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Cyclotomic analogues of finite multiple zeta values

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Abstract

We introduce the notion of finite multiple harmonic q-series at a primitive root of unity and show that these specialize to the finite multiple zeta value (FMZV) and the symmetrized multiple zeta value (SMZV) through an algebraic and analytic operation, respectively. Further, we obtain families of linear relations among these series which induce linear relations among FMZVs and SMZVs of the same form. This gives evidence towards a conjecture of Kaneko and Zagier relating FMZVs and SMZVs. Motivated by the above results, we define cyclotomic analogues of FMZVs, which conjecturally generate a vector space of the same dimension as that spanned by the finite multiple harmonic q-series at a primitive root of unity of sufficiently large degree.

1 Introduction

The purpose of this paper is to describe a connection between finite and symmetrized multiple zeta values. We explicate this connection in terms of a class of q-series evaluated at primitive roots of unity. This construction provides new evidence and a re-interpretation of a conjecture due to Kaneko and Zagier, thus relating finite and symmetrized multiple zeta values in an explicit and surprising way.

For an index $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$ with $k_1 \geq 2$ the *multiple zeta value* (MZV) is defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

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We call the number $\mathrm{wt}(\mathbf{k}) = \sum_{j=1}^{r} k_j$ the *weight* of $\mathbf{k}$, r the *depth* of $\mathbf{k}$ and by $\mathcal{Z}$ we denote the $\mathbb{Q}$-vector space spanned by all such MZVs; this is a subalgebra of $\mathbb{R}$ over $\mathbb{Q}$. It is known that MZVs of the same weight satisfy numerous $\mathbb{Q}$-linear relations. The simplest example is $\zeta(2,1) = \zeta(3)$ in weight three.

In [9], Kaneko and Zagier introduced the *finite multiple zeta value* (FMZV) $\zeta_{\mathcal{A}}(\mathbf{k})$, defined by collecting the values

$$\sum_{p>m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \mod p \tag{1.1}$$

for all primes p, as an element of the $\mathbb{Q}$-algebra $\mathcal{A} = (\prod_p \mathbb{F}_p)/(\bigoplus_p \mathbb{F}_p)$. Let $\mathcal{Z}_{\mathcal{A}}$ be the $\mathbb{Q}$-vector space spanned by all FMZVs, which is also a subalgebra of $\mathcal{A}$ over $\mathbb{Q}$. The FMZVs of the same weight also satisfy numerous $\mathbb{Q}$-linear relations, for example

$$2\zeta_{\mathcal{A}}(4,1) + \zeta_{\mathcal{A}}(3,2) = 0, \tag{1.2}$$

which was first obtained by Hoffman [6, Theorem 7.1].

The FMZVs have a conjectural correspondence with certain real numbers $\zeta_{\mathcal{S}}(\mathbf{k})$ lying in the ring $\mathcal{Z}$, which are called the *symmetrized multiple zeta values* (SMZVs). In the work [9] Kaneko and Zagier conjecture that there exists a $\mathbb{Q}$-algebra homomorphism $\varphi_{KZ} : \mathcal{Z}_{\mathcal{A}} \to \mathcal{Z}/\zeta(2)\mathcal{Z}$ such that $\zeta_{\mathcal{A}}(\mathbf{k})$ is mapped to $\zeta_{\mathcal{S}}(\mathbf{k})$ modulo $\zeta(2)\mathcal{Z}$. It is shown by Yasuda [19] that the SMZVs span the space $\mathcal{Z}$. This conjecture would imply that the FMZVs satisfy the same $\mathbb{Q}$-linear relation as the SMZVs modulo $\zeta(2)\mathcal{Z}$ and vice versa. Although it still seems hard to prove this conjecture, a few families of relations which are satisfied by the FMZVs and the SMZVs simultaneously are obtained by Murahara, Saito and Wakabayashi in [11, 15].

In the present paper, we give several linear relations satisfied both by the FMZVs and the SMZVs. This is achieved by examining the value of the finite multiple harmonic q-series

$$z_n(\mathbf{k};q) = z_n(k_1,\ldots,k_r;q) = \sum_{n>m_1>\cdots>m_r>0} \frac{q^{(k_1-1)m_1}\ldots q^{(k_r-1)m_r}}{[m_1]_q^{k_1}\ldots[m_r]_q^{k_r}} \quad (k_1,\ldots,k_r \geq 1)$$

at a n-th primitive root of unity ζ_n, where $[m]_q = (1-q^m)/(1-q)$ is the usual q-integer. We also consider the star version $z_n^\star(\mathbf{k};q)$ which is defined by allowing equality among the m_i's in the above sum. One of the main results of this paper is that the values $z_n(\mathbf{k};\zeta_n)$ and $z_n^\star(\mathbf{k};\zeta_n)$ have a natural connection with both the FMZVs and the SMZVs as follows.

The FMZVs are naturally obtained by collecting the values $z_p(\mathbf{k};\zeta_p)$ for all prime p. Note that the values $z_p(\mathbf{k};\zeta_p)$ and $z_p^\star(\mathbf{k};\zeta_p)$, for p prime, belong to the integer ring

$\mathbb{Z}[\zeta_p]$ of the cyclotomic field $\mathbb{Q}(\zeta_p)$ because the q-integer $[m]_q$ at $q=\zeta_p$ is a cyclotomic unit for $p>m>0$. Let $\mathfrak{p}_p=(1-\zeta_p)$ be the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1-\zeta_p$. Note that $\mathbb{Z}[\zeta_p]/\mathfrak{p}_p=\mathbb{F}_p$.

Theorem 1.1. *For any index* $\mathbf{k}\in(\mathbb{Z}_{\geq 1})^r$*, we have*

$$(z_p(\mathbf{k};\zeta_p) \mod \mathfrak{p}_p)_p=\zeta_{\mathcal{A}}(\mathbf{k}), \qquad (z_p^\star(\mathbf{k};\zeta_p) \mod \mathfrak{p}_p)_p=\zeta_{\mathcal{A}}^\star(\mathbf{k}),$$

with $\zeta_{\mathcal{A}}^\star(\mathbf{k})$ *being the finite multiple zeta star value defined by* (1.1) *where the condition* $p>m_1>\cdots>m_r>0$ *is replaced with* $p>m_1\geq\cdots\geq m_r>0$*.*

The SMZVs come into play by considering the case $\zeta_n=e^{2\pi i/n}$ and taking the limit of $z_n(\mathbf{k};\zeta_n)$ as $n\to\infty$. We shall later show that this limit always exists and that its real part determines an SMZV.

Theorem 1.2. *For any index* $\mathbf{k}\in(\mathbb{Z}_{\geq 1})^r$*, the limits*

$$\xi(\mathbf{k})=\lim_{n\to\infty} z_n(\mathbf{k};e^{2\pi i/n}), \qquad \xi^\star(\mathbf{k})=\lim_{n\to\infty} z_n^\star(\mathbf{k};e^{2\pi i/n})$$

exist and it holds that

$$\operatorname{Re}\xi(\mathbf{k})\equiv\zeta_{\mathcal{S}}(\mathbf{k}), \qquad \operatorname{Re}\xi^\star(\mathbf{k})\equiv\zeta_{\mathcal{S}}^\star(\mathbf{k})$$

modulo $\zeta(2)\mathcal{Z}$ *where* $\zeta_{\mathcal{S}}^\star(\mathbf{k})$ *is the symmetrized multiple zeta star value as defined in Definition 2.4 below.*

Thanks to Theorem 1.1 and Theorem 1.2, the $\mathbb{Q}$-linear relations amongst the $z_n(\mathbf{k};\zeta_n)$ and the $z_n^\star(\mathbf{k};\zeta_n)$ give $\mathbb{Q}$-linear relations for the FMZVs and the SMZVs. Let us illustrate one example. One should start from a relation for $z_n(\mathbf{k};\zeta_n)$ which does not depend on the choice of the n-th root of unity ζ_n. It can be shown that the identity

$$2z_n^\star(4,1;\zeta_n)+z_n^\star(3,2;\zeta_n)=\frac{(n^4-1)(n+5)}{1440}(1-\zeta_n)^5+\frac{n+2}{3}(1-\zeta_n)^2z_p^\star(2,1;\zeta_n) \tag{1.3}$$

holds for any $n\geq 1$ and any n-th primitive root of unity ζ_n. From (1.3) and Theorem 1.1, one obtains the relation (1.2). On the other hand, using $1-e^{2\pi i/n}=-2\pi i/n+o(1/n)$ as $n\to+\infty$ together with Theorem 1.2, we find

$$2\zeta_{\mathcal{S}}^\star(4,1)+\zeta_{\mathcal{S}}^\star(3,2)\equiv 0 \mod \zeta(2)\mathcal{Z}\,.$$

Thus we obtain a linear relation between the FMZVs and the SMZVs of the same form from the identity (1.3).

We will describe one of such relations, which is a generalization of Hoffman's identities (see [6, Theorems 4.5 and 4.6]) given by

$$\zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = -\zeta^{\star}_{\mathcal{A}}(\mathbf{k}^{\vee}) \quad \text{and} \quad \zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\zeta^{\star}_{\mathcal{A}}(\overline{\mathbf{k}}), \tag{1.4}$$

where $\mathbf{k}^{\vee}$ is the Hoffman dual of $\mathbf{k}$ (see Section 2.4.1 for definition) and $\overline{\mathbf{k}} = (k_r, \dots, k_1)$ is the reversal of $\mathbf{k} = (k_1, \dots, k_r)$. We will refer to the first identity as the Hoffman duality and to the second equation as the reversal relation. Although both relations (1.4) do not hold separately for our object $z^{\star}_n(\mathbf{k}; \zeta_n)$, their combination holds:

Theorem 1.3. *For any index* $\mathbf{k}$ *and any* n*-th primitive root of unity* ζ_n*, we have*

$$z^{\star}_n(\mathbf{k}; \zeta_n) = (-1)^{\mathrm{wt}(\mathbf{k})+1} z^{\star}_n(\overline{\mathbf{k}^{\vee}}; \zeta_n).$$

Combining Theorem 1.3 and the relation

$$z_n(\overline{\mathbf{k}}; e^{2\pi i/n}) = (-e^{2\pi i/n})^{\mathrm{wt}(\mathbf{k})}\, \overline{z_n(\mathbf{k}; e^{2\pi i/n})},$$

where the bar on the right-hand side denotes complex conjugation, we obtain the corresponding relation to (1.4) for the SMZVs.

Theorem 1.4. *For any index* $\mathbf{k}$*, we have*

$$\zeta^{\star}_{\mathcal{S}}(\mathbf{k}) \equiv -\zeta^{\star}_{\mathcal{S}}(\mathbf{k}^{\vee}) \quad \textit{and} \quad \zeta^{\star}_{\mathcal{S}}(\mathbf{k}) \equiv (-1)^{\mathrm{wt}(\mathbf{k})}\zeta^{\star}_{\mathcal{S}}(\overline{\mathbf{k}}) \mod \zeta(2)\mathcal{Z}.$$

Apart from the Kaneko–Zagier conjecture, the value $z_n(\mathbf{k}; \zeta_n)$ itself might be worth enlighting. One reason is that the product in the $\mathbb{Q}$-vector space spanned by all $z_n(\mathbf{k}; \zeta_n)$'s preserves the weight; that is, the product $z_n(\mathbf{k}; \zeta_n) z_n(\mathbf{k}'; \zeta_n)$ can be represented as a $\mathbb{Q}$-linear combination of $z_n(\mathbf{k}''; \zeta_n)$'s with $\mathrm{wt}(\mathbf{k}'') = \mathrm{wt}(\mathbf{k}) + \mathrm{wt}(\mathbf{k}')$. This is not true for generic q, because the coefficients in the expansion of $z_n(\mathbf{k}; q) z_n(\mathbf{k}'; q)$ by the q-stuffle product are in $\mathbb{Q}[1-q]$ (see [1, §2]). However in the case q being a root of unity, we have $1 - \zeta_n = 2z_n(1; \zeta_n)/(n-1)$ (see (2.4) below), and the multiplication with $1 - \zeta_n$ gives a linear operator which increases the weight of $z_n(\mathbf{k}; \zeta_n)$. Thus we deduce the homogeneity with respect to weight.

Furthermore, by numerical computations, one can check that the number of linearly independent relations over $\mathbb{Q}$ among $z_p(\mathbf{k}; \zeta_p)$'s of weight k is stable for sufficiently large prime p. In order to deal with these properties, we introduce a new object, which we call a *cyclotomic analogue of finite multiple zeta value.* For this we define the cyclotomic analogue $\mathcal{A}^{\mathrm{cyc}}$ of the ring $\mathcal{A}$ by

$$\mathcal{A}^{\mathrm{cyc}} = \left(\prod_{p:\mathrm{prime}} \mathbb{Z}[\zeta_p]/(p) \right) \Big/ \left(\bigoplus_{p:\mathrm{prime}} \mathbb{Z}[\zeta_p]/(p) \right),$$

which carries a $\mathbb{Q}$-algebra structure (see [14, Definition 3.1]). Note that the ring $\mathcal{A}^{\mathrm{cyc}}$ does not depend on the choice of the p-th primitive root of unity ζ_p. Then we define the cyclotomic analogue of FMZV and its star version by

$$Z(\mathbf{k}) = (z_p(\mathbf{k};\zeta_p) \mod (p))_p \quad \text{and} \quad Z^\star(\mathbf{k}) = (z_p^\star(\mathbf{k};\zeta_p) \mod (p))_p$$

as an element of $\mathcal{A}^{\mathrm{cyc}}$.

One can prove that the $\mathbb{Q}$-linear subspace of $\mathcal{A}^{\mathrm{cyc}}$ spanned by $Z(\mathbf{k})$'s is equal to that spanned by the star version $Z^\star(\mathbf{k})$'s. We denote this subspace by $\mathcal{Z}^{\mathrm{cyc}}$. Because of the correspondence given in Theorem 1.1 and the equality $(\mathfrak{p}_p)^{p-1} = (p)$, we have the natural algebraic projection $\varphi_{\mathcal{A}} : \mathcal{Z}^{\mathrm{cyc}} \to \mathcal{Z}_{\mathcal{A}}$ sending $Z(\mathbf{k})$ to $\zeta_{\mathcal{A}}(\mathbf{k})$. We therefore hope that the cyclotomic analogue may give a new perspective of and become a tool for analyzing the Kaneko–Zagier conjecture. In this paper, we describe the algebraic structure of $\mathcal{Z}^{\mathrm{cyc}}$ and give some numerical experiments, which support the expectation that all linear relations among $Z(\mathbf{k})$'s are obtained from the duality formula and a variant of the double shuffle relations (see Theorems 3.8 and 3.9 and Remark 3.10).

The contents of this paper are as follows. In Section 2, after developing basic facts on $z_n(\mathbf{k};\zeta_n)$, we first give the connections to FMZV and prove Theorem 1.1. After this we discuss the limit $n \to \infty$ and the connection to SMZV together with Theorem 1.2. Then we prove the duality Theorems 1.3 and 1.4 in Section 2.4. In the last section we introduce the cyclotomic analogue of FMZV and describe their algebraic structure.

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2 Finite multiple harmonic q-series at a root of unity

2.1 Definitions

In this subsection, we define the finite multiple harmonic q-series and give some examples of the value of depth one at a primitive root of unity.

We call a tuple of positive integers $\mathbf{k} = (k_1, \dots, k_r)$ an *index*. An index $\mathbf{k} = (k_1, \dots, k_r)$ is said to be *admissible* if $k_1 \geq 2$ or if it is the empty set $\emptyset$.

For shorter notation we will write a subsequence $k, k, \dots, k$ of length a in an index as $\{k\}^a$. When $a = 0$ we ignore it. For example, $(\{2\}^3) = (2, 2, 2)$, $(\{1\}^2, 3, \{1\}^1) = (1, 1, 3, 1)$ and $(\{1\}^0, 3, \{1\}^2, 2, \{1\}^0, 4) = (3, 1, 1, 2, 4)$.

We define the *weight* $\mathrm{wt}(\mathbf{k})$ and the *depth* $\mathrm{dep}(\mathbf{k})$ of an index $\mathbf{k} = (k_1, \dots, k_r)$ by

$$\mathrm{wt}(\mathbf{k}) = k_1 + \cdots + k_r, \qquad \mathrm{dep}(\mathbf{k}) = r.$$

With this notation we can define the following q-series which will be one of the main objects in this work.

Definition 2.1. *Let $n \geq 1$ be a natural number and q a complex number satisfying $q^m \neq 1$ for $n > m > 0$ (to ensure the well-definedness). For an index $\mathbf{k} = (k_1, \dots, k_r)$ we define*

$$z_n(\mathbf{k}; q) = z_n(k_1, \dots, k_r; q) = \sum_{n > m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1} \dots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \dots [m_r]_q^{k_r}}$$

and

$$z_n^\star(\mathbf{k}; q) = z_n^\star(k_1, \dots, k_r; q) = \sum_{n > m_1 \geq \cdots \geq m_r > 0} \frac{q^{(k_1-1)m_1} \dots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \dots [m_r]_q^{k_r}},$$

where $[m]_q$ is the q-integer

$$[m]_q = \frac{1 - q^m}{1 - q}$$

By agreement we set $z_n(\mathbf{k}; q) = 0$ if $\mathrm{dep}(\mathbf{k}) \geq n$ and $z_n(\emptyset; q) = z_n^\star(\emptyset; q) = 1$.

The above q-series $z_n(\mathbf{k}; q)$ was also studied by Bradley [2, Definition 4] (see also [22]). When $\mathbf{k}$ is admissible, the limit $\lim_{n\to\infty} z_n(\mathbf{k}; q)$ converges for $|q| < 1$ and it is called a q-analogue of multiple zeta values. It can be shown that $\lim_{q\to 1} \lim_{n\to\infty} z_n(\mathbf{k}; q) = \zeta(\mathbf{k})$. Their algebraic structure as well as the $\mathbb{Q}$-linear relation were studied by many authors [1, 7, 12, 13, 16, 17, 18, 21].

Using the standard decomposition

$$\frac{q^{(k_1-1)m}}{[m]_q^{k_1}}\frac{q^{(k_2-1)m}}{[m]_q^{k_2}}=\frac{q^{(k_1+k_2-1)m}}{[m]_q^{k_1+k_2}}+(1-q)\frac{q^{(k_1+k_2-2)m}}{[m]_q^{k_1+k_2-1}}\qquad (m,k_1,k_2\geq 1),$$

we see that $z_n(\mathbf{k};q)$ and $z_n^\star(\mathbf{k};q)$ are related to each other in the following way:

$$z_n^\star(\mathbf{k};q)=\sum_{\mathbf{a}} z_n(\mathbf{a};q)+\sum_{\substack{\mathbf{k}'\\ \mathrm{wt}(\mathbf{k}')<\mathrm{wt}(\mathbf{k})}} c_{\mathbf{k},\mathbf{k}'}(1-q)^{\mathrm{wt}(\mathbf{k})-\mathrm{wt}(\mathbf{k}')}z_n(\mathbf{k}';q), \tag{2.1}$$

$$z_n(\mathbf{k};q)=\sum_{\mathbf{a}}(-1)^{\mathrm{dep}(\mathbf{k})-\mathrm{dep}(\mathbf{a})} z_n^\star(\mathbf{a};q)+\sum_{\substack{\mathbf{k}'\\ \mathrm{wt}(\mathbf{k}')<\mathrm{wt}(\mathbf{k})}} \tilde{c}_{\mathbf{k},\mathbf{k}'}(1-q)^{\mathrm{wt}(\mathbf{k})-\mathrm{wt}(\mathbf{k}')}z_n^\star(\mathbf{k}';q), \tag{2.2}$$

where the sum $\sum_{\mathbf{a}}$ is over all indices of the form $(k_1\square k_2\square\cdots\square k_r)$ in which each $\square$ is '+' (plus) or ',' (comma) and $c_{\mathbf{k},\mathbf{k}'}$ and $\tilde{c}_{\mathbf{k},\mathbf{k}'}$ are integers independent on n (an algebraic setup of this translation formula was given in [7, Definition 1]). For example, it holds that

$$\begin{aligned} z_n^\star(3,2,1;q) &= z_n(3,2,1;q)+z_n(5,1;q)+z_n(3,3;q)+z_n(6;q)\\ &\quad +(1-q)\left(z_n(4,1;q)+z_n(3,2;q)+2z_n(5;q)\right)+(1-q)^2 z_n(4;q). \end{aligned}$$

As mentioned in the introduction, we will be interested in the values $z_n(\mathbf{k};q)$ and $z_n^\star(\mathbf{k};q)$ where q is equal to a primitive n-th root of unity ζ_n. Then they are well-defined as an element of the cyclotomic field $\mathbb{Q}(\zeta_n)$. The generating function of the value $z_n(k;\zeta_n)$ of depth one is given by

$$\sum_{k>0} z_n(k;\zeta_n)\left(\frac{x}{1-\zeta_n}\right)^k=\frac{nx}{1-(1+x)^n}+1, \tag{2.3}$$

which can be shown by using the basic properties of the n-th root of unity ζ_n. In particular this shows that $z_n(k;\zeta_n)\in(1-\zeta_n)^k\cdot\mathbb{Q}$ and for example we have

$$\begin{aligned} z_n(1;\zeta_n)&=\frac{n-1}{2}(1-\zeta_n)\,,\quad z_n(2;\zeta_n)=-\frac{n^2-1}{12}(1-\zeta_n)^2\,,\\ z_n(3;\zeta_n)&=\frac{n^2-1}{24}(1-\zeta_n)^3\,,\quad z_n(4;\zeta_n)=\frac{(n^2-1)(n^2-19)}{720}(1-\zeta_n)^4. \end{aligned} \tag{2.4}$$

Remark 2.2. i) The formula (2.3) implies that for $k \geq 1$

$$\frac{z_n(k;\zeta_n)}{(1-\zeta_n)^k} = -\frac{\beta_k(n^{-1})}{k!} n^k\,, \tag{2.5}$$

where $\beta_k(x) \in \mathbb{Q}[x]$ is the *degenerate Bernoulli number* defined by Carlitz in [3]. Since the limit of $\beta_k(n^{-1})$ as $n \to \infty$ is equal to the k-th Bernoulli number B_k, formula (2.5) can be viewed as a finite analogue of Euler's formula given by $\zeta(k)/(-2\pi i)^k = -B_k/2k!$ for even k.

ii) Since the q-stuffle product of the $z_n(\mathbf{k}, q)$ is an example of a quasi-shuffle product (see [5]) it follows that for all $k, r \geq 1$ the $z_n(\{k\}^r; q)$ can be written as a polynomial in $(1-q)^{kr-m} z_n(m;q)$ with $1 \leq m \leq kr$ (see [5], eq. (32)). By (2.3) this implies $z_n(\{k\}^r;\zeta_n) \in (1-\zeta_n)^{kr} \cdot \mathbb{Q}$ for any $k, r \geq 1$. Moreover, one can prove

$$z_n(\{1\}^r;\zeta_n) = \frac{1}{n}\binom{n}{r+1}(1-\zeta_n)^r\,, \quad z_n(\{2\}^r;\zeta_n) = \frac{(-1)^r}{n(r+1)}\binom{n+r}{2r+1}(1-\zeta_n)^{2r}\,,$$
$$z_n(\{3\}^r;\zeta_n) = \frac{1}{n^2(r+1)}\left(\binom{n+2r+1}{3r+2} + (-1)^r\binom{n+r}{3r+2}\right)(1-\zeta_n)^{3r}.$$

by calculating the generating function $\sum_{n>r\geq 0} z_n(\{k\}^r;\zeta_n)\, n^{k-1} X^r Y^n$. However, it might be difficult to get a simple formula of $z_n(\{k\}^r;\zeta_n)$ for $k \geq 4$ with this technique. This will be discussed in more detail in a upcoming work of the authors.

2.2 Connection with finite multiple zeta values

2.2.1 Definition of finite multiple zeta values

The finite multiple zeta values will be elements in the ring

$$\mathcal{A} = \left(\prod_{p:\text{prime}} \mathbb{F}_p\right) \Big/ \left(\bigoplus_{p:\text{prime}} \mathbb{F}_p\right).$$

Its elements are of the form $(a_p)_p$, where p runs over all primes and $a_p \in \mathbb{F}_p$. Two elements $(a_p)_p$ and $(b_p)_p$ are identified if and only if $a_p = b_p$ for all but finitely many primes p. The ring $\mathcal{A}$, which was introduced by Kontsevich [10, §2.2], carries a $\mathbb{Q}$-algebra structure by sending $a \in \mathbb{Q}$ to $(a \mod p)_p \in \mathcal{A}$ diagonally except for finitely many primes which divide the denominator of a.

Definition 2.3. *For an index* $\mathbf{k} = (k_1, \ldots, k_r)$*, we define the* finite multiple zeta value

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}(k_1, \ldots, k_r) = \left(\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_p \in \mathcal{A}$$

and its star version

$$\zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = \zeta^{\star}_{\mathcal{A}}(k_1, \ldots, k_r) = \left(\sum_{p > m_1 \geq \cdots \geq m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_p \in \mathcal{A}.$$

2.2.2 Proof of Theorem 1.1

Now we prove Theorem 1.1, which is immediate from the standard facts on the algebraic number theory (see, e.g., [20]).

Proof of Theorem 1.1. For p prime and any p-th primitive root of unity ζ_p, the ring $\mathbb{Z}[\zeta_p]$ is the ring of algebraic integers in the cyclotomic field $\mathbb{Q}(\zeta_p)$. Since the value $[m]_{\zeta_p} = (1 - \zeta_p^m)/(1 - \zeta_p)$ is a cyclotomic unit, $z_n(\mathbf{k}; \zeta_p)$ and $z_n^{\star}(\mathbf{k}; \zeta_p)$ belong to $\mathbb{Z}[\zeta_p]$.

Let $\mathfrak{p}_p = (1 - \zeta_p)$ be the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1 - \zeta_p$. Since the norm of $\mathfrak{p}_p$ is equal to p, we have $\mathbb{Z}[\zeta_p]/\mathfrak{p}_p = \mathbb{F}_p$. Now Theorem 1.1 follows from $[m]_{\zeta_p} \equiv m \mod \mathfrak{p}_p$ for $p > m > 0$. □

2.3 Connection with symmetrized multiple zeta values

2.3.1 Definition of symmetrized multiple zeta values

To define the symmetrized multiple zeta values, we recall Hoffman's algebraic setup [4] with a slightly different convention.

Let $\mathfrak{H} = \mathbb{Q}\langle e_0, e_1 \rangle$ be the noncommutative polynomial algebra of indeterminates e_0 and e_1 over $\mathbb{Q}$. Define its subalgebra $\mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}e_1$. We put $e_k = e_0^{k-1}e_1$ $(k \geq 1)$ and set for an index $\mathbf{k} = (k_1, \ldots, k_r)$

$$e_{\mathbf{k}} := e_{k_1} \cdots e_{k_r}.$$

For the empty index $\emptyset$ we set $e_{\emptyset} = 1$. The monomials $\{e_{\mathbf{k}}\}$ associated to all indices $\mathbf{k}$ form a basis of $\mathfrak{H}^1$ over $\mathbb{Q}$.

The *stuffle product* is the $\mathbb{Q}$-bilinear map $* : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ characterized by the following properties:

$$1 * w = w * 1 = w \quad (w \in \mathfrak{H}^1),$$
$$e_k w * e_{k'} w' = e_k(w * e_{k'} w') + e_{k'}(e_k w * w') + e_{k+k'}(w * w') \quad (k, k' \geq 1, w, w' \in \mathfrak{H}^1).$$

We denote by $\mathfrak{H}^1_*$ the commutative $\mathbb{Q}$-algebra $\mathfrak{H}^1$ equipped with the multiplication $*$.

As stated in [8, Proposition 1], there exists a unique $\mathbb{Q}$-algebra homomorphism $R : \mathfrak{H}^1_* \to \mathbb{R}[T]$ satisfying $R(1) = 1$, $R(e_1) = T$ and $R(e_{\mathbf{k}}) = \zeta(\mathbf{k})$ for any admissible index $\mathbf{k}$ [1]. For an index $\mathbf{k}$ we define the *stuffle regularized multiple zeta values* $R_{\mathbf{k}}(T)$ by

$$R_{\mathbf{k}}(T) := R(e_{\mathbf{k}}) \in \mathbb{R}[T].$$

Note that $R_{\emptyset}(T) = 1$ and $R_{\mathbf{k}}(T) = \zeta(\mathbf{k})$ if $\mathbf{k}$ is admissible.

Definition 2.4. *For an index* $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r$ *we define the* symmetrized multiple zeta value

$$\zeta_{\mathcal{S}}(\mathbf{k}) = \zeta_{\mathcal{S}}(k_1, \ldots, k_r) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,,k_{a-1},\ldots,k_1}(T) R_{k_{a+1},k_{a+2},\ldots,k_r}(T).$$

and its star version

$$\zeta^{\star}_{\mathcal{S}}(\mathbf{k}) = \zeta^{\star}_{\mathcal{S}}(k_1, \ldots, k_r) = \sum_{\substack{\square \text{ is either a comma `,'} \\ \text{or a plus `+'}}} \zeta_{\mathcal{S}}(k_1 \square \cdots \square k_r).$$

Kaneko and Zagier [9] showed that the symmetrized multiple zeta value does not depend on T, i.e. we have

$$\zeta_{\mathcal{S}}(k_1, \ldots, k_r) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,\ldots,k_1}(0) R_{k_{a+1},\ldots,k_r}(0) \in \mathbb{R}. \tag{2.6}$$

In general, one can prove the following lemma, which will be used in computing the limit of $z_n(\mathbf{k}; e^{2\pi i/n})$ as $n \to \infty$.

Lemma 2.5. *For any index* $\mathbf{k} = (k_1, \ldots, k_r)$*, the polynomial*

$$\sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,k_{a-1},\ldots,k_1}(T+X) R_{k_{a+1},k_{a+2},\ldots,k_r}(T-X) \tag{2.7}$$

does not depend on T*. Hence it is equal to*

$$\sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,k_{a-1},\ldots,k_1}(X) R_{k_{a+1},k_{a+2},\ldots,k_r}(-X).$$

[1] The map R is denoted by Z^* in [8].

Proof. From the definition, we see that the polynomial (2.7) is a sum of polynomials of the form

$$\pm\sum_{a=0}^{s}(-1)^a R_{\{1\}^a,\mathbf{k}}(T+X)R_{\{1\}^{s-a},\mathbf{k}'}(T-X) \tag{2.8}$$

with some admissible indices $\mathbf{k}$ and $\mathbf{k}'$.

For any index $\mathbf{k}=(k_1,\dots,k_r)$ and $s\geq 0$, it holds that

$$e_1 * (e_1^s e_{\mathbf{k}}) = (s+1)e_1^{s+1}e_{\mathbf{k}} + \sum_{a=1}^{r}(e_1^s e_{\mathbf{k}'(a)} + e_1^s e_{\mathbf{k}''(a)}) + \sum_{b=1}^{s} e_1^{b-1}e_2 e_1^{s-b}e_{\mathbf{k}}, \tag{2.9}$$

where

$$\mathbf{k}'(a) = (k_1,\dots,k_a+1,\dots,k_r), \quad \mathbf{k}''(a) = (k_1,\dots,k_a,1,k_{a+1},\dots,k_r)\,. \tag{2.10}$$

Using this one can show by induction on s that

$$R_{\{1\}^s,\mathbf{k}}(T) = \sum_{j=0}^{s} R_{\{1\}^{s-j},\mathbf{k}}(0)\frac{T^j}{j!}. \tag{2.11}$$

From this formula we see that the sum (2.8) without sign is equal to

$$\begin{aligned}
&\sum_{a=0}^{s}\sum_{j=0}^{a}\sum_{l=0}^{s-a}(-1)^a R_{\{1\}^{a-j},\mathbf{k}}(0)R_{\{1\}^{s-a-l},\mathbf{k}'}(0)\frac{(T+X)^j}{j!}\frac{(T-X)^l}{l!}\\
&=\sum_{\substack{j,l\geq 0\\ j+l\leq s}}\sum_{a=j}^{s-l}(-1)^a R_{\{1\}^{a-j},\mathbf{k}}(0)R_{\{1\}^{s-a-l},\mathbf{k}'}(0)\frac{(T+X)^j}{j!}\frac{(T-X)^l}{l!}\\
&=\sum_{m=0}^{s}\sum_{a=0}^{s-m}(-1)^a R_{\{1\}^{a},\mathbf{k}}(0)R_{\{1\}^{s-a},\mathbf{k}'}(0)\sum_{\substack{j+l=m\\ j,l\geq 0}}(-1)^j\frac{(T+X)^j}{j!}\frac{(T-X)^l}{l!}\\
&=\sum_{m=0}^{s}(-2X)^m\sum_{a=0}^{s-m}(-1)^a R_{\{1\}^{a},\mathbf{k}}(0)R_{\{1\}^{s-a},\mathbf{k}'}(0),
\end{aligned}$$

which shows that the polynomial (2.8) does not depend on T, neither does (2.7). □

2.3.2 Evaluation of the limit

Theorem 2.6. *For any non-empty index* $\mathbf{k} = (k_1, \ldots, k_r)$ *it holds that*

$$\lim_{n\to\infty} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,k_{a-1},\ldots,k_1}\Big(\frac{\pi i}{2}\Big) R_{k_{a+1},k_{a+2},\ldots,k_r}\Big(-\frac{\pi i}{2}\Big).$$

To prove Theorem 2.6, we rewrite the value $z_n(\mathbf{k}; e^{2\pi i/n})$. Let n be a positive integer. When $q = e^{2\pi i/n}$ we see that

$$\frac{1-q}{1-q^m} = e^{-\frac{\pi i}{n}(m-1)} \frac{\sin\frac{\pi}{n}}{\sin\frac{m\pi}{n}} \qquad (n > m \geq 0).$$

Therefore it holds that

$$z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \Big(e^{\frac{\pi i}{n}} \frac{n}{\pi} \sin\frac{\pi}{n}\Big)^{\mathrm{wt}(\mathbf{k})} \sum_{n>m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{e^{\frac{\pi i}{n}(k_j-2)m_j}}{\big(\frac{n}{\pi}\sin\frac{m_j\pi}{n}\big)^{k_j}}$$

for any non-empty index $\mathbf{k} = (k_1, \ldots, k_r)$. Decompose the set $\{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid n > m_1 > \cdots > m_r > 0\}$ into the disjoint union

$$\bigsqcup_{a=0}^{r} \{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid n > m_1 > \cdots > m_a > \frac{n}{2} \geq m_{a+1} > \cdots > m_r > 0\}$$

and change the summation variables m_j to $n_j = n - m_{a+1-j}$ $(1 \leq j \leq a)$ and $l_j = m_{a+j}$ $(1 \leq j \leq r-a)$. Then we find that

$$\begin{aligned} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) &= \Big(e^{\frac{\pi i}{n}} \frac{n}{\pi} \sin\frac{\pi}{n}\Big)^{\mathrm{wt}(\mathbf{k})} \\ &\times \sum_{a=0}^{r} (-1)^{\sum_{j=1}^{a} k_j} \sum_{n/2>n_1>\cdots>n_a>0} \prod_{j=1}^{a} \frac{e^{-\frac{\pi i}{n}(k_{a+1-j}-2)n_j}}{\big(\frac{n}{\pi}\sin\frac{n_j\pi}{n}\big)^{k_{a+1-j}}} \sum_{n/2\geq l_1>\cdots>l_{r-a}>0} \prod_{j=1}^{r-a} \frac{e^{\frac{\pi i}{n}(k_{a+j}-2)l_j}}{\Big(\frac{n}{\pi}\sin\frac{l_j\pi}{n}\Big)^{k_{a+j}}}. \end{aligned}$$

Motivated by the above expression we introduce the following numbers. For an index $\mathbf{k} = (k_1, \ldots, k_r)$ and a positive integer n, we define

$$\begin{aligned} A_n^-(\mathbf{k}) &= \sum_{n/2>m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{e^{-\frac{\pi i}{n}(k_j-2)m_j}}{\big(\frac{n}{\pi}\sin\frac{m_j\pi}{n}\big)^{k_j}}, \\ A_n^+(\mathbf{k}) &= \sum_{n/2\geq m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{e^{\frac{\pi i}{n}(k_j-2)m_j}}{\big(\frac{n}{\pi}\sin\frac{m_j\pi}{n}\big)^{k_j}}. \end{aligned}$$

Then we see that

$$z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \left(e^{\frac{\pi i}{n}} \frac{n}{\pi} \sin \frac{\pi}{n}\right)^{\mathrm{wt}(\mathbf{k})} \times \sum_{a=0}^{r} (-1)^{\sum_{j=1}^{a} k_j} A_n^-(k_a, k_{a-1}, \ldots, k_1) A_n^+(k_{a+1}, k_{a+2}, \ldots, k_r).$$

Now Theorem 2.6 follows from Lemma 2.5 and Proposition 2.7 below.

Proposition 2.7. *For any index* $\mathbf{k}$ *it holds that*

$$A_n^{\pm}(\mathbf{k}) = R_{\mathbf{k}} \left(\log\left(\frac{n}{\pi}\right) + \gamma \mp \frac{\pi i}{2}\right) + O\left(\frac{(\log n)^{J(\mathbf{k})}}{n}\right) \qquad (n \to +\infty), \tag{2.12}$$

where γ *is Euler's constant and* $J(\mathbf{k})$ *is a positive integer which depends on* $\mathbf{k}$*.*

Note that it suffices to prove (2.12) for $A_n^+(\mathbf{k})$ since

$$A_n^-(k_1, \ldots, k_r) = \begin{cases} \overline{A_n^+(k_1, \ldots, k_r)} & (n\text{: odd}), \\ \overline{A_n^+(k_1, \ldots, k_r)} + (-\frac{\pi i}{n})^{k_1} \overline{A_n^+(k_2, \ldots, k_r)} & (n\text{: even}), \end{cases}$$

where the bar on the right-hand side denotes complex conjugation. For this we will first consider the cases where the index $\mathbf{k}$ is admissible.

Lemma 2.8. *Let* $\mathbf{k}$ *be an admissible index. Then it holds that*

$$A_n^+(\mathbf{k}) = \zeta(\mathbf{k}) + O\left(\frac{(\log n)^{J_1(\mathbf{k})}}{n}\right) \qquad (n \to +\infty),$$

where $J_1(\mathbf{k})$ *is a positive integer which depends on* $\mathbf{k}$*.*

Proof. Set $\mathbf{k} = (k_1, \ldots, k_r)$ and define for $k \geq 1$ the function

$$g_k(x) = e^{(k-2)ix} \left(\frac{x}{\sin x}\right)^k.$$

Then it holds that $|A_n^+(\mathbf{k}) - \zeta(\mathbf{k})| \leq I_1 + I_2$, where

$$I_1 = \sum_{n/2 \geq m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{1}{m_j^{k_j}} \left| \prod_{j=1}^{r} g_{k_j}\left(\frac{m_j \pi}{n}\right) - 1 \right|,$$

$$I_2 = \sum_{m > n/2} \frac{1}{m^{k_1}} \left(\sum_{m > m_2 > \cdots > m_r > 0} \prod_{j=2}^{r} \frac{1}{m_j^{k_j}} \right).$$

Since $g_k(x) = 1+(k-2)ix+o(x)$ $(x \to +0)$, there exists a positive constant C depending on k such that $|g_k(m\pi/n)-1| \le Cm/n$ for all integers m and n satisfying $n/2 \ge m > 0$. Using the identity

$$\left(\prod_{j=1}^{r} x_j\right) - 1 = \sum_{a=1}^{r} (\prod_{j=1}^{a-1} x_j)\,(x_a - 1)$$

and the inequality $0 < (\sin x)^{-1} \le \pi/2x$ on the interval $(0, \frac{\pi}{2}]$, we see that

$$\begin{aligned} I_1 &\le \frac{C_1}{n} \sum_{a=1}^{r} \sum_{n/2 \ge m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_a^{k_a - 1} \cdots m_r^{k_r}} \\ &\le \frac{C_1}{n} \sum_{a=1}^{r} \sum_{n/2 \ge m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1-1} m_2^{k_2} \cdots m_r^{k_r}} \\ &= \frac{C_1 r}{n} \sum_{n/2 \ge m > 0} \frac{1}{m^{k_1 - 1}} \left(\sum_{m > m_2 > \cdots > m_r > 0} \prod_{j=2}^{r} \frac{1}{m_j^{k_j}} \right) \end{aligned}$$

for some positive constant C_1 which depends on $\mathbf{k}$. Using the estimation

$$\sum_{m > m_2 > \cdots > m_r > 0} \prod_{j=2}^{r} \frac{1}{m_j^{k_j}} \le \left(\sum_{s=1}^{m-1} \frac{1}{s} \right)^{r-1} \le (2 \log m)^{r-1},$$

we get

$$I_1 + I_2 \le C_2 \left(\frac{1}{n} \sum_{n/2 > m > 0} \frac{(\log m)^{r-1}}{m^{k_1 - 1}} + \sum_{m > n/2} \frac{(\log m)^{r-1}}{m^{k_1}} \right)$$

for some positive constant C_2 which depends on $\mathbf{k}$. Since $k_1 \ge 2$ it holds that

$$\sum_{n/2 > m > 0} \frac{(\log m)^{r-1}}{m^{k_1 - 1}} = O((\log n)^r), \quad \sum_{m > n/2} \frac{(\log m)^{r-1}}{m^{k_1}} = O\left(\frac{(\log n)^{r-1}}{n} \right)$$

as $n \to +\infty$. This completes the proof. □

Proof of Proposition 2.7. Lemma 2.8 implies that the equality (2.12) for $A_n^+(\mathbf{k})$ holds if $\mathbf{k}$ is admissible. Let us prove that it holds also for the index $(\{1\}^s, \mathbf{k})$ with any $s \ge 0$ and any admissible index $\mathbf{k}$.

Using the equality

$$\frac{e^{-\frac{\pi i}{n}m}}{\frac{n}{\pi}\sin\frac{m\pi}{n}}\frac{e^{\frac{\pi i}{n}(k-2)m}}{\left(\frac{n}{\pi}\sin\frac{m\pi}{n}\right)^k}=\frac{e^{\frac{\pi i}{n}(k-1)m}}{\left(\frac{n}{\pi}\sin\frac{m\pi}{n}\right)^{k+1}}-\frac{2\pi i}{n}\frac{e^{\frac{\pi i}{n}(k-2)m}}{\left(\frac{n}{\pi}\sin\frac{m\pi}{n}\right)^k},$$

for $k\geq 1$ and $n/2\geq m>0$, we see that

$$\begin{aligned}A_n^+(1)A_n^+(\{1\}^s,\mathbf{k})&=(s+1)A_n^+(\{1\}^{s+1},\mathbf{k})\\&\quad+\sum_{b=1}^{s}\left(A_n^+(\{1\}^{b-1},2,\{1\}^{s-b},\mathbf{k})-\frac{2\pi i}{n}A_n^+(\{1\}^s,\mathbf{k})\right)\\&\quad+\sum_{a=1}^{r}\left(A_n^+(\{1\}^s,\mathbf{k}'(a))+A_n^+(\{1\}^s,\mathbf{k}''(a))-\frac{2\pi i}{n}A_n^+(\{1\}^s,\mathbf{k})\right),\end{aligned}$$

where $\mathbf{k}'(a)$ and $\mathbf{k}''(a)$ are the indices defined by (2.10). Therefore we obtain the desired equality (2.12) by induction on s from (2.9) and

$$A_n^+(1)=\log\left(\frac{n}{\pi}\right)+\gamma-\frac{\pi i}{2}+O\left(\frac{1}{n}\right)\qquad(n\to+\infty).\tag{2.13}$$

Let us prove (2.13). From the definition of $A_n^+(1)$ we see that

$$A_n^+(1)=\frac{\pi}{n}\sum_{n/2\geq m>0}\left(\frac{\cos\frac{m\pi}{n}}{\sin\frac{m\pi}{n}}-i\right)=\frac{\pi}{n}\sum_{n/2\geq m>0}\frac{\cos\frac{m\pi}{n}}{\sin\frac{m\pi}{n}}-\frac{\pi i}{2}+O\left(\frac{1}{n}\right)$$

as $n\to+\infty$. Hence it suffices to show that

$$\frac{\pi}{n}\sum_{n/2\geq m>0}\frac{\cos\frac{m\pi}{n}}{\sin\frac{m\pi}{n}}=\log\frac{n}{\pi}+\gamma+O\left(\frac{1}{n}\right)\qquad(n\to+\infty).\tag{2.14}$$

Since the function $f(x)=x^{-1}-(\tan x)^{-1}$ is positive and increasing on the interval $(0,\pi)$, we see that

$$\int_0^{\frac{n-1}{2}}f\Big(\frac{\pi x}{n}\Big)\,dx\leq\sum_{n/2\geq m>0}\left(\frac{n}{\pi}\frac{1}{m}-\frac{\cos\frac{m\pi}{n}}{\sin\frac{m\pi}{n}}\right)\leq\int_1^{\frac{n}{2}+1}f\Big(\frac{\pi x}{n}\Big)\,dx.$$

Set $g(x)=\log(1+x)-\log(\cos\frac{\pi x}{2})$. By direct calculation we have

$$\int_0^{\frac{n-1}{2}}f\Big(\frac{\pi x}{n}\Big)\,dx=\frac{n}{\pi}\left(g\Big(-\frac{1}{n}\Big)+\log\frac{\pi}{2}\right),$$

$$\int_1^{\frac{n}{2}+1} f\Big(\frac{\pi x}{n}\Big)\,dx = \frac{n}{\pi}\left(g\Big(\frac{2}{n}\Big) + \log\left(\frac{n}{\pi}\sin\frac{\pi}{n}\right) + \log\frac{\pi}{2}\right).$$

Since $g(x) = x + o(x)$ $(x \to 0)$ and $\log(x^{-1}\sin x) = o(x)$ $(x \to +0)$, there exist positive constants c_1 and c_2 such that

$$\int_0^{\frac{n-1}{2}} f\Big(\frac{\pi x}{n}\Big)\,dx \geq -c_1 + \frac{n}{\pi}\log\frac{\pi}{2}, \quad \int_1^{\frac{n}{2}+1} f\Big(\frac{\pi x}{n}\Big)\,dx \leq c_2 + \frac{n}{\pi}\log\frac{\pi}{2}$$

for $n \gg 0$. Therefore we find that

$$\frac{\pi}{n}\sum_{n/2 \geq m > 0} \frac{\cos\frac{m\pi}{n}}{\sin\frac{m\pi}{n}} = \sum_{n/2 \geq m > 0} \frac{1}{m} - \log\frac{\pi}{2} + O\Big(\frac{1}{n}\Big) \qquad (n \to +\infty).$$

Using the asymptotic expansion

$$\sum_{n/2 \geq m > 0} \frac{1}{m} = \log\frac{n}{2} + \gamma + O\Big(\frac{1}{n}\Big) \qquad (n \to +\infty),$$

we get the formula (2.14). □

2.3.3 Proof of Theorem 1.2

For the later purpose we introduce the following complex numbers.

Definition 2.9. *For a non-empty index* $\mathbf{k}$ *we define*

$$\xi(\mathbf{k}) = \lim_{n\to\infty} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) \quad \text{and} \quad \xi^\star(\mathbf{k}) = \lim_{n\to\infty} z_n^\star(\mathbf{k}; e^{\frac{2\pi i}{n}})$$

and set $\xi(\emptyset) = \xi^\star(\emptyset) = 1$.

Theorem 2.6 implies that

$$\xi(k_1,\dots,k_r) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} R_{k_a,k_{a-1},\dots,k_1}\Big(\frac{\pi i}{2}\Big) R_{k_{a+1},k_{a+2},\dots,k_r}\Big(-\frac{\pi i}{2}\Big), \tag{2.15}$$

and

$$\xi^\star(k_1,\dots,k_r) = \sum_{\substack{\square \text{ is either a comma ‘,’} \\ \text{or a plus ‘+’}}} \xi(k_1 \square \cdots \square k_r), \tag{2.16}$$

which follows from (2.1) and $(1 - e^{2\pi i/n})^k z_n(\mathbf{k}; e^{2\pi i/n}) \to 0\,(n \to +\infty)$ for $k > 0$. If

$\mathbf{k} = (k_1, \ldots, k_r)$ is an index with $k_j \geq 2$ for all $1 \leq j \leq r$, we have the equalities $\xi(\mathbf{k}) = \zeta_{\mathcal{S}}(\mathbf{k})$ and $\xi^\star(\mathbf{k}) = \zeta^\star_{\mathcal{S}}(\mathbf{k})$ from Definition 2.4, and hence $\xi(\mathbf{k}), \xi^\star(\mathbf{k}) \in \mathbb{R}$.

Example 2.10. Using (2.15) one can write down the value $\xi(k)$ of depth one:

$$\xi(k) = \begin{cases} -\pi i & (k=1) \\ 2\zeta(k) & (k \geq 2,\ k \text{ is even}) \\ 0 & (k \geq 3,\ k \text{ is odd}) \end{cases}$$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. The convergence is already proved. From (2.11) we see that the coefficient of T^a in the polynomial $R_{\mathbf{k}}(T)$ lies in $\mathcal{Z}$ for any $a \geq 0$. Hence the formulas (2.6) and (2.15) imply that $\mathrm{Re}(\xi(\mathbf{k})) - \zeta_{\mathcal{S}}(\mathbf{k})$ is a polynomial of π^2 whose coefficients belong to $\mathcal{Z}$. Therefore $\mathrm{Re}(\xi(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k})$ modulo $\zeta(2)\mathcal{Z}$. The star version is then immediate from (2.16). □

2.4 Duality formula

2.4.1 Notation

For an index $\mathbf{k} = (k_1, \ldots, k_r)$ we define its *reverse* $\overline{\mathbf{k}}$ by

$$\overline{\mathbf{k}} = (k_r, k_{r-1}, \ldots, k_1).$$

Let $\tau : \mathfrak{H} \to \mathfrak{H}$ be the monoid homomorphism defined by $\tau(e_1) = e_0$ and $\tau(e_0) = e_1$. Every word $w \in \mathfrak{H}^1$ can be written as $w = w'e_1$ with $w' \in \mathfrak{H}$. Then we set $w^\vee = \tau(w')e_1 \in \mathfrak{H}^1$ and call it the *Hoffman dual* of w. We also define the Hoffman dual $\mathbf{k}^\vee$ of an index $\mathbf{k}$ by

$$e_{\mathbf{k}^\vee} = (e_{\mathbf{k}})^\vee.$$

For example, the Hoffman dual of the word e_3e_2 is given by

$$(e_3e_2)^\vee = (e_0e_0e_1e_0e_1)^\vee = \tau(e_0e_0e_1e_0)e_1 = e_1e_1e_0e_1e_1 = e_1e_1e_2e_1\,.$$

Hence $(3,2)^\vee = (1,1,2,1)$. Note that $\mathrm{wt}(\mathbf{k}^\vee) = \mathrm{wt}(\mathbf{k})$ for any index $\mathbf{k}$.

2.4.2 Proof of Theorem 1.3

We will use the following fact.

Lemma 2.11. *Suppose that $n \geq 1$ and ζ_n is a primitive n-th root of unity. Then it holds that $(-1)^n \zeta_n^{n(n+1)/2} = -1$.*

Proof of Theorem 1.3. Note that any index is uniquely written in the form

$$(\{1\}^{a_1-1}, b_1+1, \dots, \{1\}^{a_{r-1}-1}, b_{r-1}+1, \{1\}^{a_r-1}, b_r), \tag{2.17}$$

where r and $a_i, b_i\,(1 \le i \le r)$ are positive integers [2]. Denote it by $[a_1, \dots, a_r; b_1, \dots, b_r]$. Then we see that

$$\overline{[a_1, \dots, a_r; b_1, \dots, b_r]^{\vee}} = [b_r, \dots, b_1; a_r, \dots, a_1].$$

Now we fix a positive integer r and introduce the generating function

$$K(x_1, \dots, x_r; y_1, \dots, y_r) = \sum \frac{z_n^{\star}([a_1, \dots, a_r; b_1, \dots, b_r]; \zeta_n)}{(1-\zeta_n)^{a_1+\cdots+a_r+b_1+\cdots+b_r-1}} \prod_{i=1}^{r} (x_i^{a_i-1} y_i^{b_i-1}),$$

where the sum is taken over all positive integers $a_i, b_i\,(1 \le i \le r)$. Then Theorem 1.3 follows from the equality

$$K(x_1, \dots, x_r; y_1, \dots, y_r) = K(-y_r, \dots, -y_1; -x_r, \dots, -x_1). \tag{2.18}$$

Let us prove (2.18). It holds that

$$1 + \sum_{a=2}^{\infty} \sum_{B \ge m_1 \ge \cdots \ge m_{a-1} \ge A} \frac{x^{a-1}}{\prod_{i=1}^{a-1}(1-\zeta_n^{m_i})} = \prod_{i=A}^{B} \frac{1-\zeta_n^i}{1-x-\zeta_n^i}$$

for $n > B \ge A > 0$, and that

$$\sum_{b=1}^{\infty} \frac{\zeta_n^{bm}}{(1-\zeta_n^m)^{b+1}} y^{b-1} = \frac{1}{1-\zeta_n^m} \frac{\zeta_n^m}{1-\zeta_n^m(1+y)}$$

for $n > m > 0$. Using the above formulas we have

$$K(x_1, \dots, x_r; y_1, \dots, y_r) = \sum_{n > l_1 \ge \cdots \ge l_r > 0} \prod_{i=l_r}^{n-1} (1-\zeta_n^i)$$
$$\times \prod_{j=1}^{r-1} \left(\frac{\zeta_n^{l_j}}{1-\zeta_n^{l_j}(1+y_j)} \prod_{i=l_j}^{l_{j-1}} \frac{1}{1-x_j-\zeta_n^i} \right) \frac{1}{1-\zeta_n^{l_r}(1+y_r)} \prod_{i=l_r}^{l_{r-1}} \frac{1}{1-x_r-\zeta_n^i},$$

where $l_0 = n-1$. Rewrite the right-hand side above by using the partial fraction

[2] If $r = 1$, (2.17) should read as $(\{1\}^{a_1-1}, b_1)$.

expansion

$$\begin{aligned}\prod_{i=A}^{B}\frac{1}{X-\zeta_n^i}&=\sum_{i=A}^{B}\frac{1}{X-\zeta_n^i}\prod_{j=A}^{i-1}\frac{1}{\zeta_n^i-\zeta_n^j}\prod_{j=i+1}^{B}\frac{1}{\zeta_n^i-\zeta_n^j}\\&=\sum_{t=A}^{B}\frac{1}{X-\zeta_n^t}\frac{(-1)^{B-t}\zeta_n^{-\binom{B+1}{2}+At-\binom{t}{2}}}{\prod_{i=1}^{t-A}(1-\zeta_n^{-i})\prod_{i=1}^{B-t}(1-\zeta_n^{-i})}\end{aligned}$$

for $n>B\geq A>0$. Then we find that

$$\begin{aligned}&K(x_1,\dots,x_r;y_1,\dots,y_r)\\&=\sum_{n>t_1\geq l_1\geq\cdots\geq t_r\geq l_r>0}\prod_{i=l_r}^{n-1}(1-\zeta_n^i)(-1)^{\sum_{j=1}^{r}(l_{j-1}-t_j)}\zeta_n^{\sum_{j=1}^{r}(-\binom{l_{j-1}+1}{2}+l_jt_j-\binom{t_j}{2})}\\&\qquad\times\prod_{j=1}^{r}\left(\prod_{i=1}^{t_j-l_j}\frac{1}{1-\zeta_n^{-i}}\prod_{i=1}^{l_{j-1}-t_j}\frac{1}{1-\zeta_n^{-i}}\right)\\&\qquad\times\prod_{j=1}^{r-1}\left(\frac{\zeta_n^{l_j}}{1-\zeta_n^{l_j}(1+y_j)}\frac{1}{1-x_j-\zeta_n^{t_j}}\right)\frac{1}{1-\zeta_n^{l_r}(1+y_r)}\frac{1}{1-x_r-\zeta_n^{t_r}}.\end{aligned}$$

Now change the summation variable t_j and l_j to $n-l_{r+1-j}$ and $n-t_{r+1-j}$, respectively $(1\leq j\leq r)$. As a result we get the desired equality (2.18) using Lemma 2.11. □

2.4.3 Proof of Theorem 1.4

Theorem 1.4 follows from Theorem 1.2 and Theorem 2.12 below, which describes the reversal relation and the Hoffman duality for $\xi^\star(\mathbf{k})$.

Theorem 2.12. *For any index* $\mathbf{k}$*, the following relations hold.*

i) $\xi(\overline{\mathbf{k}})=(-1)^{\mathrm{wt}(\mathbf{k})}\,\overline{\xi(\mathbf{k})}$, $\xi^\star(\overline{\mathbf{k}})=(-1)^{\mathrm{wt}(\mathbf{k})}\,\overline{\xi^\star(\mathbf{k})}$

ii) $\xi^\star(\mathbf{k}^\vee)=-\,\overline{\xi^\star(\mathbf{k})}$

Here the bar on the right-hand sides denotes complex conjugation.

Proof. i) Changing the summation variable m_j to $n-m_{r+1-j}$ $(1\leq j\leq r)$, we see that

$$z_n(\overline{\mathbf{k}};e^{2\pi i/n})=(-e^{2\pi i/n})^{\mathrm{wt}(\mathbf{k})}\,\overline{z_n(\mathbf{k};e^{2\pi i/n})}.$$

Taking the limit as $n\to+\infty$, we obtain $\xi(\overline{\mathbf{k}})=(-1)^{\mathrm{wt}(\mathbf{k})}\,\overline{\xi(\mathbf{k})}$. The same calculation works also for $z_n^\star(\mathbf{k};e^{2\pi i/n})$.

ii) From Theorem 1.3 we see that $\xi^\star(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})+1}\xi^\star(\overline{\mathbf{k}^\vee})$. Combining it with the equality proved in i), we get the desired equality. □

3 Cyclotomic analogue of finite multiple zeta values

3.1 Definition and examples

As an cyclotomic analogue of the ring $\mathcal{A}$ we define

$$\mathcal{A}^{\mathrm{cyc}} = \left(\prod_{p:\mathrm{prime}} \mathbb{Z}[\zeta_p]/(p)\right) \Big/ \left(\bigoplus_{p:\mathrm{prime}} \mathbb{Z}[\zeta_p]/(p)\right).$$

Similar to $\mathcal{A}$ (see Section 2.2) the ring $\mathcal{A}^{\mathrm{cyc}}$ is a $\mathbb{Q}$-algebra.

Definition 3.1. *For an index* $\mathbf{k}$ *we define the* cyclotomic analogue of finite multiple zeta value $Z(\mathbf{k})$ *by*

$$Z(\mathbf{k}) = (z_p(\mathbf{k};\zeta_p) \mod (p))_p \in \mathcal{A}^{\mathrm{cyc}},$$

and its star version by

$$Z^\star(\mathbf{k}) = (z_p^\star(\mathbf{k};\zeta_p) \mod (p))_p \in \mathcal{A}^{\mathrm{cyc}}.$$

Recall that $\mathfrak{p}_p = (1-\zeta_p)$ is a prime ideal in $\mathbb{Z}[\zeta_p]$ and that $(p) = \mathfrak{p}_p^{p-1}$. This gives a surjective map

$$\mathbb{Z}[\zeta_p]/(p) \to \mathbb{Z}[\zeta_p]/\mathfrak{p}_p \simeq \mathbb{F}_p$$

for all prime p. Let φ be the induced $\mathbb{Q}$-algebra homomorphism

$$\begin{aligned} \varphi : \mathcal{A}^{\mathrm{cyc}} &\longrightarrow \mathcal{A}, \\ (a_p \mod (p))_p &\longmapsto (a_p \mod \mathfrak{p}_p)_p\,. \end{aligned} \tag{3.1}$$

The map φ satisfies $\varphi(Z(\mathbf{k})) = \zeta_{\mathcal{A}}(\mathbf{k})$ and $\varphi(Z^\star(\mathbf{k})) = \zeta^\star_{\mathcal{A}}(\mathbf{k})$.

Let us write down the formula for $Z(k)$ of depth one. As a kind of analogue of π in $\mathbb{R}$, let

$$\varpi = (1-\zeta_p)_p \in \mathcal{A}^{cyc}. \tag{3.2}$$

We introduce the numbers G_k ($k \geq 0$) defined by

$$\sum_{k\geq 0} G_k z^k = \frac{z}{\log(1+z)},$$

which are called *Gregory coefficients*. It is known that $G_k \neq 0$ for any $k \geq 0$ (see [23]).

Proposition 3.2. *For any $k \geq 1$, we have $Z(k) = -G_k \varpi^k \in \varpi^k \mathbb{Q}^{\times}$.*

Proof. Let

$$h_j(x) = \frac{1}{(j+1)!} \prod_{a=1}^{j} (x-a) \qquad (j \geq 1).$$

Then the generating function (2.3) can be written as

$$\sum_{k=1}^{\infty} z_n(k;\zeta_n) \left(\frac{x}{1-\zeta_n}\right)^k = -\sum_{l=1}^{\infty} \left(-\sum_{j=1}^{\infty} h_j(n) x^j\right)^l. \tag{3.3}$$

Hence, for each $k \geq 1$, there exists a unique polynomial $D_k(x) \in \mathbb{Q}[x]$ of degree at most k such that $z_n(k;\zeta_n) = D_k(n)(1-\zeta_n)^k$ for all $n \geq 1$. Then

$$z_p(k;\zeta_p) \equiv D_k(0)(1-\zeta_p)^k \mod (p)$$

for sufficiently large prime p. Therefore $Z(k) = D_k(0)\,\varpi^k$ for $k \geq 1$.

On the other hand, from (3.3) we see that

$$\sum_{k=1}^{\infty} D_k(0) z^k = -\sum_{l=1}^{\infty} \left(-\sum_{j=1}^{\infty} h_j(0) x^j\right)^l = 1 - \frac{z}{\log(1+z)}.$$

Hence $D_k(0) = -G_k$ for $k \geq 1$, which complete the proof. □

Example 3.3. We have

$$Z(1) = -\frac{1}{2}\varpi, \quad Z(2) = \frac{1}{12}\varpi^2, \quad Z(3) = -\frac{1}{24}\varpi^3, \quad Z(4) = \frac{19}{720}\varpi^4.$$

3.2 Algebraic structure

In this subsection, we examine an algebraic structure of $Z(\mathbf{k})$'s and $Z^{\star}(\mathbf{k})$'s.

Recall that the stuffle product $*$ is defined on $\mathfrak{H}^1$. We also use another stuffle product $\star$ characterized by

$$
\begin{aligned}
&1 \star w = w \star 1 = w \quad (w \in \mathfrak{H}^1),\\
&e_k w \star e_{k'} w' = e_k(w \star e_{k'} w') + e_{k'}(e_k w \star w') - e_{k+k'}(w \star w') \quad (k, k' \geq 1,\ w, w' \in \mathfrak{H}^1).
\end{aligned}
$$

In addition we need an extension of $\mathfrak{H}^1$. Let $\mathcal{C} = \mathbb{Q}[\hbar]$ be the polynomial ring of one variable $\hbar$. We set $\widehat{\mathfrak{H}}^1 = \mathcal{C} \otimes_{\mathbb{Q}} \mathfrak{H}^1$ which is viewed as a $\mathcal{C}$-module. Then the $\mathcal{C}$-bilinear maps $*_q, \star_q : \widehat{\mathfrak{H}}^1 \times \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$ are defined by

$$
\begin{aligned}
&1 *_q w = w *_q 1 = w, \qquad 1 \star_q w = w \star_q 1 = w,\\
&e_{k_1} v *_q e_{k_2} w = e_{k_1}(v *_q e_{k_2} w) + e_{k_2}(e_{k_1} v *_q w) + (e_{k_1+k_2} + \hbar\, e_{k_1+k_2-1})(v *_q w),\\
&e_{k_1} v \star_q e_{k_2} w = e_{k_1}(v \star_q e_{k_2} w) + e_{k_2}(e_{k_1} v \star_q w) - (e_{k_1+k_2} + \hbar\, e_{k_1+k_2-1})(v \star_q w)
\end{aligned}
$$

for $v, w \in \widehat{\mathfrak{H}}^1$ and $k_1, k_2 \geq 1$.

For simplicity, we introduce the following notation. Let γ be a function defined on the set of indices taking values in a $\mathbb{Q}$-module M. Then, by abuse of notation, we denote by the same letter γ the $\mathbb{Q}$-linear map $\mathfrak{H}^1 \to M$ which sends $e_{\mathbf{k}}$ to $\gamma(\mathbf{k})$. Similarly, for a function Γ taking values in a $\mathcal{C}$-module $\widehat{M}$, we denote the induced $\mathcal{C}$-linear map $\widehat{\mathfrak{H}}^1 \to \widehat{M}$ by the same letter Γ. For example, $\Gamma(e_2 *_q e_1) = \Gamma(2,1) + \Gamma(1,2) + \Gamma(3) + \hbar\,\Gamma(2)$.

We define the $\mathbb{Q}$-linear action of $\mathcal{C}$ on $\mathcal{A}^{\mathrm{cyc}}$ by $\hbar z = \varpi z$ $(z \in \mathcal{A}^{\mathrm{cyc}})$, where ϖ is given by (3.2). Then the $\mathcal{C}$-linear maps $Z, Z^\star : \widehat{\mathfrak{H}}^1 \to \mathcal{A}^{\mathrm{cyc}}$ are defined by the properties $Z(e_{\mathbf{k}}) = Z(\mathbf{k})$ and $Z^\star(e_{\mathbf{k}}) = Z^\star(\mathbf{k})$ for any index $\mathbf{k}$. They satisfy

$$
Z(v *_q w) = Z(v)Z(w), \qquad Z^\star(v \star_q w) = Z^\star(v)Z^\star(w) \tag{3.4}
$$

for any $v, w \in \widehat{\mathfrak{H}}^1$ (see [1, §2]).

Lemma 3.4. *For any index* $\mathbf{k}$*, we have*

$$
\begin{aligned}
\varpi\, Z(\mathbf{k}) &= -\frac{2}{2\,\mathrm{dep}(\mathbf{k}) + 1}\, Z(e_1 * e_{\mathbf{k}}),\\
\varpi\, Z^\star(\mathbf{k}) &= \frac{2}{2\,\mathrm{dep}(\mathbf{k}) - 1}\, Z^\star(e_1 \star e_{\mathbf{k}}).
\end{aligned}
$$

Proof. It holds that

$$
e_1 *_q e_{\mathbf{k}} = e_1 * e_{\mathbf{k}} + \hbar\,\mathrm{dep}(\mathbf{k}) e_{\mathbf{k}}, \quad e_1 \star_q e_{\mathbf{k}} = e_1 \star e_{\mathbf{k}} - \hbar\,\mathrm{dep}(\mathbf{k}) e_{\mathbf{k}}
$$

for any index $\mathbf{k}$. Now the desired formula follows from (3.4) and $Z(1) = -\varpi/2$. □

Motivated by Lemma 3.4 we define the $\mathbb{Q}$-linear maps $L, L^\star : \mathfrak{H}^1 \to \mathfrak{H}^1$ by

$$L(e_{\mathbf{k}}) = -\frac{2}{2\,\mathrm{dep}(\mathbf{k})+1}\, e_1 * e_{\mathbf{k}}, \quad L^\star(e_{\mathbf{k}}) = \frac{2}{2\,\mathrm{dep}(\mathbf{k})-1}\, e_1 \star e_{\mathbf{k}}$$

for any index $\mathbf{k}$. Note that if $\mathrm{wt}(\mathbf{k}) = k$ then $L(e_{\mathbf{k}})$ and $L^\star(e_{\mathbf{k}})$ are written as a $\mathbb{Q}$-linear combination of monomials of weight $k+1$. Using these maps we introduce the $\mathbb{Q}$-linear maps $\rho, \rho^\star : \widehat{\mathfrak{H}}^1 \to \mathfrak{H}^1$ defined by

$$\rho(\hbar^k w) = L^k(w), \quad \rho^\star(\hbar^k w) = (L^\star)^k(w) \qquad (k \geq 0,\, w \in \mathfrak{H}^1)\,,$$

with $L^0(w) = (L^\star)^0(w) = w$. Note that $\rho(v) = v$ for $v \in \mathfrak{H}^1$ and by Lemma 3.4 we get

$$Z(\rho(w)) = Z(w), \quad Z^\star(\rho^\star(w)) = Z^\star(w) \qquad (w \in \widehat{\mathfrak{H}}^1). \tag{3.5}$$

Now define the $\mathbb{Q}$-bilinear maps $\tilde{*}, \tilde{\star} : \mathfrak{H}^1 \times \mathfrak{H}^1 \to \mathfrak{H}^1$ by

$$v \,\tilde{*}\, w = \rho(v *_q w), \quad v \,\tilde{\star}\, w = \rho^\star(v \star_q w) \qquad (v, w \in \mathfrak{H}^1)$$

and define for $d \geq 0$ the space

$$\mathfrak{H}^1_d = \bigoplus_{\substack{\mathbf{k} \\ \mathrm{wt}(\mathbf{k})=d}} \mathbb{Q}\, e_{\mathbf{k}}\,,$$

which is a $\mathbb{Q}$-linear subspace of $\mathfrak{H}^1$.

Proposition 3.5. *(i) It holds that* $\mathfrak{H}^1_{d_1} \,\tilde{*}\, \mathfrak{H}^1_{d_2} \subset \mathfrak{H}^1_{d_1+d_2}$ *and* $\mathfrak{H}^1_{d_1} \,\tilde{\star}\, \mathfrak{H}^1_{d_2} \subset \mathfrak{H}^1_{d_1+d_2}$ *for* $d_1, d_2 \geq 0$.

(ii) For $v, w \in \mathfrak{H}^1$*, it holds that* $Z(v \,\tilde{*}\, w) = Z(v)Z(w)$ *and* $Z^\star(v \,\tilde{\star}\, w) = Z^\star(v)Z^\star(w)$.

Proof. (i) Note that, if we define the weight of $\hbar$ to be one, then the $\mathcal{C}$-bilinear maps $*_q$ and $\star_q$ preserve the total weight. Hence the statement follows from the property $L(\mathfrak{H}^1_d) \subset \mathfrak{H}^1_{d+1}$ and $L^\star(\mathfrak{H}^1_d) \subset \mathfrak{H}^1_{d+1}$.

(ii) This follows from (3.4) and (3.5). □

Corollary 3.6. *For positive integers* k, k'*, let* $\mathbf{k}$ *and* $\mathbf{k}'$ *be indices of weight* k *and* k'*. Then the product* $Z(\mathbf{k})Z(\mathbf{k}')$ *(resp.* $Z^\star(\mathbf{k})Z^\star(\mathbf{k}')$*) can be written as* $\mathbb{Q}$*-linear combinations of* $Z(\mathbf{a})$*'s (resp.* $Z^\star(\mathbf{a})$*'s) of weight* $k+k'$.

3.3 Linear relations

In this subsection we discuss the dimension of the $\mathbb{Q}$-vector space spanned by $Z(\mathbf{k})$'s and $Z^\star(\mathbf{k})$'s. First we note the following fact.

Proposition 3.7. *For any $k \geq 0$, it holds that $Z(\mathfrak{H}^1_k) = Z^\star(\mathfrak{H}^1_k)$ as a $\mathbb{Q}$-linear subspace of $\mathcal{A}^{\mathrm{cyc}}$.*

Proof. From (2.1) we see that $Z^\star(\mathbf{k})$ is represented as

$$Z^\star(\mathbf{k}) = \sum_{\substack{\mathbf{k}' \\ \mathrm{wt}(\mathbf{k}') \leq \mathrm{wt}(\mathbf{k})}} c_{\mathbf{k},\mathbf{k}'} \varpi^{\mathrm{wt}(\mathbf{k})-\mathrm{wt}(\mathbf{k}')} Z(\mathbf{k}'),$$

where $c_{\mathbf{k},\mathbf{k}'} \in \mathbb{Q}$. Lemma 3.4 implies that $\varpi^{\mathrm{wt}(\mathbf{k})-\mathrm{wt}(\mathbf{k}')} Z(\mathbf{k}') = Z(L^{\mathrm{wt}(\mathbf{k})-\mathrm{wt}(\mathbf{k}')}(e_{\mathbf{k}'}))$, and the weight of $L^{\mathrm{wt}(\mathbf{k})-\mathrm{wt}(\mathbf{k}')}(e_{\mathbf{k}'})$ is equal to $\mathrm{wt}(\mathbf{k})$. Hence $Z^\star(\mathbf{k}) \in Z(\mathfrak{H}^1_k)$ for any index $\mathbf{k}$ of weight k. In the same way we see that $Z(\mathbf{k}) \in Z^\star(\mathfrak{H}^1_k)$ if $\mathrm{wt}(\mathbf{k}) = k$ from (2.2) and therefore $Z(\mathfrak{H}^1_k) = Z^\star(\mathfrak{H}^1_k)$. □

For $k \geq 0$ we define the $\mathbb{Q}$-linear subspace $\mathcal{Z}^{\mathrm{cyc}}_k$ of $\mathcal{A}^{\mathrm{cyc}}$ by

$$\mathcal{Z}^{\mathrm{cyc}}_k = Z^\star(\mathfrak{H}^1_k) = Z(\mathfrak{H}^1_k).$$

There are two families of linear relations in $\mathcal{Z}^{\mathrm{cyc}}_k$. First we have the duality below as a consequence of Theorem 1.3:

Theorem 3.8. *For any index $\mathbf{k}$, it holds that*

$$Z^\star(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})+1} Z^\star(\overline{\mathbf{k}^\vee}).$$

Combining this with Proposition 3.5 (ii), we obtain a variant of the double shuffle relation [8] among $Z^\star(\mathbf{k})$'s. To describe it, we denote by δ the $\mathbb{Q}$-linear map $\delta : \mathfrak{H}^1 \to \mathfrak{H}^1$ sending $e_{\mathbf{k}}$ to $(-1)^{\mathrm{wt}(\mathbf{k})+1} e_{\overline{\mathbf{k}^\vee}}$ for any index $\mathbf{k}$. Note that the map δ is an involution on $\mathfrak{H}^1$ and with this Theorem 3.8 can be stated as $Z^\star(e_{\mathbf{k}}) = Z^\star(\delta(e_{\mathbf{k}}))$.

Theorem 3.9. *For any indices $\mathbf{k}$ and $\mathbf{k}'$, we have*

$$Z^\star(e_{\mathbf{k}} \mathbin{\tilde{\star}} e_{\mathbf{k}'} - \delta(\delta(e_{\mathbf{k}}) \mathbin{\tilde{\star}} \delta(e_{\mathbf{k}'}))) = 0.$$

Proof. This follows from Proposition 3.5 (ii) and Theorem 3.8 because

$$Z^\star(\delta(\delta(e_{\mathbf{k}}) \mathbin{\tilde{\star}} \delta(e_{\mathbf{k}'}))) = Z^\star(\delta(e_{\mathbf{k}}) \mathbin{\tilde{\star}} \delta(e_{\mathbf{k}'})) = Z^\star(\delta(e_{\mathbf{k}}))Z^\star(\delta(e_{\mathbf{k}'})) = Z^\star(e_{\mathbf{k}})Z^\star(e_{\mathbf{k}'}),$$

which is equal to $Z^\star(e_{\mathbf{k}} \mathbin{\tilde{\star}} e_{\mathbf{k}'})$. □

Remark 3.10. Using Theorem 3.8 and Theorem 3.9, we have the following upper bounds for the dimension of $\mathcal{Z}^{\mathrm{cyc}}_k$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathrm{cyc}} \leq$	1	1	1	2	2	4	5	8	12	17	27	38	57

For prime $p \geq 2$ and $k \geq 0$, we denote by $\mathcal{Z}_k^{(p)}$ the $\mathbb{Q}$-vector space spanned by $z_p(\mathbf{k}; e^{2\pi i/p})$ with $\mathrm{wt}(\mathbf{k}) = k$. Notice that $\dim_{\mathbb{Q}} \mathcal{Z}_k^{(p)} \leq p-1 = [\mathbb{Q}(\zeta_p) : \mathbb{Q}]$. Denote by d_k the numbers in the second column of the above table. By numerical experiments, we observed that for $1 \leq k \leq 12$ we have $\dim_{\mathbb{Q}} \mathcal{Z}_k^{(p)} \geq d_k$ for primes $p > d_k$ up to $p = 113$. We expect that Theorem 3.8 and Theorem 3.9 give all $\mathbb{Q}$-linear relations among the $Z^\star(\mathbf{k})$'s.

3.4 Kaneko–Zagier conjecture revisited

In this subsection we will give a new interpretation of the Kaneko–Zagier conjecture in terms of the cyclotomic analogue of finite multiple zeta values $Z(\mathbf{k})$. Let us first recall the statement of their conjecture.

Conjecture 3.11. *(Kaneko–Zagier [9]) There exists a $\mathbb{Q}$-algebra isomorphism*

$$\begin{aligned} \varphi_{KZ} : \mathcal{Z}_{\mathcal{A}} &\longrightarrow \mathcal{Z}/\zeta(2)\mathcal{Z}, \\ \zeta_{\mathcal{A}}(\mathbf{k}) &\longmapsto \zeta_{\mathcal{S}}(\mathbf{k}) \mod \zeta(2)\mathcal{Z}. \end{aligned}$$

To give a new interpretation of this conjecture, we consider the $\mathbb{Q}$-vector space spanned by all $Z(\mathbf{k})$

$$\mathcal{Z}^{\mathrm{cyc}} = Z^\star(\mathfrak{H}^1) = Z(\mathfrak{H}^1).$$

By Corollary 3.6 this is a $\mathbb{Q}$-subalgebra of $\mathcal{A}^{\mathrm{cyc}}$. The restriction of the map $\varphi : \mathcal{A}^{\mathrm{cyc}} \to \mathcal{A}$ defined in (3.1) to $\mathcal{Z}^{\mathrm{cyc}}$ gives the surjective $\mathbb{Q}$-algebra homomorphism to the $\mathbb{Q}$-algebra $\mathcal{Z}_{\mathcal{A}}$ of finite multiple zeta values denoted by

$$\varphi_{\mathcal{A}} : \mathcal{Z}^{\mathrm{cyc}} \longrightarrow \mathcal{Z}_{\mathcal{A}}\,.$$

For any index $\mathbf{k}$ it is $\varphi_{\mathcal{A}}(Z(\mathbf{k})) = \zeta_{\mathcal{A}}(\mathbf{k})$. On the other hand the relationship of the $Z(\mathbf{k})$ to the symmetrized multiple zeta values is not understood yet, but we expect the following.

Conjecture 3.12. *i) There exists a $\mathbb{Q}$-algebra homomorphism*

$$\begin{aligned} \varphi_{\mathcal{S}} : \mathcal{Z}^{\mathrm{cyc}} &\longrightarrow \mathcal{Z}/\zeta(2)\mathcal{Z}\,, \\ Z(\mathbf{k}) &\longmapsto \zeta_{\mathcal{S}}(\mathbf{k}) \mod \zeta(2)\mathcal{Z}\,. \end{aligned}$$

ii) The equality $\ker\varphi_{\mathcal{S}} = \ker\varphi_{\mathcal{A}}$ *holds.*

This conjecture is a re-interpretation of the conjecture by Kaneko and Zagier.

Theorem 3.13. *Conjecture 3.12 implies Conjecture 3.11.*

We end this paper by giving some observation on the elements of the ideal $\ker\varphi_{\mathcal{A}}$ in $\mathcal{Z}^{\mathrm{cyc}}$. As an easy consequence of the definition of $\varphi_{\mathcal{A}}$ we obtain the following.

Proposition 3.14. *We have* $\ker\varphi_{\mathcal{A}} = \mathcal{Z}^{\mathrm{cyc}} \cap \varpi\mathcal{A}^{\mathrm{cyc}}$*, where* $\varpi\mathcal{A}^{\mathrm{cyc}}$ *denotes the ideal of* $\mathcal{A}^{\mathrm{cyc}}$ *generated by* ϖ*.*

Proof. This is immediate from $\ker\varphi = \varpi\mathcal{A}^{\mathrm{cyc}}$. □

We now examine a class of elements in the ideal $\ker\varphi_{\mathcal{A}}$. Lemma 3.4 implies that $\varpi\mathcal{Z}^{\mathrm{cyc}} \subset \mathcal{Z}^{\mathrm{cyc}}$. Hence $\varpi\mathcal{Z}^{\mathrm{cyc}} \subset \ker\varphi_{\mathcal{A}}$. However we expect $\varpi\mathcal{Z}^{\mathrm{cyc}} \neq \ker\varphi_{\mathcal{A}}$. For example, by [6, Theorem 7.1] we have

$$\zeta_{\mathcal{A}}(4,1) - 2\,\zeta_{\mathcal{A}}(3,1,1) = 0. \tag{3.6}$$

Therefore $Z(4,1) - 2Z(3,1,1) \in \ker\varphi_{\mathcal{A}}$. If $\varphi_{\mathcal{A}} = \varpi\mathcal{Z}^{\mathrm{cyc}}$, this would imply that there exists a relation of the form

$$a_4(p)z_p(4,1;\zeta_p) + a_{3,1,1}(p)z_p(3,1,1;\zeta_p) = \sum_{\substack{\mathbf{k}\\ \mathrm{wt}(\mathbf{k})\leq 4}} a_{\mathbf{k}}(p)(1-\zeta_p)^{5-\mathrm{wt}(\mathbf{k})} z_p(\mathbf{k};\zeta_p) \tag{3.7}$$

for large prime p, with $a_{\mathbf{k}}(p) \in \mathbb{Z}[p]$. On the other hand we observed for prime $5 < p \leq 113$ by numerical computations that $(1-\zeta_p)^4 z_p(1;\zeta_p)$, $(1-\zeta_p)^2 z_p(2,1;\zeta_p)$, $z_p(4,1;\zeta_p)$ and $z_p(3,1,1;\zeta_p)$ seem to form a basis of the $\mathbb{Q}$-vector space spanned by all $z_p(\mathbf{k};\zeta_p)$ of weight 5. We therefore do not believe that a relation of the form (3.7) exists and expect that $Z(4,1) - 2Z(3,1,1)$ is an element in $\ker\varphi_{\mathcal{A}}$ but not in $\varpi\mathcal{Z}^{\mathrm{cyc}}$.

So far it is not known how to describe the elements in $(\ker\varphi_{\mathcal{A}})\backslash\varpi\mathcal{Z}^{\mathrm{cyc}}$ in general. For example, because of (3.6), we should have

$$z_p(4,1;\zeta_p) - 2z_p(3,1,1;\zeta_p) \overset{?}{\in} p\mathbb{Z}[\zeta_p]$$

for all large prime p, which seems to be difficult to prove.

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