IN MEMORY OF MISHA SAVELIEV: LIE ALGEBRAS, INTEGRABILITY AND PARTICLE PHYSICS

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Abstract I talk about my personal interaction with Misha Saveliev and the influence upon my research of his work with Leznov concerning the construction of explicit general solutions for the conformal Toda field theory equations associated with any simple Lie group. This result can be, and was, extended in many ways and, in particular to the affine Toda field theories associated with affine Kac-Moody algebras. These theories exhibit a rich spectrum of particles, both as quantum excitations and as solitons and their physical attributes can be directly related to mathematical structures.

Last year, in 1998, a few months before his untimely death, Misha spent some months at the UNESP Institute for Theoretical Physics, in Sao Paulo, Brazil. There his host was Luiz Agostinho Ferreira, my former research student. Unfortunately Luiz is unable to be with us today but he wrote me the citing Misha's great contributions to the theory of integrable models: "His method (with Leznov) of constructing general solutions of non-linear theories is a major achievement and has influenced all of us. As a person he was very kind. He was very open to discuss his ideas and also to listen to other's views. It was a great pleasure to have him as a collaborator". I think that is well said and I can only echo these sentiments with feeling.

I first met Misha in 1978 at a meeting in Reinhardsbrunn (then in the DDR, now united Germany). There I learnt for the first time of his work with Leznov and the various aspects of it have had a deep influence on my subsequent research. Then I was interested in self-dual or Bogomolny equations for monopole solutions in spontaneously broken gauge theories, as part of my more general interest in electromagnetic duality.

Leznov and Saveliev showed how spherically symmetric solutions in grand unified theories (with gauge group G) are related to the Toda equations associated with that group and hence soluble. I think this was one of the first links established between duality and integrability. My presentation of this work at a school in Bonn a year later probably influenced Werner Nahm in his work on deriving his monopole equations.

Originally the Toda equations had no ostensible relation with Lie algebra theory but such connections were soon appreciated. First the relation was made with finite dimensional Lie algebras and Leznov and Saveliev and collaborators extended the idea to affine Kac-Moody algebras and later to other sorts of infinite dimensional algebra as well as to superalgebras. This was my first acquaintance with affine Kac-Moody algebras and I shall say more about them later in my talk. In fact the theme of this talk will be the contribution of Misha to developing the (still not fully understood) links between Lie algebras (in the wider sense), integrability and particle physics. I shall assume a familiarity with Lie algebra theory.

Since that time in Reinhardsbrunn, more than twenty years ago, I have had the pleasure of meeting Misha regularly, twice in Swansea, twice in Cambridge, in London, in Paris, in CERN and elsewhere. In the summer of 1981 I visited him at Protvino and, as a memento, I have a photo of the two of us on the beach of a local river where we had been bathing.

Let me now explain a little of what Leznov and Saveliev did. The simplest Toda equation is the Liouville field theory in two dimensional space-time:

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=2\partial_+\partial_-\phi=-e^{\phi}.$$

As Liouville himself well knew, it exhibited conformal symmetry. Leznov and Saveliev realised that, in addition, there was an underlying $SU(2)$ Lie algebraic structure. Exploiting this they found the known general solution in the form:

$$e^{-\phi/2}=\frac{1+AB}{2\sqrt{A'B'}},\qquad A=A(x^+),B=B(x^-).$$

A and B are arbitrary functions of the light-cone coordinates $x^{\pm}=(t\pm x)/\sqrt{2}$ respectively. Leznov and Saveliev associated to any simple Lie algebra g the "conformal Toda field theory equations"

$$(\partial_t^2-\partial_x^2)\phi=-\sum_{\alpha_i\text{simple roots of }g}e^{\alpha_i\cdot\phi}.$$

The fields ϕ are to be thought of as forming a column vector with $r=\text{rank } g$ entries. These equations are still conformally invariant and their general solution can be written down similarly [Leznov and Saveliev 1979b]

$$e^{-\lambda_i\cdot\phi}=<\lambda_i|\text{group theory factors}|\lambda_i>,\quad i=1,2\dots r$$

The ket denotes the highest weight state whose weight λ_i is the fundamental weight reciprocal to the simple root α_i in the sense $2\lambda_i.\alpha_j/\alpha_j^2=\delta_{ij}$. So again the solutions are constructed using the representation theory of the Lie group G. The relevance of the Lie group stems from the fact that there are associated zero-curvature gauge potentials taking values in its Lie algebra. Again the unspecified group theoretical factors depend on the light-cone coordinates separately. When the solution is independent of time t it is relevant to describing the shape of spherically symmetric magnetic monopole solitons in GUT theories [Leznov and Saveliev 1979a].

Many extensions of this idea are possible, in particular, to affine untwisted Kac-Moody algebras, $g\to\hat{g}$. The modification involves adding to the simple roots already mentioned the "extended root" $\alpha_0=-\psi$ where ψ is the "highest root" of g. Thus added to the right hand side of the conformal Toda field theory equation above is the term $e^{-\psi\cdot\phi}$. Later this was recognised to constitute an integrable deformation in the sense of Zamolodchikov [1989]. Such a term "deforms" a conformally invariant field theory into an integrable one possessing an infinite number of conservation laws which survive as relics of the conformal symmetry. The simplest case is when $g = SU(2)$ and the result of deforming the Liouville field equation is the affine Toda field theory equation familiarly known as the sinh-Gordon equation.

Mikhailov, Olshanetsky and Perelomov [1981] interpreted these coupled equations as describing quantum fields which could create particles whose masses they calculated. The presence of mass is, of course, the signature that conformal symmetry is broken. Corrigan and his collaborators [1989, 1990] found a beautiful interpretation for the mass formula; the masses of the r particles are proportional to the components of the Perron-Frobenius vector. This is the eigenvector of the Cartan matrix of g corresponding to its lowest eigenvalue. By the Perron-Frobenius theorem, valid because the off-diagonal entries of the matrix are negative (or zero), these components never vanish and are all of the same sign, just as is appropriate for this physical interpetation.

The affine Toda field equations are integrable in the sense that again there are underlying zero curvature gauge conditions, now taking values in the affine Kac-Moody algebra. Furthermore there are an infinite number of local conservation laws. As well as the light-cone components of momentum $P^{\pm}$ which carry "spins" ± 1 there are higher conserved quantities carrying spins given by integer values equal to what are called the exponents of the Lie group G modulo its Coxeter number, h.

All this physical structure relates to a beautiful mathematical construction called the principal gradation of the affine Kac-Moody algebra. This is an alternative to what is called the homogeneous gradation that is more familiar to physicists because of its relation to current algebra. Each simple Lie algebra possesses a principal $SO(3)$ subalgebra (essentially the largest possible such subalgebra) and we shall denote its generator lying in the Cartan subalgebra as T^3. The homogeneous gradation is specified by the adjoint action of the derivation $d_H = -L_0$, where L_m are the Virasoro generators. Instead, the principal gradation is specified by the adjoint action of $d_P = T^3_0 - hL_0$.

In the principal gradation an important role is played by an infinite-dimensional subalgebra of $\hat{g}$, called the principal Heisenberg subalgebra. Its generators are labelled E_M, where the values taken by the suffix M are precisely the exponents of g modulo its Coxeter number h. The commutation relations are

$$[E_M, E_N] = xM\delta_{M+N,0}, \qquad [d_P, E_M] = M\,E_M.$$

This is very reminiscent of the oscillators in string theory (where instead the suffix runs over all integers). x is the value of the central term. Because of what we said about conservation laws, this algebra, with x put to zero would be isomorphic with the enlarged Poincaré algebra describing the space-time symmetries of the $\hat{g}$ affine Toda field theory. In particular d_P corresponds to the Lorentz boost.

It would be natural to choose as remaining generators ones that diagonalise the principal Heisenberg subalgebra under commutation. This can be done with quantities

$F(\alpha, \rho)$, (α a root of g and ρ a complex number):

$$[E_M, F(\alpha, \rho)] = \alpha.q(M)\rho^M F(\alpha, \rho).$$

The coefficient $q(M)$ are essentially eigenvectors of the Cartan matrix of g with $M = 1$ corresponding to the Perron-Frobenius case.

Actually these generators are overcomplete and obey the following relations:

$$e^{2\pi i T_0^3/h} F(\alpha, \rho) e^{-2\pi i T_0^3/h} = F(\sigma(\alpha), \rho) = F(\alpha, \rho e^{\frac{2\pi i}{h}}).$$

Here σ denotes the element of the Weyl group of g known as the Coxeter element, obtained by a product of reflections in simple roots. Its order indeed equals h. In fact the roots of g fall into r orbits of h elements each. This means that it is sufficient to consider r quantities F corresponding to one judicious choice of representative from each of these orbits. This will be each of the simple roots (up to a sign).

The Leznov-Saveliev solution can be extended to this more general situation but is especially interesting when soliton solutions are sought. Making the coupling constant imaginary turns the sinh-Gordon equation into the sine-Gordon equation whose soliton solutions are well understood. It turns out that the same procedure endows all the affine Toda field equations with the degenerate vacua which are an essential prerequisite for soliton solutions.

Amongst the general solutions, those describing a configuration of incoming solitons that collide and eventually depart are rather special as asymptotically they consist of isolated lumps of energy. Use of this boundary condition drastically simplifies the general solution of Leznov and Saveliev to yield what Misha dubbed the "solitonic specialisation":

$$e^{-i\lambda_j\cdot\phi} = \frac{< \Lambda_j | e^{-\mu E_1 x^+} g(0) e^{-E_{-1} x^-} | \Lambda_j >}{< \Lambda_0 | e^{-\mu E_1 x^+} g(0) e^{-E_{-1} x^-} | \Lambda_0 >^{m_j}}.$$

Since these expressions again involve highest weight representations the central term x takes a positive integral value corresponding to the "level" of the representation concerned. In the numerator the representation with highest weight Λ_j, a fundamental weight of $\hat{g}$, has level m_j. Thus the levels in numerator and denominator coincide. The "group element" $g(0)$ contains the soliton data. It is a product of factors $e^{QF(\alpha,\rho)}$, one for each soliton. Q specifies its coordinate and ρ its rapidity. There are r species of solitons corresponding to the r orbits under the action of the Coxeter element.

The above expression is very general (applying to any number of solitons for any of the sets of affine Toda field equations) and yet nevertheless very simple. In practice it is not difficult to evaluate explicitly (yielding the known results for sine-Gordon) and it is possible to extract a number of physical features rather directly.

This is due to the fact that the dangerous looking exponentials e^{QF} terminate when expanded as a power series. All powers of F higher than the level x vanish, whilst the highest non vanishing term, $F^x/x!$, is a vertex operator, that is, an exponential in the principal Heisenberg subalgebra. This result makes it easy to see that the solution has the correct asymptotic behavior in the past and future.

Because there are r distinct quantities F, as explained above, there are precisely r species of soliton (matching the number of particle species created by the fields). Furthermore their masses are again proportional to the components of a Perron-Frobenius vector of the Cartan matrix of g. If g is not simply-laced, the Cartan matrix is not symmetric and so has distinct left and right eigenvectors, related by exchange of roots and coroots, and so suggestive of duality [Goddard, Nuyts and Olive 1977]. The time delays suffered by solitons in collisions can be found directly from the vertex operators just mentioned.

There are many unresolved issues concerning the quantum treatment of this picture. Misha became interested in this work, based as it was on his earlier work, and carried out by my collaborators and I in 1992-94 [Olive, Turok and Underwood 1993a, 1993b, Kneipp and Olive 1993] He was interested in extensions to more general sorts of Toda field equations, and this led to our collaborative work [Olive, Saveliev and Underwood 1993]. The last scientific contact I had with him was when he sent me his ideas on the quantisation of the above picture. It is a matter of great regret for me that I did not respond before it was too late.

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