Linear vs. Piecewise-Linear Embeddability of Simplicial Complexes

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§1. Introduction

(1.1) Definitions.

In order to state our results we will first fix the definitions of the notions mentioned in the title.

SIMPLICIAL COMPLEX K: by this we mean a finite set whose members, called its *simplices,* are themselves finite sets, and which is closed under subsets. The members of the simplices of K are called K's *vertices.*

Its realization K: If K has N vertices, then by thinking of these as the canonical basis vectors of $\mathbf{R}^N$, and of each simplex as the convex hull of its vertices, one obtains a subspace of $\mathbf{R}^N$, which too will be denoted K.

LINEAR EMBEDDABILITY OF K IN $\mathbf{R}^m$: a one-one map $e : K \to \mathbf{R}^m$ (from this realization K) will be called a *linear embedding* if it is the restriction of a linear map $\mathbf{R}^N \to \mathbf{R}^m$.

Note that for $m \geq 2(dimK) + 1$, any general position linear map $\mathbf{R}^N \to \mathbf{R}^m$ will restrict to such a linear embedding of K in $\mathbf{R}^m$. Thus the cases of interest are $dimK \leq m \leq 2(dimK)$.

PIECEWISE-LINEAR EMBEDDABILITY OF K IN $\mathbf{R}^m$: this means that, for some $r \geq 0$, the rth derived $K^{(r)}$ of K embeds linearly in $\mathbf{R}^m$.

Here the *rth derived* is defined inductively by $K^{(0)} = K$ and $K^{(r)} = (K^{(r-1)})'$, where L' denotes the simplicial complex whose simplices are sets of nonempty simplices of L which are totally ordered under $\subset$.

By mapping each vertex of K' (a simplex of K) to its barycentre, one gets the linear *barycentric* embedding of K' onto K, and so, by iteration, $K^{(r)} \overset{\cong}{\to} K$.

Composing with the inverse of this barycentric subdivision map, each linear embedding $K^{(r)} \to \mathbf{R}^m$ determines a one-one *piecewise-linear embedding* $e : K \to \mathbf{R}^m$.

The notion of piecewise-linear embeddability has been much studied – see e.g. Hudson [7] and Rourke-Sanderson [8] which will be our references for all other piecewise-linear terminology – because it avoids the possible wildness of topological embeddings, but is at the same time flexibile enough to make it much easier to handle than linear (or 'simplex-wise-linear' or 'geometric') embeddability.

(1.2) Statements of results.

As an easy consequence of a theorem of Steinitz [14], 1922, it follows that a one-dimensional complex, i.e. a *graph* K^1, will embed piecewise-linearly (or even topologically) in $\mathbf{R}^2$, only if it occurs as a subcomplex of the boundary of a simplicial 3-polytope: so à *fortiori* such a K^1 must also embed linearly in $\mathbf{R}^2$. See also Wagner [17], Fáry [3], Stein [13] and Stojaković [15].

In 1969, Grünbaum [6, p.502] conjectured that, likewise, for all $n \geq 2$, the piecewise-linear embeddability of a K^n in $\mathbf{R}^{2n}$ will be sufficient to guarantee its linear embeddability in $\mathbf{R}^{2n}$. We show that this conjecture is false in the following very strong sense.

Theorem A. *For each $n \geq 2$, $r \geq 0$, there is a simplicial n-complex L which embeds piecewise-linearly in $\mathbf{R}^{2n}$, but whose rth derived $L^{(r)}$ does not embed linearly in $\mathbf{R}^{2n}$.*

By virtue of a theorem of van Kampen [16, p.152], 1932, it is known that if K^n is a *pseudomanifold,* i.e. if each of its $(n-1)$-simplices is incident to at most two n-simplices, then it embeds piecewise-linearly in $\mathbf{R}^{2n}$. Though the K^n's of Theorem A are *not* pseudomanifolds, we do have, for ambient dimension one less, the following result which exhibits a similar phenomenon on the part of some 'higher-dimensional Möbius strips'.

Theorem B. *For each $n = 2^k, k \geq 1$, there is a K^n homeomorphic to M^n, the piecewise-linear manifold-with-boundary obtained by deleting an n-ball B^n from real projective space $\mathbf{R}P^n$, such that K^n embeds piecewise-linearly, but not linearly, in $\mathbf{R}^{2n-1}$.*

The case $n = 2$ of Theorem B, *viz.* that of the ordinary Möbius strip, was dealt with by the first author in [2].

Method of proof. The constructions given below to establish Theorems B and A are based on the notion of *linking,* and follow the basic strategy already used in [2]:

First, we arrange that, under any arbitrary piecewise-linear embedding, some two spherical subcomplexes will link each other with linking number ≥ 2.

Second, we take care to triangulate these two spheres by so few vertices that, under a linear embedding, this would be impossible.

We now recall what we need about linking, for more see e.g. Rourke-Sanderson [8], pp. 68-73, and Wu [19], pp. 175-181.

LINKING NUMBER: of any oriented p.l. sphere $S^{a-1} \subset \mathbf{R}^m$, with a disjoint oriented closed p.l. manifold $M^{m-a} \subset \mathbf{R}^m$, is the *intersection number,* i.e. counts the algebraical number of intersections, of any bounding compatibly oriented general position p.l. disk D^a, $\partial D^a = S^{a-1}$, with M^{m-a}. This is done by assigning an orientation to $\mathbf{R}^m$, and counting each of these intersections as $+1$ or -1 depending on whether the local orientation of D followed by that of M agrees with that of $\mathbf{R}^m$ or not.

If this number is zero, i.e. if S^{a-1} *does not link* M^{m-a}, then $S^{a-1} \hookrightarrow M^{m-a}$ extends to a map f of D^a into $\mathbf{R}^m$ such that $f(D^a) \cap M^{m-a} = \emptyset$.

Upto sign, the linking number of $S^{a-1} \subset \mathbf{R}^m$ with a sphere $S^{m-a} \subset \mathbf{R}^m$, is same as that of S^{m-a} with S^{a-1}, and coincides with the *degree* of an associated map – cf. proof of (3.1.1) – of the join $S^m = S^{a-1} \cdot S^{m-a}$ into itself.

§2. Higher Möbius strips

(2.1) Proof of Theorem B.

As is well known the manifold-with-spherical boundary, $M^n = \mathbf{R}P^n - (intB^n)$, $\partial M^n = \partial B^n = S^{n-1}$, can be considered as a twisted line bundle over a *core* submanifold $\mathbf{R}P^{n-1} \subset M^n$.

(2.1.1) *M^n embeds piecewise-linearly in $\mathbf{R}^{2n-1}$.*

To see this we can e.g. first embed (some triangulation of) the core $\mathbf{R}P^{n-1}$ piecewise-linearly in $\mathbf{R}^{2n-2}$, and so a trivial line bundle over it into $\mathbf{R}^{2n-1}$. The assertion now follows because we can locally twist the trivial bundle, for each of the $\mathbf{R}^{n-1}$ worth of directions along $\mathbf{R}P^{n-1}$, in the corresponding direction from the $\mathbf{R}^{n-1}$ worth of directions available complementary to the embedded trivial bundle.

(2.1.2) *The bounding sphere of M^n links its core under any piecewise-linear embedding $e : M^n \to \mathbf{R}^{2n-1}$.*

We give below, for all $k \geq 2$, a geometric argument; another more algebraical proof is sketched later in (2.2).

Assume, if possible, that $e(S^{n-1})$ does not link $e(\mathbf{R}P^{n-1})$. So we can extend the embedding e to a general position map f (of some triangulation) of $\mathbf{R}P^n$ into $\mathbf{R}^{2n-1}$, such that $f(\mathbf{R}P^{n-1}) \cap f(B^n) = \emptyset$.

We will now use some well-known constructions – cf. Zeeman [20] and [9] – to modify f to a piecewise-linear embedding g of $\mathbf{R}P^n$ in $\mathbf{R}^{2n-1}$: this suffices to furnish the desired contradiction because a theorem of Thom – see e.g. Steenrod [12], p. 34 – tells us that if $n = 2^k$, then $\mathbf{R}P^n$ does not embed in $\mathbf{R}^{2n-1}$.

We begin by noting that the singularities $sing(f)$ of f constitute an, at most one-dimensional, subset of the open n-ball $\mathbf{R}P^n - \mathbf{R}P^{n-1}$. So we can find a 2-dimensional conical subset A of this open n-ball such that $A \supset sing(f)$.

In case $k \geq 3$ one has $3 + n < 2n - 1$, so in this case we can enlarge the 2-dimensional subset $f(A)$ of $f(\mathbf{R}P^n) \subset \mathbf{R}^{2n-1}$ to a 3-dimensional cone $C \subset \mathbf{R}^{2n-1}$ which meets $f(\mathbf{R}P^n)$ only in $f(A)$.

We now choose regular neighbourhoods $N(A)$ of A in $\mathbf{R}P^n$, and $N(C)$ of C in $\mathbf{R}^{2n-1}$, such that the exterior, boundary, and the interior of $N(A)$ are mapped by f into the exterior, boundary, and the interior, respectively, of $N(C)$. Note that $N(A)$ is an n-ball, while $N(C)$ is a $(2n-1)$-ball, and that f is one-one outside $int(N(A))$. So, by coning $f(\partial(N(A)))$ over an interior point of the ball $N(C)$, we obtain an embedding $g : \mathbf{R}P^n \to \mathbf{R}^{2n-1}$.

In case $k = 2$ we can, in the first instance, only ensure that the cone C meets $f(\mathbf{R}P^n)$ in finitely many points besides $f(A)$. But then, by using a preliminary modification of f near some one-dimensional tree containing this zero-dimensional singular set, we can replace f by an f' such that C meets $f'(\mathbf{R}P^n)$ only in $f'(A) = f(A)$. After that we proceed as above to modify f' to an embedding g.

(2.1.3) *The image of the bounding sphere of* M^n *has a nonzero and even* self-linking number *under any piecewise-linear embedding* $e : M^n \to \mathbf{R}^{2n-1}$.

Here, by *self-linking number* of $\partial M^n = S^{n-1}$, we mean its linking number with a disjoint isotopic $\Sigma^{n-1} \subset M^n$.

To see the above note that any general position n-disk $D^n \subset \mathbf{R}^{2n-1}$, with $\partial D^n = e(S^{n-1})$, hits the core $e(\mathbf{R}P^{n-1})$ transversely in finitely many points. By (2.1.2) we know that the algebraical number t of such intersections is nonzero.

Now push S^{n-1} uniformly, along the fibers of the line bundle M^n over $\mathbf{R}P^{n-1}$, to obtain an isotopic sphere Σ^{n-1} arbitrarily close to the core $\mathbf{R}P^{n-1}$. Then the n-disk $D^n \subset \mathbf{R}^{2n-1}$ will intersect this double cover $e(\Sigma^{n-1})$ of $e(\mathbf{R}P^{n-1})$ transversely in $2t$ points.

(2.1.4) CONSTRUCTION OF K^n: Triangulate the boundary S^{n-1} and the isotopic sphere Σ^{n-1} of (2.1.3) as boundaries ∂s^n and $\partial\sigma^n$ of n-simplices s^n and σ^n. We choose any triangulation K^n of M^n which extends – cf. Armstrong [1] – this triangulation $\partial s^n \cup \partial\sigma^n$ of $S^{n-1} \cup \Sigma^{n-1}$. For example one can choose the explicit K^n's of (2.2.5).

(2.1.5) *K^n does not embed linearly in $\mathbf{R}^{2n-1}$.*

Otherwise, there will be some general position linear map $e : \mathbf{R}^N \to \mathbf{R}^{2n-1}$, whose restriction to the realization K^n is one-one.

The e-images of the closed simplices s^n and σ^n will either not intersect, or intersect in a line segment. In the latter case, if both ends of the line segment lie on the boundary of the same closed simplex, say on $e(\partial(s^n))$, then there is no linking, because $e(s^n) \cap e(\partial\sigma^n) = \emptyset$. And, if the two ends of the line segment lie on different boundaries, then we have $card(e(s^n) \cap e(\partial\sigma^n)) = 1$.

So the linking number of S^{n-1} and Σ^{n-1}, under a linear embedding e, would be 0 or ± 1, which contradicts (2.1.3). *q.e.d.*

(2.2) Deleted joins.

Embeddability questions – see e.g. [10] and its references – are intimately related to the following notion.

DELETED JOIN K_*: subcomplex of $K \cdot \overline{K}$, the join of two disjoint copies of K, consisting of all simplices $\sigma \cdot \overline{\theta}$ such that $\sigma \cap \theta = \emptyset$, and equipped with the free $\mathbf{Z}_2$-action $\sigma \cdot \overline{\theta} \leftrightarrow \theta \cdot \overline{\sigma}$.

Remarks (2.2.1) - (2.2.3) below sketch an alternative proof of (2.1.2) via deleted joins.

(2.2.1) *If $e(S^{n-1})$ were not linking $e(\mathbf{R}P^{n-1})$ under the embedding $e : M^n \to \mathbf{R}^{2n-1}$, then there would be a continuous $\mathbf{Z}_2$-map from the deleted join T_*, of some triangulation of $\mathbf{R}P^n$, into the antipodal $(2n-1)$-sphere S^{2n-1}.*

This is not hard to check, cf. proof of (3.1.3). In fact there would also be such a $\mathbf{Z}_2$-map from the *deleted product* $T_\bullet$, i.e. the 'mid-section' of T_* consisting of all cells $\sigma \times \overline{\theta}$ such that $\sigma \cap \theta = \emptyset$, into the antipodal sphere S^{2n-2} of one dimension less.

(2.2.2) WU LEMMA. *The $\mathbf{Z}_2$-homotopy types of the deleted join and the deleted product of a simplicial complex are topological invariants of the space underlying the complex.*

This is harder – cf. Wu [19, Ch.2] for products – but it will be shown in [11] that, with some care, this important fact generalizes even to *higher deleted joins,* i.e. analogues of K_* for groups G other than $\mathbf{Z}_2$.

(2.2.3) So, using any convenient triangulation of $\mathbf{R}P^n$, $n = 2^k$, it suffices to show by a *calculation of the characteristic classes* of the free $\mathbf{Z}_2$-homotopy type $(\mathbf{R}P^n)_\bullet$, that there is no continuous $\mathbf{Z}_2$-map from it to S^{2n-2}.

This calculation, which will be included in [11], is reminiscent of, but more general than, the proof of the
BORSUK-ULAM THEOREM. *There is no continuous $\mathbf{Z}_2$-map from S^p to S^q for $p > q$.*
However for $k = 1$, the Borsuk-Ulam Theorem *itself* provides the desired contradiction because of the following remarkable fact.

(2.2.4) *The deleted join of the 6-vertex real projective plane $\mathbf{R}P^2_6$ is $\mathbf{Z}_2$-homeomorphic to the antipodal 4-sphere.*

We recall that $\mathbf{R}P^2_6$ is a $\mathbf{Z}_2$-quotient or, if one prefers, one of the two parts of a *yin-yang decomposition* — cf. Grothendieck [5] — of the regular 12-vertex 2-sphere, i.e. the ubiquitous *icosahedron.*
The above result is not hard to check. In fact the second author hopes to include in [11] a complete classification of *all* K^n's for which K_* is a closed pseudomanifold. For example, if this pseudomanifold is n-dimensional, then it has to be the *octahedral n-sphere* $(\sigma^n_n)_*$ and – see [10] – if it is $(2n+1)$-dimensional, then it has to be a join of some *Flores' spheres* $(\sigma^{2s}_{s-1})_*$. *Here and below σ^i_j denotes the j-skeleton of an i-simplex.*

(2.2.5) *The omission of the n-simplex $\overline{\sigma^n}$, from the* simplicial join across σ^n, *of any triangulation of $\mathbf{R}P^n$ and the octahedral n-sphere $(\sigma^n_n)_*$, results in a K^n which satisfies the requirements of* (2.1.4).

This is straightforward. Here, by *simplicial join* $\mathbf{R}P^n\#(\sigma_n^n)_*$ *across* σ^n we mean the operation of first omitting an open n-simplex from the first factor and σ^n from the second factor, and then glueing the remaining complexes together by identifying the boundaries of these n-simplices.

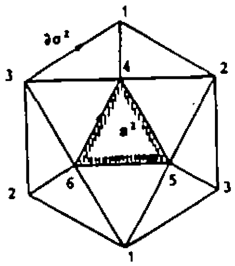
Note in particular that $(\mathbf{R}P_6^2\#(\sigma_2^2)_*) - \overline{\sigma^2}$ gives the 9-*vertex Möbius strip* [2] which fails to embed linearly in $\mathbf{R}^3$.

(2.2.6) The characteristic class computations of (2.2.3) suggest that if $\alpha(n)$ denotes the number of 1's in the binary expansion of n, then the simplicial Möbius n-strips K^n, $n \geq 2$, of (2.1) embed piecewise-linearly, but not linearly, in the space $\mathbf{R}^{2n-\alpha(n)}$.

§3. Grünbaum's conjecture

(3.1) Proof of Theorem A.
We will first consider the case $n = 2$.

Let $Mö_6$ denote the 6-*vertex Möbius strip,* i.e. $\mathbf{R}P_6^2$ minus one of its 2-simplices which will be called s^2. We note that, with appropriate orientations, $Mö_6$'s *boundary* ∂s^2 is homologous to twice its *core* $\partial\sigma^2$, where $\sigma^2 \notin \mathbf{R}P_6^2$ denotes the complementary 2-simplex $vert(\mathbf{R}P_6^2) - s^2$.



Besides $Mö_6$, we will also use a disjoint 6-simplex τ^6, one of whose 2-faces will also be called s^2, with the complementary 3-simplex $vert(\tau^6) - s^2$ denoted by φ^3.

(3.1.1) THE 2-COMPLEXES L_t. Each of these will contain a triangle called ∂s^2. For $t = 0$ we set

$$L_0 = \tau_2^6 - s^2,$$

and having defined L_t, $t \geq 0$, obtain L_{t+1} from L_t by identifying its ∂s^2 with the core $\partial\sigma^2$ of a disjoint copy of $Mö_6$. So, after this identification, the boundary ∂s^2 of $Mö_6$ becomes the ∂s^2 of L_{t+1}.

(3.1.2) *The 2-complexes L_t embed piecewise-linearly in* $\mathbf{R}^4$.

This is clear for $t = 0$.

So, assume inductively that there is a piecewise-linear embedding $e : L_t \to \mathbf{R}^4$, for some $t \geq 0$. Since $M\ddot{o}_6$ embeds piecewise-linearly even in $\mathbf{R}^3$, we can extend e to a general position piecewise-linear map $f : L_{t+1} \to \mathbf{R}^4$, with its finitely many double points (x, y) all such that $x \in L_t$ and $y \in M\ddot{o}_6$. For each such y choose a disjoint arc of $M\ddot{o}_6$ from y to its boundary ∂s^2. Removing from L_{t+1} small regular neighbourhoods of all these arcs we get a subspace X piecewise-linearly homeomorphic to L_{t+1} on which the map f is one-one.

(3.1.3) *The disjoint spheres $\partial\varphi^3$ and ∂s^2 of L_0 must link under any piecewise-linear embedding* $e : L_0 \to \mathbf{R}^4$.

By a lemma of Flores [4] the deleted join $(\tau_2^6)_*$ is an antipodal 5-sphere. So Borsuk-Ulam tells us that there can not be a continuous $\mathbf{Z}_2$-map from it to S^4.

But, S^4 has the same $\mathbf{Z}_2$-homotopy type as the join $\mathbf{R}^4 \cdot \overline{\mathbf{R}^4}$ minus its *diagonal,* i.e. all points of the type $\frac{1}{2}x + \frac{1}{2}\overline{x}$. And, there is a continuous $\mathbf{Z}_2$-map of $(L_0)_*$ into this space, *viz.* the *map* e_* defined by

$$\lambda x + (1-\lambda)\overline{y} \mapsto \lambda e(x) + (1-\lambda)\overline{e(y)}.$$

The closure of $(\tau_2^6)_* - (L_0)_*$ consists of the 5-ball $\partial\varphi^3 \cdot \overline{s_2^2}$ and its conjugate. The restriction of e_* to the boundary of this 5-ball has *degree* zero iff the linking number of the spheres $e(\partial\varphi^3)$ and $e(\partial s^2)$ is zero. So, if this were the case, e_* would extend to yield a continuous $\mathbf{Z}_2$-map $(\tau_2^6)_* \to S^4$, which is not possible.

(3.1.4) *The disjoint spheres $\partial\varphi^3$ and ∂s^2 of L_t, $t \geq 0$, must have linking number at least 2^t (in absolute value) under any piecewise-linear embedding* $e : L_t \to \mathbf{R}^4$.

We argue by induction starting from the above case $t = 0$. The triangle ∂s^2 of complex L_t, $t \geq 1$, is homologous to twice the triangle $\partial\sigma^2 \subset M\ddot{o}_6$ which was identified (3.1.1) to the triangle ∂s^2 of L_{t-1} to form L_t. So each transverse intersection under e of the latter, with a general position 3-disk spanning $e(\partial\varphi^3)$, gives rise to two intersections of the former having the same intersection number.

(3.1.5) *For any $r \geq 0$ we can choose t so big that the rth derived of $L = L_t$ does not embed linearly in $\mathbf{R}^4$.*

The number of simplices, contained in the simplicial 2 and 1-spheres occuring as the rth deriveds of $\partial\varphi^3$ and ∂s^2, is bounded in terms of r. From this it follows easily that, under any *linear* embedding of the union of these spheres in $\mathbf{R}^4$, the absolute value of the linking number is also bounded by a constant depending only on r. Choose any t such that 2^t is bigger than this number and use (3.1.4).

This concludes the proof of Theorem A for $n = 2$.

(3.1.6) *For $n \geq 3$ the above argument modifies as follows :*

(a) Instead of $M\ddot{o}_6$ we use its $(n-3)$-fold *suspension* $S^{n-3}(M\ddot{o}_6)$. Note that in it the $(n-1)$-sphere $S^{n-3}(\partial s^2)$ is homologous to twice the $(n-1)$-sphere $S^{n-3}(\partial\sigma^2)$.

(b) The n-complexes $L_{n,t}$, $t \geq 0$, are defined almost as before except for one small change. Instead of the n-skeleton of a τ^{2n+2}, minus one n-face u^n, we start with

$$L_{n,0} = (\tau_n^{2n+2} - u^n) \cup A^n,$$

where A^n is a simplicial *annulus* $S^{n-1} \times I$ having boundary $\partial A^n = \partial u^n \cup S^{n-3}(\partial s^2)$. So we have a $S^{n-3}(\partial s^2)$ in $L_{n,0}$ which is homologous to ∂u^n. For any $t \geq 1$, we now obtain $L_{n,t}$ from $L_{n,t-1}$ by identifying this $S^{n-3}(\partial s^2)$ of $L_{n,t-1}$, with the $S^{n-3}(\partial\sigma^2)$ of a disjoint copy of $S^{n-3}(M\ddot{o}_6)$.

The rest of the argument is unchanged: the piecewise-linear embeddability of these n-complexes in $\mathbf{R}^{2n}$ follows just as in (3.1.2), and the same argument as in (3.1.3) shows that the disjoint spheres $\partial\varphi^{n+1}$ and ∂u^n of $L_{n,0}$ link under any embedding in $\mathbf{R}^{2n}$, from which it follows almost as before that the linking number of $\partial\varphi^{n+1}$ and $S^{n-3}(\partial s^2)$ is $\geq 2^t$ for any embedding of $L_{n,t}$ in $\mathbf{R}^{2n}$ ... *q.e.d.*

(3.2) Concluding remarks.

We will now consider some variations of the above construction which give in particular a *generalization (3.2.3) of Theorem A* and a corollary (3.2.5) pertaining to *linear immersions.*

(3.2.1) *Examples $L_{n,t}$ analogous to those of (3.1) can be made starting from any* Kuratowski n-complex [9]

$$T^n = \tau_{n_1-1}^{2n_1} \cdot \tau_{n_2-1}^{2n_2} \cdot \dots \cdot \tau_{n_k-1}^{2n_k}, \; n_1 + \dots + n_k = n+1,$$

instead of just τ_n^{2n+2}.

For instance had we started off by setting $L_0 = \tau_1^4 \cdot \tau_0^2 - s^1 \cdot s^0$, then the analogue of (3.1.3) is that the 2-sphere $\partial\phi^2 \cdot \partial\phi^1$, formed by the vertices of L_0 not in the omitted 2-simplex $s^1 \cdot s^0$, always links the boundary of $s^1 \cdot s^0$ under any embedding of L_0 into $\mathbf{R}^4$.

(3.2.2) *Analogous constructions also give some* n*-complexes* $L_{n,m,t}$ *which embed piecewise-linearly, but not linearly in* $\mathbf{R}^m$*, for some other* n*'s and* m*'s such that* $n < m < 2n$.

We now start with different T^n's. For example, we can start with the join of $m-n$ disjoint copies of τ_0^2 (i.e. three points) and $2n-m$ disjoint copies of τ_0^0 (i.e. one point). Then the deleted join T_* is an antipodal $(m+1)$-sphere, so there is no $\mathbf{Z}_2$-map from it to S^m. Omitting an n-face from this T^n and proceeding as in (3.1.6) gives such complexes.

Their piecewise-linear embeddability in $\mathbf{R}^m$ follows from arguments analogous to those of (3.1.2) which remain valid at least under conditions like $m \geq \frac{3}{2}n + 1$ – cf. [18] – and thus we obtain examples of the above sort.

(3.2.3) *For each* $n \geq 2$, $r \geq 1$, $n < m \leq 2n$*, there is a simplicial* n*-complex which embeds piecewise-linearly in* $\mathbf{R}^m$*, but whose* r*th derived does not embed linearly in* $\mathbf{R}^m$.

Furthermore, if $n \geq 3$, we can take $n \leq m \leq 2n$ in the above.

These generalizations of Theorem A follow by using (3.2.2): e.g. one takes disjoint union of an $L_{[\frac{m}{2}],m,t}$ and a σ_n^n, etc.

We note that a finesse is required when dealing with the case $n = 2$, $m = 3$ of (3.2.3) since, by attaching $M\ddot{o}_6$'s à *la* (3.1.1), one now loses piecewise-linear embeddability. To overcome this, attach instead, at each step, an $\mathbf{R}P_6^2$ minus a 2-simplex s^2 having exactly one vertex on the attaching triangle $\partial\sigma^2$.

(3.2.4) *By iterating the construction (3.1.1) indefinitely one obtains an* infinite 2-complex L_∞*, which embeds topologically, but not piecewise-linearly, in* $\mathbf{R}^4$.

This is clear. Here, by *topologically embeddable,* we mean simply that there exists a continuous one-one map from L_∞ into $\mathbf{R}^4$.

Construction of such *finite* complexes is much harder, but might be implicit in the well-known work of R.D.Edwards and M.H.Freedman.

(3.2.5) *For each $n \geq 3$, $r \geq 0$, $max\{n,4\} \leq m < 2n$, there is a simplicial n-complex which embeds piecewise-linearly in $\mathbf{R}^m$, but whose rth derived does not even immerse linearly in $\mathbf{R}^m$.*

This follows either by considering cones over suitable examples from (3.2.3) or formulating an analogue of (3.2.3) for embeddings in S^m.

(3.2.6) *Embeddability of $\mathcal{K}$ in $\mathbf{R}^m$.* Thinking again, as in §1, of the N vertices of K, as the canonical basis vectors of $\mathbf{R}^N$, one gets a bigger (non-compact) space $\mathcal{K}$, if with each simplex of K is associated the *affine hull* of its vertices in $\mathbf{R}^N$ instead of the convex hull of its vertices.

Note that $\mathcal{K}$ collapses to K, from which it follows that the topological embeddability of K in $\mathbf{R}^m$ implies that of $\mathcal{K}$. But it is very easy to see – e.g. consider a segment and a disjoint point in $\mathbf{R}^1$ – that the *linear* embeddability of $\mathcal{K}$ in $\mathbf{R}^m$ is a strictly stronger notion than that of K.

There will be included in Chapter IV (on "Linear Embeddability") of [11] some interesting results involving this stronger notion, which incidentally makes sense not only for an ordered field like $\mathbf{R}$, but for any field whatsoever.

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