebra whose underlying Lie algebra is $\mathcal{H}_3$, and $\mathbb{R}$ is the trivial 1-dimensional left-symmetric algebra (i.e., $\mathbb{R}$ with the zero product). It is not hard to prove the following proposition (compare [10], Theorem 3.5).

Proposition 12 *Up to left-symmetric isomorphism, the complete left-symmetric structures on the Heisenberg algebra $\mathcal{H}_3$ are classified as follows: There is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the left-symmetric product is given by one of the following classes:*

(i) $e_1 \cdot e_1 = pe_3$, $e_2 \cdot e_2 = qe_3$, $e_1 \cdot e_2 = \frac{1}{2}e_3$, $e_2 \cdot e_1 = -\frac{1}{2}e_3$, *where* $p, q \in \mathbb{R}$.

(ii) $e_1 \cdot e_2 = me_3$, $e_2 \cdot e_1 = (m-1)\, e_3$, $e_2 \cdot e_2 = e_1$, *where* $m \in \mathbb{R}$.

Remark 13 *It is noticeable that the left-symmetric products on $\mathcal{H}_3$ belonging to class (i) in Proposition 12 are obtained by central extensions (in the sense fixed in Subsection 3.1) of $\mathbb{R}^2$ endowed with some complete left-symmetric structure by $\mathbb{R}$ endowed with the trivial left-symmetric product. However, the left-symmetric products on A_3 belonging to class (ii) are obtained by central extensions of the nonabelian two-dimensional Lie algebra $\mathcal{G}_2$ endowed with its unique complete left-symmetric structure by $\mathbb{R}$ endowed with the trivial left-symmetric structure.*

Now we can return to the short exact sequence (15). First, let $\sigma : \mathbb{R} \to A_4$ be a section, and set $\sigma(1) = x_0 \in A_4$, and define two linear maps λ, $\rho \in End\,(I_3)$ by putting $\lambda(y) = x_0 \cdot y$ and $\rho(y) = y \cdot x_0$, and put $\mathbf{e} = x_0 \cdot x_0$ (clearly $\mathbf{e} \in I_3$).

Let $g : \mathbb{R} \times \mathbb{R} \to I_3$ be the bilinear map defined by $g(a, b) = \sigma(a) \cdot \sigma(b) - \sigma(a \cdot b)$. It is clear that $g(a, b) = ab\mathbf{e}$, or equivalently $g(1, 1) = \mathbf{e}$, and it is obvious too (using the notation of Section 3) to verify that $\delta_2 g = 0$, i.e. $g \in Z^2_{\lambda,\rho}(\mathbb{R}, I_3)$.

The extended left-symmetric product on $I_3 \oplus \mathbb{R}$ given by (9) turns out to take the simplified form

$$(x, a) \cdot (y, b) = (x \cdot y + a\lambda(y) + b\rho(x) + ab\mathbf{e}, 0), \tag{16}$$

for all $x, y \in I_3$ and $a, b \in \mathbb{R}$.

The conditions in Theorem 3 can be simplified to the following conditions:

$$\begin{aligned}
\lambda(x \cdot y) &= \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y && (17)\\
\rho([x, y]) &= x \cdot \rho(y) - y \cdot \rho(x) && (18)\\
[\lambda, \rho] + \rho^2 &= R_{\mathbf{e}} && (19)
\end{aligned}$$

Let $\phi : \mathbb{R} \to End(\mathcal{H}_3)$ be the linear map defined by formula (3). As we mentioned in Remark 1, $\mathbb{R}$ acts on $\mathcal{H}_3$ by derivations, that is, $\phi : \mathbb{R} \to Der(\mathcal{H}_3)$. In particular, we deduce in view of (4) that $\lambda = D + \rho$ for some derivation D of $\mathcal{H}_3$. The derivations of $\mathcal{H}_3$ are given by the following lemma, whose proof is straightforward and is therefore omitted.

Lemma 14 *In a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ satisfying $[e_1, e_2] = e_3$, a derivation D of $\mathcal{H}_3$ takes the form*

$$D = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & a_1 + b_2 \end{pmatrix}.$$

On the other hand, observe that $(x, a) \in T(A_4)$ if and only if $(x, a) \cdot (y, b) = (0, 0)$ for all $(y, b) \in I_3 \oplus \mathbb{R}$, or equivalently, $x \cdot y + a\lambda(y) + b\rho(x) + ab\mathbf{e} = 0$ for all $(y, b) \in I_3 \oplus \mathbb{R}$. Since y and b are arbitrary, we conclude that this is also equivalent to say that $(L_x)_{|_{I_3}} = -a\lambda$ and $\rho(x) = -a\mathbf{e}$. In particular, an element $x \in I_3$ belongs to $T(A_4)$ if and only if $(L_x)_{|_{I_3}} = 0$ and $\rho(x) = 0$, or equivalently,

$$I_3 \cap T(A_4) = T(I_3) \cap \ker \rho. \tag{20}$$

The following lemma will be crucial for the classification of complete left-symmetric structures on $\mathcal{O}_4$.

Lemma 15 *The center* $C(A_4) = T(A_4) \cap Z(\mathcal{O}_4)$ *is nontrivial.*

Proof. In view of Proposition 12, we have to consider two cases.

Case 1. Assume that there is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the left-symmetric product of I_3 is given by : $e_1 \cdot e_1 = pe_3$, $e_2 \cdot e_2 = qe_3$, $e_1 \cdot e_2 = \frac{1}{2}e_3$, $e_2 \cdot e_1 = -\frac{1}{2}e_3$, where $p, q \in \mathbb{R}$.

Substituting $x = e_1$ and $y = e_2$ into (18), we find that the operator ρ takes the form

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

with $\gamma_3 = p\beta_1 - q\alpha_2 + \frac{1}{2}(\alpha_1 + \beta_2)$. Since $\lambda = D + \rho$ for some $D \in \mathcal{H}_3$, we use Lemma 14 to deduce that

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 & \beta_1 + b_1 & 0 \\ \alpha_2 + a_2 & \beta_2 + b_2 & 0 \\ \alpha_3 + a_3 & \beta_3 + b_3 & \gamma_3 + a_1 + b_2 \end{pmatrix}.$$

Since $(L_{e_3})_{|I_3} = 0$ and $\mathbf{e} \in I_3$, then (19) when applied to e_3 gives

$$\gamma_3^2 e_3 = e_3 \cdot \mathbf{e} = 0,$$

from which we get $\gamma_3 = 0$, i.e. $\rho(e_3) = 0$. It follows from (20) that $e_3 \in T(A_4)$. Since $Z(\mathcal{O}_4) = \mathbb{R}e_3$, we deduce that $C(A_4) = T(A_4) \cap Z(\mathcal{O}_4) \neq 0$, as required.

Case 2. Assume now that there is a basis $\{e_1, e_2, e_3\}$ of $\mathcal{H}_3$ relative to which the left-symmetric product of I_3 is given by : $e_1 \cdot e_2 = me_3$, $e_2 \cdot e_1 = (m-1)e_3$, $e_2 \cdot e_2 = e_1$, where m is a real number.

Substituting successively $x = e_1,\ y = e_2$ and $x = e_2,\ y = e_3$ into equation (18), we find that the operator ρ takes the form

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & -\alpha_2 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & m\beta_2 - (m-1)\alpha_1 \end{pmatrix}, \tag{21}$$

with $(m-1)\alpha_2 = 0$.

We claim that $\alpha_2 = 0$. To prove this, let us assume to the contrary that $\alpha_2 \neq 0$. It follows that $m = 1$, and therefore

$$\begin{aligned} \rho(e_3) &= -\alpha_2 e_1 + \beta_2 e_3 \\ \rho^2(e_3) &= -\alpha_2(\alpha_1 + \beta_2)e_1 - \alpha_2^2 e_2 + \left(\beta_2^2 - \alpha_2\alpha_3\right)e_3 \end{aligned}$$

Since $\alpha_2 \neq 0$, we deduce that e_3, $\rho(e_3)$, $\rho^2(e_3)$ form a basis of I_3. Since ρ is nilpotent (by completeness of the left-symmetric structure), it follows that $\rho^3(e_3) = 0$. In other words, ρ has the form

$$\rho = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with respect to the basis $e'_1 = -\rho(e_3)$, $e'_2 = \rho^2(e_3)$, $e'_3 = -e_3$.

Using the fact that $\alpha_1 + 2\beta_2 = 0$ which follows from the identity $\rho^3(e_3) = 0$, we see that $e'_1 \cdot e'_2 = \alpha_2^3 e'_3$, $e'_2 \cdot e'_2 = \alpha_2^3 e'_1$, and all other products are zero.

For simplicity, assume without loss of generality that $\alpha_2 = 1$. Since $\lambda = D + \rho$ for some $D \in \mathcal{H}_3$, Lemma 14 tells us that, with respect to the basis e'_1, e'_2, e'_3, the operator λ takes the form

$$\lambda = \begin{pmatrix} a_1 & b_1 & 1 \\ a_2 - 1 & b_2 & 0 \\ a_3 & b_3 & a_1 + b_2 \end{pmatrix}.$$

Applying formula (19) to e'_3 and recalling that $e'_3 \cdot \mathbf{e} = 0$ since $\mathbf{e} \in I_3$, we deduce that $a_2 = 1$ and $b_2 = a_3 = 0$. Then, substituting $x = y = e'_2$ into equation (17), we get $a_1 = b_1 = 0$. Thus, the form of λ reduces to

$$\lambda = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix}.$$

Now, by setting $\mathbf{e} = ae_1 + be_2 + ce_3$ and applying (19) to e_1, we get that $b_3 = -b$. By using (16), we deduce that the nonzero left-symmetric products are

$$\begin{aligned} e'_1 \cdot e'_2 &= e'_3, \quad e'_2 \cdot e'_2 = e'_1, \\ e'_1 \cdot e'_4 &= -e'_2, \quad e'_4 \cdot e'_2 = -be'_3 \\ e'_3 \cdot e'_4 &= e'_4 \cdot e'_3 = e'_1, \quad e'_4 \cdot e'_4 = \mathbf{e}. \end{aligned}$$

This implies, in particular, that $\dim[\mathcal{O}_4, \mathcal{O}_4] = \dim[A_4, A_4] = 2$, a contradiction. It follows that $\alpha_2 = 0$, as desired.

We now return to (21). Since $\alpha_2 = 0$, we have

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ 0 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & m\beta_2 - (m-1)\alpha_1 \end{pmatrix},$$

and since $\lambda = D + \rho$ for some $D \in \mathcal{H}_3$ then, in view of Lemma 14, the operator λ takes the form

$$\lambda = \begin{pmatrix} \alpha_1 + a_1 & \beta_1 + b_1 & 0 \\ a_2 & \beta_2 + b_2 & 0 \\ \alpha_3 + a_3 & \beta_3 + b_3 & a_1 + b_2 + m\beta_2 - (m-1)\,\alpha_1 \end{pmatrix}.$$

Once again, by applying (19) to e_3 and recalling that $e_3 \cdot \mathbf{e} = 0$ since $\mathbf{e} \in I_3$, we deduce that $(m\beta_2 - (m-1)\,\alpha_1)^2 = 0$, thereby showing that $\rho(e_3) = 0$. Now, in view of (20) we get $e_3 \in T(A_4)$, and since $Z(\mathcal{O}_4) = \mathbb{R}e_3$ we deduce that $C(A_4) = T(A_4) \cap Z(\mathcal{O}_4) \neq 0$, as desired. ■

5 Classification

We know from Section 4 that A_4 has a proper two-sided ideal I which is isomorphic to either the trivial left-symmetric algebra $\mathbb{R}$ or a 3-dimensional complete left-symmetric algebra I_3, as described in Proposition 12, whose associated Lie algebra is the Heisenberg algebra $\mathcal{H}_3$. Thus, and according to Lemma 6, the classification of all complete non-simple left-symmetric structures on $\mathcal{O}_4$ can be obtained by considering central extensions of complete left-symmetric algebras.

In case where $I \cong I_3$, we know by Lemma 15 that $C(A_4) \neq \{0\}$. Since, in our situation, $\dim Z(\mathcal{O}_4) = 1$, it follows that $C(A_4) \cong \mathbb{R}$, so that we have a central short exact sequence of complete left-symmetric algebras of the form

$$0 \to \mathbb{R} \to A_4 \to A_3 \to 0. \tag{22}$$

In general, one has that the center of a left-symmetric algebra is a part of the center of the associated Lie algebra, and therefore the following lemma is proved.

Lemma 16 *The Lie algebra associated to A_3 is isomorphic to the Lie algebra $\mathcal{E}(2)$ of the group of Euclidean motions of the plane.*

Recall that $\mathcal{E}(2)$ is solvable non-nilpotent and has a basis $\{e_1, e_2, e_3\}$ which satisfies $[e_1, e_2] = e_3$ and $[e_1, e_3] = -e_2$.

In the case where $I \cong \mathbb{R}$, we know by Lemma 10 that the Lie algebra $\mathcal{I} \cong \mathbb{R}$ associated to $I \cong \mathbb{R}$ is an ideal in $\mathcal{O}_4$. Since, by Lemma 11, $\mathcal{O}_4$ has only two proper ideals which are $Z(\mathcal{O}_4) \cong \mathbb{R}$ and $[\mathcal{O}_4, \mathcal{O}_4] \cong \mathcal{H}_3$, it follows that $\mathcal{I} \cong \mathbb{R}$ coincides with the center $Z(\mathcal{O}_4)$. We deduce from this that, if $\mathcal{J}$ denotes the Lie algebra of the left-symmetric algebra J in the short exact sequence (14), then $\mathcal{J}$ is isomorphic to $\mathcal{E}(2)$. Therefore, we have a short sequence of left-symmetric algebras which looks like (22) except that it would not necessarily be central. But, as we will see a little later, this is necessarily a central extension (i.e., $I \cong C(A_4) \cong \mathbb{R}$).

To summarize, each complete non-simple left-symmetric structure on $\mathcal{O}_4$ may be obtained by extension of a complete 3-dimensional left-symmetric algebra A_3 whose associated Lie algebra is $\mathcal{E}(2)$ by the trivial left-symmetric algebra $\mathbb{R}$.

Next, we shall determine all the complete left-symmetric structures on $\mathcal{E}(2)$. These are described by the following lemma that we state without proof (see [10], Theorem 4.1).

Lemma 17 *Up to left-symmetric isomorphism, any complete left-symmetric structure on $\mathcal{E}(2)$ is isomorphic to the following one which is given in a basis $\{e_1, e_2, e_3\}$ of $\mathcal{E}(2)$ by the relations*

$$e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2,\ e_2 \cdot e_2 = e_3 \cdot e_3 = \varepsilon e_1.$$

There are exactly two nonisomorphic conjugacy classes according to whether $\varepsilon = 0$ or $\varepsilon \neq 0$.

From now on, A_3 will denote the vector space $\mathcal{E}(2)$ endowed with one of the complete left-symmetric structures described in Lemma 17. The extended Lie bracket on $\mathcal{E}(2) \oplus \mathbb{R}$ is given by

$$[(x,a),(y,b)] = ([x,y], \omega(x,y)), \tag{23}$$

with $\omega \in Z^2(\mathcal{E}(2), \mathbb{R})$. The extended left-symmetric product on $A_3 \oplus \mathbb{R}$ is given by

$$(x,a) \cdot (y,b) = \big(x \cdot y, b\lambda_x + a\rho_y + g(x,y)\big), \tag{24}$$

with λ, $\rho : A_3 \to End(\mathbb{R}) \cong \mathbb{R}$ and $g \in Z^2_{\lambda,\rho}(A_3, \mathbb{R})$. Note here that we have identified the value of λ (resp. ρ) at an element $x \in A_3$ with the corresponding real number λ_x (resp. ρ_x) via the isomorphism $End(\mathbb{R}) \cong \mathbb{R}$.

As we have noticed in Section 3, $\mathbb{R}$ is an A_3-bimodule, or equivalently, the conditions in Theorem 3 simplify to the following conditions:

(i) $\lambda_{[x,y]} = 0$.

(ii) $\rho_{x \cdot y} = \rho_y \circ \rho_x$.

(iii) $g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x (g(y, z)) - \lambda_y (g(x, z)) - g([x, y], z)$

$-\rho_z (g(x, y) - g(y, x)) = 0$.

By using (23) and (24), we deduce from

$$[(x, a), (y, b)] = (x, a) \cdot (y, b) - (y, b) \cdot (x, a),$$

that

$$\omega(x, y) = g(x, y) - g(y, x) \quad \text{and } \lambda = \rho. \tag{25}$$

By applying identity (ii) above to $e_i \cdot e_i$, $1 \leq i \leq 3$, we deduce that $\rho = 0$, and a fortiori $\lambda = 0$. In other words, the extension A_4 is always central, i.e., $I \cong C(A_4)$ even in the case where $\mathcal{I} \cong \mathbb{R}$. It follows, according to Lemma that A_4 is complete.

In fact, we have

Claim 18 *The extension* $0 \to \mathbb{R} \to A_4 \to A_3 \to 0$ *is exact.*

Proof. We recall from Subsection 3.1 that the extension given by the short sequence (22) is exact, i.e. $i(\mathbb{R}) = C(A_4)$, if and only if $I_{[g]} = 0$, where

$$I_{[g]} = \{x \in A_3 : x \cdot y = y \cdot x = 0 \text{ and } g(x, y) = g(y, x) = 0, \text{ for all } y \in A_3\}.$$

To show that $I_{[g]} = 0$, let x be an arbitrary element in $I_{[g]}$, and put $x = ae_1 + be_2 + ce_3 \in I_{[g]}$. Now, by computing all the products $x \cdot e_i = e_i \cdot x = 0$, $1 \leq i \leq 3$, we easily deduce that $x = 0$. ∎

Our aim is to classify the complete non-simple left-symmetric structures on $\mathcal{O}(4)$, up to left-symmetric isomorphisms. By Proposition 9, the classification of the exact central extensions of A_3 by $\mathbb{R}$ is, up to left-symmetric isomorphism, the orbit space of $H^2_{ex}(A_3, \mathbb{R})$ under the natural action of $G = Aut(\mathbb{R}) \times Aut(A_3)$. Accordingly, we must compute $H^2_{ex}(A_3, \mathbb{R})$. Since $\mathbb{R}$ is a trivial A_3-bimodule, we see first from formulae (10) and (11) in Section 3 that the coboundary operator δ simplifies as follows:

$$\begin{aligned} \delta_1 h(x, y) &= -h(x \cdot y), \\ \delta_2 g(x, y, z) &= g(x, y \cdot z) - g(y, x \cdot z) - g([x, y], z), \end{aligned}$$

where $h \in L^1(A_3, \mathbb{R})$ and $g \in L^2(A_3, \mathbb{R})$.

In view of Lemma 17, there are two cases to be considered.

Case 1. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2 \rangle$.

In this case, using the first formula above for δ_1, we get

$$\delta_1 h = \begin{pmatrix} 0 & h_{12} & h_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $h_{12} = -h(e_3)$ and $h_{13} = h(e_2)$. Similarly, using the second formula above for δ_2, we verify easily that if g is a cocycle (i.e. $\delta_2 g = 0$) and $g_{ij} = g(e_i, e_j)$, then

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & -g_{23} & g_{22} \end{pmatrix},$$

that is, $g_{21} = g_{31} = 0$, $g_{32} = -g_{23}$, and $g_{33} = g_{22}$. We deduce that, in the basis above, the class $[g] \in H^2(A_3, \mathbb{R})$ of a cocycle g may be represented by a matrix of the simplified form

$$g = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\gamma & \beta \end{pmatrix}.$$

We can now determine the extended left-symmetric structure on A_4. By setting $\widetilde{e}_i = (e_i, 0)$, $1 \leq i \leq 3$, and $\widetilde{e}_4 = (0, 1)$, and using formula (24) which (since $\lambda = \rho = 0$) reduces to

$$(x, a) \cdot (y, b) = (x \cdot y, g(x, y)), \tag{26}$$

we obtain

$$\begin{aligned} \widetilde{e}_1 \cdot \widetilde{e}_1 &= \alpha \widetilde{e}_4,\ \widetilde{e}_2 \cdot \widetilde{e}_2 = \widetilde{e}_3 \cdot \widetilde{e}_3 = \beta \widetilde{e}_4 \\ \widetilde{e}_1 \cdot \widetilde{e}_2 &= \widetilde{e}_3,\ \ \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2, \\ \widetilde{e}_2 \cdot \widetilde{e}_3 &= \gamma \widetilde{e}_4,\ \ \widetilde{e}_3 \cdot \widetilde{e}_2 = -\gamma \widetilde{e}_4, \end{aligned} \tag{27}$$

and all the other products are zero. We observe here that we should have $\gamma \neq 0$, given that the underlying Lie algebra is $\mathcal{O}(4)$. We denote by $A_4(\alpha, \beta, \gamma)$

the Lie algebra $\mathcal{O}(4)$ endowed with the above complete left-symmetric product.

Let now $A_4(\alpha,\beta,\gamma)$ and $A_4(\alpha',\beta',\gamma')$ be two arbitrary left-symmetric structures on $\mathcal{O}(4)$ given as above, and let $[g]$ and $[g']$ be the corresponding classes in $H^2_{ex}(A_3,\mathbb{R})$. By Proposition 9, we know that $A_4(\alpha,\beta,\gamma)$ is isomorphic to $A_4(\alpha',\beta',\gamma')$ if and only if the exists $(\mu,\eta)\in Aut(\mathbb{R})\times Aut(A_3)$ such that for all $x,y\in A_3$, we have

$$g'(x,y)=\mu\left(g\left(\eta(x),\eta(y)\right)\right). \tag{28}$$

We shall first determine $Aut(\mathbb{R})\times Aut(A_3)$. We have $Aut(\mathbb{R})=\mathbb{R}^*$, and it is easy too to determine $Aut(A_3)$. Indeed, recall that the unique left-symmetric structure of A_3 is given by $e_1\cdot e_2=e_3,\quad e_1\cdot e_3=-e_2$, and let $\eta\in Aut(A_3)$ be given, in the basis $\{e_1,e_2,e_3\}$, by

$$\eta=\begin{pmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}.$$

From the identity $\eta(e_3)=\eta(e_1\cdot e_2)=\eta(e_1)\cdot\eta(e_2)$, we get that $c_1=0$, $c_2=-a_1b_3$, and $c_3=a_1b_2$. From the identity $-\eta(e_2)=\eta(e_1\cdot e_3)=\eta(e_1)\cdot\eta(e_3)$ we get that $b_1=0$, $b_2=a_1c_3$, and $b_3=-a_1c_2$. Since $\det\eta\neq 0$, we deduce that $a_1=\pm1$. It follows, by setting $\varepsilon=\pm1$, that $b_3=-\varepsilon c_2$ and $c_3=\varepsilon b_2$. From the identity $\eta(e_1)\cdot\eta(e_1)=\eta(e_1\cdot e_1)=0$, we obtain that $a_2=a_3=0$. Therefore, η takes the form

$$\eta=\begin{pmatrix} \varepsilon & 0 & 0\\ 0 & b_2 & c_2\\ 0 & -\varepsilon c_2 & \varepsilon b_2\end{pmatrix},$$

with $b_2^2+c_2^2\neq 0$.

We shall now apply formula (28). For this we recall first that, in the basis above, the classes $[g]$ and $[g']$ corresponding, respectively, to $A_4(\alpha,\beta,\gamma)$ and $A_4(\alpha',\beta',\gamma')$ have, respectively, the forms

$$g=\begin{pmatrix} \alpha & 0 & 0\\ 0 & \beta & \gamma\\ 0 & -\gamma & \beta\end{pmatrix}\quad\text{and}\quad g'=\begin{pmatrix} \alpha' & 0 & 0\\ 0 & \beta' & \gamma'\\ 0 & -\gamma' & \beta'\end{pmatrix}.$$

From $g'(e_1, e_1) = \mu g(\eta(e_1), \eta(e_1))$, we get

$$\alpha' = \mu\alpha, \tag{29}$$

and from $g'(e_2, e_2) = \mu g(\eta(e_2), \eta(e_2))$, we get

$$\beta' = \mu\left(b_2^2 + c_2^2\right)\beta. \tag{30}$$

Similarly, from $g'(e_2, e_3) = \mu g(\eta(e_2), \eta(e_3))$ we get

$$\gamma' = \mu\varepsilon\left(b_2^2 + c_2^2\right)\gamma. \tag{31}$$

Recall here that $\mu \neq 0$, $\gamma \neq 0$, and $b_2^2 + c_2^2 \neq 0$.

Claim 19 *Each $A_4(\alpha, \beta, \gamma)$ is isomorphic to some $A_4(\alpha', \beta', 1)$. Precisely, $A_4(\alpha, \beta, \gamma)$ is isomorphic to $A_4\left(\varepsilon\frac{\alpha}{\gamma}, \varepsilon\frac{\beta}{\gamma}, 1\right)$.*

Proof. By (29), (30), and (31), $A_4(\alpha, \beta, \gamma)$ is isomorphic to $A_4(\alpha', \beta', 1)$ if and only if there exists $\mu \in \mathbb{R}^*$ and $b, c \in \mathbb{R}$, with $b^2 + c^2 \neq 0$, such that

$$\begin{aligned} \alpha' &= \mu\alpha, \\ \beta' &= \mu\left(b^2 + c^2\right)\beta, \\ 1 &= \mu\varepsilon\left(b^2 + c^2\right)\gamma. \end{aligned}$$

Now, by taking $b^2 + c^2 = 1$ (for instance, $b = \cos\theta_0$ and $c = \sin\theta_0$ for some θ_0), the third equation yields $\mu = \frac{\varepsilon}{\gamma}$. Substituting the value of μ in the two first equations, we deduce that $\alpha' = \varepsilon\frac{\alpha}{\gamma}$ and $\beta' = \varepsilon\frac{\beta}{\gamma}$. Consequently, each $A_4(\alpha, \beta, \gamma)$ is isomorphic to $A_4\left(\varepsilon\frac{\alpha}{\gamma}, \varepsilon\frac{\beta}{\gamma}, 1\right)$. ■

Case 2. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2,\ e_2 \cdot e_2 = e_3 \cdot e_3 = e_1\rangle$.

Similarly to the first case, we get

$$\delta_1 h = \begin{pmatrix} 0 & h_{12} & h_{13} \\ 0 & h_{22} & 0 \\ 0 & 0 & h_{22} \end{pmatrix}, \quad \text{and} \quad g = \begin{pmatrix} 0 & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & -g_{23} & g_{22} \end{pmatrix},$$

where $h_{12} = -h(e_3)$, $h_{13} = h(e_2)$, $h_{22} = -h(e_1)$, and $g_{ij} = g(e_i, e_j)$. It follows that, in this case, the class $[g] \in H^2(A_3, \mathbb{R})$ of a cocycle g takes the reduced form

$$g=\left(\begin{array}{ccc}0 & 0 & 0\\ 0 & 0 & \gamma\\ 0 & -\gamma & 0\end{array}\right),\quad \gamma\neq 0.$$

By setting $\widetilde{e}_i=(e_i,0)$, $1\leq i\leq 3$, and $\widetilde{e}_4=(0,1)$, and using formula (26) we find that the nonzero relations are

$$\begin{aligned}\widetilde{e}_1\cdot\widetilde{e}_2 &= \widetilde{e}_3,\ \widetilde{e}_1\cdot\widetilde{e}_3=-\widetilde{e}_2,\ \widetilde{e}_2\cdot\widetilde{e}_2=\widetilde{e}_3\cdot\widetilde{e}_3=\widetilde{e}_1\\ \widetilde{e}_2\cdot\widetilde{e}_3 &= \gamma\widetilde{e}_4,\ \ \widetilde{e}_3\cdot\widetilde{e}_2=-\gamma\widetilde{e}_4,\end{aligned}\tag{32}$$

with $\gamma\neq 0$.

We can now state the main result of this paper.

Theorem 20 *Let A_4 be a complete non-simple real left-symmetric algebra whose associated Lie algebra is $\mathcal{O}(4)$. Then A_4 is isomorphic to one of the following left-symmetric algebras:*

(i) $A_4(s,t)$: *There exist real numbers $s,t,$ and a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathcal{O}(4)$ relative to which the nonzero left-symmetric relations are*

$$\begin{aligned}e_1\cdot e_1 &= se_4,\quad e_2\cdot e_2=e_3\cdot e_3=te_4\\ e_1\cdot e_2 &= e_3,\quad e_1\cdot e_3=-e_2,\\ e_2\cdot e_3 &= \frac{1}{2}e_4,\quad e_3\cdot e_2=-\frac{1}{2}e_4.\end{aligned}$$

The conjugacy class of $A_4(s,t)$ is given as follows: $A_4(s',t')$ is isomorphic to $A_4(s,t)$ if and only if $(s',t')=(\alpha s,\pm t)$ for some $\alpha\in\mathbb{R}^*$.

(ii) B_4 : *There is a basis $\{e_1,e_2,e_3,e_4\}$ of $\mathcal{O}(4)$ relative to which the nonzero left-symmetric relations are*

$$\begin{aligned}e_1\cdot e_2 &= e_3,\quad e_1\cdot e_3=-e_2,\ e_2\cdot e_2=e_3\cdot e_3=e_1\\ e_2\cdot e_3 &= \frac{1}{2}e_4,\quad e_3\cdot e_2=-\frac{1}{2}e_4.\end{aligned}$$

Proof. According to the discussion above, there are two cases to be considered.

Case 1. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2 \rangle$.

In this case, Claim 19 asserts that A_4 is isomorphic to some $A_4(\alpha, \beta, 1)$; and according to equations (27), we know that there is a basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of $\mathcal{O}(4)$ relative to which the nonzero relations for $A_4(\alpha, \beta, 1)$ are:

$$\begin{aligned}
\widetilde{e}_1 \cdot \widetilde{e}_1 &= \alpha \widetilde{e}_4,\ \widetilde{e}_2 \cdot \widetilde{e}_2 = \widetilde{e}_3 \cdot \widetilde{e}_3 = \beta \widetilde{e}_4 \\
\widetilde{e}_1 \cdot \widetilde{e}_2 &= \widetilde{e}_3,\ \ \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2, \\
\widetilde{e}_2 \cdot \widetilde{e}_3 &= \widetilde{e}_4,\ \ \widetilde{e}_3 \cdot \widetilde{e}_2 = -\widetilde{e}_4.
\end{aligned}$$

Now, it is clear that by setting $s = \frac{\alpha}{2}$, $t = \frac{\beta}{2}$, $e_i = \widetilde{e}_i$ for $1 \leq i \leq 3$, and $e_4 = 2\widetilde{e}_4$, we get the desired two-parameter family $A_4(s, t)$.

On the other hand, we see from equations (29), (30), and (31) that $A_4(s', t')$ is isomorphic to $A_4(s, t)$ if and only if exists $\alpha \in \mathbb{R}^*$ and $b, c \in \mathbb{R}$, with $b^2 + c^2 \neq 0$, such that

$$\begin{aligned}
s' &= \alpha s, \\
t' &= \alpha \left(b^2 + c^2\right) t, \\
1 &= \alpha \varepsilon \left(b^2 + c^2\right).
\end{aligned}$$

From the third equation, we get $b^2 + c^2 = \frac{\varepsilon}{\alpha}$; and by substituting the value of $b^2 + c^2$ in the second equation, we get $t' = \varepsilon t$. In other words, we have shown that $A_4(s', t')$ is isomorphic to $A_4(s, t)$ if and only if exists $\alpha \in \mathbb{R}^*$ such that $s' = \alpha s$ and $t' = \pm t$.

Case 2. $A_3 = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2,\ e_2 \cdot e_2 = e_3 \cdot e_3 = e_1 \rangle$.

In this case, by (32), there is a basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of $\mathcal{O}(4)$ relative to which the nonzero relations in A_4 are:

$$\begin{aligned}
\widetilde{e}_1 \cdot \widetilde{e}_2 &= \widetilde{e}_3,\ \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2,\ \widetilde{e}_2 \cdot \widetilde{e}_2 = \widetilde{e}_3 \cdot \widetilde{e}_3 = \widetilde{e}_1 \\
\widetilde{e}_2 \cdot \widetilde{e}_3 &= \gamma \widetilde{e}_4,\ \ \widetilde{e}_3 \cdot \widetilde{e}_2 = -\gamma \widetilde{e}_4,
\end{aligned}$$

with $\gamma \neq 0$.

By setting $e_i = \widetilde{e}_i$ for $1 \leq i \leq 3$, and $e_4 = 2\gamma \widetilde{e}_4$, we see that A_4 is isomorphic to B_4. This finishes the proof of the main theorem. ■

Remark 21 *Recall that a left-symmetric algebra A is called* Novikov *if it satisfies the condition*

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y$$

for all $x, y, z \in A$. Novikov left-symmetric algebras were introduced in [2] (see also [22] for some important results concerning this). We note here that $A_4(s, 0)$ is Novikov and that B_4 is not.

We note that the mapping $X \mapsto (L_X, X)$ is a Lie algebra representation of $\mathcal{O}_4$ in $\mathfrak{aff}(\mathbb{R}^4) = End(\mathbb{R}^4) \oplus \mathbb{R}^4$. By using the (Lie group) exponential maps, Theorem 20 can now be stated, in terms of simply transitive actions of subgroups of the affine group $Aff(\mathbb{R}^4) = GL(\mathbb{R}^4) \ltimes \mathbb{R}^4$, as follows.

To state it, define the continuous functions f, g, h, and k by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \qquad g(x) = \begin{cases} \frac{1-\cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

and

$$h(x) = \begin{cases} \frac{x-\sin x}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \qquad k(x) = \begin{cases} \frac{1-\cos x}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

and set

$$\begin{aligned} \Phi_t(x) &= \left(\frac{y}{2} + tz\right) g(x) - \left(\frac{z}{2} - ty\right) f(x), \\ \Psi_t(x) &= \left(\frac{y}{2} + tz\right) f(x) + \left(\frac{z}{2} - ty\right) g(x). \end{aligned}$$

Theorem 22 *Suppose that the oscillator group O_4 acts simply transitively by affine transformations on $\mathbb{R}^4$, and assume in addition that the center of O_4 acts by translations. Then, as a subgroup of $Aff(\mathbb{R}^4) = GL(\mathbb{R}^4) \ltimes \mathbb{R}^4$, O_4 is conjugate to one of the following subgroups:*

(i)

$$G_4(s,t) = \left\{ \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos x & -\sin x & 0 \\ 0 & \sin x & \cos x & 0 \\ sx & \Phi_t(x) & \Psi_t(x) & 1 \end{bmatrix} \times \\ \begin{bmatrix} x \\ yf(x) - zg(x) \\ zf(x) + yg(x) \\ w + \frac{s}{2}x^2 + (y^2+z^2)\left(\frac{h(x)}{2} + tk(x)\right) \end{bmatrix} : x, y, z, w \in \mathbb{R} \end{array} \right\},$$

where $s, t \in \mathbb{R}$. The only pairs of conjugate subgroups in $Aff(\mathbb{R}^4)$ are $G_4(s,t)$ and $G_4(\alpha s, \pm t)$ where $\alpha \in \mathbb{R}^*$.

(ii)

$$G_4 = \left\{ \begin{array}{c} \begin{bmatrix} 1 & yf(x)+zg(x) & zf(x)-yg(x) & 0 \\ 0 & \cos x & -\sin x & 0 \\ 0 & \sin x & \cos x & 0 \\ 0 & \Phi_0(x) & \Psi_0(x) & 1 \end{bmatrix} \\ \times \begin{bmatrix} x+(y^2+z^2)k(x) \\ yf(x)-zg(x) \\ zf(x)+yg(x) \\ w+\frac{(y^2+z^2)}{2}h(x) \end{bmatrix} : \ x,y,z,w \in \mathbb{R} \end{array} \right\},$$

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