Some geometry and combinatorics for the S-invariant of ternary cubics.

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Introduction.

Given a real ternary cubic form $f(x_1, x_2, x_3)$, there is a pseudo-Riemannian metric, given by the matrix $(g_{ij}) = \frac{1}{6}(\partial^2 f/\partial x_i \partial x_j)$, defined on the open subset of $\mathbf{R}^3$ where the determinant $h = \det(g_{ij})$ is non-zero. Building on previous work in [5,4], we determine the full curvature tensor of this metric in terms of h and the S-invariant of f (Theorem 1.3). In the process, we obtain various characterizations of S, which will be of use later (Lemma 1.1, Proposition 1.2).

Motivated by the theory from [5], and in particular cubic forms associated to complete intersection threefolds in the product of three projective spaces, we are led to study those cubic forms which arise as follows: We choose positive integers d_1, d_2, d_3 and $r \geq 0$ such that $d_1 + d_2 + d_3 = r + 3$, and set

$$P = (x_1 H_1 + x_2 H_2 + x_3 H_3)^3 \prod_{j=1}^{r} (a_j H_1 + b_j H_2 + c_j H_3),$$

with the a_j, b_j, c_j are non-negative, and such that the cubic $F(x_1, x_2, x_3)$, defined by taking the coefficient of the term in $H_1^{d_1} H_2^{d_2} H_3^{d_3}$ in the above formal product P, is non-degerate. Calculations from [5] suggest various conjectures concerning the invariants of such cubics. In this paper, we shall concentrate mainly on Conjecture 2.1 that, regarding S as a polynomial in the a_j, b_j and c_j, every coefficient is non-negative. Extensive computer investigations are described in support of this conjecture, and the conjecture is also reinterpreted combinatorially in terms of partitions.

In Section 3, we set Conjecture 2.1 in the context of Cayley's omega process. The corresponding result for binary quadrics is verified (Propositions 3.1 and 3.2), with the coefficients being Catalan numbers. The natural generalizations of our conjecture on ternary cubics to higher order invariants turn out to be false, and a counterexample is constructed.

In Section 4, we consider the cofactors B_{pq} of the Hessian matrix of F. In the specific case under consideration, these are polynomials in x_1, x_2, x_3 and the a_j, b_j, c_j. We derive formulae for the coefficients of these polynomials, and deduce that these coefficients are negative if $p = q$ and positive if $p \neq q$ (Theorem 4.1). From this, we deduce that the Hessian determinant H of F only has positive coefficients.

In the final section, we return to the formula for S, given in Section 1, in terms of the cofactors B_{pq} of the Hessian matrix. The fact that for the cubics F being considered, we have formulae for the coefficients of monomials in the B_{pq}, enables us to produce a useful algorithm for determining the coefficient of a given monomial in S. We run this algorithm for some critical cases, where we check that the conjectured positivity holds.

1. The S-invariant and curvature.

We consider a general non-degenerate ternary cubic form with real coefficients

$$\begin{aligned} f =& a_{300}x_1^3 + a_{030}x_2^3 + a_{003}x_3^3 + 3a_{210}x_1^2x_2 + 3a_{201}x_1^2x_3 \\ & + 3a_{120}x_1x_2^2 + 3a_{021}x_2^2x_3 + 3a_{102}x_1x_3^2 + 3a_{012}x_2x_3^2 + 6a_{111}x_1x_2x_3. \end{aligned}$$

Associated to f, we have two basic invariants S and T, one of degree 4 in the coefficients and one of degree 6 [1,3]. The S-invariant is given explicitly (see [3], page 167) by

$$\begin{aligned} S =& a_{300}a_{120}a_{021}a_{003} - a_{300}a_{120}a_{012}^2 - a_{300}a_{111}a_{030}a_{003} + a_{300}a_{111}a_{021}a_{012} \\ & + a_{300}a_{102}a_{030}a_{012} - a_{300}a_{102}a_{021}^2 - a_{210}^2a_{021}a_{003} + a_{210}^2a_{012}^2 \\ & + a_{210}a_{201}a_{030}a_{003} - a_{210}a_{201}a_{021}a_{012} + a_{210}a_{120}a_{111}a_{003} - a_{210}a_{120}a_{102}a_{012} \\ & - 2a_{210}a_{111}^2a_{012} + 3a_{210}a_{111}a_{102}a_{021} - a_{210}a_{102}^2a_{030} - a_{201}^2a_{030}a_{012} \\ & + a_{201}^2a_{021}^2 - a_{201}a_{120}^2a_{003} + 3a_{201}a_{120}a_{111}a_{012} - a_{201}a_{120}a_{102}a_{021} \\ & - 2a_{201}a_{111}^2a_{021} + a_{201}a_{111}a_{102}a_{030} + a_{120}^2a_{102}^2 - 2a_{120}a_{111}^2a_{102} + a_{111}^4. \end{aligned}$$

An indication that this invariant is closely associated with curvature was provided in [5]. We define the *index cone* in $\mathbf{R}^3$ to consist of the points at which f is positive and the indefinite metric defined by the matrix $f_{ij} = \partial^2 f/\partial x_i \partial x_j$ is of signature $(1, 2)$. The restriction of $g_{ij} = -\frac{1}{6} f_{ij}$ to the level set M given by $f = 1$ in the index cone is then a Riemannian metric, whose curvature at any point is given by the formula

$$-\frac{9}{4} + \frac{Sf^2}{4h^2},$$

where $h = \det(g_{ij}) = -H/6^3$, with H denoting the Hessian determinant of f ([5], Theorem 5.1). Strictly speaking, we do not need to include the f^2 in this formula, since by definition it has value one on the level set; however for any point in the index cone, the formula given provides the curvature at the unique point of M on the corresponding ray. This formula was both extended to higher degrees and clarified further in [4].

Consider now the pseudo-Riemannian metric on $\mathbf{R}^3$ defined by the matrix $g_{ij} = -\frac{1}{6} f_{ij}$. In the case of cubics, Theorem 3.1 of [4] reduces to the following statement: if U is an open subset of $\mathbf{R}^3$ on which the Hessian H is non-zero, and M denotes the level set in U given by $f = 1$, then the sectional curvature of U on the tangent 2-plane to M at a point is just $6^6 Sf/H^2 = Sf/h^2$. This then reproves the formula given above for the curvature of the restricted metric to M and generalises in a natural way to arbitrary degrees $d > 2$ ([4], (3.1)). It should be noted here that for ternary cubics f, the Clebsch version $S(f)$ of the S-invariant (as used in [4]) is the Aronhold S-invariant (as used in this paper) multiplied by a factor 6^4.

The point that I wish to make in this section is that, once one knows the S-invariant and the Hessian H, the whole curvature tensor of the above pseudo-Riemannian metric is given very simply. In the course of showing this, we recover various characterisations of the S-invariant, some known to Aronhold and others seemingly new.

It is shown in [4] that the curvature tensor of the pseudo-Riemannian metric has components

$$R_{ijkl} = -\frac{1}{144} \sum_{p,q} g^{pq} (f_{jlp} f_{ikq} - f_{ilp} f_{jkq}),$$

where (g^{pq}) denotes the inverse matrix to (g_{ij}). We now let B denote the adjoint matrix to $A = (f_{ij})$, and consider the function

$$G = 6^{-4} \sum_{p,q} B_{pq} (f_{11p} f_{22q} - f_{12p} f_{12q}).$$

If we let $h = \det(g_{ij}) = H/6^3$, then the above formula identifies $4hR_{1212}$ as $-G$. Moreover, a computer check verifies the following identity.

Lemma 1.1. *We have $G = Sx_3^2$.*

Remark. If we now express the cofactors B_{pq} in terms of the f_{ij} in the above formula, and then differentiate both sides by $\partial^2/\partial x_3^2$, we get the formula for S (modulo a cyclic permutation of the indices) given on page 116 of [1].

We now observe that

$$(f_{11p}f_{22q} + f_{11q}f_{22p} - 2f_{12p}f_{12q}) = \partial^2(f_{11}f_{22} - f_{12}^2)/\partial x_p \partial x_q = \partial^2 B_{33}/\partial x_p \partial x_q,$$

and so Lemma 1 may be rewritten as

$$\tfrac{1}{2}\sum_p B_{pp}(\partial^2 B_{33}/\partial x_p \partial x_p) + \sum_{p<q} B_{pq}(\partial^2 B_{33}/\partial x_p \partial x_q) = 6^4 S x_3^2,$$

or equivalently that

$$\tfrac{1}{2}\sum_{p,q} B_{pq}(\partial^2 B_{33}/\partial x_p \partial x_q) = 6^4 S x_3^2.$$

This however suggests a rather more general identity.

Proposition 1.2.

$$\tfrac{1}{2}\sum_{p,q} B_{pq}(\partial^2 B_{ij}/\partial x_p \partial x_q) = 6^4 S x_i x_j.$$

Again, once one has found the formula, it may be checked for instance simply by computer. From this last formula, it now follows that

$$\tfrac{1}{2}\sum_{p,q}(\partial^2 B_{pq}/\partial x_i \partial x_j)(\partial^2 B_{ij}/\partial x_p \partial x_q) = 6^4 S$$

for all i, j. Summing now over i, j, we obtain the more symmetric formula

$$6^{-6}\sum_{i,j,p,q}(\partial^2 B_{pq}/\partial x_i \partial x_j)(\partial^2 B_{ij}/\partial x_p \partial x_q) = \tfrac{1}{2}S.$$

If in Proposition 2 we take $(i, j) = (1, 2)$, then $B_{12} = f_{13}f_{23} - f_{12}f_{33}$, and for a given (p, q) we have

$$\partial^2 B_{12}/\partial x_p \partial x_q = f_{13p}f_{23q} + f_{13q}f_{23p} - f_{12p}f_{33q} - f_{12q}f_{33p}.$$

Thus, from the formula for curvature,

$$\tfrac{1}{2}6^{-4}\sum_{p,q} B_{pq}(\partial^2 B_{12}/\partial x_p \partial x_q) = 4hR_{1323},$$

and so $Sx_1x_2 = 4hR_{1323} = -4hR_{1332}$.

Theorem 1.3. *All components of the curvature tensor of the Hessian metric on $U \subset \mathbf{R}^3$, where U is the open subset given by the non-vanising of H, are given simply in terms the invariant S and the Hessian of f, and are all of the form $\pm\frac{1}{4}Sx_ix_j/h$ for appropriate i, j.*

Proof. Since we have formulae for R_{1212} and R_{1323}, we have the analogous formulae for R_{ijij} and R_{ijkj}. We now use the general fact that the curvature tensor is invariant under exchanging the first pair of indices with the second pair of indices, and is anti-invariant under exchanging the first pair (or second pair) of indices; in our particular case, these symmetries are clear from the above formula for the curvature tensor, taken from [4]. In this way, we obtain expressions of the required form for all the components of the curvature tensor.

Remark. Related to the formula given in (1.2) is the following formula:

$$6^{-6}\sum_{p,q} B_{pq}(\partial^2 H/\partial x_p \partial x_q) = \tfrac{1}{3}Sf,$$

which again may be checked most simply by computer.

2. Conjectural positivity of S for certain cubics arising in geometry.

In Section 5 of [5], we were interested in the cubics which occur as intersection forms for 3-dimensional complete intersections in the product of three projective spaces. We can however formalise this into a purely algebraic problem. Suppose a ternary cubic is obtained as follows : We choose positive integers d_1, d_2, d_3 and $r \geq 0$ such that $d_1 + d_2 + d_3 = r + 3$, and set

$$P = (x_1H_1 + x_2H_2 + x_3H_3)^3 \prod_{j=1}^{r}(a_jH_1 + b_jH_2 + c_jH_3),$$

with the a_j, b_j and c_j non-negative, and such that the cubic $F(x_1, x_2, x_3)$, defined by taking the coefficient of the term in $H_1^{d_1}H_2^{d_2}H_3^{d_3}$ in the above formal product P, is non-degenerate. To relate this to the geometry, note that if the a_j, b_j, c_j take non-negative integer values, then we may consider the complete intersection projective threefolds X in $\mathbf{P}^{d_1} \times \mathbf{P}^{d_2} \times \mathbf{P}^{d_3}$ given by r general trihomogeneous polynomials, with tridegrees (a_j, b_j, c_j) for $j = 1, \dots, r$. The cubic we have defined above is then the intersection form on the rank three sublattice of $H^2(X, \mathbf{Z})$ generated by the pullbacks of hyperplane classes from the three factors; by Lefshetz type arguments, this is usually the whole of $H^2(X, \mathbf{Z})$.

We now let S denote the S-invariant of F, and H the Hessian determinant of F.

Conjecture 2.1. *Regarding S as a polynomial in the a_j, b_j, c_j, every coefficient of this polynomial is non-negative.*

Conjecture 2.2. *Regarding $9H^2 - 6^6SF^2$ as a polynomial in the a_j, b_j, c_j and x_1, x_2, x_3, every coefficient of this polynomial is also non-negative.*

These conjectures imply their (weaker) geometric counterparts, in the case of X being a complete intersection threefold in the product of three projective spaces, with second betti number three, and F being its intersection form. Here, we have taken specific non-negative integral values for the degrees a_j, b_j, c_j.

Conjecture 2.3. *The intersection form of X has non-negative S-invariant.*

Conjecture 2.4. *The polynomial $9H^2 - 6^6SF^2$ in x_1, x_2, x_3 takes non-negative values on the Kähler cone of X, given by $x_1 > 0$, $x_2 > 0$, $x_3 > 0$.*

Remarks. Conjecture 2.4 may be rephrased [5] as saying that the level set $F = 1$ in the Kähler cone has non-positive curvature, a property which is thought to hold for a wide range of Kähler threefolds with $h^{1,1} = 3$. Recall that when $h^{1,1} = 3$, the curvature is given by the formula $-\frac{9}{4} + \frac{1}{4}6^6SF^2/H^2$. Conjecture 2.3 says that, for complete intersection threefolds in a product of three projective spaces, the curvature of this level set in the Kähler cone is bounded below by $-\frac{9}{4}$. This is known to fail for arbitrary compact Kähler threefolds with $h^{1,1} = 3$ [5], and the correct lower bound is then thought to be -3. The author anticipates that an argument along the lines outlined in (4.4) of [5] will eventually yield that the curvature lies between -3 and 0 for the above complete intersections, although technical difficulties remain here. It may also be the case that the possibility $S = 0$ could be ruled out for the complete intersections being considered, using a geometric argument. As yet however, the author sees no geometrical reasons why the stronger positivity statements should hold — these conjectures seem to be combinatorial in nature and rest on the computations described below.

Considered as a polynomial in the a_j, b_j, c_j, we have that S is homogeneous of degree 4 in any given set (a_j, b_j, c_j), and hence of total degree $4r = 4(d_1 + d_2 + d_3) - 12$. In fact, by inspection of the given formula for S, we see that S is of degree $4d_1 - 4$ in the variables $(a_1, \ldots, a_r)$, of degree $4d_2 - 4$ in the variables $(b_1, \ldots, b_r)$, and of degree $4d_3 - 4$ in the variables $(c_1, \ldots, c_r)$.

As explained above, the conjectures arose out of the theory developed in [5]; there is moreover now extensive computational evidence in their favour. In particular, Conjecture 2.1 has been checked using MATHEMATICA for all $d_i \leq 5$. One can of course reduce to the case where all the d_i equal some d. To give a flavour of these calculations, I can report that in the case $d = 3$ there are 209,520 non-zero terms in S, all with positive coefficients, and that a simple minded check of this takes some two hours of computer time. However, there are a very large number of symmetries, and if one takes such symmetries into account, one can reduce the calculation to less than a couple of minutes. For larger d therefore, one should factor out by these symmetries. For $d = 4$ one can then check the conjecture in a couple of hours, and for $d = 5$ in about four days. The formula for S given in Section 1 in terms of cofactors turns out to be slightly more efficient computationally than the formula in terms of the coefficients of the cubic. The programs used by the author for these checks may be found on his home page: http://www.dpmms.cam.ac.uk/~pmhw/. The case $d = 6$ seems to be beyond the range of standard computers. The author has not carried out as many calculations on the Conjecture 2.2, but it has been verified for $d_1 = 3, d_2 = d_3 = 2$, and there are stronger theoretical reasons in its support [5].

Conjecture 2.3 is also implied by another conjecture.

Conjecture 2.5. *The discriminant of F, a positive multiple of $64S^3 - T^2$, when considered as a polynomial in the a_i, b_j and c_k, only takes non-negative values for non-negative values of the variables.*

In the statement of (2.5), T denotes the basic invariant of degree 6 in the coefficients of the cubic; see [1,3] and Section 3 for more details. By consideration of the case $d = 2$, we see that the stronger statement here is not true; there are six negative terms in 33 pages of output from MATHEMATICA. These terms correspond to the monomial $a_2^6 a_3^6 b_1^6 b_3^6 c_1^6 c_2^6$, and the five other such monomials obtained by symmetry. If however we set $a_1 = 0$, $b_2 = 0$ and $c_3 = 0$ in $64S^3 - T^2$, we obtain (using MATHEMATICA)

$$27a_2^3 a_3^3 b_1^3 b_3^3 c_1^3 c_2^3 (a_2 b_3 c_1 + a_3 b_1 c_2)^4 (8a_2^2 b_3^2 c_1^2 - 11 a_2 a_3 b_1 b_3 c_1 c_2 + 8 a_3^2 b_1^2 c_2^2).$$

This is plainly non-negative for non-negative values of the variables. The conjecture for $d = 2$ then follows by symmetry. The discrimant may in this case equal zero, when for instance $a_3 = b_3 = 0$; equivalently this is the case $d_1 = d_2 = 2$ and $d_3 = 1$. Computer investigations in the case $d = 3$ support the assertion that the ratio S^3/T^2 is always bounded below by 1/64. Homogeneity of this ratio means that we need only check it for values of the variables between 0 and 1; the bound has been checked by computer for several million random choices of the variables in this range.

In this paper, we shall however concern ourselves mainly with the problem of Conjecture 2.1, that S only has non-negative coefficients, and results closely related to this.

For the case $d = 3$, one can obtain very precise information using MATHEMATICA about the coefficients. The monomials appearing in any of the 25 terms in S all appear in the expansion of a_{111}^4. There are two types of monomial which appear in a_{111}^4 but not in S (because the coefficients cancelling out) — examples of these are $a_1^4 b_2^4 c_3^4 a_4^2 b_4^2 b_5^2 c_5^2 a_6^2 c_6^2$ and $a_1^4 b_2^4 c_3^4 a_4^2 b_4 c_4 a_5 b_5^2 c_5 a_6 b_6 c_6^2$. If one considers the exponents as forming a 3×6 matrix, the first of these for instance can be written rather more clearly as

$$\begin{bmatrix} 4 & 0 & 0 & 2 & 0 & 2 \\ 0 & 4 & 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 & 2 & 2 \end{bmatrix},$$

and the second as

$$\begin{bmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{bmatrix}.$$

Matrices differing from each other by permutations of the rows and/or columns are regarded as being of the same type. There are then three types with coefficient 1, represented by matrices

$$\begin{bmatrix} 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & 0 & 3 & 0 & 1 \\ 0 & 4 & 0 & 1 & 3 & 0 \\ 0 & 0 & 4 & 0 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 4 & 0 & 0 & 0 & 2 & 2 \\ 0 & 4 & 1 & 1 & 2 & 0 \\ 0 & 0 & 3 & 3 & 0 & 2 \end{bmatrix}.$$

A similar feature occurs for higher coefficicients of there being rather a small number of types. The largest coefficients which occur are 356, 280, 214, 176, 164, 128, 106, ..., all of which correspond to only one type. The highest coefficient 356 corresponds to type

$$\begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}.$$

It makes more sense however if we ignore all monomials containing fourth powers, on the grounds that these correspond to cases with the d_i smaller. With this convention, the corresponding matrices do not have 4 in any entry. In the case $d_1 = d_2 = d_3 = 3$ as above, the smallest coefficients are then 4, 6 and 9, corresponding (respectively) to matrices

$$\begin{bmatrix} 3 & 0 & 1 & 3 & 0 & 1 \\ 1 & 3 & 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 3 & 0 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 1 & 3 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 & 3 \\ 0 & 1 & 3 & 1 & 3 & 0 \end{bmatrix}.$$

We shall also denote the first of these as $(d-1)\begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, where $d = 3$.

If we now move on to the cases $d_1 = d_2 = d_3 = d > 3$, we may ask about the coefficients corresponding to

$$(d-1)\begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix};$$

for $d = 4$, the coefficient may be calculated as 40, and for $d = 5$ as 652. For $d = 4$ and 5, a computer check verifies that this is the smallest non-zero coefficient (assuming no fourth powers) and the unique type of monomial corresponding to it, and one would conjecture that a similar statement is true for arbitrary values of $d > 2$. A formula for this coefficient for arbitrary d will be produced in Section 5.

To illustrate on the other hand that the general coefficients do get large, we give the coefficients corresponding to

$$(d-1)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix};$$

for $d = 2, 3, 4$ and 5 respectively, these are 0, 356, 70155 and 23857956.

Computer calculations suggest also a result that the cofactors B_{pq} which appeared in Section 1 satisfy the condition that B_{pq}, considered as a polynomial in the a_i, b_j, c_k and x_1, x_2, x_3, has only positive coefficients if $p \neq q$, and has only negative coefficients if $p = q$. In the geometric situation of a three dimensional complete intersection in the product of three projective spaces, with the (a_j, b_j, c_j) being assigned specific non-negative integral values, the negativity of B_{pp} corresponds to the Hodge index theorem on the surface cut out by $H_p = 0$. We shall prove the indicated results in Section 4.

We may reduce our main conjecture on the positivity of the terms in S to an equivalent problem on partitions, albeit one which seems rather difficult. We will see in the next section that the analogous problem in two variables can be solved easily. Let us consider a fixed monomial M in the a_i, b_j, c_k of the type described above, namely homogeneous of degree 4 in any given (a_j, b_j, c_j), of degree $4d_1 - 4$ in the variables $(a_1, \ldots, a_r)$, of degree $4d_2 - 4$ in the variables $(b_1, \ldots, b_r)$, and of degree $4d_3 - 4$ in the variables $(c_1, \ldots, c_r)$. If we consider one of the terms of S, say the last one a_{111}^4, and look for the coefficient N_{25} (the subscript corresponding to the order we have written down the terms of S) of M in its expansion, we can interpret this number in terms of partititions. Explicitly, it is the number of ways we can write M as a product $M_1 M_2 M_3 M_4$ of monomials, where each monomial M_α involves exactly one of $\{a_j, b_j, c_j\}$ for each j, and involves $d_1 - 1$ elements from the a_i, $d_2 - 1$ elements from the b_j and $d_3 - 1$ elements from the c_k. In general, for a given term in S, we shall be interested in splittings where each M_α will involve $d_1 - e_1$ elements from the a_i, $d_2 - e_2$ elements from the b_j and $d_3 - e_3$ elements from the c_j, with $e_1 + e_2 + e_3 = 3$. For instance, for the first term in S, we need to count those splittings for which (e_i, e_2, e_3) is $(3, 0, 0)$ for M_1, is $(1, 2, 0)$ for M_2, is $(0, 2, 1)$ for M_3 and is $(0, 0, 3)$ for M_4; the total number of such splittings will then be denoted by N_1.

In this way, for a given monomial M, we obtain 25 non-negative integers $N_1, \ldots, N_{25}$; the coefficient of this monomial in the expansion of S is then

$$\begin{aligned}
&N_1 - N_2 - N_3 + N_4 + N_5 - N_6 - N_7 + N_8 + N_9 - N_{10} + N_{11} - N_{12} \\
&- 2N_{13} + 3N_{14} - N_{15} - N_{16} + N_{17} - N_{18} + 3N_{19} - N_{20} - 2N_{21} \\
&+ N_{22} + N_{23} - 2N_{24} + N_{25}.
\end{aligned}$$

Our conjecture is that this is always non-negative.

The reader is encouraged to see this algorithm at work by checking a few simple examples. As an example, one might take $d = 2$ and the matrix

$$\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

Here one checks easily that the non-zero N_i correspond to 5 different types of splitting, with $N_8 = N_{17} = N_{23} = 1$, $N_{10} = N_{12} = N_{20} = 1$, $N_{13} = N_{21} = N_{24} = 2$, $N_{14} = N_{19} = 1$ and $N_{25} = 6$, and that the above combination is indeed 0, as calculated above.

The above algorithm may be used to check some of the simpler cases with d arbitrarily large. As an example, we can consider the case $s = 3t$, and so $d = 3t+1$, and M a monomial with matrix of exponents having $4t$ columns of the form $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $4t$ columns of the form $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and t columns of the form $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$. Ignoring the latter columns, this is really an example with $d_1 = 3t + 1$, $d_2 = 2t + 1$ and $d_3 = 3t + 1$. If we then wish for instance to find the coefficient of M in $a_{300}a_{120}a_{021}a_{003}$, we need to count the number of ways of factoring $M = M_1M_2M_3M_4$, where M_1 involves $3t-2$ of the a_j, $2t+1$ of the b_j and $3t+1$ of the c_j, etc. This however is the same as choosing the b_j appropriately. Our condition that M_1 is a monomial in a_{300} may be rephrased as saying that we take $t + 2$ of the b_j

from monomials represented by columns $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, and $t-1$ from monomials represented by columns $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$. Having dealt with the number of ways of factoring out a suitable M_1, we then count the number of ways of factoring out a suitable M_2 from what remains, which will involve choosing t of the b_j from columns $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ that we have not already used, and $t+1$ from columns $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ not aready used. In this way we see that for the particular monomial being considered,

$$N_1 = \binom{4t}{t+2}\binom{4t}{t-1}\binom{3t-2}{t}\binom{3t+1}{t+1}\binom{2t-2}{t-1}\binom{2t}{t+1}.$$

Similarly

$$N_2 = \binom{4t}{t+2}\binom{4t}{t-1}\binom{3t-2}{t}\binom{3t+1}{t}\binom{2t-2}{t-1}\binom{2t+1}{t+2}.$$

The final term is

$$N_{25} = \binom{4t}{t}^2\binom{3t}{t}^2\binom{2t}{t}^2.$$

In this way we obtain (as a function of t) the relevant coefficient of the S-invariant. With the aid of MATHEMATICA, one can then simplify the formula to the surprisingly simple form

$$(4t)!^2\left(\frac{t!}{(t-1)!(t+1)!} - \frac{1}{t!^2}\right)^4.$$

In particular, one notes that it is positive. Evaluating this formula for t taking values $0, 1, 2, 3, 4, \ldots$, one obtains values for the coefficient of the monomial in the S-invariant to be $1, 36, 78400$, 533610000, $6363107150400, \ldots$. The first two of these values coincide with previously calculated numbers, and the rest are new. In Section 4, we introduce another (and better) way of deriving this formula.

We have now reduced our problem to an explicit but non-trivial combinatorial question, and another such reduction appears in Section 5. Much easier however to calculate is, for a given $p = 1, \ldots, 25$, the sum of the $N_p(M)$ over the possible monomials M. We may assume that $d_1 = d_2 = d_3 = d$ and set $s = d - 1$. For $p = 1$ for instance, the above sum is then

$$\left(\frac{(3s)!}{(s-2)!(s+1)!^2}\right)^2\left(\frac{(3s)!}{s!(s-1)!(s+1)!}\right)^2,$$

and for $p = 25$, it is just $((3s)!/s!^3)^4$. Taking the sum of these 25 terms with coefficients as above is however clearly the same as evaluating S when all the a_i, b_j, c_k are set to one. In this case, the cubic F is given by taking the coefficient of the term in $H_1^{s+1}H_2^{s+1}H_3^{s+1}$ in the formal product

$$P = (x_1H_1 + x_2H_2 + x_3H_3)^3(H_1 + H_2 + H_3)^{3s},$$

and then an easy check gives

$$F = \frac{(3s)!}{(s+1)!^2 s!}\Big(s(s-1)x_1^3 + 3s(s+1)x_1^2x_2 + \ldots + 6(s+1)^2x_1x_2x_3\Big).$$

Forming the S-invariant of this cubic visibly recovers the sum of the above 25 explicit terms with coefficients as given in the formula for S. However, this S-invariant is easy to calculate by computer, and the answer is found to be

$$S = \left(\frac{(3s)!}{s!^3}\right)^4 \frac{(3s+1)^2}{(s+1)^6}.$$

The moral of this calculation is that for large s, there is, averaging over all monomials, a large amount of cancellation going on when calculating S, since for any particular term in S, the sum over all monomials is of order $\left(\frac{(3s)!}{s!^3}\right)^4$. Thus, the reason for the conjectured positivity of the coefficients is not that one of the positive terms in S (for instance a_{111}^4) dominates, but is rather more delicate.

3. The Cayley omega process.

There are three more ways to compute S which might be considered, all of which are described in [3]. One is given by the symbolic method, one in terms of a Lie algebra action, and one in terms of the Cayley operator Ω. The most suggestive of these ways seems to be the last, which we now describe. In the 3-dimensional case, this is described as follows. Suppose we have independent sets of variables (x_i, y_i, z_i) for $i = 1, 2, 3$. The Cayley Ω operator ([2], p. 234) is the third order partial differential operator

$$\Omega = \det\begin{pmatrix} \partial/\partial x_1 & \partial/\partial y_1 & \partial/\partial z_1 \\ \partial/\partial x_2 & \partial/\partial y_2 & \partial/\partial z_2 \\ \partial/\partial x_3 & \partial/\partial y_3 & \partial/\partial z_3 \end{pmatrix}.$$

Given functions $Q_i(x, y, z)$, where $i = 1, 2, 3$, we can form the product

$$Q_1(x_1, y_1, z_1)Q_2(x_2, y_2, z_2)Q_3(x_3, y_3, z_3),$$

and then operate on this by Ω^r for some $r > 0$ and take the *trace* of the result, which by definition means setting all the x_i to be x, all the y_i to be y and all the z_i to be z. The function we obtain is called the rth *transvectant* of the functions Q_1, Q_2, Q_3, and is denoted $(Q_1, Q_2, Q_3)^{(r)}$. For instance, the Hessian determinant of a function $F(x, y, z)$ may be checked to be some multiple of $\frac{1}{6}(F, F, F)^{(2)}$. We can also consider the invariant defined by $(F^2, F^2, F^2)^{(6)}$, which involves operating by Ω^6 on the product of the squares. In the case of F being a ternary cubic, one discovers that the result is a multiple (namely $2^{14}3^9 5$) of the T invariant, the fundamental invariant of degree 6 in the coefficients of the cubic, as defined in [3].

There is a variant of this which involves polarising the form. Let us consider the case of a ternary cubic F, and let G denote the corresponding trilinear form in variables (x_i, y_i, z_i) for $i = 1, 2, 3$. Explicitly, G is

$$\tfrac{1}{6}(x_1\partial/\partial x + y_1\partial/\partial y + z_1\partial/\partial z)(x_2\partial/\partial x + y_2\partial/\partial y + z_2\partial/\partial z)(x_3\partial/\partial x + y_3\partial/\partial y + z_3\partial/\partial z)F.$$

Because G is now symmetric in the three sets of variables, we can just take a power of G and then apply some power of the Ω-process, and then if necessary take the trace (opposite of polarisation). By symmetry, it is clear that this will yield zero for odd powers of Ω.

The Hessian determinant H of F may be recovered as the trace of $3\Omega^2(G^3)$. By the general theory of the invariants of ternary cubics, or direct computation, we know that $\Omega^2(G^2) = 0$. More generally, for all $r \geq 3$, a computer calculation verifies that the trace of $\Omega^2(G^r)$ is

$$\frac{1}{36} r(r-1)^2(r-2) H\, F^{r-3}.$$

In view of Theorem 4.2 below, for the special type of ternary cubic introduced in Section 2, all these polynomials have only positive coefficients.

In this notation, the invariant $\Omega^4(G^4)$ of the cubic is a multiple (namely $13824 = 24^3$) of the S-invariant for F. The construction described here is essentially equivalent to that described in Chapter 4 of [3], except that it should be noted that the explicit expression for Ω on page 167 of [3] is incorrect, as is the constant, which appears as 18630 rather than 13824. Clearly $\Omega^4(G^r)$ is zero for $r < 4$, and a computer calculation yields that the trace of $\Omega^4(G^r)$ for $r \geq 4$ is given by the formula

$$6^3 r(r-1)(r-2)^2(r-3)^2 F^{r-6}\left(\frac{4}{3}(r-3)SF^2 + (r-4)(r-5)h^2\right),$$

where as ususual $h = 6^{-3}H$. This identity, and the one in the previous paragraph, are best proved by reducing the cubic to canonical form $x^3 + y^3 + z^3 + 6\lambda\, xyz$ by means of a change of coordinates, given by an element of $GL(3,\mathbf{C})$, and observing that both sides transform in the same way; even with this reduction, a computer is the practical way to proceed. For the particular ternary cubics under consideration, if we can prove Conjecture 2.1, it will follow using (4.2) that all these polynomials have only positive coefficients. In a similar vein, I remark that if we denote by $\tilde{H}$ the trilinear form obtained by polarizing H, then a computer calculation yields that $24\,\Omega^2(G^2\Omega^2(G^3)) = S\,G$, and so the trace of $\Omega^2(\tilde{H}G^2)$ is 12^3SF. This identity may be compared with the last formula in Section 1.

More generally, we may wish to consider the invariants given by $\Omega^{2m}(G^{2m})$. The fact that this recipe does provide invariants is classical, and follows for instance by Theorem 4.3.7 of [3]; the resulting invariant may however be zero. We have seen that for $m = 1, 2$, we obtain zero and the appropriate multiple of S. The invariant $\Omega^6(G^6)$ is checked by computer to be a multiple (namely $2^{12}3^5 5^2$) of $-T$. For higher m, we know in general that $\Omega^{2m}(G^{2m})$ is a polynomial in S and $-T$, and the author expects the coefficients always to be positive (for $m = 4, 5, 6$, one obtains positive multiples of S^2, $-ST$ and $T^2 + 217S^3$ respectively). This theory may be generalised in an obvious way to homogeneous polynomials F of degree n in n variables. The Cayley Ω-process generalises in the obvious way ([2], page 234); if G denotes the polarisation of F, an n-linear form, we can consider the invariants $\Omega^{2m}(G^{2m})$ for $m = 1, 2, \ldots$. We can also restrict attention to polynomials F given in an analogous way to the particular cubics studied above.

To clarify these ideas, let us first consider the simple case of binary quadrics $F = ax^2 + 2bxy + cy^2$. Then G is the bilinear form $ax_1x_2 + bx_1y_2 + bx_2y_1 + cy_1y_2$. The operator Ω in this case is just $\partial/\partial x_1\,\partial/\partial y_2 - \partial/\partial x_2\,\partial/\partial y_1$. An easy calculation shows that $\Omega^2(G^2) = 4(b^2 - ac)$, the negative of the discrimant δ. For general $m > 0$, $\Omega^{2m}(G^{2m})$ has to be a multiple of $(b^2 - ac)^m$; by explicit calculation this multiple is seen to be positive.

Proposition 3.1. *For general $m > 0$,*

$$\Omega^{2m}(G^{2m}) = (2m)!^2(b^2 - ac)^m.$$

Proof. We prove this by induction on m. A straightforward calculation verifies that

$$\Omega(G^{2m}) = 2m(2m-1)(x_1y_2 - x_2y_1)(b^2 - ac)G^{2m-2}.$$

Suppose now $L(x_1, y_1; x_2, y_2)$ is any bihomogeneous polynomial, homogeneous of degree $p > 0$ in each of the two sets of variables. A further straightforward calculation (using

Euler's formula) verifies that

$$\Omega(x_1y_2 - x_2y_1)L = 2(p+1)L + (x_1y_2 - x_2y_1)\Omega L.$$

Applying Ω to both sides, we obtain

$$\Omega^2(x_1y_2 - x_2y_1)L = 2((p+1)+p)\Omega L + (x_1y_2 - x_2y_1)\Omega^2 L.$$

By induction, this yields the formula

$$\Omega^{p+1}(x_1y_2 - x_2y_1)L = (p+1)(p+2)\Omega^p L.$$

Applying this to $L = G^{2m-2}$, we deduce

$$\Omega^{2m-1}(x_1y_2 - x_2y_1)G^{2m-2} = 2m(2m-1)\Omega^{2m-2}G^{2m-2}.$$

Thus applying Ω^{2m-1} to the first identity of the proof, along with the inductive hypothesis, now yields the result claimed.

We now consider quadrics of our given special type, namely quadrics F obtained by taking the bidegree (d_1, d_2) part of

$$P = (xH_1 + yH_2)^2(a_1H_1 + b_1H_2)\ \dots\ (a_rH_1 + b_rH_2),$$

where $d_1 + d_2 = r + 2$. The fact that the invariant $(b^2 - ac)$ only takes non-negative values follows from the Hodge index theorem. However, a more precise statement is true.

Proposition 3.2. *With the quadratic polynomial F as defined above, the invariant $-\delta$, considered as a polynomial in the variables (a_j, b_j), has non-negative coefficients, which are all Catalan numbers.*

Proof. We may reduce to the case $r = 2s$ and $d_1 = d_2 = s + 1$. A given monomial M appearing in $b^2 - ac$ is quadratic in each pair of variables (a_j, b_j). Moreover, its total degree in the a_i variables will equal its total degree in the b_j, namely $2s$. We may suppose now that the monomial M in question contains exactly $2p$ monomials of the form a_ib_i, and otherwise only involves the a_j^2 and b_k^2 (equal number of both). The coefficient of the given monomial in the polynomial b^2 is then seen to be the total number of ways of splitting M into a product of two monomials which are linear in each pair (a_j, b_j), and of total degree s in the a_i, respectively b_j. This is readily seen to be the number of ways of choosing p of the $2p$ monomials a_jb_j in M for which we take a_j rather than b_j, and is therefore just the binomial coefficient $\binom{2p}{p}$.

A similar argument shows that the coefficient of M in the polynomial ac is the number of ways of choosing $p+1$ of the $2p$ monomials a_jb_j in M for which we take a_j rather than b_j, and is therefore just the binomial coefficient $\binom{2p}{p+1}$. The coefficient of M in $b^2 - ac$ is therefore the Catalan number

$$c_p = \binom{2p}{p} - \binom{2p}{p+1} = \frac{1}{p+1}\binom{2p}{p}.$$

Returning now to the case of ternary cubics ($n = 3$), we have explained in the previous section why we might expect positivity of the coefficients for $\Omega^4(G^4)$ (or equivalently S) for the specific type of cubics being studied. We should also remark that S is not necessarily a *positive* polynomial, in the sense of only taking positive values irrespective of the signs of the variables, as may be seen from the following example. We take $d_1 = 3$, $d_2 = d_3 = 2$, and the matrix, with columns given by the (a_j, b_j, c_j), to be

$$\begin{pmatrix} -2 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

The corresponding cubic is checked by computer to have S-invariant -20.

The next case will be $\Omega^6(G^6)$, and this has been computed by the author in the case $d_1 = d_2 = d_3 = 2$. The result in this case is a polynomial with 332 positive coefficients and 12 *negative* coefficients (for example, there is a term $-2^{13}3^5 5^2\ a_1^3 b_1^3 a_2^3 c_2^3 b_3^3 c_3^3$). Thus we cannot in general hope for all the terms to have positive coefficients. One might however hope that the polynomial only takes non-negative values for non-negative values of the variables, as in the case of Conjecture 2.5, so that the T-invariant for the cubics in question always took non-positive values for non-negative values of the variables. This however also turns out in general to be false. If for instance, we take the complete intersection threefold in $\mathbf{P}^4 \times \mathbf{P}^5 \times \mathbf{P}^6$ given by 12 polynomials, trihomogeneous of degree one in each set of variables, the T-invariant is calculated as $2^{12}3^9 5^3 7^6 11^6 13^2$. Another case of interest is the analogous complete intersection in $\mathbf{P}^3 \times \mathbf{P}^5 \times \mathbf{P}^8$ given by 13 polynomials, trihomogeneous of degree one in each set of variables, where the T-invariant is actually zero. This, together with Conjecture 2.5, predicts the range of values for S^3/T^2 (or equivalently, the range of values for the j-invariant) we expect to be covered by cubics of the type being studied.

For the case of quartics F in four variables, one checks easily that $\Omega^2(G^2) = 0$. For the specific quartics, given analogously to the cubics studied above, a computation in the case $d_1 = d_2 = d_3 = d_4 = 2$ for $\Omega^4(G^4)$ reveals some of the coefficients to be negative.

4. The cofactors of the Hessian matrix.

In this Section, we study further the cofactors B_{pq} of the Hessian matrix of our ternary cubic F, where it will be more convenient here to denote the variables as x_1, x_2, x_3 rather than x, y, z. Recall that these cofactors were related to the S-invariant by means of various expressions for S described in Section 1; we shall return to this aspect in Section 5. In particular, for the special type of cubics we have studied in the last two sections, the B_{pq} may be considered as polynomials in the a_j, b_j, c_j and x_1, x_2, x_3. In this Section, we confirm the expectations mentioned in Section 2, reinforced by Proposition 5, concerning the signs of their coefficients; this in turn will show that the Hessian determinant H only has positive coefficients. This latter fact might be expected because of the Hodge Index Theorem, which would imply the weaker statement that H takes non-negative values for non-negative values of its variables.

Theorem 4.1. *The polynomials B_{pp} only have negative coefficients, and the polynomials B_{pq} for $p \neq q$ only have positive coefficients.*

Proof. For the first part, we may consider $B_{33} = f_{11}f_{22} - f_{12}^2$. For a general cubic f, we have

$$\frac{1}{36}B_{33} = (a_{300}x_1 + a_{210}x_2 + a_{201}x_3)(a_{120}x_1 + a_{030}x_2 + a_{021}x_3) - (a_{210}x_1 + a_{120}x_2 + a_{111}x_3)^2.$$

The fact that, in our particular case, this polynomial is non-positive for all non-positive values of the variables follows from the Hodge index theorem again. We however prove the more precise result that the coefficients are all negative.

Let us consider for instance the term in x_1x_2; we prove that its coefficient

$$a_{300}a_{030} - a_{210}a_{120},$$

considered as a polynomial in the a_i, b_j, c_k, has only negative coefficients. Without loss of generality, we can assume that $d_1 = d_2 = d_3 = d$, and we set $s = d - 1$. Then the polynomial in question is of degree 2 in each set of variables (a_j, b_j, c_j), and is of degree $2s - 1$ in the a_i, degree $2s - 1$ in the b_j, and degree $2s + 2$ in the c_k. On the other hand,

a_{300} (respectively, a_{210}) is of degree $s-2$ (respectively, $s-1$) in the a_i, degree $s+1$ (respectively, s) in the b_j, and degree $s+1$ (respectively, $s+1$) in the c_k, with analogous statements for a_{030} and a_{120}.

Let us now consider a monomial of the appropriate degrees in the (a_j, b_j, c_j), and ask about its coefficient as a term in $a_{300}a_{030} - a_{210}a_{120}$. We suppose that the monomial in question consists of p_1, respectively p_2, p_3, occurrences (for various j) of a_j^2, respectively b_j^2, c_j^2, and $\tilde{u}$, respectively $\tilde{v}$, $\tilde{w}$, occurrences of a_jb_j, respectively a_jc_j, b_jc_j. As in Proposition 5, we shall see that only the mixed cases will be of relevance. Note that $2p_1+\tilde{u}+\tilde{v} = 2s-1$, $2p_2+\tilde{u}+\tilde{w} = 2s-1$ and $2p_3+\tilde{v}+\tilde{w} = 2s+2$.

The coefficient of the monomial in $a_{300}a_{030}$ is given by counting the number of ways of expressing it as a monomial in a_{300} times a monomial in a_{030}, and similarly for its coefficient in $a_{210}a_{120}$. To obtain the first factor in the former case, involves choosing $s-p_1-2 = \frac{1}{2}(\tilde{u}+\tilde{v}-3)$ of the a_jb_j and a_jc_j appearing for which we choose the a_j, $s-p_2+1 = \frac{1}{2}(\tilde{u}+\tilde{w}+3)$ of the a_jb_j and b_jc_j for which we choose the b_j, and $s-p_3+1 = \frac{1}{2}(\tilde{v}+\tilde{w})$ of the a_jc_j and b_jc_j for which we choose the c_j. Note here the necessary parity condition that either $\tilde{u}$ is odd and $\tilde{v}, \tilde{w}$ are even, or the other way round. We shall deal with the first case; the other case follows similarly.

We set $\tilde{u} = 2u+1$, $\tilde{v} = 2v$ and $\tilde{w} = 2w$. The possible factorizations are then given by choosing k of the $2u+1$ occurrences of a_jb_j for which we choose the a_j, choosing $u+v-1-k$ occurrences of a_jc_j for which we choose the a_j, and finally $w-u+1+k$ occurrences of the b_jc_j for which we choose the b_j, the rest then being determined. Thus the number of ways of doing this, and hence the coefficient of the monomial in $a_{300}a_{030}$, is

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \binom{2v}{v+u-(k+1)} \binom{2w}{w+u-(k+1)}.$$

Similarly, the coefficient of the monomial in $a_{210}a_{120}$ is seen to be

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \binom{2v}{v+u-k} \binom{2w}{w+u-k}.$$

Thus we need to verify the negativity of

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \left(\binom{2v}{v+u-(k+1)} \binom{2w}{w+u-(k+1)} - \binom{2v}{v+u-k} \binom{2w}{w+u-k} \right).$$

This sum may however be rearranged as

$$
\begin{aligned}
&-\binom{2v}{v+u}\binom{2w}{w+u}+\binom{2v}{v+u+2}\binom{2w}{w+u+2}\\
&-\sum_{k=1}^{2u+1}\binom{2v}{v+u-k}\binom{2w}{w+u-k}\left(\binom{2u+1}{k}-\binom{2u+1}{k-1}\right).
\end{aligned}
$$

The first line of this rearranged sum is now clearly non-positive. In the summation, the term $\left(\binom{2u+1}{k}-\binom{2u+1}{k-1}\right)$ is antisymmetric about $u+1$, and in fact equals

$$\frac{2(u+1-k)}{2u+2}\binom{2u+2}{k}.$$

If we therefore pair these antisymmetric terms, and use the fact that for $j>0$, we have

$$\binom{2v}{v+j-1}\geq\binom{2v}{v+j+1},\quad \binom{2w}{w+j-1}\geq\binom{2w}{w+j+1},$$

the claimed inequality follows.

For the term in x_1x_3, we need to show that the polynomial $a_{300}a_{021}+a_{201}a_{120}-2a_{210}a_{111}$ only has negative terms. In fact, we prove this for the two polynomials $a_{300}a_{021}-a_{210}a_{111}$ and $a_{201}a_{120}-a_{210}a_{111}$. Let us consider a particular monomial appearing in these polynomials; with the notation as above, the parities on $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ will differ from before. Since $2p_1+\tilde{u}+\tilde{v}=2s-1$, $2p_2+\tilde{u}+\tilde{w}=2s$ and $2p_3+\tilde{v}+\tilde{w}=2s+1$, we have either $\tilde{v}$ odd and $\tilde{u}$, $\tilde{w}$ even, or the other way round. Considering for instance the case $\tilde{v}=2v+1$, $\tilde{u}=2u$ and $\tilde{w}=2w$, we can run through a similar argument to that given above, and find that the coefficient of the given monomial in $a_{300}a_{021}-a_{210}a_{111}$ is

$$\sum_{k=0}^{2v+1}\binom{2v+1}{k}\binom{2w}{w+v-k}\left(\binom{2u}{u+v-(k+1)}-\binom{2u}{u+v-k}\right).$$

We now observe that the bracket in this summation is antisymmetric about $k=v-\frac{1}{2}$, and then pairing off terms proves the result in an analogous way to before. Similarly, the coefficient of the given monomial in $a_{201}a_{120}-a_{210}a_{111}$ is

$$\sum_{k=0}^{2v+1}\binom{2v+1}{k}\binom{2u}{u+v-k}\left(\binom{2w}{w+v-(k+1)}-\binom{2w}{w+v-k}\right),$$

and the same argument goes through, switching the roles of u and w.

For the term in x_1^2, we need to show that the polynomial $a_{300}a_{120}-a_{210}^2$ only has negative terms. For a monomial to appear in this polynomial, we have yet another parity condition, namely that $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ are all even, or are all odd. The reader is left to check the negativity. By symmetry, the only other term we need to consider is that in x_3^2; here we need that the polynomial $a_{201}a_{021} - a_{111}^2$ only has negative terms. The parity condition here is the same as for x_1^2, and the reader is left to verify the details of the negativity.

We now need to consider the cofactors B_{pq} with $p \neq q$. We shall only explicitly verify the x_3^2 terms here, and leave the others to the reader. Note in passing that in the formula for $6^4 S x_3^2$ from Section 1, we may consider instead the identity given simply by the terms in x_3^2, and so it will be the x_3^2 terms in the above cofactors which will occur in the algorithm we describe in Section 5. We check these terms for B_{12} and B_{13}, the rest then following from considerations of symmetry. Let us start with $\frac{1}{36}B_{12}$, which is

$$\begin{aligned}\frac{1}{36}(f_{13}f_{23} - f_{12}f_{33}) = ((&a_{201}x_1 + a_{111}x_2 + a_{102}x_3)(a_{111}x_1 + a_{021}x_2 + a_{012}x_3) \\ &- (a_{210}x_1 + a_{120}x_2 + a_{111}x_3)(a_{102}x_1 + a_{012}x_2 + a_{003}x_3)),\end{aligned}$$

whose term in x_3^2 is

$$a_{102}a_{012} - a_{111}a_{003}.$$

For $\frac{1}{36}B_{13} = \frac{1}{36}(f_{12}f_{23} - f_{13}f_{22})$, we have instead the polynomial

$$a_{111}a_{012} - a_{102}a_{021}.$$

The latter we already know has only positive terms from our calculations on the x_1x_3 term for B_{33}, where we saw that the polynomial $a_{201}a_{120} - a_{210}a_{111}$ only had negative terms (simply switch the first and last indices). For a given monomial to appear in the first polynomial, we need parities that $\tilde{u}$ is odd and $\tilde{v}$, $\tilde{w}$ even, or the other way round. For the monomial to appear in the second polynomial, we need parities that $\tilde{v}$ is odd and $\tilde{u}$, $\tilde{w}$ even, or the other way round.

For the former, namely $a_{102}a_{012} - a_{111}a_{003}$, the by now familiar calculation shows that the coefficient of our monomial, say in the case $\tilde{u} = 2u + 1$ odd and $\tilde{v} = 2v$, $\tilde{w} = 2w$ even, is the sum

$$\sum_{k=0}^{2u+1} \binom{2u+1}{k} \binom{2v}{v+u-k} \left(\binom{2w}{w+u-k} - \binom{2w}{w+u-(k-1)} \right).$$

The bracketed term is now antisymmetric about $k = u + \frac{1}{2}$, and pairing the terms again, we see that the sum is positive.

Theorem 4.2. *For the cubics under consideration, the Hessian determinant H is a polynomial in the a_j, b_j, c_j and x_1, x_2, x_3, all of whose coefficients are positive.*

Proof. Recall that, for any $n \times n$ matrix A with $n > 2$, we have $\mathrm{Adj}(\mathrm{Adj}A) = \det(A)\, A$. Applying this, with $A = (F_{ij})$, we deduce that

$$F_{12}H = -B_{33}B_{12} + B_{23}B_{13}.$$

Theorem 4.1 then implies that $F_{12}H$, a polynomial in the a_j, b_j, c_j and x_1, x_2, x_3, only has positive coefficients, where we may without loss of generality assume that F_{12} is non-trivial.

An easy argument shows however that if f, g are polynomials in a finite set of variables, with f non-trivial, such that f and fg only have positive coefficients, then the same is true for g. To see this, choose an order for the variables, and then order the monomials lexicographically. Now pick the largest monomial in f, and the largest monomial (if it exists) whose coefficient in g is negative; the product of these terms would yield a monomial in fg with negative coefficient. Applying this, since F_{12} only has positive coefficients, we deduce that the same holds for H.

5. More combinatorics of the S-invariant.

The fact that we have explicit formulae for the coefficients in both the polynomials $\partial^2 B_{33}/\partial x_p \partial x_q$ and $\partial^2 B_{pq}/\partial x_3^2$, provides an explicit recipe for calculating the coefficients in S directly. From Section 1, we know that

$$\tfrac{1}{4}\sum_{p,q}(\partial^2 B_{pq}/\partial x_3^2)(\partial^2 B_{33}/\partial x_p \partial x_q) = 6^4 S.$$

The tridegrees of the terms in $\partial^2 B_{33}/\partial x_p \partial x_q$ are

$$\begin{pmatrix} (2s-2, 2s, 2s+2) & (2s-1, 2s-1, 2s+2) & (2s-1, 2s, 2s+1) \\ (2s-1, 2s-1, 2s+2) & (2s, 2s-2, 2s+2) & (2s, 2s-1, 2s+1) \\ (2s-1, 2s, 2s+1) & (2s, 2s-1, 2s+1) & (2s, 2s, 2s) \end{pmatrix},$$

and those of $\partial^2 B_{pq}/\partial x_3^2$ the complementary degrees with respect to $4s$; for instance the tridegree of $\partial^2 B_{12}/\partial x_3^2$ is $(2s+1, 2s+1, 2s-2)$. The recipe for calculating the coefficient of a given allowable monomial M in now clear. Consider all factorisations $M = M_1 M_2$ of M, where the M_i are quadratic in each set of variables (a_j, b_j, c_j), and where M_2 has one of the tridegrees listed above for $\partial^2 B_{33}/\partial x_p \partial x_q$, with M_1 having the complementary tridegree. The M_i give rise to numbers $\tilde{u}_i, \tilde{v}_i, \tilde{w}_i$, where $i = 1, 2$, from which we have an explicit expression for the coefficient of M_i in the relevant entry of the matrix in question. Adding the products of these two coefficients as we range over the factorizations gives us the coefficient of M in S.

We illustrate this with the example given in Section 2; namely we consider the case $s = 3t$, and so $d = 3t+1$, and M a monomial with matrix of exponents having $4t$ columns of the form $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $4t$ columns of the form $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and t columns of the form $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$. Note that for all factorizations $M = M_1M_2$, we have $\tilde{v}_1 = 0 = \tilde{v}_2$. A factorization is determined by specifying for how many of the $a_j^3 b_j$ one takes a_j^2 in M_1, and for how many of the $b_j c_j^3$ one takes c_j^2; if these numbers are denoted by k, l respectively, then $\tilde{u}_1 = 4t-k$, $\tilde{w}_1 = 4t-l$, $\tilde{u}_2 = k$, $\tilde{w}_2 = l$. Note that M_2 then has tridegree $(8t-k, 2t+k+l, 8t-l)$. Thus the only pairs (k, l) of relevance will be $(2t, 2t)$, $(2t, 2t-1)$, $(2t, 2t-2)$, $(2t+1, 2t-1)$, $(2t+1, 2t-2)$ and $(2t+2, 2t-2)$. We consider each pair in turn; the fact that $\tilde{v} = 0$ simplifies the algebra considerably. The case $(2t, 2t)$ corresponds to the x_3^2 term in B_{33}; the coefficient of the monomial in $B_{33}/36$ is checked to simplify to

$$\binom{2t}{t}\left(\binom{2t}{t-1} - \binom{2t}{t}\right).$$

The case $(2t, 2t-1)$ corresponds to the x_2x_3 term in B_{33}; the relevant coefficient is

$$2\binom{2t-1}{t}\left(\binom{2t}{t-1} - \binom{2t}{t}\right).$$

The case $(2t, 2t-2)$ corresponds to the x_2^2 term in B_{33}; the relevant coefficient is

$$\binom{2t-2}{t-1}\left(\binom{2t}{t-1} - \binom{2t}{t}\right).$$

The case $(2t+1, 2t-1)$ corresponds to the x_1x_3 term; the relevant coefficient is

$$\binom{2t-1}{t}\left(\binom{2t+1}{t-1} - \binom{2t+1}{t}\right).$$

The case $(2t+1, 2t-2)$ corresponds to the x_1x_2 term; the relevant coefficient is

$$\binom{2t-2}{t-1}\left(\binom{2t+1}{t-1} - \binom{2t+1}{t}\right).$$

The case $(2t+2, 2t-2)$ corresponds to the x_1^2 term; the relevant coefficient is

$$\binom{2t-2}{t-1}\left(\binom{2t+2}{t} - \binom{2t+2}{t+1}\right).$$

Now we need the corresponding x_3^2 terms in B_{pq}. We already know that $(k,l) = (2t, 2t)$ corresponds to the x_3^2 term in B_{33} with coefficient of the monomial in $B_{33}/36$ being

$$\binom{2t}{t}\left(\binom{2t}{t-1} - \binom{2t}{t}\right).$$

We check that $(2t, 2t-1)$ corresponds to the x_3^2 term in $B_{23} = f_{12}f_{13} - f_{11}f_{23}$, namely $36(a_{111}a_{102} - a_{201}a_{012})$, and that the coefficient required is

$$\binom{2t+1}{t}\left(\binom{2t}{t} - \binom{2t}{t-1}\right);$$

the pair $(2t, 2t-2)$ corresponds to the x_3^2 term of $B_{22} = f_{11}f_{33} - f_{13}^2$, namely $36(a_{201}a_{003} - a_{102}^2)$, and the coefficient is

$$\binom{2t}{t+1}\binom{2t+2}{t+2} - \binom{2t}{t}\binom{2t+2}{t+1};$$

the pair $(2t+1, 2t-1)$ corresponds to the x_3^2 term of $B_{13} = f_{12}f_{23} - f_{13}f_{22}$, namely $36(a_{111}a_{012} - a_{102}a_{021})$, and the coefficient is

$$\binom{2t-1}{t-1}\binom{2t+1}{t+1} - \binom{2t-1}{t-1}\binom{2t+1}{t} = 0;$$

the pair $(2t+1, 2t-2)$ corresponds to the x_3^2 term of $B_{12} = f_{13}f_{23} - f_{12}f_{33}$, namely $36(a_{102}a_{012} - a_{111}a_{003})$, and the coefficient is

$$\binom{2t-1}{t-1}\left(\binom{2t+2}{t+1} - \binom{2t+2}{t+2}\right);$$

finally $(2t+2, 2t-2)$ corresponds to the x_3^2 term of $B_{11} = f_{22}f_{33} - f_{23}^2$, namely $36(a_{021}a_{003} - a_{012}^2)$, and the coefficient is

$$\binom{2t-2}{t-1}\left(\binom{2t+2}{t} - \binom{2t+2}{t+1}\right).$$

We now have all the information we need to calculate S from the above formula, where of course for a given (k,l) we shall need to weight the contribution by $\binom{4t}{k}\binom{4t}{l}$. Putting all this together, we get a formula for the relevant coefficient of the S-invariant as a function of t; with the help of MATHEMATICA, one checks that this simplifies to the same surprisingly simple formula that we obtained in Section 2.

Another case calculated by the author is that when $s = 4t$, and so $d = 4t+1$, and the monomial M has a matrix of exponents having $5t$ columns $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, t columns $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$,

$5t$ columns $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, and t columns $\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$. In a factorization $M = M_1 M_2$, we suppose that for precisely k of the $a_j^3 b_j$ we have taken a_j^2 in M_1 and for precisely k' of the $a_j b_j^3$ we have taken b_j^2. Similarly, we suppose that for precisely l of the $b_j c_j^3$ we have taken c_j^2 and for precisely l' of the $b_j^3 c_j$ we have taken b_j^2. We set $\tilde{k} = k - k'$ and $\tilde{l} = l - l'$. Consideration of the tridegrees shows that the only pairs $(\tilde{k}, \tilde{l})$ of relevance will be $(2t, 2t)$, $(2t, 2t-1)$, $(2t, 2t-2)$, $(2t+1, 2t-1)$, $(2t+1, 2t-2)$ and $(2t+2, 2t-2)$. For each pair $(\tilde{k}, \tilde{l})$, we need to consider $\tilde{k} \le k \le \tilde{k} + t$ and $\tilde{l} \le l \le \tilde{l} + t$. If $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ refer to M_2 and have the same meaning as before, we have $\tilde{u} = k + k' = 2k - \tilde{k}$, $\tilde{v} = 0$ and $\tilde{w} = l + l' = 2l - \tilde{l}$. The formulae we get are more complicated that those of the previous example, but we do nonetheless obtain an explicit formula for the coefficient of M in S. For $t = 1, 2$ and 3, the value of this coefficient is respectively 1600, 59340960 and 859033894118400, the $t = 1$ case coinciding with a previously calculated value.

Example 5.1. Finally, we shall consider the case where $s = d - 1$ and the monomial M has a matrix of exponents

$$(d-1)\begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}.$$

The coefficient was calculated for $d \le 5$ in Section 2, and the monomial was conjectured to have the smallest coefficient (assuming no fourth powers) for any given value of s. For this reason, it is an obvious crucial case in which to verify our main conjecture. In a factorization $M = M_1 M_2$, we suppose that for precisely k of the $a_j^3 b_j$ we have taken a_j^2 in M_1, for precisely l of the $a_j c_j^3$ we have taken c_j^2 in M_1, and for precisely m of the $b_j^3 c_j$ we have taken b_j^2. Consideration of tridegrees shows that the only pairs $(k - l, m - l)$ of relevance are $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 1)$, $(1, 2)$ and $(2, 2)$. For a given choice of (k, l, m), the corresponding triple $(\tilde{u}, \tilde{v}, \tilde{w})$ associated with M_2 is just (k, l, m). Because the $\tilde{v}$ is no longer zero in general, the formula for the coefficient (as a function of s) that we obtain involves triple summations. The formula may be found in the Appendix to this paper. Although MATHEMATICA does not reduce this formula to a simple form, it is nevertheless an explicit formula, which has been checked to give positive values for $s \le 501$. The proof of positivity for general s presumably follows by suitably rearranging the sums which occur in the formula. The values for $s = 1, 2, 3, 4, 5, 6, 7, 8$ are respectively 1, 4, 40, 652, 13174, 308464, 8158021 and 23830660; the first four of these correspond to previously calculated values. The last of these should be compared with the case $t = 2$ in the previous example. The fact that the numbers generated tend to have large prime factors (for instance 8158021 is prime) suggests that there is no simple form of the formula.

A proof of the positivity of the coefficient for the case of a general monomial still seems some way off. I restrict myself to the comment that the formulae we derived for the coefficients of monomials in the cofactors can all be expressed as the difference between two reasonably simple hypergeometric series of the form ${}_3F_2$ — in some of the special cases worked out above, they were the difference of even simpler terms. The theory of hypergeometric series may therefore feature in a proof of the conjecture.

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Appendix.

The formula for the coefficient A of M in S, with M as in Example 5.1, is given as $A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6$, where the A_i are defined (as functions of s) as follows:

$$A_1 = \sum_{l=0}^{s}\sum_{j=0}^{l}\sum_{i=0}^{s-l}\binom{s}{l}^3\binom{l}{j}^2\left(\binom{l}{j+1}-\binom{l}{j}\right)\binom{s-l}{i}^2\left(\binom{s-l}{i+1}-\binom{s-l}{i}\right).$$

$$A_2 = \sum_{l=0}^{s-1}\sum_{j=0}^{l}\binom{s}{l}^2\binom{s}{l+1}\binom{l}{j}\left(\binom{l}{j+1}\binom{l+1}{j+2}+\binom{l}{j+1}\binom{l+1}{j+1}-2\binom{l}{j}\binom{l+1}{j+1}\right)$$
$$\sum_{i=0}^{s-l}\binom{s-l}{i}\left(\binom{s-l}{i}\binom{s-l-1}{i-1}-\binom{s-l}{i+1}\binom{s-l-1}{i}\right).$$

$$A_3 = \sum_{l=0}^{s-2} \sum_{j=0}^{l} \binom{s}{l}^2 \binom{s}{l+2} \binom{l}{j} \left(\binom{l}{j+1} \binom{l+2}{j+2} - \binom{l}{j} \binom{l+2}{j+1} \right)$$
$$\sum_{i=0}^{s-l} \binom{s-l}{i} \binom{s-l-2}{i-1} \left(\binom{s-l}{i+1} - \binom{s-l}{i} \right).$$
$$A_4 = \sum_{l=0}^{s-1} \sum_{j=0}^{l+1} \binom{s}{l+1}^2 \binom{s}{l} \left(\binom{l}{j+1} \binom{l+1}{j+1} + \binom{l}{j} \binom{l+1}{j+1} - 2 \binom{l+1}{j} \binom{l}{j-1} \right)$$
$$\binom{l+1}{j} \sum_{i=0}^{s-l-1} \binom{s-l-1}{i} \binom{s-l}{i+1} \left(\binom{s-l-1}{i} - \binom{s-l-1}{i+1} \right).$$
$$A_5 = \sum_{l=0}^{s-2} \sum_{j=0}^{l} \binom{s}{l} \binom{s}{l+1} \binom{s}{l+2} \binom{l+1}{j} \left(\binom{l}{j+1} \binom{l+2}{j+2} - \binom{l}{j} \binom{l+2}{j+1} \right)$$
$$\sum_{i=0}^{s-l-1} \binom{s-l-1}{i} \binom{s-l}{i+1} \left(\binom{s-l-2}{i} - \binom{s-l-2}{i-1} \right).$$
$$A_6 = \sum_{l=0}^{s-2} \sum_{j=1}^{l+2} \binom{s}{l+2}^2 \binom{s}{l} \binom{l+2}{j} \left(\binom{l}{j-2} \binom{l+2}{j-1} - \binom{l}{j-1} \binom{l+2}{j} \right)$$
$$\sum_{i=0}^{s-l-2} \binom{s-l-2}{i} \binom{s-l}{i+1} \left(\binom{s-l-2}{i-1} - \binom{s-l-2}{i} \right).$$

If we take the formula for S in terms of cofactors, but write it as a sum over $p \leq q$, these numbers represent the coefficients of M in the terms with $(p, q) = (3, 3)$, $(2, 3)$, $(2, 2)$, $(1, 3)$, $(1, 2)$ and $(1, 1)$, respectively. If we take as an example $s = 4$ in the given formulae, the above numbers are $A_1 = 5804$, $A_2 = -3048$, $A_3 = 2352$, $A_4 = -4552$, $A_5 = -2256$, $A_6 = 2352$ and $A = 652$. In fact, for the monomial M of this example, we have $A_3 = A_6$ for all s; this latter identity may be seen by writing A_3 in terms of $l' = s-2-l$, rearranging the sums over i and j, and then comparing with the formula for A_6.