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Preprint Series 2011 (77)

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MPIM 11-77

BIQUANTIZATION OF SYMMETRIC PAIRS AND THE QUANTUM SHIFT

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Abstract. The biquantization of symmetric pairs was studied in [7] in terms of Kontsevich-like graphs. This note, also in view of recent results in [4], amends a minor mistake that did not spoil the main results of the paper. The mistake consisted in ignoring a regular term in the boundary contribution of some propagators. On the other hand, its correction brings back the quantum shift, present in the approaches by the orbit method, that was otherwise puzzlingly missing. In addition a detailed comparison of the two, equivalent, ways of defining biquantization working on the upper half plane or on one quadrant is presented, as well as a more conceptual approach to biquantization and the due corrections of some results of [7] in view of the aforementioned correction by the quantum shift.

1. Introduction

In [7] the biquantization of symmetric pairs was studied in terms of Kontsevich-like graphs. A puzzling result was the absence of the quantum shift, otherwise present in the treatments using the orbit method, by the natural character of the adjoint representation of $\mathfrak{k}$ on $\mathfrak{g}/\mathfrak{k} = \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the symmetric pair under consideration. It turns out that due to a (fortunately minor) mistake in [7] the quantum shift is actually there. Apart from this, the mistake does not spoil the other results of the paper.

The whole construction of [7, Section 1.6] relies on the 4-colored propagators introduced in [5] for the Poisson sigma model with two branes. It was recently observed by G. Felder and the second author in the preparation of [4], that, unlike in Kontsevich [12], the boundary contributions of the 4-colored propagators on the first quadrant for the collapse of the two endpoints may have a regular term in addition to the usual singular one. The regular term turns out simply to be the differential of the angle of the position where the two points collapsed, measured with respect to the origin, up to a sign, which depends on the boundary conditions themselves (roughly speaking, if we consider the same boundary conditions on the two half-lines bounding the first quadrant, then the sign is positive, while, for different boundary conditions on the two half-lines, we have a negative sign). Recall that these propagators are constructed from the Euclidean propagator (the differential of the angle of the line joining the two points) by reflecting the second argument with respect to the two boundaries of the first quadrant (producing four distinct closed 1-forms on the compactified configuration space of two points in the interior of the first quadrant) and then summing them up with signs; concretely,

$$\omega^{\varepsilon_1,\varepsilon_2}(z_1,z_2) = \frac{1}{2\pi}\left[\mathrm{d}\,\arg(z_2-z_1) + \varepsilon_1 \mathrm{d}\,\arg(z_2-\overline{z}_1) + \varepsilon_2 \mathrm{d}\,\arg(z_2+\overline{z}_1) + \varepsilon_1\varepsilon_2 \mathrm{d}\,\arg(z_2+z_1)\right],$$

where $\varepsilon_i = \pm$, $i = 1, 2$.

The contribution where the second argument is reflected w.r.t. the origin (corresponding to the situation where the second argument is reflected w.r.t. both boundaries of the first quadrant) is responsible for the regular term in all four situations, see Figure 1.

The first author acknowledges partial support by SNF Grant 200020_131813/1.

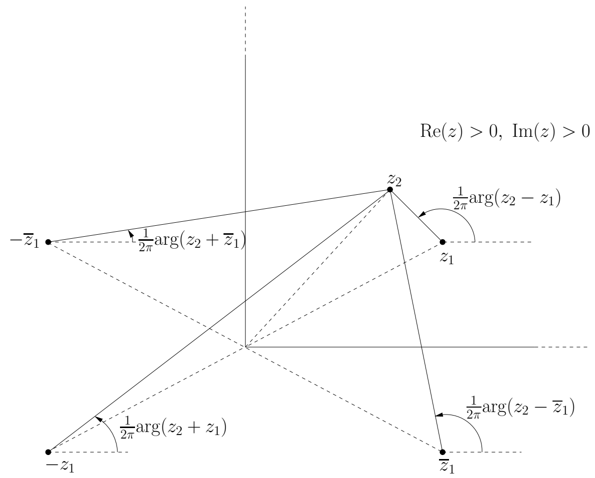

Figure 1 - A geometric explanation of the "singular term" and the "regular term"

The presence of this regular term was mistakenly neglected in [7]. Its net effect is that more boundary contributions have to be taken into account and extra terms are needed for cancellation. It turns out [4] that it is enough to allow for the presence of short loops and to assign each of them the regular term. This also has the pleasant effect of restoring the quantum shift. Some by-products, in particular in [7, Sections 4 and 5], have to be modified accordingly.

Regular terms also appear in the 8-colored propagators introduced in [7, Section 6] for the three brane case that is needed to show the independence from the choice of polarization. Also in this case the introduction of short loops, consistent with what observed above, saves the game: we will review these aspects, as well as their relationship with the Harish-Chandra homomorphism, in a forthcoming paper.

2. Notation and conventions

We work over a ground field $\mathbb{K}$, which may be $\mathbb{R}$ or $\mathbb{C}$. We consider a finite-dimensional symmetric pair $\mathfrak{g}$ over $\mathbb{K}$, *i.e.* a Lie algebra $\mathfrak{g}$, endowed with a Lie algebra automorphism σ, which is additionally an involution. In particular, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$, resp. $\mathfrak{p}$, is the eigenspace w.r.t. the eigenvalue $+1$, resp. -1, of σ. For a Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{h}^\perp$ its annihilator.

As $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{k} = \mathfrak{p}$ is a $\mathfrak{k}$-module, we introduce the short-hand notation

$$\delta(\bullet) = \frac{1}{2}\mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}_{\mathfrak{k}}(\bullet)),$$

see *e.g.* also [16, 17].

3. Biquantization in the framework of the 2-brane formality

We consider a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$; further, we consider two Lie subalgebras $\mathfrak{h}_i$, $i = 1, 2$.

To these data, we associate a Poisson manifold X and two coisotropic submanifolds U_i, $i = 1, 2$, as follows: we set $X = \mathfrak{g}^*$, endowed with the linear Kirillov–Kostant–Souriau Poisson bivector π, and $U_i = \mathfrak{h}_i^\perp$.

We want to regard biquantization as analyzed in [7] in the more general framework of the 2-brane Formality Theorem [4, Theorem 7.2]: thus, before entering into the details of biquantization, we need to review in some detail the main result of [4] and draw a bridge between it and the computations in [7].

3.1. On compactified configuration spaces.

For the upcoming discussion of the 4-colored propagators, we need to fix certain issues regarding compactified configuration spaces: in particular, we discuss two types of compactified configuration spaces, which arise in the context of biquantization, and we prove that they are in fact diffeomorphic. We observe that the following discussion may be viewed as an extension of certain computations in [4].

3.1.1. *The compactified configuration space of points in $\mathbb{H}^+ \sqcup \mathbb{R}$.* For two non-negative integers m, n, we consider the (open) configuration space $C^+_{n,m}$ of n distinct points in the complex upper half-plane $\mathbb{H}^+$ and m ordered points on the real axis $\mathbb{R}$. Its precise definition is

$$C^+_{n,m} = \left\{(z_1, \dots, z_n, x_1, \dots, x_m) \in (\mathbb{H}^+)^n \times \mathbb{R}^m : \ z_i \neq z_j, \ i \neq j, \ x_1 < \cdots < x_m\right\}/G_2,$$

where G_2 is the two-dimensional real Lie group $\mathbb{R}^+ \ltimes \mathbb{R}$, acting on $\mathbb{H}^+ \sqcup \mathbb{R}$ by rescalings and real translations. Provided $2n + m - 2 \geq 0$, $C^+_{n,m}$ is a smooth manifold of dimension $2n + m - 2$; it is obviously oriented.

We further consider the open configuration space

$$C_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \ z_i \neq z_j, \ i \neq j\} / G_3,$$

where G_3 is the 3-dimensional real Lie group $\mathbb{R}^+ \ltimes \mathbb{C}$, acting on $\mathbb{C}$ by rescalings and complex translations. It is obvious that, provided $2n-3 \geq 0$, C_n is a smooth manifold of dimension $2n-3$, which admits an obvious orientation from $\mathbb{C}^n$ and from the obvious orientability of G_3.

Kontsevich [12, Subsection 5.1] provides compactifications $\mathcal{C}^+_{n,m}$ and $\mathcal{C}_n$ of $C^+_{n,m}$ and C_n respectively in the sense of Fulton–MacPherson [9] (to be more precise, the smooth version of the algebraic compactification of [9], exploited in detail by Axelrod–Singer [2]): both compactified configuration spaces admit structures of smooth manifolds with corners (*i.e.* locally modeled on $(\mathbb{R}^+)^k \times \mathbb{R}^l$), and as such they admit boundary stratifications.

We observe that the permutation group $\mathfrak{S}_n$ acts naturally on C_n, and it can be proved that its action extends to $\mathcal{C}_n$: in particular, we may consider more general compactified configuration spaces $\mathcal{C}_A$, for a finite subset of $\mathbb{N}$. Because of similar reasons, we may consider compactified configuration spaces $\mathcal{C}^+_{A,B}$, for a finite subset A and a finite, ordered subset B of $\mathbb{N}$.

The stratifications of $\mathcal{C}^+_{n,m}$ and $\mathcal{C}_n$ admit a beautiful description in terms of trees [12, Subsection 5.1]. We first consider the boundary stratification of $\mathcal{C}_n$: for simplicity, we illustrate the boundary strata of codimension 1, namely such boundary strata are labeled by subsets A of $[n] = \{1, \dots, n\}$ of cardinality $2 \leq |A| \leq n$,

$$\partial_A \mathcal{C}_n \cong \mathcal{C}_A \times \mathcal{C}_{([n] \smallsetminus A) \sqcup \{\bullet\}},$$

where the first, resp. second, factor on the right-hand side of the previous identification represents the configuration of distinct points in $\mathbb{C}$ labeled by A which collapse together in $\mathbb{C}$ to a single point $\bullet$, resp. the final configuration of points after the collapse.

The boundary strata of codimension 1 of $\mathcal{C}^+_{n,m}$ are of two types, namely,

i) there exists a subset A of $[n]$, of cardinality $2 \leq |A| \leq n$, such that

$$\partial_A \mathcal{C}^+_{n,m} \cong \mathcal{C}_A \times \mathcal{C}^+_{([n] \smallsetminus A) \sqcup \{\bullet\}, m},$$

where the first, resp. second, factor on the right-hand side of the previous identification describes the collapse of the points in $\mathbb{H}^+$ labeled by A to a single point $\bullet$ in $\mathbb{H}^+$, resp. the final configuration of points after the collapse;

ii) there exist a subset A of $[n]$ and an ordered subset of $[m]$ consisting of consecutive non-negative integers, such that $0 \leq |A| \leq n$, $0 \leq |B| \leq m$, $1 \leq |A| + |B| \leq n + m - 1$, for which we have

$$\partial_{A,B} \mathcal{C}^+_{n,m} \cong \mathcal{C}^+_{A,B} \times \mathcal{C}^+_{[n] \smallsetminus A, ([m] \smallsetminus B) \sqcup \{\bullet\}},$$

where the first, resp. second, factor on the right-hand side of the previous identification describes the collapse of the points in $\mathbb{H}^+$ labeled by A and the consecutive, ordered points on $\mathbb{R}$ labeled by B to a single point $\bullet$ in $\mathbb{R}$, resp. the final configuration of points after the collapse.

3.1.2. *The compactified configuration space of points in $Q^{+,+} \sqcup i\mathbb{R}^+ \sqcup \mathbb{R}^+$.* For three non-negative integers l, m and n, we consider the (open) configuration space $C^+_{n,k,l}$ of n distinct points in the first quadrant $Q^{+,+}$, k ordered points on the positive imaginary axis $i\mathbb{R}^+$ and l ordered points on the positive real axis $\mathbb{R}$. We observe that the order of the points on the positive imaginary axis is the opposite of the intuitive one, *i.e.* $ix \leq iy$ if and only if $y \leq x$, for x, y in $\mathbb{R}$.

The precise definition of $C^+_{n,k,l}$ is

$$\begin{aligned} C^+_{n,k,l} = \big\{&(z_1, \dots, z_n, ix_1, \dots, ix_k, y_1, \dots, y_l) \in (Q^{+,+})^n \times (i\mathbb{R}^+)^k \times (\mathbb{R}^+)^l : \ z_i \neq z_j, \ i \neq j, \\ &x_k < \dots < x_1, \ y_1 < \dots < y_l\big\} / G_1, \end{aligned}$$

where $G_1 = \mathbb{R}^+$ acts on $Q^{+,+} \sqcup i\mathbb{R}^+ \sqcup \mathbb{R}^+$ by rescalings. Provided $2n + k + l - 1 \geq 0$, $C^+_{n,k,l}$ is a smooth manifold of dimension $2n + k + l - 1$. It inherits an obvious orientation from the natural orientation of $(Q^{+,+})^n \times (i\mathbb{R}^+)^k \times (\mathbb{R}^+)^l$ and the one of $\mathbb{R}^+$.

We may provide a compactification $\mathcal{C}^+_{n,k,l}$ in the sense of Fulton–MacPherson [9] of $C^+_{n,k,l}$ in a way similar to Kontsevich: the compactified configuration space $\mathcal{C}^+_{n,k,l}$ admits a structure of smooth manifold with corners.

We now consider the boundary strata of $\mathcal{C}^+_{n,k,l}$ of codimension 1, which are of the three following types:

i) there exists a subset A of $[n]$, of cardinality $2 \leq |A| \leq n$, such that

$$\partial_A \mathcal{C}^+_{n,k,l} \cong \mathcal{C}_A \times \mathcal{C}^+_{([n] \smallsetminus A) \sqcup \{\bullet\}, k, l},$$

where the first, resp. second, factor on the right-hand side of the previous identification describes the collapse of the points in $Q^{+,+}$ labeled by A to a single point $\bullet$ in $Q^{+,+}$, resp. the final configuration of points after the collapse;

ii) there exist a subset A of $[n]$ and an ordered subset of $[k]$, resp. $[l]$, consisting of consecutive non-negative integers, such that $0 \leq |A| \leq n$, $0 \leq |B| \leq k$, resp. $0 \leq |B| \leq l$, for which we have

$$\partial_{A,B}\mathcal{C}^+_{n,k,l} \cong \mathcal{C}^+_{A,B} \times \mathcal{C}^+_{[n]\smallsetminus A,([k]\smallsetminus B)\sqcup\{\bullet\},l}, \text{ resp. } \partial_{A,B}\mathcal{C}^+_{n,k,l} \cong \mathcal{C}^+_{A,B} \times \mathcal{C}^+_{[n]\smallsetminus A,k,([l]\smallsetminus B)\sqcup\{\bullet\}}$$

where the first, resp. second, factor on the right-hand side of the previous identification describes the collapse of the points in $Q^{+,+}$ labeled by A and the consecutive, ordered points on $i\mathbb{R}^+$ or $\mathbb{R}^+$ labeled by B to a single point $\bullet$ in $i\mathbb{R}^+$ or $\mathbb{R}^+$, resp. the final configuration of points after the collapse.

iii) there exist a subset A of $[n]$ and an ordered subset $B = B_1 \sqcup B_2$ of $[k] \sqcup [l]$, for which B_1 and B_2 are ordered subsets of consecutive points in $[k]$ and $[l]$ respectively, such that $k \in B_1$ if $B_1 \neq \emptyset$, $1 \in B_2$ if $B_2 \neq \emptyset$, $0 \leq |A| \leq n$, $0 \leq |B| \leq k + l$, resp. $1 \leq |A| + |B| \leq n + k + l - 1$, for which we have

$$\partial_{A,B_1,B_2}\mathcal{C}^+_{n,k,l} \cong \mathcal{C}^+_{A,B_1,B_2} \times \mathcal{C}^+_{[n]\smallsetminus A,[k]\smallsetminus(B\cap[k]),[l]\smallsetminus(B\cap[l])},$$

where the first, resp. second, factor on the right-hand side of the previous identification describes the collapse of the points in $Q^{+,+}$ labeled by A and the consecutive, ordered points on $i\mathbb{R}^+$ and $\mathbb{R}^+$ labeled by $B = B_1 \sqcup B_2$ to the origin of the axes, resp. the final configuration after the collapse.

3.1.3. *The relationship between $\mathcal{C}^+_{n,m}$ and $\mathcal{C}^+_{n,k,l}$.* First of all, we consider the open configuration spaces $C^+_{n,m}$ and $C^+_{n,k,l}$, where $m = k + l + 1$.

We observe that the holomorphic function $z \mapsto z^2$ on $\mathbb{C}$, when restricted to $Q^{+,+} \sqcup i\mathbb{R}^+ \sqcup \mathbb{R}^+$, gives rise to a biholomorphism to $\mathbb{H}^+ \sqcup (\mathbb{R} \smallsetminus \{0\})$, whose inverse we denote by $z \mapsto \sqrt{z}$: in fact, we have to choose a well-suited branch-cut for the complex square root, *e.g.* we cut out from the complex plane the negative imaginary axis plus the origin.

For m, k and l as before, we choose the $k+1$-st point on $\mathbb{R}$. Then, there is an obvious map from $C^+_{n,m}$ to $C^+_{n,k,l}$, which is defined by the following explicit formula:

$$\begin{aligned} &C^+_{n,m} \ni [(z_1, \dots, z_n, x_1, \dots, x_{k+1}, \dots, x_m)] \mapsto \\ &\mapsto \left[\left(\sqrt{z_1 - x_{k+1}}, \dots, \sqrt{z_n - x_{k+1}}, \sqrt{x_1 - x_{k+1}}, \dots, \sqrt{x_k - x_{k+1}}, \sqrt{x_{k+1} - x_{k+1}}, \dots, \sqrt{x_m - x_{k+1}}\right)\right] \in C^+_{n,k,l}. \end{aligned} \tag{1}$$

First of all, we observe that, because of the order on the points on the real axis, the difference $x_i - x_{k+1}$ is strictly negative, resp. positive, if $1 \leq i \leq k$, resp. $k + 2 \leq i \leq m$: thus, because of the said choice of a complex square root, $\sqrt{x_i - x_{k+1}} = i\sqrt{x_{k+1} - x_i}$, if $1 \leq i \leq k$, or $\sqrt{x_i - x_{k+1}} = \sqrt{x_i - x_{k+1}}$, if $k + 2 \leq i \leq m$, where now both square roots on the right-hand side of both equalities are real, positive numbers. Again, the order on the points x_i, $1 \leq i \leq k$ implies that $\sqrt{x_{k+1} - x_i} > \sqrt{x_{k+1} - x_{i+1}}$, therefore, the natural order on x_i, $1 \leq i \leq k$, is mapped to the natural order on $i\sqrt{x_{k+1} - x_i}$ discussed in Subsubsection 3.1.2. We may depict the morphism (1) graphically *via*

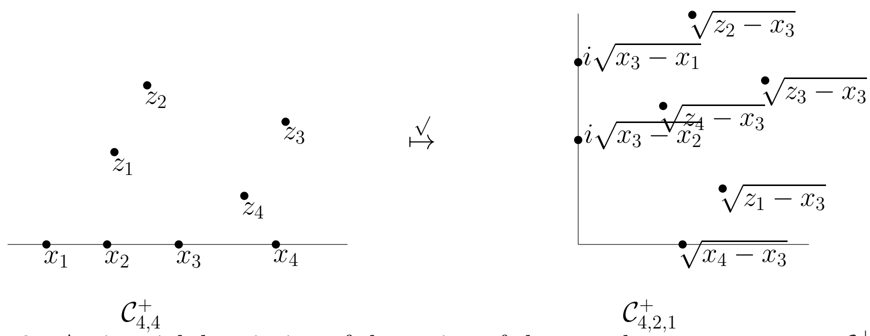

Figure 2 - A pictorial description of the action of the complex square root on $\mathcal{C}^+_{4,4}$

An easy computation proves that the above morphism is well-defined, *i.e.* it does not depend on the choice of representatives; furthermore, the morphism is obviously smooth, and is in fact a diffeomorphism, whose inverse is

$$C^+_{n,k,l} \ni [(z_1, \dots, z_n, ix_1, \dots, ix_k, y_1, \dots, y_l)] \mapsto \left[\left(z_1^2, \dots, z_n^2, -x_1^2, \dots, -x_k^2, 0, y_1^2, \dots, y_l^2\right)\right] \in C^+_{n,m}.$$

The important point is that the complex square function and the chosen inverse (the above complex square root) extend to smooth functions between the compactified configuration spaces $\mathcal{C}^+_{n,m}$ and $\mathcal{C}^+_{n,k,l}$.

Proposition 3.1. *For non-negative integers n, m, k, l, such that $m = k + l + 1$, the smooth manifolds with corners $\mathcal{C}^+_{n,m}$ and $\mathcal{C}^+_{n,k,l}$ are diffeomorphic* via *the choice of a complex square root with branch cut $i\mathbb{R}^- \sqcup \{0\}$.*

Proof. We prove that the diffeomorphism (1) extends to a diffeomorphism on the compactified configuration spaces by computing its expression w.r.t. local coordinates for the relevant boundary strata of codimension 1. In fact, as sketched in [12, Subsection 5.2], the boundary strata of higher codimension correspond to products with more than two factors of compactified configuration spaces of the same kind, representing configuration of points collapsing together, be it in the complex upper half-plane or on the real axis, resp. in the first quadrant or on the positive complex or real axis or on the origin.

It suffices therefore to prove the claim on the interior of the boundary strata of codimension 1 of $\mathcal{C}^+_{n,m}$ and $\mathcal{C}^+_{n,k,l}$: these have been characterized explicitly in Subsubsections 3.1.1 and 3.1.2. Furthermore, without loss of generality, we may assume $A=[i]$ and $B=[j]$.

We have to prove that the map (1) maps diffeomorphically the interior of boundary strata of codimension 1 of $\mathcal{C}^+_{n,m}$ to the interior of boundary strata of codimension 1 of $\mathcal{C}^+_{n,k,l}$.

We consider first the boundary stratum of type i) of $\mathcal{C}^+_{n,m}$ labeled by $A=[i]$, for $2\leq i\leq n$. Local coordinates of the interior $C_i\times\mathcal{C}^+_{n-i+1,m}$ are provided by

$$C_i\times C^+_{n-i+1,m}\ni\left(\left(e^{i\varphi},z_1,\dots,z_{i-2}\right),\left(e^{it},w_1,\dots,w_{n-i},x_1,\dots,x_k,0,x_{k+2},\dots,x_m\right)\right),$$

where φ is in $[0,2\pi)$, t in $(0,\pi)$, z_i in $\mathbb{C}$, w_i in $\mathbb{H}^+$, and all points in $\mathbb{C}$ and $\mathbb{H}^+$ are distinct, while the points on the real axis are lexicographically (strictly) ordered. On the other hand, the interior of the boundary stratum $\mathcal{C}_i\times\mathcal{C}^+_{n-i+1,k,l}$ is described by the following local coordinates:

$$C_i\times C^+_{n-i+1,k,l}\ni\left(\left(e^{i\varphi},z_1,\dots,z_{i-2}\right),\left(e^{it},w_1,\dots,w_{n-i},ix_1,\dots,ix_k,y_1,\dots,y_l\right)\right),$$

where φ is in $[0,2\pi)$, t in $(0,\frac{\pi}{2})$, z_i in $\mathbb{C}$, w_i in $Q^{+,+}$, and all points in $\mathbb{C}$ and $Q^{+,+}$ are distinct, and $x_1>\cdots x_k>0$, $0<y_1<\cdots<y_l$.

For $\varepsilon>0$ sufficiently small, local coordinates for $\mathcal{C}^+_{n,m}$, resp. $\mathcal{C}^+_{n,k,l}$, near the interior of the boundary stratum $\mathcal{C}_i\times\mathcal{C}^+_{n-i,m}$, resp. $\mathcal{C}_i\times\mathcal{C}^+_{n-i+1,k,l}$, are given by

$$\begin{aligned}&\left[\left(e^{it},e^{it}+\varepsilon e^{i\varphi},e^{it}+\varepsilon z_1,\dots,e^{it}+\varepsilon z_{i-2},w_1,\dots,w_n,x_1,\dots,x_k,0,x_{k+2},\dots,x_m\right)\right],\ \text{resp.}\\&\left[\left(e^{it},e^{it}+\varepsilon e^{i\varphi},e^{it}+\varepsilon z_1,\dots,e^{it}+\varepsilon z_{i-2},w_1,\dots,w_n,ix_1,\dots,ix_k,y_1,\dots,y_l\right)\right].\end{aligned}$$

We apply the morphism (1) to the first of the previous expressions, getting

$$\left[\left(\sqrt{e^{it}},\sqrt{e^{it}+\varepsilon e^{i\varphi}},\sqrt{e^{it}+\varepsilon z_1},\dots,\sqrt{e^{it}+\varepsilon z_{i-2}},\sqrt{w_1},\dots,\sqrt{w_n},i\sqrt{-x_1},\dots,i\sqrt{-x_k},\sqrt{x_{k+2}},\dots,\sqrt{x_m}\right)\right].\tag{2}$$

We rewrite the terms in the previous expression containing the infinitesimal parameter ε using the fact that the chosen complex square root is holomorphic on $\mathbb{H}^+$, thus getting

$$\sqrt{e^{it}+\varepsilon z}=e^{i\frac{t}{2}}+\frac{\varepsilon z}{2e^{i\frac{t}{2}}}+\mathcal{O}(\varepsilon^2)=e^{i\frac{t}{2}}+\frac{\varepsilon}{2}e^{-i\frac{t}{2}}z+\mathcal{O}(\varepsilon^2),\ t\in(0,\pi),\ z\in\mathbb{C}.$$

To compare expressions, we may neglect terms in the expansion of order strictly higher than 2: rescaling by $\frac{1}{2}$ the infinitesimal parameter ε, Expression (2) can be rewritten as

$$\left[\left(e^{i\frac{t}{2}},e^{i\frac{t}{2}}+\varepsilon e^{i\left(\varphi-\frac{t}{2}\right)},e^{i\frac{t}{2}}+\varepsilon e^{-i\frac{t}{2}}z_1,\dots,e^{i\frac{t}{2}}+\varepsilon e^{-i\frac{t}{2}}z_{i-2},\sqrt{w_1},\dots,\sqrt{w_n},i\sqrt{-x_1},\dots,i\sqrt{-x_k},\sqrt{x_{k+2}},\dots,\sqrt{x_m}\right)\right],$$

whence it follows immediately that the morphism (1) maps the interior of $\mathcal{C}_i\times\mathcal{C}^+_{n-i+1,m}$ diffeomorphically to the interior of $\mathcal{C}_i\times\mathcal{C}^+_{n-i+1,k,l}$, where the diffeomorphism is explicitly the product of the morphism (1) from $C^+_{n-i+1,m}$ to $C^+_{n-i+1,k,l}$ with the obvious diffeomorphism of C_i given by

$$C_i\ni[(z_1,\dots,z_i)]\mapsto\left[\left(\frac{z_1}{\sqrt{w_1-x_{k+1}}},\dots,\frac{z_i}{\sqrt{w_1-x_{k+1}}}\right)\right]\in C_i,$$

where w_1 and x_{k+1} are taken from $C^+_{n-i+1,m}$.

We now consider the interior of the boundary stratum $\mathcal{C}^+_{i,B}\times\mathcal{C}^+_{n-i,([m]\smallsetminus B)\sqcup\{\bullet\}}$, where B is an ordered subset of $[m]$ consisting of consecutive elements, and we assume that $1\leq i+|B|\leq n+m-1$. We have to further distinguish between two situations: $|B|=0$ (and consequently $1\leq i\leq n$), and $|B|\neq 0$.

We consider the situation $|B|=0$, and we further distinguish between the case, where the new point $\bullet$ on the real axis (corresponding to the cluster of points labeled by $[i]$ in $\mathbb{H}^+$ approach $\mathbb{R}$) lies on the left or on the right of the distinguished point x_{k+1}. We do the explicit computations only in the case, where $\bullet$ is on the left of x_{k+1}, leaving the other case to the reader.

If $\bullet$ lies on the left of x_{k+1}, we may safely assume that $\bullet = x$ lies on the left of x_1: then, local coordinates for the interior of $\mathcal{C}^+_{i,0}\times\mathcal{C}^+_{n-i,m+1}$ are given by

$$C^+_{i,0}\times C^+_{n-i,m+1}\ni\left((i,z_1,\dots,z_{i-1}),\left(e^{it},w_1,\dots,w_{n-i-1},x,x_1,\dots,x_k,0,x_{k+2},\dots,x_m\right)\right),$$

where t in $(0,\pi)$, z_i and w_j are in $\mathbb{H}^+$, and all points in $\mathbb{H}^+$ are distinct, while the points on the real axis are lexicographically (strictly) ordered. Similarly, local coordinates for the interior of $\mathcal{C}^+_{i,0}\times\mathcal{C}^+_{n-i,m+1}$ are given by

$$C^+_{i,0}\times C^+_{n-i,k+1,l}\ni\left((i,z_1,\dots,z_{i-1}),\left(e^{it},w_1,\dots,w_{n-i-1},ix,ix_1,\dots,ix_k,y_1,\dots,y_l\right)\right),$$

where t in $(0,\frac{\pi}{2})$, z_i in $\mathbb{H}^+$ and w_i in $Q^{+,+}$, all points in $\mathbb{H}^+$ and $Q^{+,+}$ are distinct, $x>x_1>\cdots>x_k>0$ and $0<y_1<\cdots<y_l$.

For $\varepsilon>0$ sufficiently small, local coordinates for $\mathcal{C}^+_{n,m}$, resp. $\mathcal{C}^+_{n,k,l}$, near the interior of the boundary stratum $\mathcal{C}^+_{i,0}\times\mathcal{C}^+_{n-i,m+1}$, resp. $\mathcal{C}^+_{i,0}\times\mathcal{C}^+_{n-i,k+1,l}$, are given by

$$\left[\left(x+\varepsilon i,x+\varepsilon z_1,\dots,x+\varepsilon z_{i-1},e^{it},w_1,\dots,w_{n-i-1},x_1,\dots,x_k,0,x_{k+2},\dots,x_m\right)\right],\ \text{resp.}$$
$$\left[\left(ix+\varepsilon,ix-i\varepsilon z_1,\dots,ix-i\varepsilon z_{i-1},e^{it},w_1,\dots,w_{n-i-1},ix_1,\dots,ix_k,y_1,\dots,y_l\right)\right]$$

The image of the first expression w.r.t. the morphism (1) is simply

$$\left[\left(\sqrt{x+\varepsilon i},\sqrt{x+\varepsilon z_1},\dots,\sqrt{x+\varepsilon z_{i-1}},\sqrt{e^{it}},\sqrt{w_1},\dots,\sqrt{w_{n-i-1}},i\sqrt{-x_1},\dots,i\sqrt{-x_k},\sqrt{x_{k+2}},\dots,\sqrt{x_m}\right)\right].\tag{3}$$

We consider the first i entries in the previous expression: once again, using the holomorphy of the chosen complex square root, and recalling that $x<x_1<0$ and that ε is chosen sufficiently small, we find

$$\sqrt{x+\varepsilon i}=\sqrt{x}+\frac{\varepsilon i}{2\sqrt{x}}+\mathcal{O}(\varepsilon^2)=i\sqrt{-x}+\frac{\varepsilon}{2\sqrt{-x}}+\mathcal{O}(\varepsilon^2),$$
$$\sqrt{x+\varepsilon z_j}=i\sqrt{-x}-\frac{i\varepsilon}{2}\frac{z_j}{\sqrt{-x}}+\mathcal{O}(\varepsilon^2),\ 1\le j\le i-1.$$

Once again, rescaling by $\frac{1}{2}$ the infinitesimal parameter ε, and neglecting terms of order higher than 1 w.r.t. ε in the above expressions, we may rewrite Expression (3) as

$$\left[\left(i\sqrt{-x}+\varepsilon,i\sqrt{-x}+\varepsilon\frac{z_1}{\sqrt{-x}},\dots,i\sqrt{-x}+\varepsilon\frac{z_{i-1}}{\sqrt{-x}},e^{i\frac{t}{2}},\sqrt{w_1},\dots,\sqrt{w_{n-i-1}},i\sqrt{-x_1},\dots,i\sqrt{-x_k},\sqrt{x_{k+2}},\dots,\sqrt{x_m}\right)\right],$$

and it is easy to see that the morphism (1) maps $C^+_{i,0}\times C^+_{n-i,m+1}$ diffeomorphically to $C^+_{i,0}\times C^+_{n-i,k+1,l}$, and the induced morphism is precisely given by the product of the morphism (1) from $C^+_{n-i,m+1}$ to $C^+_{n-i,k+1,l}$ with the obvious diffeomorphism of $C^+_{i,0}$ given by

$$C^+_{i,0}\ni[(z_1,\dots,z_i)]\mapsto\left[\left(\frac{z_1}{\sqrt{x_{k+1}-x}},\dots,\frac{z_i}{\sqrt{x_{k+1}-x}}\right)\right]\in C^+_{i,0},$$

where x denotes the first point on the real axis in lexicographical order, and x_{k+1} is the special point on real axis.

For the situation $|B|\neq 0$, we need to distinguish between two cases, namely *i)* B contains $k+1$ or *ii)* b does not contain $k+1$ (in which case, either the minimum of B is greater or equal than $k+2$ or the maximum of B is less or equal than k).

We first consider the case, where B contains $k+1$, and we assume $A=[i]$ and $B=[p,q]=\{p,\dots,q\}$, where $1\le p\le k+1\le q\le m$; we further write $B=\{k\}\sqcup B_1\sqcup B_2$, where $B_1=[p,k]$ and $B_2=[k+2,q]$ (of course, B_1 and/or B_2 may be empty). The interior of the corresponding boundary stratum of $C^+_{n,m}$, resp. $C^+_{n,k,l}$, is $C^+_{i,B}\times C^+_{n-i,([m]\smallsetminus B)\sqcup\{\bullet\}}$, resp. $C^+_{i,B_1,B_2}\times C^+_{n-i,[k]\smallsetminus B_1,[l]\smallsetminus B_2}$, and corresponding local coordinates are given by

$$\begin{aligned}C^+_{i,B}\times C^+_{n-i,([m]\smallsetminus B)\sqcup\{\bullet\}}&\ni\left(\left(e^{it_1},z_1,\dots,z_{i-1},x_1,\dots,x_{k-p+1},0,x_{k-p+3},\dots,x_{q-p+1}\right),\right.\\&\qquad\left.\left(e^{it_2},w_1,\dots,w_{n-i-1},x'_1,\dots,x'_{p-1},0,x'_{p+1},\dots,x'_{m-2k-q}\right)\right),\\C^+_{i,B_1,B_2}\times C^+_{n-i,[k]\smallsetminus B_1,[l]\smallsetminus B_2}&\ni\left(\left(e^{it_1},z_1,\dots,z_{i-1},ix_1,\dots,ix_{k-p+1},y_1,\dots,y_{q-k-1}\right),\right.\\&\qquad\left.\left(e^{it_2},w_1,\dots,w_{n-i-1},ix'_1,\dots,ix'_{p-1},y'_1,\dots,y'_{l-q+k+1}\right)\right),\end{aligned}$$

where t_i, $i=1,2$, is in $(0,\pi)$, resp. $(0,\frac{\pi}{2})$, all points in $\mathbb{H}^+$, resp. $Q^{+,+}$, are distinct in the first, resp. second, expression. In the first, resp. second, expression, the x_i and x'_i are lexicographically ordered, resp. $x_1>\cdots x_{k-p+1}>0$, $x'_1>\cdots x'_{p-1}>0$, $0<y_1<\cdots y_{q-k-1}$ and $0<y'_1<\cdots<y'_{l-q+k+1}$.

Choosing a positive number ε sufficiently small as before, we may write local coordinates of $\mathcal{C}^+_{n,m}$, resp. $\mathcal{C}^+_{n,k,l}$, near the interior of the boundary stratum $\mathcal{C}^+_{i,B} \times \mathcal{C}^+_{n-i,([m]\smallsetminus B)\sqcup\{\bullet\}}$, resp. $\mathcal{C}^+_{i,B_1,B_2} \times \mathcal{C}^+_{n-i,[k]\smallsetminus B_1,[l]\smallsetminus B_2}$, namely

$$\begin{aligned}&\big[\big(\varepsilon e^{it_1}, \varepsilon z_1, \dots, \varepsilon z_{i-1}, e^{it_2}, w_1, \dots, w_{n-i-1},\\ &x'_1, \dots, x'_{p-1}, \varepsilon x_1, \dots, \varepsilon x_{k-p+1}, 0, \varepsilon x_{k-p+3}, \dots, \varepsilon x_{q-p+1}, x'_{p+1}, \dots, x'_{m-2k-q}\big)\big], \text{ resp.}\\ &\big[\big(\varepsilon e^{it_1}, \varepsilon z_1, \dots, \varepsilon z_{i-1}, e^{it_2}, w_1, \dots, w_{n-i-1},\\ &ix'_1, \dots, ix'_{p-1}, \varepsilon i x_1, \dots, \varepsilon i x_{k-p+1}, 0, \varepsilon y_1, \dots, \varepsilon y_{q-k+1}, y'_1, \dots, y'_{l-q+k+1}\big)\big].\end{aligned}$$

We now apply the morphism (1) to the first of the two previous expressions, getting

$$\begin{aligned}&\Big[\Big(\sqrt{\varepsilon e^{it_1}}, \sqrt{\varepsilon z_1}, \dots, \sqrt{\varepsilon z_{i-1}}, \sqrt{e^{it_2}}, \sqrt{w_1}, \dots, \sqrt{w_{n-i-1}},\\ &i\sqrt{-x'_1}, \dots, i\sqrt{-x'_{p-1}}, i\sqrt{-\varepsilon x_1}, \dots, i\sqrt{-\varepsilon x_{k-p+1}}, \sqrt{\varepsilon x_{k-p+3}}, \dots, \sqrt{\varepsilon x_{q-p+1}}, \sqrt{x'_{p+1}}, \dots, \sqrt{x'_{m-2k-q}}\Big)\Big] =\\ &\Big[\Big(\sqrt{\varepsilon} e^{i\frac{t_1}{2}}, \sqrt{\varepsilon}\sqrt{z_1}, \dots, \sqrt{\varepsilon}\sqrt{z_{i-1}}, e^{i\frac{t_2}{2}}, \sqrt{w_1}, \dots, \sqrt{w_{n-i-1}},\\ &i\sqrt{-x'_1}, \dots, i\sqrt{-x'_{p-1}}, \sqrt{\varepsilon} i\sqrt{-x_1}, \dots, \sqrt{\varepsilon} i\sqrt{-x_{k-p+1}}, \sqrt{\varepsilon}\sqrt{x_{k-p+3}}, \dots, \sqrt{\varepsilon}\sqrt{x_{q-p+1}}, \sqrt{x'_{p+1}}, \dots, \sqrt{x'_{m-2k-q}}\Big)\Big],\end{aligned}$$

from which we read immediately that the morphism (1) maps diffeomorphically $C^+_{i,B} \times C^+_{n-i,([m]\smallsetminus B)\sqcup\{\bullet\}}$ to $C^+_{i,B_1,B_2} \times C^+_{n-i,[k]\smallsetminus B_1,[l]\smallsetminus B_2}$.

Finally, we consider the case $|B| \neq 0$, such that the maximum of $B = [j]$ with $j \leq k$. The interior of the corresponding boundary stratum of $C^+_{n,m}$, resp. $C^+_{n,k,l}$, is $C^+_{i,j} \times C^+_{n-i,m-j+1}$, resp. $C^+_{i,j} \times C^+_{n-i,k-j+1,l}$, and corresponding local coordinates are given by

$$\begin{aligned}C^+_{i,j} \times C^+_{n-i,m-j+1} &\ni \big((i, z_1, \dots, z_{i-1}, x_1, \dots, x_j), (e^{it}, w_1, \dots, w_{n-i-1}, x'_1, \dots, x'_{k-j+1}, 0, x'_{k-j+2}, \dots, x'_{m-j+1})\big), \text{ resp.}\\ C^+_{i,j} \times C^+_{n-i,k-j+1,l} &\ni \big((i, z_1, \dots, z_{i-1}, x_1, \dots, x_j), (e^{it}, w_1, \dots, w_{n-i-1}, ix'_1, \dots, ix'_{k-j+1}, y'_1, \dots, y'_l)\big),\end{aligned}$$

where t is in $(0, \pi)$, resp. $(0, \frac{\pi}{2})$, all points in $\mathbb{H}^+$ and $Q^{+,+}$ are distinct in both expressions. In the first, resp. second, expression, the x_i and x'_i are lexicographically ordered, resp. $x'_1 > \cdots x'_{k-j+1} > 0$ and $0 < y'_1 < \cdots < y'_l$.

Choosing a positive number ε sufficiently small, we now write local coordinates of $\mathcal{C}^+_{n,m}$, resp. $\mathcal{C}^+_{n,k,l}$, near the interior of the boundary stratum $\mathcal{C}^+_{i,j} \times \mathcal{C}^+_{n-i,m-j+1}$, resp. $\mathcal{C}^+_{i,j} \times \mathcal{C}^+_{n-i,k-j+1,l}$, namely

$$\begin{aligned}&\big[\big(x'_1 + \varepsilon i, x'_1 + \varepsilon z_1, \dots, x'_1 + \varepsilon z_{i-1}, e^{it}, w_1, \dots, w_{n-i-1}, x'_1, x'_1 + \varepsilon x_1, \dots, x'_1 + \varepsilon x_j, x'_2, \dots, x'_{k-j+1}, 0,\\ &x'_{k-j+2}, \dots, x'_{m-j+1}\big)\big], \text{ resp.}\\ &\big[\big(ix'_1 + \varepsilon, ix'_1 - i\varepsilon z_1, \dots, ix'_1 - i\varepsilon z_{i-1}, e^{it}, w_1, \dots, w_{n-i-1}, ix'_1, ix_1 - i'\varepsilon x_1, \dots, ix'_1 - i\varepsilon x_j, ix'_2, \dots, ix'_{k-j+1}, y'_1, \dots, y'_l\big)\big].\end{aligned}$$

If we apply the morphism (1) to the first of the two previous expressions, we get

$$\begin{aligned}&\Big[\Big(\sqrt{x'_1 + \varepsilon i}, \sqrt{x'_1 + \varepsilon z_1}, \dots, \sqrt{x'_1 + \varepsilon z_{i-1}}, \sqrt{e^{it}}, \sqrt{w_1}, \dots, \sqrt{w_{n-i-1}},\\ &\sqrt{x'_1}, \sqrt{x'_1 + \varepsilon x_1}, \dots, \sqrt{x'_1 + \varepsilon x_j}, \sqrt{x'_2}, \dots, \sqrt{x'_{k-j+1}}, \sqrt{x'_{k-j+2}}, \dots, \sqrt{x'_{m-j+1}}\Big)\Big].\end{aligned}$$

Once again, we find

$$\begin{aligned}\sqrt{x'_1 + \varepsilon i} &= i\sqrt{-x'_1} + \frac{\varepsilon}{2}\frac{1}{\sqrt{-x'_1}} + \mathcal{O}(\varepsilon^2), \qquad \sqrt{x'_1 + \varepsilon i z_e} = i\sqrt{-x'_1} - i\frac{\varepsilon}{2}\frac{z_e}{\sqrt{-x'_1}} + \mathcal{O}(\varepsilon^2),\\ \sqrt{x'_1 + \varepsilon x_e} &= i\sqrt{-x'_1} - i\frac{\varepsilon}{2}\frac{x_e}{\sqrt{-x'_1}} + \mathcal{O}(\varepsilon^2),\end{aligned}$$

and using the same arguments as in the previous computations, we see that the morphism (1) maps $C^+_{i,j} \times C^+_{n-i,m-j+1}$ diffeomorphically to $C^+_{i,j} \times C^+_{n-i,k-j+1,l}$. □

3.2. **The choice of propagators.** We now discuss the propagators needed for the computations in the framework of (bi)quantization. In particular, we discuss in detail the 4-colored propagators: we will mainly work here with the 4-colored propagators as introduced originally in [5], and used extensively in [7]. The point is that we will view the biquantization techniques in [7] in the framework of the 2-brane formality of [4]. In [4], the authors preferred to work with the 4-colored propagators on $\mathcal{C}^+_{2,1}$, in order to use the (simpler) compactified configuration spaces $\mathcal{C}^+_{n,m}$ of Kontsevich's type: in order to tie in with the computations in [7], we want to establish a more precise relationship than the one sketched in [4] about the 4-colored propagators in [7] and in [4].

3.2.1. *The Kontsevich propagator.* We consider a pair (z_1, z_2) of distinct points in $\mathbb{H}^+$, and we associate to it a closed 1-form by the formula

$$\omega(z_1,z_2) = \frac{1}{2\pi}\left[\mathrm{d}\arg(z_1-z_2) - \mathrm{d}\arg(\overline{z}_1 - z_2)\right] = \frac{1}{2\pi}\left[\mathrm{d}\arg(z_1-z_2) + \mathrm{d}\arg(z_1-\overline{z}_2)\right]. \tag{4}$$

In Formula (4), the function $\arg(z)$ denotes the Euclidean angle of the complex number z: it can be made into a smooth function by restricting its domain of definition on $\mathbb{C} \smallsetminus i\mathbb{R}^-$. In particular, the restriction of $\arg(z)$ on $\mathbb{H}^+$ is a smooth function. However, we want to consider $\omega(z_1, z_2)$ as a closed 1-form on $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta$, Δ being the diagonal in $\mathbb{H}^+ \times \mathbb{H}^+$: as such, $\omega(z_1, z_2)$ is the sum of a closed form on $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta$ and of an exact 1-form, where the corresponding function is $\arg(z_1 - \overline{z}_2)/2\pi$. We observe, for the sake of later computations (see [19] for a very nice application of this idea), that the closed 1-form can be made into a truly exact 1-form by restricting the domain of definition to

$$\{(z_1,z_2) \in (\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta :\ \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \Rightarrow \mathrm{Im}(z_1) > \mathrm{Im}(z_2)\}.$$

It is not difficult to prove that the 1-form (4) descends to $C^+_{2,0}$; a bit more involved is the proof that it extends to a smooth 1-form ω on the compactified configuration space $\mathcal{C}^+_{2,0}$. The function $\eta(z_1, z_2) = \arg(z_1 - \overline{z}_2)/2\pi$ also descends to $C^+_{2,0}$ and extends to a smooth function on $\mathcal{C}^+_{2,0}$.

Lemma 3.2. *The closed* 1*-form* (4) *determines a smooth, closed* 1*-form ω on $\mathcal{C}^+_{2,0}$, which further enjoys the following properties:*

i)

$$\omega|_{\mathcal{C}_2 \times \mathcal{C}^+_{1,0}} = \mathrm{d}\varphi,$$

where $\mathrm{d}\varphi$ denotes (improperly) the normalized volume form of $\mathcal{C}_2 \cong S^1$;

ii)

$$\omega|_{\mathcal{C}^+_{1,0} \times \mathcal{C}^+_{1,1}} = 0,$$

where $\mathcal{C}^+_{1,0} \times \mathcal{C}^+_{1,1}$ denotes the boundary stratum of $\mathcal{C}^+_{2,0}$ corresponding to the approach of the first argument z_1 to $\mathbb{R}$.

The function $\eta(z_1, z_2)$ determines a smooth function η on $\mathcal{C}^+_{2,0}$, which restricts on the boundary stratum $\mathcal{C}_2 \times \mathcal{C}^+_{1,0}$ to the constant function $\pi/2$; observe that $\mathcal{C}^+_{1,0} \cong \{i\}$.

The 1-form ω is usually called Kontsevich's angle form [12, Subsection 6.2]: it will be useful, for certain computations, to recall that Kontsevich's angle function is the sum of a closed 1-form and of an exact 1-form, constructed by means of the function η.

We finally observe that the natural involution $(z_1, z_2) \overset{\tau}{\mapsto} (z_2, z_1)$ of $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta$ yields an involution τ of $\mathcal{C}^+_{2,0}$: we may then consider two Kontsevich's angle forms $\omega^\pm$ defined through

$$\omega^+ = \omega,\ \omega^- = \tau^*(\omega).$$

The angle forms $\omega^\pm$ have been first introduced in [5,6]: they have opposite boundary conditions when one of their arguments approaches $\mathbb{R}$, as can be easily deduced from Lemma 3.2.

3.2.2. *The* 4*-colored propagators on $\mathcal{C}^+_{2,1}$.* We consider a triple (z_1, z_2, x) in $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta \times \mathbb{R}$.

There is a natural smooth projection from $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta \times \mathbb{R}$ to $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta$, thus we may consider the pull-back $\omega^{+,+}$ of the closed 1-form $\omega^+(z_1, z_2)$ to $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta \times \mathbb{R}$. We set $\omega^{-,-}(z_1, z_2, x)$ to be the pull-back of ω^- w.r.t. the very same projection.

We recall the complex square root discussed in Subsubsection 3.1.3: as already remarked, it is a biholomorphism from $\mathbb{H}^+$ to $Q^{+,+}$, and we associate to a triple (z_1, z_2, x) in $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta \times \mathbb{R}$ a pair $(\sqrt{z_1 - x}, \sqrt{z_2 - x})$ in $(Q^{+,+} \times Q^{+,+}) \smallsetminus \Delta$ (compare with the morphism of Proposition 3.1). We then set

$$\begin{aligned}
\omega^{+,-}(z_1,z_2,x) = \frac{1}{2\pi}\Big[&\mathrm{d}\arg\big(\sqrt{z_1-x}-\sqrt{z_2-x}\big) + \mathrm{d}\arg\Big(\overline{\sqrt{z_1-x}} - \sqrt{z_2-x}\Big) - \\
&-\mathrm{d}\arg\Big(\overline{\sqrt{z_1-x}} + \sqrt{z_2-x}\Big) - \mathrm{d}\arg\big(\sqrt{z_1-x}+\sqrt{z_2-x}\big)\Big] \\
\omega^{-,+}(z_1,z_2,x) = \frac{1}{2\pi}\Big[&\mathrm{d}\arg\big(\sqrt{z_1-x}-\sqrt{z_2-x}\big) - \mathrm{d}\arg\Big(\overline{\sqrt{z_1-x}} - \sqrt{z_2-x}\Big) + \\
&+\mathrm{d}\arg\Big(\overline{\sqrt{z_1-x}} + \sqrt{z_2-x}\Big) - \mathrm{d}\arg\big(\sqrt{z_1-x}+\sqrt{z_2-x}\big)\Big].
\end{aligned}$$

The two 1-forms $\omega^{+,-}$ and $\omega^{-,+}$ are smooth and obviously closed on $(\mathbb{H}^+ \times \mathbb{H}^+) \smallsetminus \Delta \times \mathbb{R}$.

We need to characterize more explicitly the compactified configuration space $\mathcal{C}^+_{2,1}$ (for whose more precise description we refer to [4, Section 5]): here, we content ourselves to describe all boundary strata of codimension 1, which we depict as follows

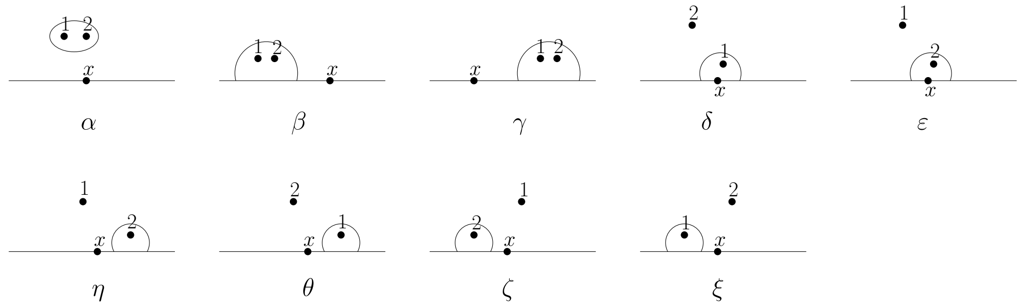

Figure 3 - The boundary strata of codimension 1 of $\mathcal{C}^+_{2,1}$

We observe that the boundary stratum α corresponds to $\mathcal{C}_2 \times \mathcal{C}^+_{1,1}$, the boundary strata β and γ to two copies of $\mathcal{C}^+_{2,0} \times \mathcal{C}^+_{0,2}$, the boundary strata δ and ε to two copies of $\mathcal{C}^+_{1,1} \times \mathcal{C}^+_{1,1}$, and the boundary strata η, θ, ζ and ξ to four copies of $\mathcal{C}^+_{1,0} \times \mathcal{C}^+_{1,2}$. When it is clear from the context, we will omit to write the projections π_i, $i = 1, 2$, from the these spaces to the each of the factors. We finally recall, once again, that the function $\arg(z)$ is well-defined and smooth on $\mathbb{H}^+$: in particular, the function η from Lemma 3.2, Subsubsection 3.2.1, yields a smooth function (denoted again by η) on $\mathcal{C}^+_{1,1}$, when the second argument approaches $\mathbb{R}$. In more down-to-earth terms, $\eta = \arg(z-x)/2\pi$.

It is not difficult to prove that the 4 1-forms $\omega^{+,+}$, $\omega^{+,-}$, $\omega^{-,+}$ and $\omega^{-,-}$ descend to smooth, closed 1-forms on $C^+_{2,1}$. In fact, as the following Lemma shows (for whose proof we refer to [4, Lemma 5.4]), these in turn extend to smooth, closed 1-forms on the compactified configuration space $\mathcal{C}^+_{2,1}$.

Lemma 3.3. *The* 1*-forms* $\omega^{+,+}$*,* $\omega^{+,-}$*,* $\omega^{-,+}$ *and* $\omega^{-,-}$ *determine smooth, closed* 1*-forms on the compactified configuration space* $\mathcal{C}^+_{2,1}$*, which enjoy the following properties:*

i)

$$\omega^{+,+}|_\alpha = \mathrm{d}\varphi,\ \omega^{+,-}|_\alpha = \mathrm{d}\varphi - \mathrm{d}\eta,\ \omega^{-,+}|_\alpha = \mathrm{d}\varphi - \mathrm{d}\eta,\ \omega^{-,-}|_\alpha = \mathrm{d}\varphi,$$

where $\mathrm{d}\varphi$ *is the normalized volume form of* $\mathcal{C}_2 \cong S^1$*.*

ii)

$$\begin{aligned} &\omega^{+,+}|_\beta = \omega^+, \quad \omega^{+,-}|_\beta = \omega^+, \quad \omega^{-,+}|_\beta = \omega^-, \quad \omega^{-,-}|_\beta = \omega^- \quad \textit{and} \\ &\omega^{+,+}|_\gamma = \omega^+, \quad \omega^{+,-}|_\gamma = \omega^-, \quad \omega^{-,+}|_\gamma = \omega^+, \quad \omega^{-,-}|_\gamma = \omega^-, \end{aligned}$$

where $\omega^\pm$ *have to be understood on* $\mathcal{C}^+_{2,0}$*.*

iii)

$$\begin{aligned} \omega^{+,+}|_\delta = \omega^{+,-}|_\delta = \omega^{-,+}|_\delta = 0, \\ \omega^{+,-}|_\varepsilon = \omega^{-,+}|_\varepsilon = \omega^{-,-}|_\varepsilon = 0. \end{aligned}$$

iv)

$$\begin{aligned} \omega^{+,-}|_\eta = \omega^{-,-}|_\eta = 0, \quad \omega^{+,+}|_\theta = \omega^{-,+}|_\theta = 0, \\ \omega^{-,+}|_\zeta = \omega^{-,-}|_\zeta = 0, \quad \omega^{+,+}|_\xi = \omega^{+,-}|_\xi = 0. \end{aligned}$$

3.2.3. *The* 4*-colored propagators on* $\mathcal{C}^+_{2,0,0}$. We now define on $(Q^{+,+} \times Q^{+,+}) \smallsetminus \Delta$ 4 closed, smooth 1-forms, which, by an (apparent) abuse of notation, are denoted by $\omega^{\pm,\pm}$: namely, we set

$$\begin{aligned} \omega^{+,+} &= \frac{1}{2\pi}\left[\mathrm{d}\ \arg(z_1 - z_2) - \mathrm{d}\ \arg(\overline{z}_1 - z_2) - \mathrm{d}\ \arg(\overline{z}_1 + z_2) + \mathrm{d}\ \arg(z_1 + z_2)\right], \\ \omega^{+,-} &= \frac{1}{2\pi}\left[\mathrm{d}\ \arg(z_1 - z_2) + \mathrm{d}\ \arg(\overline{z}_1 - z_2) - \mathrm{d}\ \arg(\overline{z}_1 + z_2) - \mathrm{d}\ \arg(z_1 + z_2)\right], \\ \omega^{-,+} &= \frac{1}{2\pi}\left[\mathrm{d}\ \arg(z_1 - z_2) - \mathrm{d}\ \arg(\overline{z}_1 - z_2) - \mathrm{d}\ \arg(\overline{z}_1 + z_2) + \mathrm{d}\ \arg(z_1 + z_2)\right], \\ \omega^{+,+} &= \frac{1}{2\pi}\left[\mathrm{d}\ \arg(z_1 - z_2) + \mathrm{d}\ \arg(\overline{z}_1 - z_2) + \mathrm{d}\ \arg(\overline{z}_1 + z_2) + \mathrm{d}\ \arg(z_1 + z_2)\right], \end{aligned}$$

for an element (z_1, z_2) of $(Q^{+,+} \times Q^{+,+}) \smallsetminus \Delta$.

We first observe that the last three summands in the previous 1-forms are exact 1-forms: namely, as has been previously remarked, the function $\arg(z)$ is smooth and well-defined on $\mathbb{C} \smallsetminus (i\mathbb{R}^- \sqcup \{0\})$, hence the three functions appearing in the last three summands of the previous formulæ are well-defined and smooth on $(Q^{+,+} \times Q^{+,+}) \smallsetminus \Delta$.

It is not difficult to prove that the closed 1-forms $\omega^{\pm,\pm}$ descends to smooth, closed 1-forms on the open configuration space $C^+_{2,0,0}$, and that these in turn determine smooth, closed 1-forms $\omega^{\pm,\pm}$ on the compactified configuration space $\mathcal{C}^+_{2,0,0}$.

Because of the results of Subsubsection 3.1.3, we already know that there is a diffeomorphism between $\mathcal{C}^+_{2,0,0}$ and $\mathcal{C}^+_{2,1}$, which smoothly extends to the compactified configuration spaces the diffeomorphism

$$C^+_{2,1} \ni [(z_1, z_2, x)] \mapsto [(\sqrt{z_1 - x}, \sqrt{z_2 - x})] \in C^+_{2,0,0}.$$

We leave it to the reader to reinterpret on $\mathcal{C}^+_{2,0,0}$ the boundary strata of codimension 1 of $\mathcal{C}^+_{2,1}$.

It is not difficult to prove that the pull-backs w.r.t. the morphism (1) from $\mathcal{C}^+_{2,1}$ to $\mathcal{C}^+_{2,0,0}$ of the 1-forms $\omega^{\pm,\pm}$ on $\mathcal{C}^+_{2,0,0}$ are exactly the 1 forms $\omega^{\pm,\pm}$ introduced in Subsubsection 3.2.2: *e.g.* for $\omega^{+,+}$, we have the obvious identity

$$\omega^{+,+} = \frac{1}{2\pi}\left[\mathrm{d}\arg(z_1^2 - z_2^2) - \mathrm{d}\arg(\overline{z}_1^2 - z_2^2)\right],$$

whence the claim follows. Similar arguments work for the other cases.

According to the boundary stratification of $\mathcal{C}^+_{2,0,0}$, we have the following variant of Lemma 3.3.

Lemma 3.4. *The* 1*-forms* $\omega^{+,+}$, $\omega^{+,-}$, $\omega^{-,+}$ *and* $\omega^{-,-}$ *determine smooth, closed* 1*-forms on the compactified configuration space* $\mathcal{C}^+_{2,0,0}$, *which enjoy the following properties:*

i)

$$\omega^{+,+}|_\alpha = \mathrm{d}\varphi + \mathrm{d}\eta,\ \omega^{+,-}|_\alpha = \mathrm{d}\varphi - \mathrm{d}\eta,\ \omega^{-,+}|_\alpha = \mathrm{d}\varphi - \mathrm{d}\eta,\ \omega^{-,-}|_\alpha = \mathrm{d}\varphi + \mathrm{d}\eta,$$

where $\mathrm{d}\varphi$ *is the normalized volume form of* $\mathcal{C}_2 \cong S^1$, *and* $\eta = \arg(z)/2\pi$ *is a well-defined, smooth function on* $\mathcal{C}^+_{1,0,0}$.

ii)

$$\begin{aligned} &\omega^{+,+}|_\beta = \omega^+, \quad \omega^{+,-}|_\beta = \omega^+, \quad \omega^{-,+}|_\beta = \omega^-, \quad \omega^{-,-}|_\beta = \omega^- \quad \textit{and} \\ &\omega^{+,+}|_\gamma = \omega^+, \quad \omega^{+,-}|_\gamma = \omega^-, \quad \omega^{-,+}|_\gamma = \omega^+, \quad \omega^{-,-}|_\gamma = \omega^-, \end{aligned}$$

where $\omega^\pm$ *have to be understood on* $\mathcal{C}^+_{2,0}$.

iii)

$$\begin{aligned} \omega^{+,+}|_\delta &= \omega^{+,-}|_\delta = \omega^{-,+}|_\delta = 0, \\ \omega^{+,-}|_\varepsilon &= \omega^{-,+}|_\varepsilon = \omega^{-,-}|_\varepsilon = 0. \end{aligned}$$

iv)

$$\begin{aligned} \omega^{+,-}|_\eta = \omega^{-,-}|_\eta = 0, \quad \omega^{+,+}|_\theta = \omega^{-,+}|_\theta = 0, \\ \omega^{-,+}|_\zeta = \omega^{-,-}|_\zeta = 0, \quad \omega^{+,+}|_\xi = \omega^{+,-}|_\xi = 0. \end{aligned}$$

We observe that the 1-forms $\omega^{\pm,\pm}$, be they defined either on $\mathcal{C}^+_{2,1}$ or on $\mathcal{C}^+_{2,0,0}$, satisfy the same boundary conditions *ii*), *iii*) and *iv*); on the other hand, the behavior of the 4-colored propagators on the boundary strata $\mathcal{C}_2 \times \mathcal{C}^+_{1,1}$ and $\mathcal{C}_2 \times \mathcal{C}^+_{1,0,0}$ are quite different. This can be traced back to the proof of Proposition 3.1, when analyzing the shape of the morphism (1) on the boundary stratum $\mathcal{C}_2 \times \mathcal{C}^+_{1,1}$. Still, we have to be careful about these (seemingly) different boundary conditions for the 4-colored propagators $\omega^{\pm,\pm}$: namely, the fact that the 4-colored propagators, quite opposite to Kontsevich's angle form, can be written as a sum of a regular and of a singular term (the 1-form living on $\mathcal{C}^+_{1,1}$ or $\mathcal{C}^+_{1,0,0}$ and on $\mathcal{C}_2$ respectively) produces a significant change in the application of Stokes' Theorem, which is the fundamental tool for proving the 2-brane Formality Theorem, from which biquantization follows.

3.3. **Formality Theorems.** In this Subsection, we recall the 2-brane Formality Theorem of [4], from which we will derive the biquantization techniques we apply later on. Although the main computations of this Subsection are already contained in [4], we review them in some detail because of the following reasons: first, the 2-brane Formality Theorem has been proved using superpropagators along the same patterns of [6], and superpropagators are better suited for keeping track of all different colors of propagators w.r.t. the treatment in [7], and second, because we deserve here a more careful treatment than in [4] of the 1-loop correction arising because of the aforementioned regular term in the 4-colored propagators. We thus profit of the space here to correct a slight mistake in [4, Subsection 7.1] (in the sense that the computations therein are correct, but a subtle point has been missed regarding the multidifferential operator associated to the 1-loop correction, which we illustrate here in detail) and, more importantly, to correct a more serious mistake in [7], where the regular part of the restriction to the boundary stratum $\mathcal{C}_2 \times \mathcal{C}^+_{1,1}$ or $\mathcal{C}_2 \times \mathcal{C}^+_{1,0,0}$

is missing completely. The correction term arising from the presence of the regular part is responsible for a quantum shift, which will be illustrated explicitly in Section 4, which is predicted by representation-theoretic arguments and was otherwise absent.

We also prove a version of [12, Lemmata 7.3.1.1, 7.3.3.1] for the 4-colored propagators: such vanishing lemmata are central in some computations in [7] regarding the Harish–Chandra homomorphism. The main idea of the proof is, once again, Stokes' Theorem, but of course here we have to be a bit more careful and slightly change the final argument.

3.3.1. *Admissible graphs.* Before entering into the technicalities of the 1-brane and 2-brane Formality Theorems, we need to spend some words on admissible graphs.

For a pair of non-negative integers (n, m), such that $2n+m-2 \geq 0$, we consider the set $\mathcal{G}_{n,m}$ of admissible graphs of type (n, m): the integer n, resp. m, refers to the number of vertices of the first, resp. second type, *i.e.* vertices in $\mathbb{H}^+$, resp. on $\mathbb{R}$. An admissible graph Γ of type (n, m) in the framework of the 1-brane Formality Theorem [4,6] is an oriented graph, which may admit double edges, *i.e.* given any two vertices (v_1, v_2), there can be more than one edge connecting v_1 to v_2, and edges departing from vertices of the second type; it does not possess short loops, *i.e.* there can no edge in Γ with coincident initial and final point. The presence of multiple edges and edges departing from $\mathbb{R}$ is in opposition to the definition of admissible graphs of type (n, m) as in [12].

Further, for a triple of non-negative integers (n, k, l), such that $2n + k + l - 1 \geq 0$, we consider the set $\mathcal{G}_{n,k,l}$ of admissible graphs of type (n, k, l), where n is the number of vertices of the first type (*i.e.* in $Q^{+,+}$), k, resp. l, is the number of vertices of the first type on $i\mathbb{R}^+$, resp. $\mathbb{R}^+$. A general element Γ of $\mathcal{G}_{n,k,l}$ is an oriented graphs with n, resp. $k+l$, vertices of the first, resp. second type, which may admit multiple edges, edges departing from $i\mathbb{R}^+ \sqcup \{0\} \sqcup \mathbb{R}^+$ and even short loops.

We observe that we may also equivalently consider, for $m = k + l + 1$, the set $\mathcal{G}_{n,m}$ of admissible graphs of type (n, m), consisting of oriented graphs with n, resp. m, vertices of the first, resp. second type (*i.e.* lying in $\mathbb{H}^+$ and on $\mathbb{R}$ respectively), such that one vertex of the first type is marked and which admit multiple edges, edges departing from $\mathbb{R}$ and short loops: the notation is abused, but it will be clear from the context if we allow elements of $\mathcal{G}_{n,m}$ to possess or not short loops, which is the only additional feature that the admissible graphs for the 2-brane Formality Theorem admit w.r.t. the ones in the 1-brane Formality Theorem. The algebraic counterpart of the geometric results of Subsubsection 3.1.3 is the fact that we may freely pass from $\mathcal{G}_{n,m}$ to $\mathcal{G}_{n,k,l}$, for $m = k + l + 1$, by noting that the vertex labeled by $k + 1$ on $\mathbb{R}$ corresponds to the origin $\{0\}$.

3.3.2. *Superpropagators.* We now pick an admissible graph Γ of type (n, m) for the 1-brane Formality Theorem of [6]. As Γ is of type (n, m), its vertices correspond to a point of $\mathcal{C}^+_{n,m}$, and an edge e determines a natural projection $\pi_e : \mathcal{C}^+_{n,m} \to \mathcal{C}^+_{2,0}$.

If we pick an admissible graph Γ of $\mathcal{G}_{n,m}$ in the framework of the 2-brane Formality Theorem, then the vertices of Γ still define a configuration of points in $\mathcal{G}_{n,m}$. An edge e defines, as in the previous situation, either a natural projection $\pi_e : \mathcal{C}^+_{n,m} \to \mathcal{C}^+_{2,1}$, if $e = (v^i_e, v^f_e)$, $v^i_e \neq v^f_e$, or $\pi_e : \mathcal{C}^+_{n,m} \to \mathcal{C}^+_{1,1}$, if e is a short loop. The point in $\mathbb{R}$ in either $\mathcal{C}^+_{2,1}$ or $\mathcal{C}^+_{1,1}$ is the marked point of $\mathcal{C}^+_{n,m}$. If, equivalently, we consider the corresponding admissible graph Γ of type (n, k, l), then an edge e of Γ determines either a projection $\pi_e : \mathcal{C}^+_{n,k,l} \to \mathcal{C}^+_{2,0,0}$ or $\pi_e : \mathcal{C}^+_{n,k,l} \to \mathcal{C}^+_{1,0,0}$.

We now consider the vector space $X = \mathbb{K}^d$ and two linear (or affine) subspaces U_i, $i = 1, 2$, for which we assume there is a direct sum decomposition

$$X = (U_1 \cap U_2) \overset{\perp}{\oplus} (U_1^\perp \cap U_2) \overset{\perp}{\oplus} (U_1 \cap U_2^\perp) \overset{\perp}{\oplus} (U_1 + U_2)^\perp, \tag{5}$$

w.r.t. a chosen inner product over X. Clearly, we have

$$U_1 = (U_1 \cap U_2) \overset{\perp}{\oplus} (U_1 \cap U_2^\perp),\ U_2 = (U_1 \cap U_2) \overset{\perp}{\oplus} (U_1^\perp \cap U_2).$$

We choose linear coordinates $\{x_i\}$ on X which are adapted to the orthogonal decomposition (5), *i.e.* there are two non-disjoint subsets I_i, $i = 1, 2$, of $[d]$, such that

$$[d] = (I_1 \cap I_2) \sqcup (I_1 \cap I_2^c) \sqcup (I_1^c \cap I_2) \sqcup (I_1^c \cap I_2^c),$$

w.r.t. which $\{x_i\}$ is a set of linear coordinates on $U_1 \cap U_2$, $U_1 \cap U_2^\perp$, $U_1^\perp \cap U_2$ or $(U_1 + U_2)^\perp$, if the index i belongs to $I_1 \cap I_2$, $I_1 \cap I_2^c$, $I_1^c \cap I_2$ or $I_1^c \cap I_2^c$ respectively. Accordingly, for I either one of the previous subsets of $[d]$, and e an edge of admissible graph Γ of type (n, m), we set

$$\tau^I_e = \sum_{i \in I} \iota^{(v^i_e)}_{\mathrm{d}x_i} \partial^{(v^f_e)}_{x_i} \in \mathrm{End}\Big(T_{\mathrm{poly}}(X)^{\otimes(m+n)}\Big),\ T_{\mathrm{poly}}(X) = \mathrm{S}(X^*) \otimes \wedge^\bullet X,$$

and $\partial_{x_i}^{(v)}$ denotes the action of the differential operator on the copy of $T_{\rm poly}(X)$ sitting at the v-th position, and similarly for $\iota_{\mathrm{d}x_i}^{(v)}$. We observe that τ_e^I is well-defined and has degree -1 w.r.t. the natural grading on $T_{\rm poly}(X)$.

We now set

$$\begin{aligned} A &= \mathrm{S}(U_1^*)\otimes\wedge(X/U_1) = \mathrm{S}(U_1^*)\otimes\wedge(U_1^\perp\cap U_2)\otimes\wedge(U_1+U_2)^\perp,\\ B &= \mathrm{S}(U_2^*)\otimes\wedge(X/U_2) = \mathrm{S}(U_2^*)\otimes\wedge(U_1\cap U_2^\perp)\otimes\wedge(U_1+U_2)^\perp,\\ K &= \mathrm{S}((U_1\cap U_2)^*)\otimes\wedge(U_1+U_2)^\perp. \end{aligned}$$

It is clear that A and B both admit a (trivial) structure of A_∞-algebra, and K is naturally an A-B-bimodule.

Wr.t. the previously introduced notation, the relevant superpropagators are then given by

$$\omega_e^A = \pi_e^*(\omega^+)\otimes\left(\tau_e^{I_1\cap I_2}+\tau_e^{I_1\cap I_2^c}\right)+\pi_e^*(\omega^-)\otimes\left(\tau_e^{I_1^c\cap I_2}+\tau_e^{I_1^c\cap I_2^c}\right), \tag{6}$$

$$\omega_e^B = \pi_e^*(\omega^+)\otimes\left(\tau_e^{I_1\cap I_2}+\tau_e^{I_1^c\cap I_2}\right)+\pi_e^*(\omega^-)\otimes\left(\tau_e^{I_1\cap I_2^c}+\tau_e^{I_1^c\cap I_2^c}\right), \tag{7}$$

$$\omega_e^K = \pi_e^*(\omega^{+,+})\otimes\tau_e^{I_1\cap I_2}+\pi_e^*(\omega^{+,-})\otimes\tau_e^{I_1\cap I_2^c}+\pi_e^*(\omega^{-,+})\otimes\tau_e^{I_1^c\cap I_2}+\pi_e^*(\omega^{-,-})\otimes\tau_e^{I_1^c\cap I_2^c}, \tag{8}$$

for an edge $e=(v_e^i,v_e^f)$, $v_e^i\neq v_e^f$, of an admissible graph Γ of type (n,m).

We observe that the superpropagators (6), (7) and(8) are closed 1-forms on $\mathcal{C}_{2,0}^+$ and $\mathcal{C}_{2,1}^+$ with values in $\mathrm{End}\big(T_{\rm poly}(X)^{\otimes(m+n)}\big)$ (of course, A, B and K may be viewed as subalgebras of $T_{\rm poly}(X)$). Equivalently, we may regard the superpropagator (8) as a closed 1-form on $\mathcal{C}_{2,0,0}^+$ with values in $\mathrm{End}\big(T_{\rm poly}(X)^{\otimes(m+n)}\big)$.

Lemma 3.2, Subsubsection 3.2.1, implies the following useful boundary conditions for the superpropagators (6) and (7):

i) their restrictions to the boundary stratum $\mathcal{C}_{2,0}^+\times\mathcal{C}_{1,0}^+$ equal

$$\omega_e^A|_{\mathcal{C}_{2,0}^+\times\mathcal{C}_{1,0}^+}=\omega_e^B|_{\mathcal{C}_{2,0}^+\times\mathcal{C}_{1,0}^+}=\mathrm{d}\varphi\otimes\tau_e^{[d]};$$

ii) their restrictions to the boundary stratum $\mathcal{C}_{1,0}^+\times\mathcal{C}_{1,1}^+$ corresponding to the approach of the first, resp. second, argument to $\mathbb{R}$ equal

$$\begin{aligned} \omega_e^A|_{\mathcal{C}_{1,0}^+\times\mathcal{C}_{1,1}^+} &= \pi_e^*(\omega^-)\otimes\left(\tau_e^{I_1^c\cap I_2}+\tau_e^{I_1^c\cap I_2^c}\right), \quad \text{resp.}\quad \omega_e^A|_{\mathcal{C}_{1,0}^+\times\mathcal{C}_{1,1}^+}=\pi_e^*(\omega^+)\otimes\left(\tau_e^{I_1\cap I_2}+\tau_e^{I_1\cap I_2^c}\right),\\ \omega_e^B|_{\mathcal{C}_{1,0}^+\times\mathcal{C}_{1,1}^+} &= \pi_e^*(\omega^-)\otimes\left(\tau_e^{I_1\cap I_2^c}+\tau_e^{I_1^c\cap I_2^c}\right), \quad \text{resp.}\quad \omega_e^A|_{\mathcal{C}_{1,0}^+\times\mathcal{C}_{1,1}^+}=\pi_e^*(\omega^+)\otimes\left(\tau_e^{I_1\cap I_2}+\tau_e^{I_1^c\cap I_2}\right). \end{aligned}$$

In particular, we see why admissible graphs appearing in the 1-brane Formality Theorem may admit edges departing from $\mathbb{R}$, see for more details [5, 6].

We now concentrate on the boundary conditions for the superpropagator (8) on $\mathcal{C}_{2,1}^+$: Lemma 3.3 yields

i) the restriction of the superpropagator (8) to the boundary stratum α of $\mathcal{C}_{2,1}^+$ equals

$$\omega_e^K|_\alpha=\mathrm{d}\varphi\otimes\tau_e^{[d]}-\mathrm{d}\eta\otimes\left(\tau_e^{I_1\cap I_2^c}+\tau_e^{I_1^c\cap I_2}\right);$$

ii) the restriction of the superpropagator (8) to the boundary strata β and γ equals

$$\omega_e^K|_\beta=\omega_e^A,\ \omega_e^K|_\gamma=\omega_e^B;$$

iii) the restriction of the superpropagator (8) to the boundary strata δ and ε equals

$$\omega_e^K|_\delta=\pi_e^*(\omega^{-,-})\otimes\tau_e^{I_1^c\cap I_2^c},\ \omega_e^K|_\varepsilon=\pi_e^*(\omega^{+,+})\otimes\tau_e^{I_1\cap I_2};$$

iv) the restriction of the superpropagator (8) to the boundary strata *eta*, θ, ζ and ξ equals

$$\begin{aligned} \omega_e^K|_\eta &= \pi_e^*(\omega^{+,+})\otimes\tau_e^{I_1\cap I_2^c}+\pi_e^*(\omega^{-,+})\otimes\tau_e^{I_1^c\cap I_2}, \qquad \omega_e^K|_\theta=\pi_e^*(\omega^{+,-})\otimes\tau_e^{I_1\cap I_2^c}+\pi_e^*(\omega^{-,-})\otimes\tau_e^{I_1^c\cap I_2^c},\\ \omega_e^K|_\zeta &= \pi_e^*(\omega^{+,+})\otimes\tau_e^{I_1\cap I_2}+\pi_e^*(\omega^{+,-})\otimes\tau_e^{I_1\cap I_2^c}, \qquad \omega_e^K|_\xi=\pi_e^*(\omega^{-,+})\otimes\tau_e^{I_1^c\cap I_2}+\pi_e^*(\omega^{-,-})\otimes\tau_e^{I_1^c\cap I_2^c}. \end{aligned}$$

If we choose the superpropagator (8) on $\mathcal{C}_{2,0,0}^+$, it satisfies the same boundary conditions, with the exception of the first one, which takes the form

$$\omega_e^K|_\alpha=\mathrm{d}\varphi\otimes\tau_e^{[d]}+\mathrm{d}\eta\otimes\left(\tau_e^{I_1\cap I_2}+\tau_e^{I_1^c\cap I_2^c}-\tau_e^{I_1\cap I_2^c}-\tau_e^{I_1^c\cap I_2}\right).$$

For the sake of simplicity, we write $\tau_e^+=\tau_e^{I_1\cap I_2}+\tau_e^{I_1^c\cap I_2^c}$ and $\tau_e^-=\tau_e^{I_1^c\cap I_2}+\tau_e^{I_1\cap I_2^c}$.

We observe that the boundary conditions of type *iii)* and *iv)* explain why the admissible graphs appearing in the 2-brane Formality Theorem admit edges departing from $\mathbb{R}$; when considering such admissible graphs in $\mathbb{Q}^{+,+}\sqcup$

$i\mathbb{R}^+ \sqcup \mathbb{R}^+ \sqcup \{0\}$, we observe that the boundary conditions $iii)$ imply that such graphs admit edges departing from or arriving at the origin.

We now deal with the so-called superloop propagator: its origin will be explained carefully in the proof of the 2-brane Formality Theorem, which will come later on. For the time being, we content ourselves by noting that the superloop propagator appear only first in the 2-brane Formality Theorem as a consequence of the boundary condition $i)$ satisfied by the superpropagator (8), more precisely it arises because of the "regular term" containing the form $\mathrm{d}\eta$.

With the same notation as before, the superloop propagator associated to a short loop e of an admissible graph Γ of type (n,m) is defined as the closed 1-form on $\mathcal{C}^+_{2,1}$ with values in $\mathrm{End}\big(T_{\mathrm{poly}}(X)^{(m+n)}\big)$

$$\omega_e^K = \frac{1}{2}\pi_e^*(\mathrm{d}\eta) \otimes (\mathrm{div}^+_{(v)} - \mathrm{div}^-_{(v)}),\ e=(v,v),$$

where

$$\mathrm{div}^+_{(v)} = \sum_{k\in(I_1\cap I_2)\sqcup(I_1^c\cap I_2^c)} \iota^{(v)}_{\mathrm{d}x_k}\partial^{(v)}_{x_k},\quad \mathrm{div}^-_{(v)} = \sum_{k\in(I_1^c\cap I_2)\sqcup(I_1\cap I_2^c)} \iota^{(v)}_{\mathrm{d}x_k}\partial^{(v)}_{x_k}.$$

We observe that the superloop propagator is exact: this fact will be used in all subsequent computations. Notice that the superloop propagator on $\mathcal{C}^+_{2,0,0}$ is defined by the same formula without the rescaling by $1/2$ (because of the morphism (1) from $\mathcal{C}^+_{2,1}$ to $\mathcal{C}^+_{2,0,0}$).

3.3.3. *The formality morphisms.* We consider X, U_1 and U_2 as before, to which we associate the graded vector spaces A, B and K. Using the superpropagators (6), (7) and (8), and keeping in mind the notation in the previous Subsubsections, we set

$$\mathcal{O}^A_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m) = \mu^B_{n+m}\left(\int_{\mathcal{C}^+_{n,m}}\prod_{e\in E(\Gamma)}\omega^A_e(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m)\right), \tag{9}$$

$$\mathcal{O}^B_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m) = \mu^B_{n+m}\left(\int_{\mathcal{C}^+_{n,m}}\prod_{e\in E(\Gamma)}\omega^B_e(\gamma_1|\cdots|\gamma_n|b_1|\cdots|b_m)\right), \tag{10}$$

$$\mathcal{O}^K_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_k|k|b_1|\cdots|b_l) = \mu^K_{m+n}\left(\int_{\mathcal{C}^+_{n,m}}\prod_{e\in E(\Gamma)}\omega^K_e(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_k|k|b_1|\cdots|b_l)\right), \tag{11}$$

where γ_i, $i=1,\dots,n$, are elements of $T_{\mathrm{poly}}(X)$, a_i and b_i are elements of A and B respectively, k is an element of K; $E(\Gamma)$ is the set of edges of an admissible graph Γ of type (n,m); μ^A, μ^B and μ^K denotes the multiplication map on $T_{\mathrm{poly}}(X)$, followed by the projection onto A, B and K respectively.

Since Γ may have multiple edges, there is a combinatorial subtlety to be taken into account: in all previous formulæ, whenever there are multiple edges between two vertices (v^i, v_f), for $v^i \neq v_f$, we must divide by the factorial of the number of such edges. We observe that short loops cannot be multiple edges, as the superpropagator for a short loop squares obviously to 0.

We also observe that the product on formulæ (9), (10) and (11) are well-defined and do not depend on the order of the factors: namely, the total degree of any superpropagator appearing in these formulæ is 0, as the 1-form piece has (form) degree 1, while the multidifferential operator piece has degree -1.

Using the multidifferential operators defined in (9), (10) and (11), we set

$$\mathcal{U}^n_A(\gamma_1|\cdots|\gamma_n)(a_1|\dots|a_m) = (-1)^{\left(\sum_{i=1}^n|\gamma_i|-1\right)m}\sum_{\Gamma\in\mathcal{G}_{n,m}}\mathcal{O}^A_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_m), \tag{12}$$

$$\mathcal{U}^n_B(\gamma_1|\cdots|\gamma_n)(a_1|\dots|a_m) = (-1)^{\left(\sum_{i=1}^n|\gamma_i|-1\right)m}\sum_{\Gamma\in\mathcal{G}_{n,m}}\mathcal{O}^B_\Gamma(\gamma_1|\cdots|\gamma_n|b_1|\cdots|b_m), \tag{13}$$

$$\mathcal{U}^n_K(\gamma_1|\cdots|\gamma_n)(a_1|\cdots|a_k|k|b_1|\cdots|b_l) = (-1)^{\left(\sum_{i=1}^n|\gamma_i|-1\right)m}\sum_{\Gamma\in\mathcal{G}_{n,m}}\mathcal{O}^K_\Gamma(\gamma_1|\cdots|\gamma_n|a_1|\cdots|a_k|k|b_1|\cdots|b_l), \tag{14}$$

$$\mathrm{d}^{k,l}_K(a_1|\cdots|a_k|k|b_1|\cdots|b_l) = \sum_{\Gamma\in\mathcal{G}_{0,m}}\mathcal{O}^K_\Gamma(a_1|\cdots|a_k|k|b_1|\cdots|b_l), \tag{15}$$

with the above notation.

Some observations are necessary here. The morphisms (12) and (13) and (14) are multilinear maps from $T_{\mathrm{poly}}(X)$ to the multidifferential operators on A and B respectively; the morphisms (14) and (15) are multilinear maps from $T_{\mathrm{poly}}(X)$ to the multidifferential operators from $A^{\otimes k}\otimes K\otimes B^{\otimes l}$ to K. All multidifferential operators appearing in the previous formulæ are non-trivial only if the number of edges of the admissible graphs of type (n,m) equals

$2n+m-2$: since to each edge of an admissible graph is associated a contraction operator (which lowers degrees by 1), it follows immediately that the morphisms 12, (13), (14) have degree $2-n$, and that the morphism (15) has degree $2-m$.

We refer to [4, Section 3] for a short introduction to A_∞-categories in the present framework (see [11, 13] for more details on A_∞-categories and related issues), which is needed for the statement of the main theorem ($1+2$-brane Formality Theorem) of the present Section. We only recall that $T_{\mathrm{poly}}(X)$ has a structure of dg (short for differential graded) Lie algebra with trivial differential and Schouten–Nijenhuis bracket (extending the natural Lie bracket on polynomial vector fields on X); similarly, the Hochschild cochain complex of an A_∞-category $\mathcal{A}$ (roughly, an abelian category, whose spaces spaces of morphisms admit the structure of A_∞-algebras and A_∞-bimodules) is also a dg Lie algebra with Hochschild differential (the A_∞-structure itself) and Gerstenhaber bracket (which is well-defined an any sort of Hochschild cochain complex).

Theorem 3.5. *We may regard A, B as A_∞-algebras, whose only non-trivial Taylor component is given by the corresponding natural (graded) commutative products: then, the morphisms* (15) *fit into the Taylor components of a non-trivial A_∞ A-B-bimodule structure over K, which restricts to the natural A left- and B-right module structures on K.*

Furthermore, the morphisms (12), (13) *and* (14) *fit into the Taylor components of an L_∞-morphism $\mathcal{U}$ from $T_{\mathrm{poly}}(X)$ to the (completed) Hochschild cochain complex of the A_∞-category $\mathcal{A}$ with two objects U_i, $=1,2$, and spaces of morphisms given by*

$$\mathrm{Hom}_{\mathcal{A}}(U_1,U_1)=A,\ \mathrm{Hom}_{\mathcal{A}}(U_1,U_1)=B,\ \mathrm{Hom}_{\mathcal{A}}(U_1,U_2)=K,\ \mathrm{Hom}_{\mathcal{A}}(U_1,U_1)=\{0\},$$

with the respective A_∞-structures. Finally, the L_∞-morphism $\mathcal{U}$ extends to an L_∞-quasi-isomorphism by suitably completing the graded vector spaces A, B, K.

Proof. The first claim has been proved in detail in [4, Proposition 6.5], to which we refer.

The second claim splits into three claims, namely $\mathcal{U}$ consists of three morphisms $\mathcal{U}_A$, $\mathcal{U}_B$ and $\mathcal{U}_K$, where $\mathcal{U}_A$, resp. $\mathcal{U}_B$, is a pre-L_∞-morphism from $T_{\mathrm{poly}}(A)$, resp. $T_{\mathrm{poly}}(B)$, to the (completed) Hochschild cochain complex of A, resp. B, and $\mathcal{U}_K$ is a collection of maps from $T_{\mathrm{poly}}(X)$ to the mixed component $\mathrm{C}^\bullet(A,B,K)$ of the (completed) Hochschild cochain complex of the above A_∞-category $\mathcal{A}$. Here, we have used the (non-canonical) identification of dg Lie algebras $T_{\mathrm{poly}}(X)=T_{\mathrm{poly}}(A)=T_{\mathrm{poly}}(B)$.

The fact that $\mathcal{U}_A$ and $\mathcal{U}_B$ are L_∞-morphisms has been proved in detail in [6]; they extend to L_∞-quasi-isomorphisms by suitably completing A and B.

The fact that the morphism $\mathcal{U}_K$ satisfies the required L_∞-identities has been proved in detail in [4, Theorem 7.2]: we profit nonetheless for discussing an incorrect issue in the proof regarding the superloop propagator. The superloop propagator, which has been defined above, is manifestly different from the one considered in [4, Subsection 7.1]: the point is that the actual superloop propagator is the correct one. We may repeat the proof of [4, Theorem 7.2] *verbatim* until the discussion of boundary strata of codimension 1 of the form $\mathcal{C}_A\times C^+_{([n]\smallsetminus A)\sqcup\{\bullet\},m}$, where $|A|=2$: the following discussion on how the corresponding integral contribution looks like is precisely the same, *i.e.* the only situation that matter arise when there are at least one and at most two edges connecting the two vertices labeled by A, *i.e.* pictorially

Figure 4 - The four possible loop-free subgraphs Γ_A yielding non-trivial boundary contributions of type i)

We are interested only in the contributions from the last three subgraphs (which we denote collectively by Γ_A). Taking into account the fact that the second graph Γ_A has 2 multiple edges (thus recalling the normalization factor 1/2), its contribution equals

$$\int_{\mathcal{C}_2}\omega^K_{\Gamma_A}=-\pi_e^*(\mathrm{d}\eta)\otimes\tau_e^{[d]}\tau_e^-=\frac{1}{2}\pi_e^*(\mathrm{d}\eta)\otimes\tau_e^{[d]}(\tau_e^+-\tau_e^-),$$

where π_e is here the projection w.r.t. the "phantom" short loop arising from the contraction of the vertices of the subgraph Γ_A. The novelty w.r.t. the corresponding computations in the proof of [4, Theorem 7.2] lies in the re-writing of the second term in the previous chain of equalities; of course, we have used the obvious fact that $(\tau_e^{[d]})^2=0$. The fourth graph in Figure 4 yields a similar contribution. The third graph, on the other hand, yields the contribution

$$\int_{\mathcal{C}_2}\omega^K_{\Gamma_A}=-\pi_e^*(\mathrm{d}\eta)\otimes\tau_{e_1}^{[d]}\tau_{e_2}^--\pi_e^*(\mathrm{d}\eta)\otimes\tau_{e_2}^{[d]}\tau_{e_1}^-=\frac{1}{2}\pi_e^*(\mathrm{d}\eta)\otimes\tau_{e_1}^{[d]}(\tau_{e_2}^+-\tau_{e_2}^-)+\frac{1}{2}\pi_e^*(\mathrm{d}\eta)\otimes\tau_{e_2}^{[d]}(\tau_{e_1}^+-\tau_{e_1}^-),$$

where $e_1 = (i,j)$, $e_2 = (j,i)$, and e is (improperly) the "phantom" short loop arising from the contraction of the two vertices i, j. Here, we have used the obvious fact that $\tau_{e_1}^{[d]}\tau_{e_2}^{[d]} = -\tau_{e_2}^{[d]}\tau_{e_1}^{[d]}$.

The factor $1/2$ before the function η on $\mathcal{C}_{1,1}^{+}$ (which we have tacitly omitted) is compatible with the fact that the pull-back of η from $\mathcal{C}_{1,0,0}^{+}$ to $\mathcal{C}_{1,1}^{+}$ is precisely the rescaled function η on $\mathcal{C}_{1,1}^{+}$.

Therefore, the same arguments as in the corresponding part of the proof of [4, Theorem 7.2] show that the right compensation for the contributions coming from the last three graphs in Figure 4 is given precisely by the superloop propagator ω_e^K, which differ from the superloop propagator chosen in the proof of [4, Theorem 7.2] in its multidifferential operator part: the trick is to prove that we may rewrite the multidifferential operator parts of the contributions coming from the last three graphs in Figure 4 using the difference $\tau_e^+ - \tau_e^-$, which is exactly the term appearing if we do the computations using the compactified configuration spaces $\mathcal{C}_{n,k,l}^{+}$ instead of $\mathcal{C}_{n,m}^{+}$. □

3.3.4. *Biquantization as a consequence of Theorem 3.5.* We consider now the particular situation $X = \mathfrak{g}^*$, for $\mathfrak{g}$ a finite-dimensional Lie algebra over $\mathbb{K}$, and for a given Lie subalgebra $\mathfrak{h}$ thereof, we set $U_1 = X$ and $U_2 = \mathfrak{h}^\perp$, the annihilator of $\mathfrak{h}$ in $\mathfrak{g}$. We observe that, later on, we will consider U_2 to be the affine space $\lambda + \mathfrak{h}^\perp$, where λ is a character of $\mathfrak{h}$: the results of the previous Subsubsection still hold true in this situation.

For the sake of explicit computations, we choose a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$, *i.e.* we choose a subspace $\mathfrak{p}$ of $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. We observe that, in general, $\mathfrak{p}$ is not $\mathfrak{h}$-invariant w.r.t. the restriction of the adjoint representation. Still, in the case of symmetric pairs $(\mathfrak{k}, \mathfrak{p})$, $\mathfrak{p}$ is a $\mathfrak{k}$-module.

We thus apply [4, Theorem 7.2] to this situation (we only observe that, in this framework, we do not consider completed algebras, as in [4]: still, Theorem 7.2 holds true, the only difference is that we have to drop the property of the L_∞-morphism to be an L_∞-quasi-isomorphism): we may view the Poisson structure on X as a Maurer–Cartan (shortly, form now on, MC) element of $T_{\mathrm{poly}}(X)$, and its image w.r.t. the L_∞-morphism from [4, Theorem 7.2] is a MC element in the Hochschild cochain complex (with mixed component completed) of the A_∞-category $\mathrm{Cat}_\infty(A, B, K)$, with objects U_i, $i = 1, 2$.

Using the previous prescriptions, we have

$$A = \mathrm{S}(\mathfrak{g}),\ B = \mathrm{S}(\mathfrak{p}) \otimes \wedge \mathfrak{h}^*,\ K = \mathrm{S}(\mathfrak{p});$$

a bit improperly, we sometimes write $\mathfrak{p} = \mathfrak{g}/\mathfrak{h}$ (as it is an identification only of vector spaces, obviously not of $\mathfrak{h}$-modules).

Since $\mathfrak{g}$ is a Lie algebra, $X = \mathfrak{g}^*$ is a Poisson manifold with linear Poisson bivector π, and U_1 and U_2 are coisotropic submanifolds thereof. The linear Poisson structure on X determines a Maurer–Cartan element of $T_{\mathrm{poly}}(X)$: for a choice of a formal parameter $\hbar$, the image of $\hbar\pi$ w.r.t. the L_∞-morphism $\mathfrak{U}$ from Theorem 3.5 is a Maurer–Cartan element $\mathcal{U}(\hbar\pi)$ in the (completed) Hochschild cochain complex of the A_∞-category $\mathcal{A}$, which is a concept needing some unraveling.

The Hochschild cochain complex of $\mathcal{A}$ splits into three terms, namely the Hochschild cochain complex of A, the one of B and a graded vector space which contains A, B and K: general elements of the mixed term $\mathrm{C}^\bullet(A, B, K)$ are multilinear maps from $A^{\otimes k} \otimes K \otimes B^{\otimes l}$ to K. From the general theory of Hochschild cochain complexes it is known that Maurer–Cartan elements of Hochschild cochain complexes correspond to A_∞-structures on the underlying graded vector spaces: in our situation, a Maurer–Cartan element is precisely a structure of A_∞-algebra on both A and B, and a corresponding structure of A_∞-A-B-bimodule on K, or, equivalently, to an A_∞-structure on the category $\mathcal{A}$.

Now we consider $A_\hbar = A[\![\hbar]\!]$, and similarly for $B_\hbar$ and $K_\hbar$: it is clear that the structure of A_∞-category on $\mathcal{A}$ extends to $\hbar$-linearly to $\mathcal{A}_\hbar$, whose objects are the same objects of $\mathcal{A}$, but whose morphism spaces are replaced by $A_\hbar$, $B_\hbar$ and $K_\hbar$ endowed with the $\hbar$-linearly extended A_∞-structure μ. We have a natural Hochschild differential $\mathrm{d_H}$ on the Hochschild cochain complex of $\mathcal{A}_\hbar$, given by the adjoint representation of μ w.r.t. the Gerstenhaber bracket. The element $\mathcal{U}(\hbar)$ satisfies the Maurer–Cartan equation

$$\mathrm{d_H}\mathcal{U}(\hbar\pi) + \frac{1}{2}\left[\mathcal{U}(\hbar\pi), \mathcal{U}(\hbar\pi)\right] = \frac{1}{2}\left[\mu + \mathcal{U}(\hbar\pi), \mu + \mathcal{U}(\hbar\pi)\right] = 0,$$

i.e. $\mu + \mathcal{U}(\hbar\pi)$ is a Maurer–Cartan element for $\mathcal{A}_\hbar$, which deforms (w.r.t. the formal parameter $\hbar$) the "classical" A_∞-structure on $\mathcal{A}$.

Since $A_\hbar$ is concentrated in degree 0 by construction, and μ_A (the component of μ in the Hochschild cochain complex of A) is the obvious $\hbar$-linear commutative, associative product on $A_\hbar$, then $\mu_A + \mathcal{U}_A(\hbar\pi)$ is an associative product $\star_{A_\hbar}$ on $A_\hbar$, which deforms non-trivially μ_A: $(A_\hbar, \star_{A_\hbar})$ is a deformation quantization of (A, μ_A) in the sense of [12].

The image of π in $T_{\mathrm{poly}}(A)$ w.r.t. the dg Lie algebra isomorphism $T_{\mathrm{poly}}(X) \cong T_{\mathrm{poly}}(A)$ (depending on a choice of $\mathfrak{p}$) is a Maurer–Cartan element in $T_{\mathrm{poly}}(A)$, which is a sum of a three polyvector fields, π_0, π_1 and π_2, where π_i is an i-th polyvector field of polynomial degree $2 - i$. We observe that $A = \mathrm{S}(\mathfrak{g})$ and $B = \mathrm{S}(\mathfrak{p} \oplus \mathfrak{h}^*[-1])$, thus it makes sense to speak about polynomial degree for elements of A and B; $[\bullet]$ is the degree-shifting functor (hence, the polynomial grading of B does not coincide with the internal grading coming from the functor $[-1]$), *e.g.* the internal degree of π_i is 1, $i = 0, 1, 2$. As a Maurer–Cartan element of $T_{\mathrm{poly}}(A)$, π defines a P_∞-structure on B, in other words, π defines a Poisson algebra structure on B up to homotopy: exemplarily, π_0 is a homological vector field over B, whose cohomology identifies with the Chevalley–Eilenberg cohomology of the $\mathfrak{h}$-module $\mathrm{S}(\mathfrak{g}/\mathfrak{h})$, which in turn inherits from π_1 (which is a bivector field of internal degree 1) a structure of graded Poisson algebra. We notice that, in degree 0, this corresponds to the well-known fact that Poisson reduction endows the commutative algebra $\mathrm{S}(\mathfrak{g}/\mathfrak{h})^{\mathfrak{h}}$ with a Poisson structure coming from the natural one on $A = \mathrm{S}(\mathfrak{g})$. The Maurer–Cartan element $\mu_B + \mathcal{U}_B(\hbar\pi)$ is an A_∞-structure on $B_\hbar$, deforming the obvious A_∞-structure on B: thus, a P_∞-algebra structure on B produces an A_∞-structure *via* the graded version of deformation quantization, see also [6]. We observe that the A_∞-structure $\mu_B + \mathcal{U}_B(\hbar\pi)$ is the sum of (possibly) infinitely many components of different internal degree: in particular, the component of internal degree 2 is an element of $B_\hbar$ of degree 2, the curvature of the A_∞-structure. If it non-trivial, then we cannot talk about the cohomology of A_∞-algebra $(B_\hbar, \mu_B + \mathcal{U}_B(\hbar\pi))$, and some problems may arise: luckily, in the present framework, the curvature vanishes, see *e.g.* [5, 7] and later on. We finally observe that the term of order 1 w.r.t. $\hbar$ of the A_∞-structure on $B_\hbar$ is precisely the ($\hbar$-shifted) P_∞-structure on $B_\hbar$: thus, if we select its vector field piece, we get the $\hbar$-shifted Chevalley–Eilenberg differential on $B_\hbar$.

Finally, the mixed component $\mu_K + \mathcal{U}_K(\hbar\pi)$ determines a deformation of the A_∞-A-B-bimodule μ_K structure on K: we do not spend here much words, because we will deal with $\mu_K + \mathcal{U}_K(\hbar\pi)$ in the rest of the paper, at least in degree 0. We only observe that, through μ_K, we may re-prove classical Koszul duality between A and B (both are graded quadratic algebras), and its deformation quantization permits to extend the Koszul duality to the deformed case $(A_\hbar, \star_\hbar)$ and $(B_\hbar, \mu_B + \mathcal{U}_B(\hbar\pi))$.

Biquantization as in [7] is the specialization to degree 0 of the data presented above. In particular, $(A_\hbar, \star_{A_\hbar})$ is an A_∞-algebra concentrated in degree 0, hence its cohomology equals itself; the piece of $B_\hbar$ of degree 0 equals $\mathrm{S}(\mathfrak{p}) \cong \mathrm{S}(\mathfrak{g}/\mathfrak{h})$ endowed with a differential $\mathrm{d}^0_{B_\hbar}$ and with an associative product $\star_{B_\hbar}$ up to homotopy. Finally, $K_\hbar$ is also concentrated in degree 0, hence its cohomology w.r.t. $\mathrm{d}^{0,0}_{K_\hbar}$ (the $(0,0)$-component of $\mu_K + \mathcal{U}_K(\hbar\pi)$) equals itself, hence $K_\hbar$ becomes w.r.t. $d^{1,0}_{K_\hbar} = \star_L$ a left $(A_\hbar, \star_{A_\hbar})$- and w.r.t. $\mathrm{d}^{0,1}_{K_\hbar} = \star_R$ a right $(\mathrm{H}^0(B_\hbar), \star_{B_\hbar})$-module (the latter also because of the vanishing of the curvature of the A_∞-structure $\mu_B + \mathcal{U}(\hbar\pi)$).

Later on, still in the framework of finite-dimensional Lie algebras and Lie subalgebras thereof, we will consider the more general framework, where both $A_\hbar$ and $B_\hbar$ are A_∞-algebras with no curvature, and $K_\hbar$ is a graded A_∞-$A_\hbar$-$B_\hbar$-bimodule, hence the 0-th cohomologies $\mathrm{H}^0(A_\hbar)$, $\mathrm{H}^0(B_\hbar)$ become associative algebras and $\mathrm{H}^0(K_\hbar)$ is an $\mathrm{H}^0(A_\hbar)$-$\mathrm{H}^0(B_\hbar)$-bimodule.

3.3.5. *Symmetries of the 4-colored propagators.* For later purposes, we now exhibit certain symmetries of the 2-colored and 4-colored propagators, which we now discuss in some detail.

The complex upper half-plane $\mathbb{H}^+$ has two obvious symmetries, namely the reflection w.r.t. the imaginary axis $i\mathbb{R}$, given by $z \overset{\sigma}{\mapsto} -\overline{z}$, and the inversion w.r.t. the unit half-circle, given by $z \overset{\tau}{\mapsto} 1/\overline{z}$: both maps extend to $\mathbb{H}^+ \sqcup \mathbb{R}$, and they define two orientation-reversing involutions σ and τ of it. Equivalently, $\mathcal{Q}^{+,+} \sqcup i\mathbb{R}^+ \sqcup \{0\} \sqcup \mathbb{R}^+$ admits two orientation-reversing involutions σ and τ, where $z \overset{\sigma}{\mapsto} i\overline{z}$ and $z \overset{\tau}{\mapsto} \frac{1}{\overline{z}}$.

It is not difficult to prove that σ and τ descend both to involutions of $C^+_{n,m}$ and $C^+_{n,k,l}$, and that, using the same techniques as in the proof of Proposition 3.1, Subsubsection 3.1.3, σ and τ extend to involutions of the compactified configuration spaces $\mathcal{C}^+_{n,m}$ and $\mathcal{C}^+_{n,k,l}$. We observe that σ and τ are orientation-preserving, resp. -reversing, if and only if $n + m - 1$ is even, resp. odd.

We then have the following technical Lemma about the behavior of the 2-colored and 4-colored propagators w.r.t. the action of σ and τ.

Lemma 3.6. *The 2-colored and 4-colored propagators behave as follows w.r.t. the involutions σ and τ on the respective compactified configuration spaces $\mathcal{C}^+_{2,0}$ and $\mathcal{C}^+_{2,1}$:*

$$\sigma^*(\omega^+) = -\omega^+, \qquad \sigma^*(\omega^-) = -\omega^-, \qquad \tau^*(\omega^+) = -\omega^+ + 2\pi_1^*(\mathrm{d}\eta), \qquad \tau^*(\omega^-) = -\omega^- + 2\pi_2^*(\mathrm{d}\eta),$$

$$\sigma^*(\omega^{+,+}) = -\omega^{+,+}, \qquad \sigma^*(\omega^{+,-}) = -\omega^{-,+}, \qquad \sigma^*(\omega^{-,+}) = -\omega^{+,-}, \qquad \sigma^*(\omega^{-,-}) = -\omega^{-,-},$$

$$\tau^*(\omega^{+,+}) = -\omega^{+,+} + 2\pi_1^*(\mathrm{d}\eta), \qquad \tau^*(\omega^{+,-}) = -\omega^{+,-}, \qquad \tau^*(\omega^{-,+}) = -\omega^{-,+}, \qquad \tau^*(\omega^{-,-}) = -\omega^{-,-} + 2\pi_2^*(\mathrm{d}\eta),$$

where now π_i, $i = 1, 2$, denotes the two natural projections from $\mathcal{C}^+_{2,0}$ onto $\mathcal{C}^+_{1,0}$ or from $\mathcal{C}^+_{2,1}$ to $\mathcal{C}^+_{1,1}$. Similar formulæ hold true for the 4-colored propagators on $\mathcal{C}_{2,0,0}$, keeping in track a rescaling before the exact 1-form η.

3.3.6. *Kontsevich's Vanishing Lemmata.* We now need a Vanishing Lemma for the 4-colored propagators, reminiscent of the Vanishing Lemmata in [12, Subsubsubsection 7.3.3.1]. We observe that Kontsevich's Vanishing Lemmata in [12, Subsubsubsection 7.3.3.1] are key ingredients in the proof of the globalization of its L_∞-Formality-quasi-isomorphism: in this sense, the Vanishing Lemma we are going to state and prove here (the main application being for later computations regarding the generalized Harish-Chandra homomorphism) play also a central *rôle* in the globalization of the 2-brane L_∞-Formality-quasi-isomorphism of [4], but do not indulge here on this point, referring to upcoming work for more details.

We consider the three natural projections π_{ij}, $i \leq i << \leq 3$, from $\mathcal{C}^+_{3,0}$ onto $\mathcal{C}^+_{2,0}$, which smoothly extend the projections $[(z_1, z_2, z_3)] \overset{\pi_{ij}}{\to} [(z_i, z_j)]$ to the corresponding compactified configuration spaces. The typical fiber of the projection π_{ij} is 2-dimensional, and it is not difficult to verify that it is a smooth manifold with corners (hence, it admits a natural stratification, whose description, at least in codimension 1, will be made explicit later on). We improperly denote by the same symbol the natural projection π_{ij}, $i \leq i << \leq 3$, from $\mathcal{C}^+_{3,1}$ onto $\mathcal{C}^+_{2,1}$, which this times extends the projection $[(z_1, z_2, z_3, x)] \overset{\pi_{ij}}{\mapsto} [(z_i, z_j, x)]$: again, its fiber is a smooth 2-dimensional manifold with corners.

For any two smooth 1-forms η_i, $i = 1, 2$, on $\mathcal{C}^+_{2,0}$ or $\mathcal{C}^+_{2,1}$, we define a smooth function on $\mathcal{C}^+_{2,0}$ or $\mathcal{C}^+_{2,1}$ *via* the integral

$$\Omega(\eta_1, \eta_2) = \pi_{13,*}(\pi^*_{12}(\eta_1) \wedge \pi^*_{23}(\eta_2)) = \int_{z_2 \in \mathcal{Q}^{+,+} \smallsetminus \{z_1, z_3\}} \eta_1(z_1, z_2) \wedge \eta_2(z_2, z_3), \tag{16}$$

where $\pi_{ij,*}$ denotes integration along the fiber of the projection π_{ij}.

Lemma 3.7. *The function $\Omega(\eta_1, \eta_2)$ vanishes, whenever $\eta_1 = \eta_2$ is either one of the 2-colored propagators or either one of the 4-colored propagators.*

Proof. The claim for the 2-colored propagators ω^+ and ω^- is precisely the content of the vanishing lemmata in [12, Subsubsubsection 7.3.3.1], to which we refer for a proof. We observe that the proof below for the 4-colored propagators applies with minor changes (*e.g.* the final argument involves the involution σ and not τ) applies to the statement for 2-colored propagators.

For symmetry reasons, it suffices to prove the claim for the 4-colored propagators $\omega^{+,+}$ and $\omega^{+,-}$.

We first prove the claim for $\eta_1 = \eta_2 = \omega^{+,+}$: the idea is to show first that $\Omega(\eta_1, \eta_2)$ is a constant function, and then to use Lemma 3.6 to prove that, for a well-suited choice of arguments, $\Omega(\eta_1, \eta_2)$ equals minus itself.

We therefore compute the exterior derivative of $\Omega(\eta_1, \eta_2)$: we make use of generalized Stokes' Theorem, and the fact that $\eta_1 = \eta_2$ is a closed 1-form, yields

$$\mathrm{d}\Omega(\eta_1, \eta_2) = \pi^\partial_{13,*}(\pi^*_{12}(\eta_1)\pi^*_{23}(\eta_2)),$$

where $\pi^\partial_{13,*}$ denotes integration along the codimension 1-boundary strata of the fiber of π_{13}: there are five such boundary strata, which correspond to *i*) the point labeled by z_2 approaching either $\mathbb{R}$ on the left or on the right of the marked point on $\mathbb{R}$ or the marked point itself, or to *ii*) the point labeled by z_2 approaching either the point labeled by z_1 or z_2.

In the three situations in *i*), the corresponding contribution vanishes in view of Lemma 3.3, *iii*) and *iv*).

In both situations described in *ii*), the boundary fibration is trivial, namely $\mathcal{C}_2 \times \mathcal{C}^+_{2,1}$: an easy computation in local coordinates for $\mathcal{C}^+_{3,1}$ near the boundary strata in *i*) shows that there are no orientation signs appearing, and we finally get, using Lemma 3.3, *i*),

$$\mathrm{d}\Omega(\eta_1, \eta_2) = \int_{\mathcal{C}^+_2} \mathrm{d}\varphi\ \omega^{+,+} + \int_{\mathcal{C}^+_2} \omega^{+,+}\mathrm{d}\varphi = 0.$$

Therefore, $\Omega(\eta_1, \eta_2)$ is a constant function on $\mathcal{C}^+_{2,1}$, whose value is completely determined by a choice of a point in $\mathcal{C}^+_{2,1}$, and a natural choice is $(i, 2i, 0)$. Since $i\mathbb{R}$ is the fixed point set of σ, σ preserves $\Omega(\eta_1, \eta_2)$, whence

$$\Omega(\eta_1, \eta_2) = \sigma^*(\Omega(\eta_1, \eta_2) = -\pi_{13,*}(\sigma^*(\pi^*_{12}(\eta_1)) \wedge \sigma^*(\pi^*_{23}(\eta_1)) = -\Omega(\eta_1, \eta_2),$$

where the minus sign in the second equality arises because s is orientation-reversing on the first quadrant, while the third equality is a consequence of Lemma 3.6.

The very same arguments can be applied in the situation $\eta_1 = \eta_2 = \omega^{-,-}$.

We consider now the case $\eta_1 = \eta_2 = \omega^{+,-}$. The computation of the exterior derivative of $\Omega(\eta_1, \eta_2)$ in this case is similar to the previous one: we only observe that the boundary condition for boundary strata of type *i*) let appear a regular term $\mathrm{d}\eta$, whose contribution to integration is trivial. The arguments for dealing with the other strata are, once again, a consequence of Lemma 3.3, *iii*) and *iv*).

Therefore, $\Omega(\eta_1, \eta_2)$ is uniquely determined by a given point in $\mathcal{C}^+_{2,1}$: quite differently from the previous case, we choose a pair of points lying on the unit circle. Recalling that the unit circle is the fixed point *locus* of the involution τ, we get in this case

$$\Omega(\eta_1, \eta_2) = \tau^*(\Omega(\eta_1, \eta_2) = -\pi_{13,*}(\tau^*(\pi^*_{12}(\eta_1)) \wedge \tau^*(\pi^*_{23}(\eta_1)) = -\Omega(\eta_1, \eta_2),$$

by the very same arguments as in the previous case, because t is orientation-reversing and because of Lemma 3.6. □

As an application of Lemma 3.7, we briefly sketch the vanishing of the curvature of the A_∞-algebra $B_\hbar$, with the previously introduced notations. As already mentioned, the curvature of $B_\hbar$ is the piece of degree 2 in $B_\hbar$ of the Maurer–Cartan element $\mu_B + \mathcal{U}_B(\hbar\pi)$: more explicitly, the curvature is given by the formal power series

$$\mathcal{U}(\hbar\pi)_0 = \sum_{n\geq 1}\frac{1}{n!}\mathcal{U}_B^n(\underbrace{\hbar\pi|\cdots|\hbar\pi}_{n}) = \sum_{n\geq 1}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}_{n,0}}\mathcal{O}_\Gamma^B(\underbrace{\hbar\pi|\cdots|\hbar\pi}_{n}) = \sum_{n\geq 1}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}_{n,0}}\mu_n^B\left(\int_{\mathcal{C}_{n,0}^+}\prod_{e\in E(\Gamma)}\omega_e^B(\underbrace{\hbar\pi|\cdots|\hbar\pi}_{n})\right).$$

For an admissible graph Γ in $\mathcal{G}_{n,0}$, the rightmost integral is non-trivial only if the degree of the integrand equals $2n-2$. To each vertex of the first type of Γ is associated a copy of the linear Poisson bivector $\hbar\pi$, hence from each vertex depart exactly two arrows: each arrow, by definition, corresponds to a derivation and a contraction. In particular, the differential operator $\mathcal{O}_\Gamma^B$ has degree $2n-2$: since $\hbar\pi$ is linear, the polynomial degree of the object on the rightmost part of the previous chain of equalities is $n-(2n-2) = -n+2$, whence $1\leq n\leq 2$. When $n=1$, $\mu_1^B(\hbar\pi)$ vanishes because of the coisotropy of U_2. For $n=2$, there is only one possible admissible graph Γ of type $(2,0)$, namely the loop graph connecting the two vertices of the first type: the corresponding operator $\mathcal{O}_\Gamma^B$ vanishes because of Lemma 3.7.

3.4. **Quantum reduction algebras.** The present Subsection presents the results of [7, Section 2] using a slightly different perspective, coherent with the approach to biquantization we have introduced in the previous Subsection: however, we think it useful to review many results mainly because of the notation, which will be then used extensively in the rest of the paper.

Thus, we will mostly concentrate on the dg vector space $B_\hbar$, where we denote by $\mathrm{d}_{B_\hbar}$ its differential (*i.e.* the piece of degree 1 of its A_∞-structure): we have already observed that $\mathrm{d}_{B_\hbar} = \hbar\mathrm{d}_{\mathrm{CE}} + \mathcal{O}(\hbar^2)$, where $\mathrm{d}_{\mathrm{CE}} = \pi_0$ is the Chevalley–Eilenberg differential on $B_\hbar$. The admissible graphs Γ appearing in $\mathrm{d}_{B_\hbar}$ are of type $(n,1)$, possibly with multiple edges and admitting edges departing from $\mathbb{R}$: each edge e of an element Γ of $\mathcal{C}_{n,1}$ is the sum of two colored edges, namely e^+ and e^- according to Formula (7), Subsubsection 3.3.2. The properties of the superpropagator (7) imply that an edge e of Γ arriving to the only vertex of the second type has color $+$, while an edge departing from it has color $-$.

One of the main tools we will use throughout the present Subsection is Lemma 3.7, Subsubsection 3.3.6. Namely, we assume v is a vertex of the first type of Γ of type $(n,1)$ with two edges at it of the form $e_1 = (\bullet_1, v)$ and $e_2 = (v, \bullet_2)$, where $\bullet_i$, $i=1,2$, denotes some other vertex (notice that now we allow $\bullet_1 = \bullet_2$): then the configuration at v is either (e_1^+, e_1^-) or (e_1^-, e_2^+). Of course, since to each edge of the first type of Γ is associated a copy of the linear Poisson bivector $\hbar\pi$, we assume that from v as above departs a third edge, which does not join any other vertex of Γ: this edge has color $-$. This "phantom" edge is the "edge to ∞", using the terminology of [7]: dimensional reasonings imply that each admissible graph Γ of type $(n,1)$ admit precisely one vertex of the first type with a phantom edge.

3.4.1. *Symmetric pairs and, more generally, Lie subalgebras of trivial extension class.* We consider here the case of a symmetric pair $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, or, more generally, of a Lie subalgebra $\mathfrak{h}\subseteq\mathfrak{g}$ admitting an $\mathfrak{h}$-invariant complementary subspace $\mathfrak{p}$: we observe that this is equivalent to the triviality of the extension class α of the short exact sequence of $\mathfrak{h}$-modules $\mathfrak{h}\hookrightarrow\mathfrak{g}\twoheadrightarrow\mathfrak{g}/\mathfrak{h}$.

By definition, a symmetric pair is a pair $(\mathfrak{g},\sigma)$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{K}$ and σ is an involutive Lie algebra automorphism: thus, $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ is the direct sum of the $+1$-eigenspace $\mathfrak{k}$ and the -1-eigenspace $\mathfrak{p}$ of σ. In particular, we have the Cartan relations

$$[\mathfrak{k},\mathfrak{k}]\subseteq\mathfrak{k},\ \ [\mathfrak{k},\mathfrak{p}]\subseteq\mathfrak{k},\ \ [\mathfrak{p},\mathfrak{p}]\subseteq\mathfrak{k}. \tag{17}$$

In particular, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a $\mathfrak{k}$-module. The graded algebra B in the case of a symmetric pair equals $B = \mathrm{S}(\mathfrak{k})\otimes\wedge^\bullet(\mathfrak{p})$.

Of course, if the above extension class α vanishes, then $\mathfrak{g}$ admits simply a decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$, with relations $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$ and $[\mathfrak{h},\mathfrak{p}]\subseteq\mathfrak{p}$.

The claim is that in the case of a symmetric pair $(\mathfrak{g},\sigma)$ or of a pair $(\mathfrak{g},\mathfrak{h})$ with trivial extension class the quantized differential $\mathrm{d}_{B_\hbar}$ equals simply the ($\hbar$-shifted) Chevalley–Eilenberg differential $\hbar\mathrm{d}_{\mathrm{CE}}$ on $B_\hbar = \mathrm{C}^\bullet(\mathfrak{h},\mathrm{S}(\mathfrak{g}/\mathfrak{h}))[\![\hbar]\!]$.

Namely, we consider an admissible graph Γ of type $(n,1)$, $n\geq 2$, appearing in Formula (13), Subsubsection 3.3.3, and we know from the above considerations that Γ possesses a vertex v of the first type with a phantom edge e_{gh} and two edges $e_1 = (\bullet_1, v)$ and $e_2 = (v,\bullet_2)$. The configuration at the vertex is either $(e_1^+, e_2^-, e_{\mathrm{gh}}^-)$ or $(e_1^-, e_2^+, e_{\mathrm{gh}}^-)$: in Lie algebraic terms, these two configurations correspond to $[\mathfrak{h},\mathfrak{p}]\subseteq\mathfrak{h}$ and $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{p}$ respectively, which is a contradiction to the Cartan relations, for $\mathfrak{g}$ a symmetric pair, or to the fact that $\mathfrak{k}$ has a $\mathfrak{k}$-invariant complement. This implies that only the contributions of order 1 w.r.t. $\hbar$ matter, whence the claim.

3.4.2. *Cohomology of degree* 0. The content of the present Subsubsection presents some arguments for dealing with the classification of admissible graphs appearing in the computation of the differential $\mathrm{d}_{B_\hbar}$, which will also appear in other contexts related to biquantization.

We consider here the case of a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with no assumptions on the extension class of $\mathfrak{h} \subseteq \mathfrak{g}$: thus, we only assume to have picked out some complementary subspace $\mathfrak{p}$ for explicit computations.

Proposition 3.8. *The admissible graphs of type of* $(n, 1)$*,* $n \geq 1$*, appearing in the restriction of the differential* $\mathrm{d}_{B_\hbar}$ *to* $\mathrm{S}(\mathfrak{p})$ *are either of type Bernoulli (*i.e. *connected graphs with one root and one phantom edge), or of type wheel (*i.e. *connected graphs whose edges between the vertices of the first type form an oriented loop with one phantom edge), or of mixed type (*i.e. *a Bernoulli-type graphs attached to a wheel-type graph), see also Figure 5.*

Proof. We consider, for $n \geq 1$, an admissible graph of type $(n, 1)$, and we denote by p the number of edges of Γ arriving at the vertex of the second type.

Degree reasons imply that such a graph admits one phantom edge, hence the actual edges connecting vertices of Γ are $2n - 1$. Dimensional reasons imply also that $p \geq 1$: namely, if p were 0, since $n \geq 1$ by assumption, there would be a 1-dimensional submanifold of $\mathcal{C}^+_{n,1}$ over which there is nothing to integrate (a subset of $\mathbb{R}$), hence integration would yield 0.

The admissible graph Γ is connected in the sense that, if we remove from it the edges to the vertex of the second type, we obtain a connected graph in the strict sense of the world, as $p \geq 1$. The connectedness of Γ is also a consequence of dimensional reasons: if $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then either Γ_1 or Γ_2 would have a phantom edge. W.l.o.g. we assume Γ_1 has a phantom edge, hence Γ_2 does not, which means that all its edges provide differential forms to be integrated: if n_i is the number of vertices of the first type of Γ_i, $i = 1, 2$, then the integral over $\mathcal{C}^+_{n_2,1}$ of the differential form associated to Γ_2 vanishes, as its degree is $2n_2$, while the dimension of $\mathcal{C}^+_{n_2,1}$ is $2n_2 - 1$.

The boundary conditions for the superpropagator (7) imply immediately that edges arriving at the vertex of second type are all colored by $+$, whence $p \leq (n - 1) + 1 = n$: in fact, from the vertex of the first type from which departs the phantom edge departs another edge, which may or may not arrive at the vertex of the second type.

On the other hand, there are $2n - 1 - p$ edges from vertices of the second type arriving at (distinct) vertices of the first type: this implies that the polynomial degree of the differential operator on $B_\hbar$ associated to Γ is $n - (2n - 1 - p) = -n + 1 + p \geq 0$, whence $p \geq n - 1$: it follows then immediately $p = n - 1$ or $p = n$.

We first consider the case $p = n$: from every vertex of the first type of Γ departs one edge to the vertex of the second type, Γ is connected, has a phantom edge and there is a single vertex of the first type, which is the final point of none of its edges. Such an admissible graph is obviously of Bernoulli type.

Then, assume $p = n - 1$: the only vertex from which does not depart an edge to the vertex of the second type may or may not be the vertex from which departs the phantom edge. In the first case, the connectedness of Γ implies that it is of wheel-type, while in the second case, it must be a wheel type, from which departs an edge hitting the root of a Bernoulli-type graph. Here, the root of a Bernoulli-type graph is the only vertex of the second type, which is the final point of none of its edges. □

Here is a pictorial representation of the three types of graphs appearing in $\mathrm{d}_{B_\hbar}$ according to the previous Proposition:

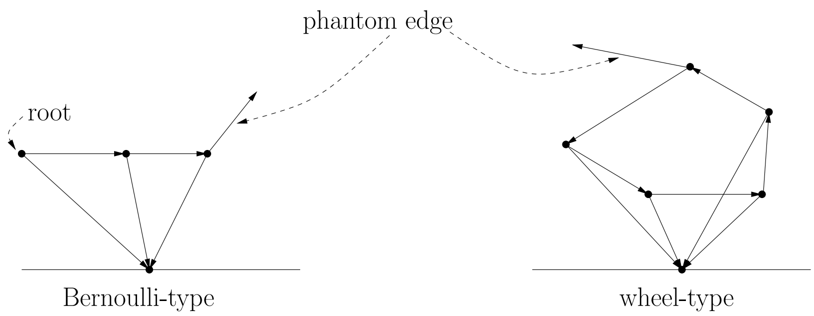

Figure 5 - Bernoulli-type and wheel-type graphs

Although for most of the computations in this framework we do not really need it, we want to understand the differential $\mathrm{d}_{B_\hbar}$ on the whole of $B_\hbar$: as the next proposition shows, the results of Proposition 3.8 with a slight addition suffice.

Proposition 3.9. *The admissible graphs of type* $(n, 1)$ *appearing in* $\mathrm{d}_{B_\hbar}$ *are either the admissible graphs of Proposition 3.8 or connected graphs which are brackets of two Bernoulli-type graphs (*i.e. *there is an edge departing from the* *vertex of the second type to a vertex of the first type, whose two outgoing edges arrive at the roots of two Bernoulli-type graphs, see Figure 6).*

Proof. We first observe that, since we are considering $\mathrm{d}_{B_\hbar}$ on the whole of $B_\hbar$, admissible graphs Γ can have edges departing from the vertex of the second type.

We first assume that that all edges departing from the vertex of the second type are phantom edges: in this case, we are reduced to the very same analysis as in the proof of Proposition 3.8.

We now assume that the admissible graph Γ of type $(n,1)$ possesses k edges departing from the vertex of the second type and arriving to k vertices of the first type (these vertices are distinct because of the linearity of the Poisson bivector $\hbar\pi$). Because of degree reasons, Γ has $k+1$ phantom edges departing from vertices of the first type. Dimensional reasons imply further that no vertex of the first type may have two phantom edges. We denote by p the number of edges departing from vertices of the first type and arriving to the vertex of the second type: the very same arguments as in the proof of Proposition 3.8 imply that $p \geq n$. On the other hand, the polynomial degree of the differential operator associated to Γ equals $(n-k)-(2n-(k+1)-p) = -n+1+p \geq 0$, whence either $p = n-1$ or $p = n$.

We first consider the case $p = n$: in this situation, from every vertex of the first type departs exactly one edge to the vertex of the second type. Thus, Γ is the disjoint union of exactly one Bernoulli-type graph and of either wheel-like graphs or Bernoulli-wheel-type graphs or Bernoulli-type graphs, whose root is the endpoint of an edge departing from the vertex of the second type (at least such a graph appears here, as $k \geq 1$ by assumption). Since Γ is a disjoint union of subgraphs, the corresponding integral factors into integrals over compactified configuration spaces of the form $\mathcal{C}^+_{m,1}$, for $1 \leq m \leq n$. For a subgraph of the last type as above, the corresponding integral vanishes, because the degree of the integrand is $2m$, while the dimension of the fiber is $2m-1$.

We consider now the case $p = n-1$: the corresponding differential operator is translation-invariant. If Γ is the disjoint union of connected subgraphs, at least one of which is a Bernoulli-type graph, whose root is the final point of an edge departing from the vertex of the second type, the last argument in the previous paragraph yields triviality of the corresponding differential operator. A subgraph like the one we have analyzed always appear, if there is a vertex of the first type of Γ, from which departs one edge to the vertex of the second type and which is the endpoint of an edge issued from the said vertex. As $p = n-1$, there can be exactly one edge departing from the vertex of the second type and arriving at the only vertex of the first type, from which no edge depart to $\mathbb{R}$: these two edges meet two distinct roots (again, because $p = n-1$) of Bernoulli-type graphs. Therefore, Γ is the disjoint union of subgraphs as in Proposition 3.8 and of a connected subgraph of the said type. □

Here is a pictorial representation of the new type of connected graphs appearing in $\mathrm{d}_{B_\hbar}$ in higher degrees according to the previous Proposition:

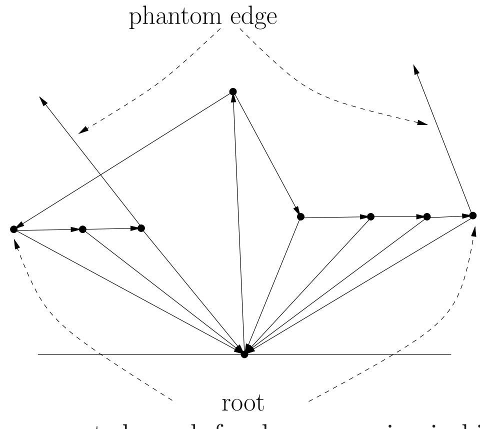

Figure 6 - A connected graph for $\mathrm{d}_{B_\hbar}$ appearing in higher degrees

We consider now a general admissible graph Γ of type $(n,1)$ appearing in $\mathrm{d}_{B_\hbar}$: we may consider the involution σ of $\mathcal{C}^+_{n,1}$ from Subsubsection 3.3.5. It preserves, resp. reverses, orientation if n is even, resp. odd; the pull-back of integrand in Formula (10) w.r.t. σ equals itself up to a global -1-sign, hence if n is even, the contributions to $\mathrm{d}_{B_\hbar}$ from admissible graphs of type $(n,1)$ with n even are trivial. We may therefore write $\mathrm{d}_{B_\hbar}$ as

$$\mathrm{d}_{B_\hbar} = \hbar \mathrm{d}_{\mathrm{CE}} + \hbar^3 \mathrm{d}_3 + \hbar^5 \mathrm{d}_5 + \cdots ,$$

where only odd powers of $\hbar$ appear. Therefore, if f is a general element of $B_\hbar$, then f is $\mathrm{d}_{B_\hbar}$-closed if and only if both its even and odd part w.r.t. $\hbar$ are $\mathrm{d}_{B_\hbar}$-closed.

3.5. **Products on quantum reduction algebras.** After having discussed in some detail the relevant quantum reduction algebras which we will encounter in the sequel, we are now interested in a detailed discussion of the existence of associative products on quantum reduction algebras.

We consider the dg vector space $B_\hbar$ in its full generality for a finite-dimensional Lie algebra $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{h}$ thereof, to which we associate a dg algebra B, whose deformation quantization $B_\hbar$ is a flat A_∞-algebra: in particular, this means that the graded vector space $\mathrm{H}^\bullet(B_\hbar)$ is endowed with an associate product. More precisely, the A_∞-structure $\mu_B + \mathcal{U}_B(\hbar\pi)$, where π denotes here the P_∞-structure on B Fourier-dual to the Poisson bivector on $X = \mathfrak{g}^*$, consists of infinitely many Taylor components

$$\mathcal{U}_B(\hbar\pi)^n : B_\hbar^{\otimes n} \to B_\hbar[2-n],\ n \geq 1,$$

which satisfy an infinite series of quadratic identities between them. For our purposes, we need only know that $\mathcal{U}_B(\hbar\pi)^1 = \mathrm{d}_{B_\hbar}$, and that $\mu_B + \mathcal{U}_B(\hbar\pi)^2$ defines a $\mathbb{K}$-bilinear pairing of degree 0 on $B_\hbar$, which is compatible with $\mathrm{d}_{B_\hbar}$ and which is associative up to a the homotopy $\mathcal{U}_B(\hbar\pi)^3$ w.r.t. $\mathrm{d}_{B_\hbar}$. Therefore, $\mu_B + \mathcal{U}_B(\hbar\pi)^2$ descends to the quantum reduction space $\mathrm{H}^\bullet(B_\hbar)$ to an associative product, which we denote for simplicity by $\star_{B_\hbar}$: its restriction to $\mathrm{H}^0(B_\hbar)$ defines an obvious deformation of the commutative product on B^0.

We refer to [7, Section 3] for a careful description of the deformed product $\star_{B_\hbar}$ on $\mathrm{H}^0(B_\hbar)$ in the case of a symmetric pair $(\mathfrak{g}, \sigma)$ (with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$) and of a character χ of the Lie subalgebra $\mathfrak{k}$. We only observe that in [7, Section 3], the authors use the notation $\star_{\mathrm{CF},\lambda} = \star_{B_\hbar}$.

4. Applications of biquantization in Lie theory for symmetric pairs

In the present Section, we discuss the first relevant applications of biquantization, as discussed in detail in the previous Section, to concrete problems in Lie theory in the framework of a symmetric space $(\mathfrak{g}, \sigma)$ with standard Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

In fact, the results presented here are the revisited versions of the results of [7, Section 4] taking into account the more precise and correct approach to biquantization presented in the previous Section.

4.1. **A comparison between the quantum deformed product and Rouvière's product.** We consider the quantum reduction algebra $(\mathrm{H}^0(B_\hbar), \star_{B_\hbar})$ for a symmetric pair $(\mathfrak{g}, \sigma)$ with the standard Cartan decomposition. On the other hand, we consider the quantum deformed algebra $(A_\hbar, \star_{A_\hbar})$ and the A_∞-$A_\hbar$-$B_\hbar$-bimodule $K_\hbar$: unraveling the A_∞-identities for $K_\hbar$, we see that $K_\hbar$, which is concentrated in degree 0, becomes actually an $(A_\hbar, \star_{A_\hbar})$-$(\mathrm{H}^0(B_\hbar), \star_{B_\hbar})$-bimodule, and we denote by $\star_L$ and $\star_R$ the respective left $(A_\hbar, \star_{A_\hbar})$- and right $(\mathrm{H}^0(B_\hbar), \star_{B_\hbar})$-action on $K_\hbar$.

More explicitly, in the present framework, we have $A_\hbar = \mathrm{S}(\mathfrak{g})[\![\hbar]\!]$, $B_\hbar^0 = K_\hbar = \mathrm{S}(\mathfrak{p})[\![\hbar]\!]$. We observe that the linearity of the Poisson bivector π on $X = \mathfrak{g}^*$ has an interesting by-product, as already remarked in [12, Subsubsection 8.3.1]: although there are infinitely many bidifferential operators appearing in the deformed product $\star_{A_\hbar}$, the action of the infinite series $\star_{A_\hbar}$, for $\hbar = 1$, on $\mathrm{S}(\mathfrak{g}) \otimes \mathrm{S}(\mathfrak{g})$ is well-defined, as can be proved by inspecting the integral weights and counting degrees. Similar arguments hold true also for the components of the A_∞-structure $\mu_B + \mathcal{U}_B(\hbar\pi)$ on $B_\hbar$ and for the A_∞-$A_\hbar$-$B_\hbar$-bimodule structure on $K_\hbar$. As a consequence, we may consider the A_∞-algebras $A_\hbar$ and $B_\hbar$ and the A_∞-$A_\hbar$-$B_\hbar$-bimodule $K_\hbar$ as polynomial deformations w.r.t. $\hbar$, and in particular we may safely consider the value of the parameter $\hbar = 1$.

Therefore, we consider the associative algebras $(A, \star_A)$, the A_∞-algebra $(B, \mu_B + \mathcal{U}_B(\pi))$, with corresponding quantum reduction algebra $(\mathrm{H}^0(B), \star_B)$, and the $(A, \star_A)$-$(\mathrm{H}^0(B), \star_B)$-bimodule $(K, \star_L, \star_R)$. The deformed differential d_B on B^0 is now a differential operator from $\mathrm{S}(\mathfrak{p})$ to $\mathrm{S}(\mathfrak{p}) \otimes \mathfrak{k}^*$ of infinite order, whose action is well-defined. In a similar way, the pairing $\mu_B + \mathcal{U}_B(\pi)^2$ is a well-defined bidifferential operator on $\mathrm{S}(\mathfrak{p})$ of infinite order, and so $\star_L$ and $\star_R$ (we should more precise on $\star_R$, as we should speak of the infinite-order bidifferential operator on $\mathrm{S}(\mathfrak{p})$ coming from the A_∞-A-B-bimodule structure on K).

The deformed product $\star_A$ on $A = \mathrm{S}(\mathfrak{g})$ has been explicitly characterized in [3, 12]. More precisely, we consider the following function on $\mathfrak{g}$,

$$q(x) = \det_{\mathfrak{g}} \left(\frac{\sinh\left(\frac{\mathrm{ad}(x)}{2}\right)}{\frac{\mathrm{ad}(x)}{2}} \right), \tag{18}$$

which is analytic in a neighborhood of 0. It can be expanded in a power series of the polynomials $c_n(x) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(x)^n)$, $n \geq 1$. Alternatively, it may be viewed as an element of the completed symmetric algebra $\widehat{\mathrm{S}}(\mathfrak{g}^*)$ of the dual of $\mathfrak{g}$, and as such, as an invertible, $\mathfrak{g}$-invariant, infinite-order differential operator with constant coefficients acting on A. Similar arguments hold true also if we consider its square root $\sqrt{q}$: we denote by $\partial_{\sqrt{q}}$ the corresponding invertible, $\mathfrak{g}$-invariant, infinite-order differential operator on A. We further denote by β the Poincaré–Birkhoff–Witt (shortly, PBW) isomorphism from $\mathrm{S}(\mathfrak{g})$ to $\mathrm{U}(\mathfrak{g})$, *i.e.* β is the symmetrization morphism

$$\mathrm{S}(\mathfrak{g}) \ni x_1 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \in \mathrm{U}(\mathfrak{g}),\ x_j \in \mathfrak{g},\ j = 1, \dots, n.$$

Then, the deformed product $\star_A$ on $\mathrm{S}(\mathfrak{g})$ is related with the product in $\mathrm{U}(\mathfrak{g})$ *via*

$$\beta\big(\partial_{\sqrt{q}}(f_1) \star_A \partial_{\sqrt{q}}(f_2)\big) = \beta(\partial_{\sqrt{q}}(f_1)) \cdot \beta(\partial_{\sqrt{q}}(f_2)),\ f_i \in A,\ i = 1, 2, \tag{19}$$

and $\cdot$ denotes here the product in $\mathrm{U}(\mathfrak{g})$.

The way the operator $\partial_{\sqrt{q}}$ arises in the framework of deformation quantization has been elucidated in detail in [12, Subsubsections 8.3.1, 8.3.2 and 8.3.3], combining the results therein with [15]. We also refer to [3, Part II] for a complete overview of the applications of deformation quantization as in [12] in Lie theory.

The motivation for the following computations lies in the comparison in Identity (19) between the UEA $\mathrm{U}(\mathfrak{g})$ and the quantum deformed algebra $(A, \star_A)$, which are related precisely by the "strange" automorphism $\partial_{\sqrt{q}}$ of the symmetric algebra $\mathrm{S}(\mathfrak{g})$, which appears also in Duflo's Theorem: namely, the composition $\beta \circ \partial_{\sqrt{q}}$ defines an algebra isomorphism between $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$ and the center of $\mathrm{U}(\mathfrak{g})$. The main point is that Kontsevich's deformed product $\star_A$ contains bidifferential operators, which are represented in terms of wheel-like graphs: such graphs are precisely responsible for the appearance of the "strange" automorphism $\partial_{\sqrt{q}}$.

Quite similarly, in the case of a symmetric pair $(\mathfrak{g}, \sigma)$, we may consider the associative algebra $(\mathrm{H}^0(B), \star_B)$, where $\mathrm{H}^0(B) = \mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$. It is worth observing that Poisson reduction methods yield a Poisson structure on $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ simply by restriction, and the product $\star_B$ defines a deformation quantization of $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ in the sense of Kontsevich. On the other hand, for any choice of a character χ of $\mathfrak{k}$ (*i.e.* a 1-dimensional $\mathfrak{k}$-representation on $\mathbb{K}$), the PBW isomorphism β induces a direct sum decomposition $\mathrm{U}(\mathfrak{g}) = \beta(\mathrm{S}(\mathfrak{p})) \oplus \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{k}^{-\chi}$, where $\mathfrak{k}^{-\chi}$ denotes the affine subspace of $\mathrm{U}(\mathfrak{g})$ spanned by elements of the form $x - \chi(x)$, x in $\mathfrak{k}$. Then, Rouvière defines also a "deformation quantization" of the Poisson algebra $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ *via* the formula

$$\beta(f_1 \# f_2) = \beta(f_1) \cdot \beta(f_2) \text{ modulo } \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{k}^{-\chi},\ f_i \in \mathrm{S}(\mathfrak{p})^{\mathfrak{k}},\ i = 1, 2. \tag{20}$$

The PBW isomorphism (of vector spaces) is obviously $\mathfrak{g}$-invariant, hence it is automatically $\mathfrak{k}$-invariant: therefore, β restricts to a $\mathfrak{k}$-invariant isomorphism of vector spaces from $\mathrm{S}(\mathfrak{p})$ to $\beta(\mathrm{S}(\mathfrak{p})) \subseteq \mathrm{U}(\mathfrak{g})$. In particular, from the above decomposition $\mathrm{U}(\mathfrak{g}) = \beta(\mathrm{S}(\mathfrak{p})) \oplus \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{k}^{-\chi}$, it follows immediately that the right-hand side of Identity (20) defines a unique bilinear pairing $\#$ on $\mathrm{S}(\mathfrak{p})$, which restricts to an associative product on $\mathfrak{k}$-invariant elements. We are now going to compare the products $\star_B$ and $\#$ *via* biquantization techniques. As one could naturally guess from Identity (19), the two products on $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ do not coincide, but are related to each other in a similar fashion, *i.e.* through a "relative" counterpart of Duflo's "strange" automorphism. The novelty of the approach through biquantization is the fact that we use it to compare $\star_B$ on $\mathrm{H}^0(B)$ with $\star_A$ on A; the rest of the proof, *i.e.* the comparison of different automorphisms of $\mathrm{S}(\mathfrak{p})$ similar in shape to Duflo's "strange" automorphism, is really similar to the proof presented in [12, Subsection 3.1] with due modifications.

Of course, the upcoming discussion can be generalized to the framework of some Lie subalgebra $\mathfrak{h}$ of any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$: we will discuss generalizations of the results presented here elsewhere, in particular in relationship with equivalences of categories of representations of Lie algebras and corresponding subalgebras and with the relative Duflo conjecture.

4.1.1. *A version of Duflo's "strange" automorphism for symmetric pairs.* Using the previous notation and conventions, we define the following operator

$$\mathcal{A}(f) = f \star_L 1,\ f \in A = \mathrm{S}(\mathfrak{g}) \tag{21}$$

from A to $K = \mathrm{S}(\mathfrak{p})$. Of course, we could have first defined the operator $\mathcal{A}_\hbar$ from $A_\hbar$ to $K_\hbar$, for $\hbar$ a formal parameter. By its very construction, $\mathcal{A}_\hbar$ is a deformation of the surjective projection from A to K, which we denote by π. By its very construction, $\mathcal{A}_\hbar = \pi \circ \mathbb{A}_\hbar$, where π is extended $\hbar$-linearly to $A_\hbar$, while $\mathbb{A}_\hbar$ is a formal series of differential operators on A, where $\mathbb{A}_0 = \mathrm{id}$, and $\mathbb{A}_n$ (the coefficient of degree n w.r.t. $\hbar$) is a differential operator of order n. In particular, it is clear that $\mathbb{A}_\hbar$ is invertible. By the same arguments as before, we may safely set $\hbar = 1$, and thus we get an invertible differential operator $\mathbb{A}$ on A of infinite order.

We consider $(A, \star_A)$ as a left $(A, \star_A)$-module: then, $\mathcal{A}$ is a surjective morphism of $(A, \star_A)$-modules from $(A, \star_A)$ to $(K, \star_L)$, whence $K \cong A/I$, where $I = \mathrm{Ker}(\mathcal{A})$.

By the very definition of $\mathcal{A}$, $I = A \star_A \mathbb{A}^{-1}(\mathfrak{k})$: in fact, $A \star_A \mathbb{A}^{-1}(\mathfrak{k}) \subseteq I$ by its very construction. To prove the opposite inclusion, we re-introduce momentarily the formal parameter $\hbar$. For $\hbar = 0$, the ideal $I = \langle \mathfrak{k} \rangle$ of A is finitely-generated. In fact, as I is the two-sided ideal of A, viewed here as a commutative algebra, generated by the ideal $\mathfrak{k}$, viewed here as an ideal of linear functions on $X = \mathfrak{g}^*$. In particular, there is a surjective morphism $A^{\oplus \dim \mathfrak{k}} \to I \to 0$ of A-modules: by the previous argument, there is a morphism of left $A_\hbar$-modules $A \star_{A_\hbar} \mathbb{A}_\hbar^{-1}(\mathfrak{k}) \to I_\hbar$, which is a formal $\hbar$-deformation of the surjective morphism $A^{\oplus \dim \mathfrak{h}} \to I$, whence the surjectivity of the deformed morphism follows. Therefore, $I_\hbar = A_\hbar \star_{A_\hbar} \mathbb{A}_\hbar^{-1}(\mathfrak{k})$, whence the claim follows by setting safely $\hbar = 1$.

It remains to compute $\mathbb{A}^{-1}(\mathfrak{k})$.

Writing $\mathbb{A} = \mathrm{id} + \sum_{n\geq 1} \mathbb{A}_n$, $\mathbb{A}_n$, $n \geq 1$, has order n by construction; furthermore, $\mathbb{A}_n$, $n \geq 1$, has no constant term. Namely, $\mathbb{A}_n$ is specified by differential operators associated to admissible graphs in $\mathcal{G}_{n,1}$: recalling from the previous Section the construction of the differential operator associated to Γ admissible of type $(n,1)$, if Γ has no edge pointing to the only vertex of the second type, the corresponding integral weight vanishes by a dimensional argument.

The operator $\mathbb{A}^{-1}$ is completely determined by the power series expansion of $\mathbb{A}$: once again, it is of the form $\mathrm{id} + \sum_{n\geq 1} \widetilde{\mathbb{A}}_n$, where $\widetilde{\mathbb{A}}_n$ has no constant term, for $n \geq 1$, as follows by an easy computation.

We thus compute $\mathbb{A}(k)$, for k a general element of $\mathfrak{k}$. Because of degree reasons, see also [12, Subsubsection 8.3.1], $\mathbb{A}(k) = k + \mathbb{A}_1(k)$. Further, it is readily checked that $\mathbb{A}_1$ is the sum of two differential operators, associated to the following admissible graphs of type $(1,2)$

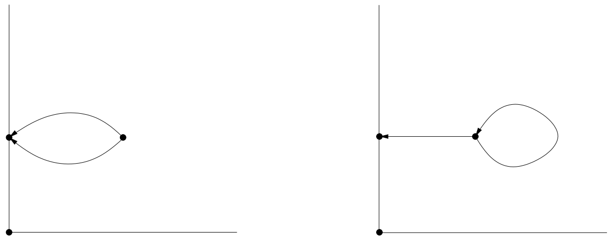

Figure 7 - The only two admissible graphs of type $(1,2)$ appearing in $\mathbb{A}_1$

The contribution of the first graph is trivial, because k is viewed as a linear function on X.

We consider the second graph: we want to observe that such a graph did not appear in the computations perfomed in [7]. First of all, the corresponding integral weight is

$$\int_{\mathcal{C}^+_{1,2}} \rho\omega^{+,-}, \tag{22}$$

omitting wedge products. In fact, in the superpropagator ω_e, only two of the 4-colored propagators are non-trivial, namely $\omega^{+,+}$ and $\omega^{+,-}$ by construction; since it acts as a derivation on an element of $\mathfrak{k}$, by its very definition, the part with $\omega^{+,+}$ vanishes.

Lemma 4.1. *The integral* $\int_{\mathcal{C}^+_{1,2}} \mathrm{d}\eta \wedge \omega^{+,-}$ *equals* $\frac{1}{4}$.

Proof. The integral weight associated to the previous admissible graph is $\int_{\mathcal{C}^+_{1,2}} \mathrm{d}\eta\omega^{+,-}$, where we have suppressed wedge products between the forms in the integrand.

The 1-form ρ is exact, whence

$$\int_{\mathcal{C}^+_{1,2}} \mathrm{d}\eta\omega^{+,-} = \int_{\partial C^+_{1,2}} \eta\omega^{+,-},$$

where we have used notation from Subsection 3.2. Hence, it suffices to compute all boundary contributions to evaluate the integral.

The boundary strata of $\mathcal{C}^+_{1,2}$ are of the type $\mathcal{C}^+_{A,B} \times \mathcal{C}^+_{1\smallsetminus A,[2]\smallsetminus B\sqcup\{\bullet\}}$, for A a subset of [1] and B an ordered subset of [2], such that $0 \leq |A| \leq 1$, $0 \leq |B| \leq 2$ and $|A| + |B| \leq 2$. Dimensional arguments imply that there are only two types of such boundary strata, $\mathcal{C}^+_{0,2} \times \mathcal{C}^+_{1,1}$ and $\mathcal{C}^+_{0,3} \times \mathcal{C}^+_{1,0}$, which correspond to five different situations.

We consider the boundary stratum $\mathcal{C}^+_{0,2} \times \mathcal{C}^+_{1,1}$, which corresponds to the situation where i_1) the point on the positive real axis approaches the origin, ii_1) the point in the interior of the first quadrant collapses to the point on the positive real axis or iii_1) the point 1 approaches the origin. The boundary conditions for $\omega^{-,+}$ yield triviality of the contributions i_1) and iii_1); the second one yields $\frac{1}{4}\int_{\mathcal{C}^+_{1,1}} \omega^+ = \frac{1}{4}$, and we have already included orientation signs.

The boundary stratum $\mathcal{C}^+_{0,3} \times \mathcal{C}^+_{1,0}$ corresponds to the point 1 approaching either the positive imaginary axis or the positive real axis: in the first case, the corresponding contribution vanishes by means of the boundary conditions for $\omega^{+,-}$, while in the second case, the function η vanishes when its argument approaches the real axis. □

Recalling now the construction of the superpropagators in biquantization from Subsubsection 3.3.2, the differential operator corresponding to the second graph in Figure 7 is

$$\frac{1}{4}\left[\mathrm{tr}_\mathfrak{p}(\mathrm{ad}_\mathfrak{k}(\bullet)) - \mathrm{tr}_\mathfrak{k}(\mathrm{ad}_\mathfrak{k}(\bullet))\right] = \delta(\bullet) - \frac{1}{4}\mathrm{tr}_\mathfrak{g}(\mathrm{ad}(\bullet)),$$

whence $\mathbb{A}(k) = k + \delta(k) - \frac{1}{4}\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(k))$.

Therefore, we have

$$\mathbb{A}^{-1}(\mathbb{A}(k)) = k = \mathbb{A}^{-1}(k) + \delta(k) - \frac{1}{4}\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(k)),\ \text{whence}\ \mathbb{A}^{-1}(k) = k - \delta(k) + \frac{1}{4}\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}(k)),\ \ k \in \mathfrak{k}.$$

We observe that we have used the fact that the terms of $\mathbb{A}_{\hbar}^{-1}$ of degree higher or equal than 1 are differential operators without constant term.

Putting all previous arguments together, we have the identification of right $(A, \star_A)$-modules

$$I = A \star_A \mathfrak{k}^{-\delta + \frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ \mathrm{ad}}.$$

Further, we consider the restriction of $\mathcal{A}$ to K, viewed here as a subalgebra of A. First of all, any admissible graph Γ of type $(n, 2)$ yielding a possibly non-trivial contribution to $\mathcal{A}$ has exactly $2n$ edges because of dimensional reasons. To any edge e of Γ corresponds a superpropagator ω_e, whose components are only of type $(+,+)$ or $(+,-)$: it follows immediately from the boundary conditions for both of them that Γ has no edge departing from the only vertex on the positive real axis, and that any edge pointing to this vertex has a corresponding superpropagator with color $(+,+)$. This excludes immediately double edges pointing to the only vertex of Γ on the positive real axis. In particular, this means that any admissible graph Γ of type $(n, 2)$ yielding a non-trivial contribution to $\mathcal{A}$ in the present situation must satisfy the following rule: from any vertex of the first of Γ departs at most one edge pointing to the only vertex on the positive real axis (of course, because of the presence of short loops, the initial and final point of such an edge may coincide). We assume therefore that $p \leq n$ edges have the only vertex of the second type on the positive real axis as endpoint. If $p < n$, there are then $2n - p$ edges, whose endpoints are both vertices of the first type (double edges and short loops are allowed). Since every edge is associated to a derivation, the polynomial degree associated to the vertices of the first type of Γ is $-n + p$ (counting n because of the linearity of the Poisson structure and $p - 2n$ derivations), which is strictly negative, leading to a contradiction. Therefore, form any vertex of the first type of Γ depart exactly one edge to the only vertex of the positive real axis and one to a vertex of the first type; the polynomial degree of the corresponding differential operator on K is immediately 0. Because of the linearity of the Poisson structure, exactly one edge has a vertex of the first type as final point, whence admissible graphs of type $(n, 2)$ contributing non-trivially to $\mathcal{A}$ are disjoint unions of wheel-type graphs; we observe that the 1-wheel may in principle appear. Further, only wheel-like graphs with n even contribute (possibly) non-trivially to $\mathcal{A}$. Namely, by the previous argument, from every vertex of the first type departs exactly one edge to the only vertex of the second type on the positive real axis, whose color is $(+,+)$ and whose operator-valued part corresponds to derivation and contraction w.r.t. $\mathfrak{p}$. The Cartan relation $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ implies that at each vertex of the first type in a wheel-like graph Γ, the edge arriving at such a vertex must have color either $(+,+)$ or $(+,-)$, while the edge departing from it on th wheel must have opposite color. In other words, the edges of the cycle in a wheel-like graph Γ must have alternating colours $(+,+)$ and $(+,-)$: this, in turn, excludes immediately n-wheels with n odd.

Summarizing all previous arguments, the restriction of the operator $\mathcal{A}$ to K, which we denote (improperly) by the same symbol, defines an invertible, translation-invariant differential operator on K. Its symbol, regarded as an element of the completed symmetric algebra $\widehat{\mathrm{S}}(\mathfrak{p})$ and defined through $j_{\mathcal{A}}(x) = e^{-x}\mathcal{A}(e^x)$, has the explicit form

$$j_{\mathcal{A}}(x) = \exp\left(\sum_{n\geq 1} W_{2n}^{\mathcal{A}} \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2n}(x))\right),\ \ x \in \mathfrak{p},$$

where $W_{2n}^{\mathcal{A}}$, $n \geq 1$, denotes the integral weight of following wheel-like graph:

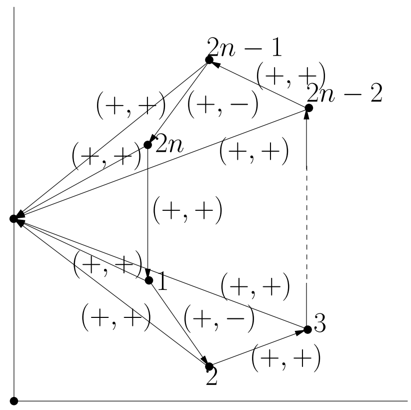

Figure 8 - The wheel-like graph with integral weight $W_{2n}^{\mathcal{A}}$

We observe that $j_{\mathcal{A}}$ is analytic in a neighborhood of 0 in $\mathfrak{p}$. We observe that the Cartan relations for the symmetric pair $(\mathfrak{g},\sigma)$ imply immediately that, for a general element x of $\mathfrak{p}$, $\mathrm{ad}(x)^2$ is a well-defined endomorphism of $\mathfrak{p}$, thus all even powers of the adjoint representation restricted to $\mathfrak{p}$: hence the above expression is well-defined. Finally, all previous computations imply also the direct sum decomposition of $(A,\star_A)$:

$$A = K \oplus (A \star_A \mathfrak{k}^{-\delta+\frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}}).$$

On the other hand, $K = B^0 = \mathrm{S}(\mathfrak{p})$ by definition. We may therefore consider the endomorphism $\mathcal{B}$ of K defined through

$$\mathcal{B}(f) = 1 \star_R f,\ \ f \in K = B^0 = \mathrm{S}(\mathfrak{p}). \tag{23}$$

We may repeat almost *verbatim* the previous arguments to evaluate explicitly the operator $\mathcal{B}$: it is an invertible, translation-invariant differential operator on K of infinite order, with symbol in $\widehat{\mathrm{S}}(\mathfrak{p})$ given by

$$j_{\mathcal{B}}(x) = \exp\left(\sum_{n\geq 1} W_{2n}^{\mathcal{B}}\mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}^{2n}(x))\right),\ x \in \mathfrak{p},$$

where $W_{2n}^{\mathcal{B}}$, $n \geq 1$, denotes the integral weight of following wheel-like graph:

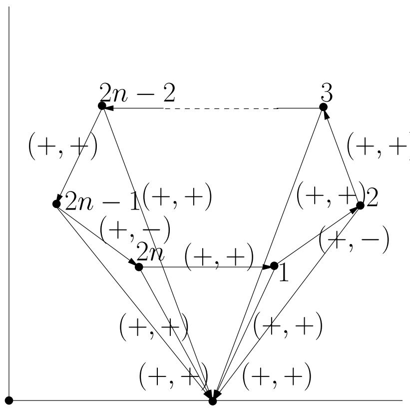

Figure 9 - The wheel-like graph with integral weight $W_{2n}^{\mathcal{B}}$

Once again, notice that $j_{\mathcal{B}}$ is analytic in a neighborhood of 0 in $\mathfrak{p}$.

At this point one could wonder whether or not the integral weights $W_{2n}^{\mathcal{A}}$ and $W_{2n}^{\mathcal{B}}$ coincide (which would imply that $j_{\mathcal{A}} = j_{\mathcal{B}}$). First of all, we observe that in the Formulæ for $j_{\mathcal{A}}$ and $j_{\mathcal{B}}$, only weights of even wheels appear because of the vanishing of the the differential operators corresponding to odd wheel-like graphs. In fact, *e.g.* the 1-wheels $\mathcal{W}_1^{\mathcal{A}}$ and $\mathcal{W}_1^{\mathcal{B}}$ are both computable and yield distinct results.

This can be seen either by computing separately both integral weights or by computing *e.g.* $W_1^{\mathcal{A}}$ and finding then a relationship between $W_1^{\mathcal{A}}$ and $W_1^{\mathcal{B}}$.

First of all, the integral weight $W_1^{\mathcal{A}}$ is explicitly

$$W_1^{\mathcal{A}} = \int_{\mathcal{C}_{1,2}^+} \rho\omega^{+,+}.$$

It can be computed by the same technique used in Lemma 4.1. Of course, there are certain differences to be taken into account, namely, the different boundary conditions satisfied by the 4-colored propagator $\omega^{+,+}$. Here, the only boundary strata of $\mathcal{C}_{1,2}^+ \cong \mathcal{C}_{1,1,0}^+$ which yield non-trivial contributions are *i*) the stratum corresponding to the approach of the point in $Q^{+,+}$ to the only point on $i\mathbb{R}^+$ and *ii*) the approach of the only point on $i\mathbb{R}^+$ to the origin. Both integrals are readily computed, as well as their orientation signs, which then yield the desired result.

We now consider the following admissible graph of type $(2,1)$:

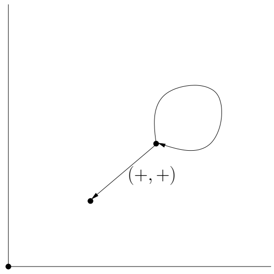

Figure 10 - The "aerial" 1-wheel

It represents a 2-form on the 3-dimensional smooth manifold with corners $\mathcal{C}_{2,1}^+$, therefore, in virtue of Stokes' Theorem,

$$\int_{\mathcal{C}_{2,1}^+} \mathrm{d}(\mathrm{d}\eta\omega^{+,+}) = 0 = \int_{\partial C_{2,1}^+} \mathrm{d}\eta\omega^{+,+}.$$

The boundary strata of codimension 1 of $\mathcal{C}_{2,1}^+$ have been illustrated explicitly in Subsubsection 3.2.2: either because of the boundary conditions for $\omega^{+,+}$ and ρ or because of dimensional reasons, it is not difficult to verify that only three boundary strata yield non-trivial contributions, namely the strata α, θ and ζ.

The boundary stratum θ, resp. ζ, yields precisely $W_1^{\mathcal{B}}$, resp. $W_1^{\mathcal{A}}$; on the other hand, it is not difficult to verify that the boundary stratum α yields the non-trivial contribution $1/4$, as can be verified by a direct computation. Thus, in general, we cannot expect the weights $W_n^{\mathcal{A}}$ and $W_n^{\mathcal{B}}$ to coincide.

4.1.2. *Explicit comparison of the products of Rouvière and Cattaneo–Felder.* The operator $\mathcal{A}$ is surjective, and its restriction to $K = \mathrm{S}(\mathfrak{p}) \subseteq A$ is an automorphism, while $\mathcal{B}$ is an automorphism of K, whence

$$1 \star_R f = \mathcal{B}(f) = \mathcal{A}(\mathcal{A}^{-1}(\mathcal{B}(f))) = \mathcal{A}^{-1}(\mathcal{B}(f)) \star_L 1,\ \ f \in K = \mathrm{S}(\mathfrak{p}).$$

We now recall from Subsubsection 3.4.1 that the quantum reduction algebra (at $\hbar = 1$) $\mathrm{H}^0(B) = \mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$. We thus consider two elements f_i, $i = 1, 2$, of $\mathrm{H}^0(B)$, endowed with the deformed associative product $\star_B$; then $(K, \star_R)$ becomes a right $(\mathrm{H}^0(B), \star_B)$-module, and the latter module structure is compatible with the left $(A, \star_A)$-module structure on $(K, \star_L)$, whence

$$\begin{aligned} &1 \star_R (f_1 \star_B f_2) = \big(\mathcal{A}^{-1}(\mathcal{B}(f_1 \star_B f_2))\big) \star_L 1 = \\ =&(1 \star_R f_1) \star_R f_2 = \big(\mathcal{A}^{-1}(\mathcal{B}(f_1)) \star_L 1\big) \star_R f_2 = \mathcal{A}^{-1}(\mathcal{B}(f_1)) \star_L (1 \star_R f_2) = \mathcal{A}^{-1}(\mathcal{B}(f_1)) \star_L \big(\mathcal{A}^{-1}(\mathcal{B}(f_2)) \star_L 1\big) = \\ =& \big(\mathcal{A}^{-1}(\mathcal{B}(f_1)) \star_A \mathcal{A}^{-1}(\mathcal{B}(f_2))\big) \star_L 1, \end{aligned}$$

whence from the previous computations follows

$$(\mathcal{A}^{-1} \circ \mathcal{B})(f_1 \star_B f_2) = (\mathcal{A}^{-1} \circ \mathcal{B})(f_1) \star_A (\mathcal{A}^{-1} \circ \mathcal{B})(f_2) \text{ modulo } A \star_A \mathfrak{k}^{-\delta + \frac{1}{4}\mathrm{tr}_{\mathfrak{g}} \circ \mathrm{ad}}. \tag{24}$$

We apply the operator $\beta \circ \partial_{\sqrt{q}}$ on both sides of Identity (24) and because of Identity (19), we get

$$(\beta \circ \partial_{\sqrt{q}} \circ \mathcal{A}^{-1} \circ \mathcal{B})(f_1 \star_B f_2) = (\beta \circ \partial_{\sqrt{q}} \circ \mathcal{A}^{-1} \circ \mathcal{B})(f_1) \cdot (\beta \circ \partial_{\sqrt{q}} \circ \mathcal{A}^{-1} \circ \mathcal{B})(f_2) \text{ modulo } \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{k}^{-\delta + \frac{1}{4}\mathrm{tr}_{\mathfrak{g}} \circ \mathrm{ad}},$$

for f_i in $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$, $i = 1, 2$.

We introduce the function J on $\mathfrak{p}$ defined *via*

$$j(x) = \det_{\mathfrak{p}}\left(\frac{\sinh(\mathrm{ad}(x))}{\mathrm{ad}(x)}\right),\ \ x \in \mathfrak{p}.$$

This function is the determinant of the exponential map for the symmetric pair $(\mathfrak{g}, \sigma)$: it can be written as a formal power series of traces in $\mathfrak{p}$ of even powers of the restriction to $\mathfrak{p}$ of the adjoint representation of $\mathfrak{g}$. Similarly to what has been done before, we define $\partial_{\sqrt{j}}$ as the invertible, translation-invariant, $\mathfrak{k}$-invariant differential operator of infinite order on $\mathrm{S}(\mathfrak{p})$, whose symbol is exactly the square root of j.

We define a modified version of the previously introduced Rouvière's product *via*

$$\beta\big(\partial_{\sqrt{j}}(f_1 \# f_2)\big) = \beta(\partial_{\sqrt{j}}(f_1)) \cdot \beta(\partial_{\sqrt{j}}(f_2)) \text{ modulo } \mathrm{U}(\mathfrak{g}) \cdot \mathfrak{k}^{-\delta + \frac{1}{4}\mathrm{tr}_{\mathfrak{g}} \circ \mathrm{ad}}.$$

We observe that the left-hand side of the previous Identity is well-defined, because of the $\mathfrak{k}$-invariance and invertibility of $\partial_{\sqrt{j}}$.

The key point is now the following identity, which puts into relationship the functions q, J, $j_{\mathcal{A}}$ and $j_{\mathcal{B}}$:

$$j_{\mathcal{A}}(x)\sqrt{j(x)} = j_{\mathcal{B}}(x)\sqrt{q(x)},\ \ \forall x \in \mathfrak{p}. \tag{25}$$

The previous identity is the relative version, in the case of a symmetric pair $(\mathfrak{g},\sigma)$, of the results of [12, Subsubsection 8.3.4]. Its proof is a consequence of [7, Lemma 14 and Proposition 12]: it is result which is left unaltered by the changes to biquantization which have been previously discussed.

Theorem 4.2. *For a general symmetric pair* $(\mathfrak{g},\sigma)$*, Rouvière's product* $\#$ *on* $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ *coincides with the product* $\star_B$ *on* $\mathrm{H}^0(B)=\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$, i.e.

$$\beta\big(\partial_{\sqrt{j}}(f_1\star_B f_2)\big)=\beta\big(\partial_{\sqrt{j}}(f_1)\big)\cdot\beta\big(\partial_{\sqrt{j}}(f_2)\big)\ \textit{modulo}\ \mathrm{U}(\mathfrak{g})\cdot\mathfrak{k}^{-\delta+\frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}},$$

for f_i*,* $i=1,2$*, a general element of* $\mathrm{H}^0(B)=\mathrm{S}(\mathfrak{p})^{\mathfrak{g}}$*.*

We observe that the characters δ and $\delta-1/4\ \mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}$ differ precisely by a character of the Lie algebra $\mathfrak{g}$ itself (the trace of its adjoint representation) restricted to a character of the Lie subalgebra $\mathfrak{h}$; as a consequence, $1/4\ \mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}$ yields an $\mathfrak{h}$-equivariant map from $\mathfrak{g}$ to the base field $\mathbb{K}$ (*i.e.* a linear functional on $\mathfrak{g}$ which vanishes on the subspace $[\mathfrak{h},\mathfrak{g}]$), thanks to which we may actual consider only the character δ, instead of the sum $\delta-1/4\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}$.

More precisely, to the symmetric pair $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ we may associate a sub-symmetric pair $\mathfrak{g}_1=\mathfrak{k}_1\oplus\mathfrak{p}$, where $\mathfrak{k}_1=[\mathfrak{p},\mathfrak{p}]$; it is clear that $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$.

The reason is that, to pass from the expression on the right-hand side of the Identity in Theorem 4.2 to the one on the left-hand side, we produce terms which are actually in $\mathrm{U}(\mathfrak{g})\cdot\mathfrak{k}_1^{-\delta+\frac{1}{4}\ \mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}}$, because we reverse two elements in $\mathfrak{p}$, actually producing an element of $\mathfrak{k}_1$ (whose commutator with elements of $\mathfrak{p}$ remains in $\mathfrak{p}$), and it is obvious that the second summand in the character vanishes on $\mathfrak{k}_1$, being a character of $\mathfrak{g}$.

If we now consider a non-trivial character χ of $\mathfrak{k}$, we may associate $X=U_1=\mathfrak{g}^*$, $U_2=\chi+\mathfrak{k}^{\perp}$. Obviously, $A=\mathrm{S}(\mathfrak{g})$, $B^0=K=\mathrm{S}(\mathfrak{g})/\langle\mathfrak{k}^{-\chi}\rangle\cong\mathrm{S}(\mathfrak{p})$, the last isomorphism being induced by an affine morphism.

The arguments of Subsubsection 4.1.1 can be repeated almost *verbatim.* The only relevant difference is that the kernel of the surjective module homomorphism $\mathcal{A}$ from $(A,\star_A)$ to $(K,\star_L)$ is identified with $A\star_A\mathfrak{k}^{-\chi-\delta+\frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}}$. Further, the restriction of $\mathcal{A}$ to B^0 and the operator $\mathcal{B}$ have the same shape as previously.

Still, there is an associative product $\star_B$ on $\mathrm{H}^0(B)\cong\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$: of course, now the product $\star_B$ depends explicitly on the character χ. Identity (24) is consequently modified as

$$(\mathcal{A}^{-1}\circ\mathcal{B})(f_1\star_B f_2)=(\mathcal{A}^{-1}\circ\mathcal{B})(f_1)\star_A(\mathcal{A}^{-1}\circ\mathcal{B})(f_2)\ \text{modulo}\ A\star_A\mathfrak{k}^{-\chi-\delta+\frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}},$$

from which we deduce

$$\beta\big(\partial_{\sqrt{j}}(f_1\star_B f_2)\big)=\beta\big(\partial_{\sqrt{j}}(f_1)\big)\cdot\beta\big(\partial_{\sqrt{j}}(f_2)\big)\ \text{modulo}\ \mathrm{U}(\mathfrak{g})\cdot\mathfrak{k}^{-\chi-\delta+\frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}},$$

for any two $\mathfrak{k}$-invariant elements of $\mathrm{S}(\mathfrak{p})$. Of course, once again, we may safely remove the character $1/4\ \mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}$ in the previous identity.

Remark 4.3. The product $\star_B$ is commutative on $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ because of the symmetry of the function $E(X,Y)$, see [7, Lemma 11], whence the algebra $(\mathrm{U}(\mathfrak{g})/\mathrm{U}(\mathfrak{g})\cdot\mathfrak{k}^{-\delta})^{\mathfrak{k}}$ is also commutative. Now the arguments of [7, Subsubsection 4.2.1] are no longer correct because of the contribution of short loop terms that make the symmetry argument inefficient. Therefore, Theorem 5 in [7], which states the commutativity of $(\mathrm{U}(\mathfrak{g})/\mathrm{U}(\mathfrak{g})\cdot\mathfrak{k}^{-z\delta})^{\mathfrak{k}}$, for any real number z, is also no longer correct.

4.2. **Differential operators expressed *via* exponential coordinates in a symmetric pair.** We consider the triple $X=U_1=\mathfrak{g}^*$ and $U_2=\mathfrak{k}^{\perp}$, for a symmetric pair $(\mathfrak{g},\sigma)$. Therefore, we have two associative algebras $(A,\star_A)$, $(\mathrm{H}^0(B),\star_B)$, and a $(A,\star_A)$=$(\mathrm{H}^0(B),\star_B)$-bimodule K, where $A=\mathrm{S}(\mathfrak{g})$, $\mathrm{H}^0(B)=\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$ and $K=\mathrm{S}(\mathfrak{p})$.

Through the function q on $\mathfrak{g}$, we define the Kashiwara–Vergne density function $D(x,y)$, for (x,y) a general element of $\mathfrak{g}\times\mathfrak{g}$, *via*

$$D(x,y)=\frac{\sqrt{q(x)}\sqrt{q(y)}}{\sqrt{q(\mathrm{BCH}(x,y))}},$$

where the Baker–Campbell–Hausdorff formula $\mathrm{BCH}(x,y)$ is defined by $\exp(x)\exp(y)=\exp(\mathrm{BCH}(x,y))$, for (x,y) in a neighborhood of $(0,0)$. The Kashiwara–Vergne density function has been introduced in to formulate the famous Kashiwara–Vergne conjecture, a general statement for finite-dimensional Lie algebras about deformations of the Baker–Campbell–Hausdorff formula for the product of exponentials in a Lie group: the Kashiwara–Vergne conjecture leads to a proof of Duflo's Theorem about the center of $\mathrm{U}(\mathfrak{g})$ in terms of the $\mathfrak{g}$-invariant symmetric algebra $\mathrm{S}(\mathfrak{g})^{\mathfrak{g}}$. The Kashiwara–Vergne conjecture has been proved in the general case in using deformation quantization techniques to find a suitable deformation of the BCH formula; recently, a different approach to the Kashiwara–Vergne conjecture using Drinfel'd associators has been found, and a relationship between the latter approach and the former *via* deformation quantization has been elucidated. We will also discuss a relative Kashiwara–Vergne conjecture in the framework of symmetric pairs later on.

For x, y general elements of $\mathfrak{p}$, we denote by e^x and e^y the exponential of x and y, viewed as linear functions on $X = \mathfrak{g}^*$. Then, we

$$e^x \star_A e^y = \frac{D(x,y)}{D(P(x,y),K(x,y))} e^{P(x,y)} \star_A e^{K(x,y)}, \quad P = P(x,y) \in \mathfrak{p}, \quad K = K(x,y) \in \mathfrak{k},$$

and the power series P and K are defined *via* the exponential map for symmetric spaces. More precisely, it has been proved that symmetric spaces admit an exponential map, which is a diffeomorphism from a neighborhood of 0 in $\mathfrak{p}$ into its image in G/K, where G is a symmetric pair (in the sense of Lie groups) and K is the fixed point set of an involution σ of G (which is a Lie group automorphism): it is simply the restriction of the exponential map of $\mathfrak{g}$ to right K-cosets in G. For x, y in a sufficiently small neighborhood of 0 in $\mathfrak{p}$, such that $\exp(x)$ and $\exp(y)$ both exist, we have $\exp(x)\exp(x) = \exp(P(x,y))\exp(K(x,y))$.

An important observation at this point is that both P and K, as previously defined, are power series in the free Lie algebra generated by x, y: in particular, the Cartan relations for $(\mathfrak{g},\sigma)$ imply that K is an element of $\mathfrak{k}_1 = [\mathfrak{p},\mathfrak{p}]$.

For x, y in a sufficiently small neighborhood of 0 of $\mathfrak{p}$ as before, we consider the expression $e^{K(x,y)} \star_L 1 = \mathcal{A}(e^{K(x,y)})$. First of all, $\mathcal{A}(e^K)$ is a constant element of $K = \mathrm{S}(\mathfrak{p})$. By its very definition, $K = K(x,y)$ is an element of $\mathfrak{k}_1$: in the computation of a summand $K^n \star_L 1$, only the part of the differential operator acting on K^n of degree n survives, because either of degree reasons or of the fact that the restriction of K as a linear function to $\mathfrak{k}^\perp$ vanishes. Using the involution s of the preceding Subsection together with the computations of Subsection 4.1, it is easy to prove that $K \star_L 1 = \delta(K) - 1/4\,\mathrm{tr}_\mathfrak{g}(\mathrm{ad}(K)) = \delta(K)$, because, as observed before, K belongs to $\mathfrak{k}_1$. We consider, more generally, graphs appearing in the computation of $K^n \star_2 1$, $n \geq 2$. We may actually repeat almost *verbatim* the arguments about the shape of the graphs appearing in $\mathcal{A}$, $\mathcal{B}$: the same arguments imply that short loops may appear only at vertices of the first type, which are linked to the only vertex of the second type on the positive real axis corresponding to K^n through a single edge, while more complicated graphs are wheels. The Cartan relations for the symmetric pair $(\mathfrak{g},\sigma)$ imply that the rays of such wheels are colored by $(+,-)$, while the wheel itself has all edges either colored by $(+,+)$ or $(+,-)$. In particular, it follows that $\mathcal{A}(e^K)$ consists of an infinite series of wheels, where the short loop graph is considered as the 1-wheel: in this case, the 1-wheel contribution appears explicitly.

Lemma 4.4. *For x a general element of $\mathfrak{k}$, the function $\mathcal{A}(e^x) = e^x \star_L 1$ satisfies*

$$\mathcal{A}(e^x) = \sqrt{q(x)}\, e^{\delta(x) - \frac{1}{4}\mathrm{tr}_\mathfrak{g}(\mathrm{ad}(x))};$$

in particular, if x belongs to the Lie subalgebra $\mathfrak{k}_1$, the exponent on the right-hand side of the previous equality simplifies to $\delta(x)$.

Proof. To compute $\mathcal{A}(e^x)$, for x in $\mathfrak{k}$, it suffices to replace K in the previous computations by x.

The rest of the proof follows along the same lines of the proof of [7, Lemma 15], but we have to observe now that

$$\mathcal{A}(e^x) = e^x \star_L 1 = \exp\left(\delta(x) - \frac{1}{4}\mathrm{tr}_\mathfrak{g}(\mathrm{ad}(x)) + \sum_{n\geq 2} w_n^\mathfrak{k}\,\mathrm{tr}_\mathfrak{k}(\mathrm{ad}(x)^n) + \sum_{n\geq 2} w_n^\mathfrak{p}\,\mathrm{tr}_\mathfrak{p}(\mathrm{ad}(x)^n)\right),$$

for certain integral weights $w_n^\mathfrak{k}$ and $w_n^\mathfrak{p}$. More precisely, such weights are associated to the colored wheel-like graphs

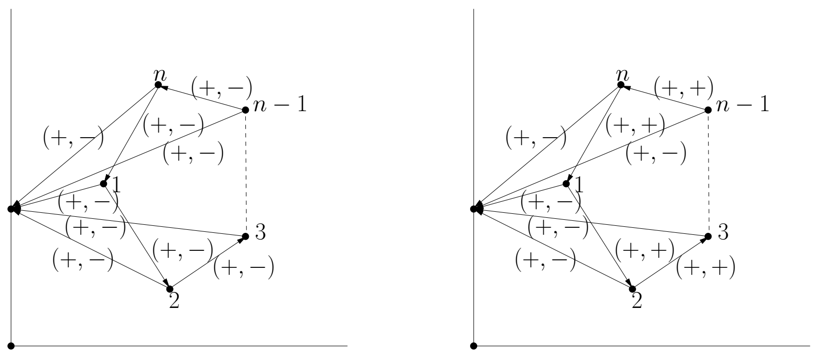

Figure 11 - The possible wheel-like graphs appearing in $\mathcal{A}(e^x)$, for x in $\mathfrak{k}$

Further, also $\sqrt{q(x)}$ can be written in a similar form, with the only difference that it does not contain terms proportional to the trace of the adjoint representations of $\mathfrak{k}$ on itself or on $\mathfrak{p}$.

Therefore, the very same computations of [7, Lemma 15] imply the above identity. □

We observe that we may get rid of the factor $e^{\delta-\frac{1}{4}\mathrm{tr}_{\mathfrak{g}}\circ\mathrm{ad}}$, viewed as an element of the completion $\widehat{\mathrm{S}}(\mathfrak{k})$ simply by applying the biquantization techniques to the modified triple $X = U_1 = \mathfrak{g}$, $U_2 = -\delta + 1/4\ \mathrm{tr}_{\mathfrak{g}} \circ \mathrm{ad} + \mathfrak{k}^{\perp}$. To the previous triple are associated two associative algebra $(A, \star_A)$ and $(\mathrm{H}^0(B), \star_B)$ and a bimodule K, where $A = \mathrm{S}(\mathfrak{g})$, $B^0 = K = \mathrm{S}(\mathfrak{p})$, and $H^0(B) = \mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$. Notice that the product $\star_B$ on $\mathrm{H}^0(B)$, for B defined by the modified triple, does not coincide with $\star_B$ for the initial triple; similarly, the operator $\mathcal{A}$ on A for the modified triple also does not coincide with the operator $\mathcal{A}$ for the initial triple, as their kernels do not obviously coincide. Still, both their restrictions to B^0 coincide, thus also their symbols, and similarly for $\mathcal{B}$. On the other hand, as already observed, for the above modified triple we have the identity

$$\mathcal{A}(e^x) = e^x \star_L 1 = \sqrt{q(x)},\ x \in \mathfrak{k}.$$

We consider the following expression, using the notation from above,

$$\begin{aligned}(e^x \star_A e^y) \star_L 1 &= \frac{D(x,y)}{D(P,K)}\left(e^P \star_A e^K\right) \star_L 1 = \frac{\sqrt{q(x)}\sqrt{q(y)}}{\sqrt{q(P)}\sqrt{q(K)}} e^P \star_L (e^K \star_L 1) = \frac{\sqrt{q(x)}\sqrt{q(y)}}{\sqrt{q(P)}} j_{\mathcal{A}}(P) e^P = \\ &= \frac{\sqrt{q(x)}\sqrt{q(y)}}{\sqrt{j(P)}} j_{\mathcal{B}}(P) e^P = \\ &= e^x \star_L (e^y \star_L 1) = j_{\mathcal{A}}(y) e^x \star_L e^y = \frac{j_{\mathcal{A}}(y)}{j_{\mathcal{B}}(y)} e^x \star_L (1 \star_R e^y),\ x, y \in \mathfrak{p}.\end{aligned}$$

Comparing the expression on the second line with the rightmost expression on the third line, we find

$$e^x \star_L (1 \star_R e^y) = \frac{\sqrt{q(x)}\sqrt{j(y)} j_{\mathcal{B}}(P)}{\sqrt{j(P)}} e^P$$

We may view the expressions on both sides of the previous identity as analytic functions on a sufficiently small neighborhood of $(0,0)$ in $\mathfrak{p} \times \mathfrak{p}$.

We consider further an element T of $\mathrm{H}^0(B) = \mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$, for which we obtain

$$\begin{aligned}e^x \star_L (1 \star_R T) &= (e^x \star_L 1) \star_R T = j_{\mathcal{A}}(x) e^x \star_R T = \\ &= T_y(e^x \star_L (1 \star_R e^y))\,|_{y=0} = T_y\left(\frac{\sqrt{q(x)}\sqrt{j(y)} j_{\mathcal{B}}(P)}{\sqrt{j(P)}} e^P\right)\Bigg|_{y=0},\end{aligned}$$

where we regard in the second line T as a differential operator on $\mathfrak{p}^*$ w.r.t. the variable y.

Therefore, using Identity (25), we get the following expression,

$$e^x \star_R T = T_y\left(\frac{\sqrt{j(x)}\sqrt{j(y)} j_{\mathcal{B}}(P)}{\sqrt{j(P)} j_{\mathcal{B}}(P)} e^P\right)\Bigg|_{y=0},$$

whose right-hand side is, according to the arguments of [14, Section 6]. precisely the differential operator $\beta(\partial_{sqrtj}(T))$ expressed *via* exponential coordinates on the symmetric space G/K, for G, K connected, simply connected Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ respectively, up to a modification by the analytic function $\sqrt{j}/j_{\mathcal{B}}$ on $\mathfrak{p}$. A deep consequence of the fact that the product $\star_B$ coincides with Rouvière's product, together with a result about the existence of polarizations compatible with the structure of symmetric pair, for which we refer to [16, 17], implies that the symbol $j_{\mathcal{B}}$ is constant and thus equal to 1.

Therefore, the expression $e^x \star_L T$, for T in $\mathrm{S}(\mathfrak{p})^{\mathfrak{k}}$, viewed here as an element of K, truly expresses in terms of biquantization the differential operator T through exponential coordinates on G/K.

4.3. **Deformation of the Baker–Campbell–Hausdorff formula for symmetric pairs.** As already briefly remarked in the previous Subsection, the problem of deforming the BCH formula and the BCH density function D (see above) for a finite-dimensional Lie algebra $\mathfrak{g}$ is related to Duflo's Theorem *via* the Kashiwara–Vergne conjecture. As the KV conjecture relies on the exponential map on Lie algebras, it seems natural to formulate a similar conjecture for a general symmetric pair, because also symmetric pairs admit an exponential map. We refer to [14] for more details on the KV conjecture for symmetric pairs.

The exponential map for a symmetric pair $(\mathfrak{g}, \sigma)$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a well-defined diffeomorphism $\exp_{G/K}$ from a sufficiently small neighborhood of 0 in $\mathfrak{p}$ to its image in G/K, for G, K as in the previous Subsection: it is induced from the exponential map $\exp_G$ of $\mathfrak{g}$. Standard manipulations, see [14, Section 2] for more details, imply that the exponential map for the symmetric pair $(\mathfrak{g}, \sigma)$ yields a BCH formula $\mathrm{BCH}_{\mathfrak{p}}$ *via*

$$\exp_G(2\mathrm{BCH}_{\mathfrak{p}}(x,y)) = \exp_G(x)\exp_G(2y)\exp_G(x),$$

for (x,y) in a sufficiently small neighborhood of $(0,0)$ in $\mathfrak{p} \times \mathfrak{p}$. The element $\mathrm{BCH}_{\mathfrak{p}}(x,y)$ belongs to $\mathfrak{p}$, and is expressible in terms of even iterated brackets of x and y.

The KV conjecture for symmetric pairs expresses a deformation of the function $\mathrm{BCH}_\mathfrak{p}$ in terms of $\mathfrak{k}$-adjoint vector fields on $\mathfrak{p}\times\mathfrak{p}$; we refer once again to [14] for a complete introduction to this issue and for a discussion of some consequences. There is also another claim, which expresses the corresponding variation in terms of the said $\mathfrak{k}$-adjoint vector fields of the density function for the symmetric pair, which is defined as

$$D_\mathfrak{p}(x,y)=\frac{\sqrt{j(x)}\sqrt{j(y)}}{\sqrt{j(\mathrm{BCH}_\mathfrak{p}(x,y))}},$$

again for (x,y) in a sufficiently small neighborhood of $(0,0)$, where all functions make sense. The deformation of $D_\mathfrak{p}$ contains traces of the adjoint $\mathfrak{k}$-action on $\mathfrak{p}$.

We are now going to illustrate how biquantization techniques yield two different deformations of $\mathrm{BCH}_\mathfrak{p}$ and $D_\mathfrak{p}$ in the sense elucidated above. These two deformations can be characterized in terms of the A_∞-structures on A, B and K through the quadratic relations between the corresponding Taylor coefficients: this is not immediately recognizable from the approach we take, which is in turn essentially motivated by the results of [1,18].

We consider the triple $X=U_1=\mathfrak{g}^*$ and $U_2=\delta-1/4\,\mathrm{tr}_\mathfrak{g}\circ\mathrm{ad}+\mathfrak{k}^\perp$ and the corresponding algebras A, $\mathrm{H}^0(B)$ and A-$\mathrm{H}^0(B)$-bimodule K.

4.3.1. *First deformation.* First of all, we may consider, for (x,y) in a sufficiently small neighborhood of $(0,0)$ in $\mathfrak{p}\times\mathfrak{p}$, where $\mathrm{BCH}_\mathfrak{p}$ and $D_\mathfrak{p}$, as well as $\sqrt{q}$, $\sqrt{j}$, $j_\mathcal{A}$ and $j_\mathcal{B}$ are well-defined, the function

$$\begin{aligned}(e^x\star_A e^y)\star_L 1&=\frac{\sqrt{q(x)}\sqrt{q(y)}}{\sqrt{j(\mathrm{BCH}_\mathfrak{p}(x,y))}}e^{\mathrm{BCH}_\mathfrak{p}(x,y)}=\\&=e^x\star_L(e^y\star_L 1)=j_\mathcal{A}(y)e^x\star_L e^y,\end{aligned}$$

where we have used results of the previous Subsection, setting $P(x,y)=\mathrm{BCH}_\mathfrak{p}(x,y)$.

On the other hand, we may also consider

$$e^x\star_L(1\star_R e^y)=j_\mathcal{B}(y)e^x\star_L e^y=e^x\star_L e^y,$$

where we have used once again the aforementioned fact that $j_\mathcal{B}\equiv 1$, whence, recalling Identity (25),

$$e^x\star_L e^y=\frac{\sqrt{q(x)}\sqrt{j(y)}}{\sqrt{j(\mathrm{BCH}_\mathfrak{p}(x,y))}}e^{\mathrm{BCH}_\mathfrak{p}(x,y)}$$

Finally, we also have

$$(e^x\star_L 1)\star_R e^y=j_\mathcal{A}(x)e^x\star_R e^y.$$

The problem is therefore, how to relate $(e^x\star_L 1)\star_R e^y$ with $e^x\star_L(1\star_R e^y)$ in order to draw a bridge between the previous two formulæ: the two expressions do not coincide, as both actions are compatible to each other only in cohomology. But the A_∞-nature of B and of the bimodule K permit to control explicitly the failure for the compatibility between $\star_L$ and $\star_R$: in facts, we find

$$(e^x\star_L 1)\star_R e^y-e^x\star_L(1\star_R e^y)=\mathrm{d}_K^{1,1}(e^x,1,\mathrm{d}_B(e^y))=\mathrm{d}_K^{1,1}(e^x,1,e^y\mathrm{d}_B(y)) \tag{26}$$

where d_B is the Chevalley–Eilenberg differential on the complex $B=\mathrm{S}(\mathfrak{p})\otimes\wedge(\mathfrak{k}^*)$, and $\mathrm{d}_K^{1,1}$ denotes the $(1,1)$-Taylor component of the A_∞-bimodule structure on K. More precisely,

$$\mathrm{d}_K^{1,1}(a_1|k|b_1)=\sum_{n\geq 0}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}_{n,3}}\mu_{n+3}^K\left(\int_{\mathcal{C}^+_{n,3}}\prod_{e\in E(\Gamma)}\omega_e^K(\underbrace{\pi|\cdots|\pi}_{n}|a_1|k|b_1)\right),$$

and dimensional arguments imply that b_1 must be an element of B^1, otherwise the previous expression is trivial.

Identity (26) may be derived in a slightly different way, which makes apparent the fact that it embodies the KV deformation problem sketched at the beginning of the present Subsection. Namely, for $n\geq 1$, we consider the forgetful projection $\pi_{n,1,1}$ from $\mathcal{C}^+_{n,1,1}$ to $\mathcal{C}^+_{0,1,1}$: in this situation, we prefer to consider the compactified configuration spaces $\mathcal{C}^+_{n,1,1}$ to highlight the fact that we consider the functions e^x, 1 and e^y to be put on the positive imaginary axis, at the origin and on the positive real axis respectively. We observe that the fiber of $\pi_{n,1,1}$ at a generic point of $\mathcal{C}^+_{0,1,1}$ is an orientable compact smooth manifold with corners of dimension $2n$, hence we may consider the push-forward (or integration along the fiber) $\pi_{n,1,1,*}$ w.r.t. $\pi_{n,1,1}$.

In particular, we may consider the expression

$$\sum_{n\geq 0}\frac{1}{n!}\sum_{\Gamma\in\mathcal{G}_{n,3}}\mu_{n+3}^K\left(\pi_{n,1,1,*}\left(\prod_{e\in E(\Gamma)}\omega_e^K\right)(\underbrace{\pi|\cdots|\pi}_{n}|e^x|1|e^y)\right),$$

for (x,y) as above.

First of all, for (x,y) as above, the previous expression is a smooth function on $\mathcal{C}^+_{0,1,1}$: it is a consequence of the fact that, for $n \geq 1$, the push-forward $\pi_{n,1,1,*}$ selects the piece of the integrand of (form) degree bigger or equal than $2n$. To the three vertices of the second type are associated functions, while to each vertex of the first type of an admissible graph Γ of type $(n,3)$ is associated a copy of the linear Poisson bivector π: as a consequence, the form degree of each integrand must be precisely $2n$, whence the first claim. We may further divide the previous expression by $j_{\mathcal{A}}(x)$: this "normalization" is due to previous computations, and it does not affect the following computations.

Because of the fact that π is a linear Poisson bivector, we may use the arguments in [3, Chapter 2] or [10] to prove that the "normalized" function on $\mathcal{C}^+_{0,1,1}$ given by the previous expression can be re-written in the form

$$D^1_{\mathfrak{p}}(x,y)e^{\mathrm{BCH}^1_{\mathfrak{p}}(x,y)}, \tag{27}$$

where the exponent $\mathrm{BCH}^1_{\mathfrak{p}}(x,y)$ is a smooth $\mathfrak{p}$-valued function on $\mathcal{C}^+_{0,1,1}$, corresponding to the connected graphs of Lie type, whose unique root is in $\mathfrak{p}$, while $D^1_{\mathfrak{p}}(x,y)$ is a smooth $\mathbb{K}$-valued function on $\mathcal{C}^+_{0,1,1}$, corresponding to graphs of Lie type with roots in $\mathfrak{k}$ and wheel-like graphs (possibly with graphs of type Lie attached to their spokes). Both $\mathrm{BCH}^1_{\mathfrak{p}}(x,y)$ and $D^1_{\mathfrak{p}}(x,y)$ are weighted sums over the graphs highlighted right above, where now the corresponding integral weights are smooth functions on $\mathcal{C}^+_{0,1,1}$.

The deformation formulæ we are interested into can be therefore computed by taking the exterior derivative of (27) as a function on $\mathcal{C}^+_{0,1,1}$: we may therefore apply the generalized Stokes Theorem for integration along the fiber of $\pi_{n,1,1}$ in the first integral formula for (27). For any admissible graph Γ of type $(n,3)$ as above, the corresponding integrand is a closed form, whence it suffices to consider the corresponding integral along the boundary strata of codimension 1 of the generic fiber. For $n \geq 1$, a general boundary stratum of codimension 1 in the generic fiber corresponds either *i)* to the collapse of points in $Q^{+,+}$ labeled by a subset A of $[n]$ of cardinality $2 \leq |A| \leq n$ to a single point in $Q^{+,+}$, or *ii)* to the approach of points in $Q^{+,+}$ labeled by a subset A of $[n]$ of cardinality $0 \leq |A| \leq n$ to $i\mathbb{R}^+$, to the origin or to $\mathbb{R}^+$. Notice that no boundary stratum appears, where either the point on $i\mathbb{R}^+$ or $\mathbb{R}^+$ approaches the origin (this is because we are considering the boundary strata of codimension 1 of the generic fiber).

Standard dimensional arguments, Kontsevich's Vanishing Lemma [12, Lemma] and the boundary conditions for the 4-colored propagators $\omega^{+,+}$ and $\omega^{+,-}$ imply that the only boundary strata yielding non-trivial contributions correspond to boundary strata of type *ii)*, where points in $Q^{+,+}$ labeled by $A \subseteq [n]$ approach the point on $\mathbb{R}^+$. We refer to [18] or [3, Chapter 2] for similar computations.

The sum over all such contributions yields a smooth $\widehat{\mathrm{S}}(\mathfrak{p})$-valued 1-form on $\mathcal{C}^+_{0,1,1}$, depending on (x,y) as above, whose integral over $\mathcal{C}^+_{0,1,1}$ identifies with the right-hand side of Identity (26).

The generalized Stokes Theorem, see *e.g.* the computations in [18], implies that the previous 1-form specifies a smooth $\mathfrak{k}$-valued 1-form $\omega_1(x,y)$ satisfying the identities

$$\begin{aligned}\mathrm{dBCH}^1_{\mathfrak{p}}(x,y) &= \langle [y,\omega_1(x,y)]\,,\partial_y \mathrm{BCH}^1_{\mathfrak{p}}(x,y)\rangle,\\ \mathrm{d}D^1_{\mathfrak{p}}(x,y) &= \langle [y,\omega_1(x,y)]\,,\partial_y D^1_{\mathfrak{p}}(x,y)\rangle + \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}(y)\partial_y\omega_1(x,y))\,D^1_{\mathfrak{p}}(x,y),\end{aligned}$$

where we have used the notation

$$\langle [y,\omega_1(x,y)],\partial_y \mathrm{BCH}^1_{\mathfrak{p}}(x,y)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{BCH}^1_{\mathfrak{p}}(x,y+t[y,\omega_1(x,y)])\big|_{t=0},$$

and analogously for other similar expressions in the previous identities.

Finally, we consider the function (27) on $\mathcal{C}^+_{0,1,1}$, which is a smooth, compact 1-dimensional manifold with corners: its two boundary strata of codimension 1 correspond to the approach of the point either on $i\mathbb{R}^+$ or on $\mathbb{R}^+$ to the origin. If we choose the smooth section of $\mathcal{C}^+_{0,1,1}$ which corresponds to fixing the point on $i\mathbb{R}^+$ to i, then $\mathcal{C}^+_{0,1,1} \cong [0,\infty]$: the boundary point $\{0\}$, resp. $\{\infty\}$, corresponds to the approach of the point on $\mathbb{R}^+$, resp. on $i\mathbb{R}^+$, to the origin. Taking into account the "normalization" w.r.t. $j_{\mathcal{A}}(x)$ in the function (27) and the computations at the beginning of the Subsubsection, the value of the function (27) at 0 yields the values of both $\mathrm{BCH}^1_{\mathfrak{p}}$ and $D^1_{\mathfrak{p}}(x,y)$ at 0, which are precisely $\mathrm{BCH}_{\mathfrak{p}}(x,y)$ and $D_{\mathfrak{p}}(x,y)$, whence the function (27) produces a genuine deformation of the BCH formula and corresponding density for the symmetric pair $(\mathfrak{g},\sigma)$. The value of the "normalized" function (27) at ∞ is also readily computed, namely it is simply $e^x \star_R e^y$.

4.3.2. *Second deformation.* We begin by considering, for (x,y) as in the previous Subsubsection, the two expressions $(1 \star_R e^x) \star_R e^y$ and $1 \star_R (e^x \star_B e^y)$. We observe that neither the pairing $\star_B$ is associative, nor it is compatible with the pairing $\star_R$ in both expressions. Since, as observe before, $j_{\mathcal{B}} \equiv 1$, the first expression equals $e^x \star_R e^y$, while the second equals $e^x \star_B e^y$.

The fact that $\star_R$ and $\star_B$ are the $(0,1)$- and 2-Taylor components of the A_∞-structures on K and on B, we get

$$(1 \star_R e^x) \star_R e^y - 1 \star_R (e^x \star_B e^y) = e^x \star_R e^y - e^x \star_B e^y = \mathrm{d}_K^{1,1}(1|\mathrm{d}_B(e^x)|e^y) + \mathrm{d}_K^{1,1}(1|e^x|\mathrm{d}_B(e^y)). \tag{28}$$

We observe that, due to the choice of the triple $X = U_1 = \mathfrak{g}^*$ and $U_2 = \delta - 1/4\ \mathrm{tr}_\mathfrak{g} \circ \mathrm{ad} + \mathfrak{k}^\perp$, the pairing $\star_B$ depends on the (natural) character $\delta - 1/4\ \mathrm{tr}_\mathfrak{g} \circ \mathrm{ad}$.

We now define, for (x,y) as above, a smooth function on $\mathcal{C}^+_{0,0,2}$ *via*

$$\sum_{n\geq 0} \frac{1}{n!} \sum_{\Gamma \in \mathcal{G}_{n,3}} \mu^K_{n+3}\left(\pi_{n,0,2,*}\left(\prod_{e\in E(\Gamma)} \omega^K_e\right)(\underbrace{\pi|\cdots|\pi}_{n}|1|e^x|e^y)\right),$$

where, for $n \geq 1$, $\pi_{n,0,2}$ is the forgetful projection from $\mathcal{C}^+_{n,0,2}$ onto $\mathcal{C}^+_{0,0,2}$, and $\pi_{n,0,2,*}$ denotes the push-forward w.r.t. $\pi_{n,0,2}$. Again, because of the fact that π is a linear Poisson bivector, the arguments of [3, Chapter 2] or [10] yield an explicit expression for the previous function on $\mathcal{C}^+_{0,0,2}$ in the shape

$$D^2_\mathfrak{p}(x,y) e^{\mathrm{BCH}^2_\mathfrak{p}(x,y)}. \tag{29}$$

The $\mathfrak{p}$-valued function $\mathrm{BCH}^2_\mathfrak{p}(x,y)$ and the $\mathbb{K}$-valued function $D^2_\mathfrak{p}(x,y)$ are as in Formula (27), with due modifications in the integral weights.

As in the previous Subsubsection, the exterior derivative of the function (29) may be computed by means of the generalized Stokes Theorem in a way similar to the sketch of the computation of the exterior derivative of the function (27). The relevant contributions come from the boundary strata of codimension 1 of the generic fiber of $\pi_{n,0,2}$, for $n \geq 1$: such strata correspond to either *i*) the collapse of points in $Q^{+,+}$ labeled by a subset A of $[n]$ of cardinality $2 \leq |A| \leq n$ to a single point in $Q^{+,+}$, or *ii*) the approach of points in $Q^{+,+}$ labeled by $A \subseteq [n]$, $0 \leq |A| \leq n$, to $i\mathbb{R}^+$, the origin, or $\mathbb{R}^+$. As we are considering a generic fiber, we do not consider strata, where the two points on $\mathbb{R}^+$ approach to each other or where the first point on $\mathbb{R}^+$ approaches the origin. Standard arguments imply that the only non-trivial contributions come from boundary strata of type *ii*), corresponding to the approach of points in $Q^{+,+}$ either to the first or to the second point on $\mathbb{R}^+$.

More precisely, the sum over all such contributions yields a smooth $\widehat{\mathrm{S}}(\mathfrak{p})$-valued 1-form on $\mathcal{C}^+_{0,0,2}$, whose integral over $\mathcal{C}^+_{0,0,2}$ is precisely the rightmost expression in the chain of identities (28). The previous 1-form yields in turn a $\mathfrak{k}\times\mathfrak{k}$-valued 1-form on $\mathcal{C}^+_{0,0,2}$ $\omega_2(x,y) = (\omega_2^1(x,y), \omega_2^2(x,y))$, which obeys the identities

$$\begin{aligned} \mathrm{dBCH}^2_\mathfrak{p}(x,y) &= \langle [x, \omega_2^1(x,y)], \partial_x \mathrm{BCH}^2_\mathfrak{p}(x,y)\rangle + \langle [y, \omega_2^2(x,y)], \partial_y \mathrm{BCH}^2_\mathfrak{p}(x,y)\rangle, \\ \mathrm{d}D^1_\mathfrak{p}(x,y) &= \langle [x, \omega_2^1(x,y)], \partial_x D^2_\mathfrak{p}(x,y)\rangle + \langle [y, \omega_2^2(x,y)], \partial_y D^2_\mathfrak{p}(x,y)\rangle + \\ &\quad + \mathrm{tr}_\mathfrak{p}\left(\mathrm{ad}(x)\partial_x\omega_2^1(x,y) + \mathrm{ad}(y)\partial_y\omega_2^2(x,y)\right) D^2_\mathfrak{p}(x,y). \end{aligned}$$

The previous results imply due modifications of the results of [7, Subsubsection 4.4.3]: we observe that changes are caused by the fact that we have chosen at the beginning the modified triple $X = U_1 = \mathfrak{g}^*$ and $U_2 = \delta - 1/4\ \mathrm{tr}_\mathfrak{g} \circ \mathrm{ad} + \mathfrak{k}^\perp$, thus introducing in many formulæ the (natural) character $\delta - 1/4\ \mathrm{tr}_\mathfrak{g} \circ \mathrm{ad}$ of $\mathfrak{k}$.

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