A Local Courant Theorem on Real Analytic Manifolds

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Abstract

Let M be a closed real analytic Riemannian manifold. We estimate from below the volume of a nodal domain component in an arbitrary ball, provided that this component enters the ball deeply enough. The proof combines a generalized form of Hadamard's Three Circles Theorem due to Nadirashvili, Rapid Growth of Eigenfunctions in Narrow Domains and the Donnelly-Fefferman Growth Bound. The estimates are almost sharp and improve the estimates obtained from the smooth case by Donnelly-Fefferman, Chanillo-Muckenhoupt and Lu.

1 Introduction and Main Results

Let (M, g) be a closed C^∞-Riemannian Manifold of dimension n. Let Δ be the Laplace–Beltrami operator on M. We consider the eigenvalue equation

$$\Delta\varphi_\lambda = \lambda\varphi_\lambda \tag{1.1}$$

The null set $\{\varphi_\lambda = 0\}$ is called the λ-*nodal set* and any connected component of the set $\{\varphi_\lambda \neq 0\}$ is called a λ-*nodal domain*. Throughout this paper $C_1, C_2, \ldots$ will denote positive constants depending only on the metric g.

In [DF90] H. Donnelly and C. Fefferman prove a local version of Courant's Nodal Domain Theorem. Their estimates were later improved by S. Chanillo and B. Muckenhoupt and by G. Lu.

Theorem 1.2 ([DF90, CM91, Lu93])**.** *Let φ_λ be as above. Let B be any ball in M, and let Ω_λ be a connected component of $\{\varphi_\lambda \neq 0\} \cap B$. If $\Omega_\lambda \cap \frac{1}{2}B \neq \emptyset$ then*

$$\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B)} \geq \frac{C_1}{(\sqrt{\lambda})^{\alpha(n)}(\log\lambda)^{4n}} \ ,$$

where $\alpha(n) = 4n^2 + n/2$. *Here,* $\frac{1}{2}B$ *is a concentric ball of half the radius of* B.

Given a ball $B \subseteq M$, this theorem gives an upper bound on the number of "deep" components of nodal domains in B, where a *deep* component is a component which intersects $\frac{1}{2}B$. For this reason, this theorem is called a "Local Courant Theorem".

The present paper is concerned with the Local Courant Theorem in the case where (M, g) is a closed *real analytic* Riemannian Manifold. We show

Theorem 1.3. *Let* (M, g) *be a closed real analytic Riemannian manifold. Let* φ_λ *be as above. Let* $B \subseteq M$ *be an arbitrary ball of radius* R, *and let* Ω_λ *be a deep connected component of* $\{\varphi_\lambda \neq 0\} \cap B$. *Then*

$$\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B)} \geq \frac{C_2}{(\sqrt{\lambda})^{2n-2}(R^*)^{n-1}(\log \lambda)^{n-1}} \ ,$$

where $R^* = \max\{R, 1/\sqrt{\lambda}\}$.

When we consider balls of arbitrary radius we get the bound

$$\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B)} \geq \frac{C_3}{(\sqrt{\lambda})^{2n-2}(\log \lambda)^{n-1}} \ ,$$

which is much better than the known bound for the general smooth case given in Theorem 1.2. An interesting special case is when $R < 1/\sqrt{\lambda}$. Then, we get the lower bound $C/(\sqrt{\lambda}\log\lambda)^{n-1}$, which will be shown to be sharp up to the logarithmic factor. A simple example on a flat torus shows that the sharp estimate for large balls cannot be bigger than $C/(\sqrt{\lambda})^n$. It would be very interesting to understand whether this is the correct bound for balls of arbitrary radius.

The proof of Theorem 1.3 relies on three properties of eigenfunctions described below. Our main innovation comes in replacing the Propagation of Smallness property used in [DF90] by a different much more simple Propagation of Smallness property due to Nadirashvili. In particular, we succeed to eliminate the difficult Carleman type estimates from the proof of Theorem 1.2 in the real analytic case.

Propagation of Smallness. If an eigenfunction is small on a set E contained in a ball B, and $|E|$ is large then the eigenfunction is also small on

B. In our work this principle takes the form of a generalized Hadamard's Three Circles Theorem on real analytic manifolds due to Nadirashvili. We explain this principle in Section 3. The sharp estimate in the Generalized Hadamard Theorem is the main source from which we get the improvement in Theorem 1.3 relative to Theorem 1.2.

Rapid Growth in Narrow Domains. If an eigenfunction vanishes on the boundary of a domain which is long and narrow then the eigenfunction must grow exponentially fast along the direction in which the domain is long. We emphasize that this is true on any *smooth* manifold. This property has been extensively developed and investigated by Landis ([Lan63]) for a certain class of solutions of second order elliptic equations. The version we found in [Lan63] cannot be directly applied to eigenfunctions. A version of it for eigenfunctions but with weaker estimates was proved in ([DF90]). In Section 3 we formulate a sharp version of this property for solutions of second order elliptic equations which can be applied to eigenfunctions. We prove this version in Section 5. The proof combines the ideas from [Lan63] and [DF90]. We replace some arguments from [DF90] by more elementary ones.

Donnelly-Fefferman Growth Bound. For any two concentric balls $B_{r_1} \subseteq B_{r_2} \subseteq M$ of radii r_1, r_2 rescpectively, one has ([DF88])

$$\frac{\sup_{B_{r_2}} |\varphi_\lambda|}{\sup_{B_{r_1}} |\varphi_\lambda|} \leq \left(\frac{r_2}{r_1}\right)^{C_1\sqrt{\lambda}} .$$

In this paper we follow the principle that on balls of small radius with respect to the wavelength $1/\sqrt{\lambda}$ a λ-eigenfunction is almost harmonic. This principle was developed in [DF88], [DF90] and [Nad91]. After rescaling an eigenfunction φ_λ in a ball of radius $\sim 1/\sqrt{\lambda}$, one arrives at a solution φ of a second order self adjoint elliptic operator L in the unit ball $B_1 \subseteq \mathbb{R}^n$, where L has coefficients bounded independently of λ, and φ has bounded growth in the unit ball in terms of λ. This principle is explained in more details in Section 2.

Organization of the paper: In Section 2 we rescale the problem for small balls to a problem on the unit ball in $\mathbb{R}^n$. In Section 3 we explain Propagation of Smallness and Rapid Growth in Narrow Domains. In Section 4 we prove the Local Courant Theorem for small balls rescaled in the unit ball of $\mathbb{R}^n$.

In Section 5 we prove Rapid Growth in Narrow Domains. In Section 6 we apply Donnelly-Fefferman Growth Bound in order to prove Theorem 1.3. In Section 7 we give two examples. The first one is a sequence of spherical harmonics which demonstrate that Theorem 1.3 is sharp up to a logarithmic factor for balls of radius smaller than the wavelength. The second one is a sequence of eigenfunctions on a flat torus which shows that the estimate in Theorem 1.3 cannot be true for large balls.

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2 Passing to the Wavelength Scale

In this section we apply rescaling in order to move from balls $B \subseteq M$ of small radius compared with the wavelength to the unit ball $B_1 \subseteq \mathbb{R}^n$. More details are given in [Man].

Let L be the second order elliptic operator with coefficients defined in the unit ball B_1 by

$$Lu := -\partial_i(a^{ij}\partial_j u) - \varepsilon_0 q u \ , \tag{2.1}$$

where a^{ij}, q are smooth functions, a^{ij} is symmetric and ε_0 is a small positive number. We assume the following ellipticity bounds

$$\kappa_1|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \kappa_2|\xi|^2. \tag{2.2}$$

We also suppose

$$\|a^{ij}\|_{C^1(\overline{B_1})} \leq K, \ |q| \leq K \ , \tag{2.3}$$

and we consider the equation

$$Lu = 0 \ . \tag{2.4}$$

Equation (2.4) with real analytic coefficients will be denoted by (2.4.RA).

Given a function φ on the unit ball and $0 < r < 1$, we define its (r-)growth exponent by

$$\beta_r(\varphi) := \log\left(\frac{\sup_{B_1}|\varphi|}{\sup_{B_r}|\varphi|}\right) . \tag{2.5}$$

The following theorem shows that if a solution φ has a deep positivity component Ω of small volume, then it grows rapidly in B_1. Here, we should emphasize that the growth of φ is measured not only in Ω, but globally in B_1.

Theorem 2.6. *Let φ satisfy (2.4.RA). Suppose $\varphi(0) > 0$, and let $\Omega \subseteq B$ be the connected component of $\{\varphi > 0\}$ which contains 0. Then,*

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_1)} \geq \frac{C_1}{(\beta^*_{\rho_0}(\varphi)\log\beta^*_{\rho_0}(\varphi))^{n-1}},$$

where ρ_0, C_1 depend on κ_1, κ_2, K, n, and where $\beta^*_r = \max\{\beta_r, 3\}$.

In Section 6 we will see that Theorem 2.6 implies Theorem 1.3.

3 Properties of Eigenfunctions

As we explained in the introduction the proof of Theorem 2.6 involves the following ingredients:

- **Propagation of Smallness** due to Nadirashvili. This is in fact a generalization of Hadamard's Three Circles Theorem. Given a subset $E \subseteq B_1$, we define its "radius" by

  $$r(E) := \left(\frac{\mathrm{Vol}(E)}{\mathrm{Vol}(B_1)}\right)^{1/n} . \tag{3.1}$$

  Theorem 3.2 ([Nad76]). *Let φ satisfy (2.4.RA), and assume that $\sup_{B_1}|\varphi| \leq 1$. Let $E \subseteq B_R$. If $\sup_E|\varphi| \leq r(E)^\gamma$ then $\sup_{B_R}|\varphi| \leq (c_0R)^{\sigma\gamma}$, whenever $\gamma > \gamma_0$. γ_0, σ, c_0 depend on κ_1, κ_2, K, n.*

  We notice that this theorem is meaningful only for $R < 1/c_0$.

  In the case where $E = B_R$ and φ satisfies (2.4) this theorem was proved by Gerasimov in [Ger66]. Nadirashvili extended Gerasimov's result for operators with real analytic coefficients by replacing the innermost circle by a general set.

We would like to remark that the proof in [Nad76] is for harmonic functions. When one goes through the proof, one sees that the only point where the harmonicity of φ is used is an interior elliptic regularity estimate which is true for solutions of any second order elliptic operator with real analytic coefficients (See [Hör64, Theorem 7.5.1].

- **Rapid Growth in Narrow Domains**: This property tells that if a solution φ has a deep positivity component Ω of small volume, then φ grows rapidly in Ω.

  Theorem 3.3. *Let φ satisfy (2.4). Suppose that $\varphi(0) > 0$, and let Ω be the connected component of $\{\varphi > 0\}$ containing 0. Let $0 < r_0 \leq 1/2$. If $\mathrm{Vol}(\Omega \cap B_r)/\mathrm{Vol}(B_r) \leq \eta^n$ for all $r_0 < r < 1$, then*

  $$\frac{\sup_\Omega \varphi}{\sup_{\Omega \cap B_{r_0}} \varphi} \geq \left(\frac{1}{r_0}\right)^{\frac{C_1}{\eta^{n/(n-1)}}} .$$

  We emphasize that this theorem is true also in the smooth case. We bring a proof of it in Section 5. One should compare this theorem with [Lan63, Theorem 4.1] and [DF90, pp. 651–652].

4 Proof of Theorem 2.6

In this section we combine Theorem 3.2 and Theorem 3.3 in order to prove Theorem 2.6. We first give the idea of the proof.

Sketch of proof: Rapid growth in narrow domains implies that if $|\varphi| \leq 1$ and $\mathrm{Vol}(\Omega)$ is very small, then φ should be very small on a set E located near the center of the ball B. Then we apply the propagation of smallness principle in order to say that φ is also very small on a ball containing E. Thus, φ must have a large growth exponent β.

We now move to the full proof. The following theorem is a first version of Theorem 2.6.

Theorem 4.1. *Under the assumptions in Theorem 2.6*

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_1)} \geq \left(\frac{C_1 \varepsilon}{\beta^*_{\rho_0}(\varphi)}\right)^{(n-1)/(1-\varepsilon)} .$$

for all $0 < \varepsilon < 1/2$.

Proof of Theorem 4.1. Let $\eta = r(\Omega)$ be the "radius" of Ω as defined in (3.1). We may assume

$$\eta < (\varepsilon/\gamma_1)^{(n-1)/(n(1-\varepsilon))}, \tag{4.2}$$

where γ_1 is a large constant to be specified below. Otherwise, the theorem becomes trivial. Let $c_0 > 1$ be as in Theorem 3.2, and let $\rho_0 = 1/(c_0 \mathrm{e})$. Define

$$r_0 := \sup \left\{ r : \frac{\mathrm{Vol}(\Omega \cap B_r)}{\mathrm{Vol}(B_r)} \geq \left(\frac{\eta}{\rho_0^2} \right)^{n - n\varepsilon} \right\} . \tag{4.3}$$

Observe that

$$r(\Omega \cap B_{r_0}) = r_0 \left(\frac{\eta}{\rho_0^2} \right)^{1-\varepsilon} = r_0 \left(\frac{r(\Omega)}{\rho_0^2} \right)^{1-\varepsilon} \geq r_0 \left(\frac{r(\Omega \cap B_{r_0})}{\rho_0^2} \right)^{1-\varepsilon} .$$

Hence,

$$r_0 \leq \rho_0^{2-2\varepsilon} r(\Omega \cap B_{r_0})^{\varepsilon} \leq \rho_0 r(\Omega \cap B_{r_0})^{\varepsilon} \leq \eta^{\varepsilon} . \tag{4.4}$$

In particular, $r_0 \leq \rho_0$. Theorem 3.3 and Inequality (4.4) give together

$$\frac{\sup_{\Omega \cap B_{r_0}} |\varphi|}{\sup_{\Omega} |\varphi|} \leq r_0^{C_1 (\rho_0^2/\eta)^{n(1-\varepsilon)/(n-1)}} \leq r(\Omega \cap B_{r_0})^{C_2 \varepsilon / \eta^{n(1-\varepsilon)/(n-1)}} . \tag{4.5}$$

By assumption (4.2), the exponent in the right hand side of (4.5) is large enough in order to apply Theorem 3.2 with $E = \Omega \cap B_{r_0}$. Hence,

$$\frac{\sup_{B_{\rho_0}} |\varphi|}{\sup_{B_1} |\varphi|} \leq (c_0 \rho_0)^{\varepsilon C_3 / \eta^{n(1-\varepsilon)/(n-1)}} = \mathrm{e}^{-\varepsilon C_3 / \eta^{n(1-\varepsilon)/(n-1)}} .$$

In other words,

$$\beta_{\rho_0} \geq \frac{\varepsilon C_3}{\eta^{n(1-\varepsilon)/(n-1)}} ,$$

which is equivalent to

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_1)} \geq \left(\frac{\varepsilon C_3}{\beta_{\rho_0}} \right)^{(n-1)/(1-\varepsilon)} .$$

□

Proof of Theorem 2.6. Let $A = \beta^*_{\rho_0}/C_1$. If $A \leq 10$, then Theorem 4.1 shows that $\mathrm{Vol}(\Omega)$ is bounded from below by a positive constant. This immediately implies Theorem 2.6. Otherwise, take $\varepsilon = 1/\log A$ in Theorem 4.1. Observe that

$$\left(\frac{\varepsilon}{A}\right)^{1/(1-\varepsilon)} \geq \frac{\varepsilon}{A}\left(\frac{\varepsilon}{A}\right)^{2\varepsilon} = \frac{\mathrm{e}^{2\varepsilon\log\varepsilon-2}}{A\log A} \geq \frac{\mathrm{e}^{-3}}{A\log A} \ .$$

□

5 Rapid Growth in Narrow Domains

In this section we prove Theorem 3.3. It will follow from the classical growth Lemma:

Let φ satisfy (2.4). Let $B^y_R = B(y,R) \subseteq B_1$. Suppose $\varphi(y) > 0$, and let Ω_y be the connected component of $\{\varphi > 0\} \cap B^y_R$ which contains y. The Growth Lemma is:

Lemma 5.1 ([Lan63, Theorem 4.1], [DF90, pp. 651–652])**.** *For all $A > 1$ there exists $\gamma(A)$ such that if*

$$\frac{\mathrm{Vol}(\Omega_y)}{\mathrm{Vol}(B^y_R)} \leq \gamma(A) \ ,$$

then

$$\frac{\sup_{\Omega_y}|\varphi|}{\sup_{\Omega_y\cap B^y_{R/2}}|\varphi|} \geq A \ .$$

In particular, γ does not depend on R, neither on y.

We give a proof of this lemma in Section 5.1. Its proof is based on ideas from the proof in [DF90], where we replaced several arguments by more elementary ones.

As a corollary of the preceding lemma, we obtain a first version of rapid growth in narrow domains:

Theorem 5.2. *Let φ satisfy (2.4). Let Ω be a connected component of $\{\varphi > 0\}$ which intersects $B_{1/2}$. Let $\eta > 0$ be small enough. If*

$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_1)} \leq \eta^n \ ,$$

then

$$\frac{\sup_\Omega |\varphi|}{\sup_{\Omega\cap B_{1/2}} |\varphi|} \geq e^{\frac{C_1}{\eta^{n/(n-1)}}} .$$

Proof of Theorem 5.2. We notice that Ω touches ∂B_1. Otherwise, since ε_0 is small, by the maximum principle ([GT83, Cor. 3.8]) φ is identically 0 in Ω, which contradicts the definition of Ω.

We decompose $B_1 \setminus B_{1/2}$ into N equally distanced spherical layers, where N will be chosen below. Let $t_k = (1/2+k/(2N))$, $k = 0 \ldots N$. Let $A_0 = B_{1/2}$, and $A_k = B_{t_k} \setminus B_{t_{k-1}}$ for $k = 1 \ldots N$. Set $\widetilde{A_0} = A_0 \cup A_1$, $\widetilde{A_N} = A_{N-1} \cup A_N$, and

$$\widetilde{A_k} = A_{k-1} \cup A_k \cup A_{k+1}$$

for $1 \leq k \leq N-1$. There exist $\geq N/2$ values of $1 \leq k \leq N$ for which

$$\mathrm{Vol}(\Omega \cap \widetilde{A_k}) \leq 6\mathrm{Vol}(\Omega)/N .$$

Let l be any one of these values. Let $y \in \Omega \cap A_l$, and let $R = 1/(2N)$. Consider the ball $B^y_R = B(y, R)$. $B^y_R \subseteq \widetilde{A}_l$, and we check that

$$\frac{\mathrm{Vol}(\Omega \cap B^y_R)}{\mathrm{Vol}(B^y_R)} \leq \frac{\mathrm{Vol}(\Omega \cap \widetilde{A}_l)}{\mathrm{Vol}(B_1)(1/(2N))^n} \leq \frac{6\mathrm{Vol}(\Omega)(2N)^n}{N\mathrm{Vol}(B_1)} \leq 6(2\eta)^n N^{n-1} . \quad (5.3)$$

Set $A = 10e/9$ and take $N = \lfloor(\gamma(A)/(6(2\eta)^n))^{1/(n-1)}\rfloor$. Inequality (5.3) and the Growth Lemma applied in B^y_R with $\Omega_y \subseteq \Omega \cap B^y_R$ show that

$$\sup_{\Omega\cap\widetilde{A_l}} \varphi \geq \sup_{\Omega\cap B^y_{R/2}} \varphi \geq A\varphi(y) .$$

Since this is true for all $y \in \Omega \cap A_l$, we get

$$\sup_{\Omega\cap B_{l+1}} \varphi \geq \sup_{\Omega\cap\widetilde{A_l}} \varphi \geq A \sup_{\Omega\cap A_l} \varphi . \quad (5.4)$$

Now we apply the following maximum principle:

Theorem 5.5 ([GT83, Corollary 3.8])**.**

$$\sup_{\Omega\cap A_k} \varphi \geq 0.9 \sup_{\Omega\cap B_k} \varphi ,$$

whenever ε_0 (cf. (2.1)) is small enough.

Hence, from (5.4) we obtain

$$\sup_{\Omega\cap B_{l+1}} \varphi \geq 0.9A \sup_{\Omega\cap B_l} \varphi = \mathrm{e} \sup_{\Omega\cap B_l} \varphi \ .$$

And since this is true for $\geq N/2$ values of k we finally have

$$\frac{\sup_\Omega \varphi}{\sup_{\Omega\cap B_{1/2}} \varphi} \geq \mathrm{e}^{N/2} \geq \mathrm{e}^{C_1/\eta^{n/(n-1)}} \ .$$

□

An iteration of Theorem 5.2 gives Theorem 3.3:

Proof of Theorem 3.3. Let N be a positive integer for which $(1/2)^{N+1} < r_0 \leq (1/2)^N$. $N = \lfloor \log(1/r_0)/\log 2 \rfloor$. Set $t_k = (1/2)^k$. It follows by scaling from Theorem 5.2 that

$$\frac{\sup_{\Omega\cap B_{t_k}} |\varphi|}{\sup_{\Omega\cap B_{t_{k+1}}} |\varphi|} \geq \mathrm{e}^{C_1/\eta^{n/(n-1)}} \ ,$$

for all $0 \leq k \leq N-1$. The point is that the bounds (2.2)-(2.3) on the operator L remain true after rescaling. Hence,

$$\frac{\sup_\Omega |\varphi|}{\sup_{\Omega\cap B_{r_0}} |\varphi|} \geq \frac{\sup_\Omega |\varphi|}{\sup_{\Omega\cap B_{t_N}} |\varphi|} \geq \mathrm{e}^{C_1 N/\eta^{n/(n-1)}}$$

$$\geq \mathrm{e}^{C_1 \log(1/r_0)/(2\eta^{n/(n-1)}\log 2)} = \left(\frac{1}{r_0}\right)^{\frac{C_2}{\eta^{n/(n-1)}}} \ .$$

□

5.1 Proof of the Growth Lemma

Proof of Lemma 5.1. Let $g(t)$ be a smooth function defined on $\mathbb{R}$ with the following properties

- $g(t) = 0$ for $t \leq 1$,
- $g(t) = t - 2$ for $t \geq 3$,
- $g''(t) \geq 0$.

Let $\delta > 0$ be small and let $g_\delta(t) = \delta g(t/\delta)$. Let $\varphi_\delta = (g_\delta \circ \varphi) \cdot \chi_{\Omega_y}$, where χ_{Ω_y} is the characteristic function of Ω_y. φ_δ is a smooth function with compact support in Ω_y. We notice that $0 \leq \varphi\chi_{\Omega_y} - \varphi_\delta \leq 2\delta$. In particular, $\varphi_\delta \to \varphi\chi_{\Omega_y}$ uniformly as $\delta \to 0$. We now calculate $L\varphi_\delta$:

$$L\varphi_\delta = \chi_{\Omega_y}\varepsilon_0 q\varphi \cdot (g'_\delta \circ \varphi) - \varepsilon_0 q\varphi_\delta \ .$$

Let us denote the right hand side by f.

Lemma 5.6.

$$\|f\|_{L^n(B)} \leq 2\varepsilon_0\delta K|\Omega_y|^{1/n}$$

We postpone the proof of this lemma to the end of this section. Recall the following local maximum principle:

Theorem 5.7 ([GT83, Theorem 9.20])**.** *Suppose $Lu \leq f$ in $B^y_R \subseteq B_1$. Then,*

$$\sup_{B^y_{R/2}} u \leq \frac{C_1}{|B^y_R|}\int_{B^y_R} u^+ \, dx + C_2\|f\|_{L^n(B^y_R)} \ ,$$

where C_1, C_2 depend only on κ_1, κ_2 and K.

Applying the local maximum principle to φ_δ gives

$$\sup_{B^y_{R/2}} \varphi_\delta \leq \frac{C_1}{|B^y_R|}\int_{B^y_R} \varphi_\delta \, dx + 2C_2\delta\varepsilon_0 K|\Omega_y|^{1/n} \ .$$

Letting $\delta \to 0$ we obtain that

$$\sup_{\Omega_y \cap B^y_{R/2}} \varphi \leq \frac{C_1}{|B^y_R|}\int_{\Omega_y} \varphi \, dx \leq \frac{C_1|\Omega_y|}{|B^y_R|}\sup_{\Omega_y} \varphi$$

Thus, we may take $\gamma(A) = 1/(C_1 A)$. $\square$

To complete the proof of the Growth Lemma 5.1 it remains to prove Lemma 5.6.

proof of Lemma 5.6. When $\varphi \geq 3\delta$, $f = 2\varepsilon_0\delta q\chi_{\Omega_y}$, and when $\varphi \leq \delta$, $f = 0$. We notice that $g_\delta(t) \geq t - 2\delta$ and $g'_\delta(t) \leq 1$. Hence, when $\delta \leq \varphi \leq 2\delta$, $f \leq 2\varepsilon_0\delta q\chi_{\Omega_y}$, and when $2\delta \leq \varphi \leq 3\delta$, $f \leq \varepsilon_0 q\varphi\chi_{\Omega_y} - \varepsilon_0 q(\varphi - 2\delta)\chi_{\Omega_y} = 2\varepsilon_0\delta q\chi_{\Omega_y}$.

We have shown that $|f| \leq 2\varepsilon_0\delta q\chi_{\Omega_y}$. Integration gives

$$\|f\|_{L^n(B^y_R)} \leq 2\varepsilon_0\delta K|\Omega_y|^{1/n}.$$

$\square$

6 Proof of Theorem 1.3

In this section we show how Theorem 1.3 is implied from Theorem 2.6.

Proof of Theorem 1.3. Let $B \subseteq M$ be a ball of radius $R \leq \sqrt{\varepsilon_0/\lambda}$. Suppose that φ_λ is positive on Ω_λ. Let $y \in \Omega_\lambda \cap \frac{1}{2}B$. Consider the ball $B^y_{R/2}$. By scaling we arrive at a function φ defined in the unit ball $B_1 \subseteq \mathbb{R}^n$. φ satisfies (2.4.RA), and $\varphi(0) > 0$. φ also satisfies the growth bound:

Theorem 6.1 ([DF88]).

$$\beta_r(\varphi) \leq \sqrt{\lambda}\log(C_1/r) \ .$$

If we substitute in Theorem 2.6 the Donnelly-Fefferman Growth Bound we get

$$\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B)} \geq C_2\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B^y_{R/2})} \geq \frac{C_3}{(\sqrt{\lambda}\ \log\lambda)^{n-1}} \ .$$

Let now B be a ball of radius $R > \sqrt{\varepsilon_0/\lambda}$. Recall the Faber-Krahn Inequality:

Theorem 6.2. *Let $\mathcal{A}_\lambda$ be a λ-nodal domain. Then*

$$\mathrm{Vol}(\mathcal{A}_\lambda) \geq \frac{C_2}{(\sqrt{\lambda})^n} \ .$$

Hence, if $\overline{\Omega_\lambda} \subseteq B$, then

$$\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B)} \geq \frac{C_3}{(R\sqrt{\lambda})^n} \geq \frac{C_4}{(\sqrt{\lambda})^n} \ ,$$

which completes the proof.

So, we may assume Ω_λ touches ∂B. We decompose $B \setminus \frac{1}{2}B$ into spherical layers, each of width $\sqrt{\varepsilon_0/\lambda}$. In each spherical layer we can find a ball B' of radius $(\sqrt{\varepsilon_0/\lambda})/2$ such that Ω_λ cuts $\frac{1}{2}B'$. By the preceding step the total volume of Ω_λ is

$$\mathrm{Vol}(\Omega_\lambda) \geq \sum_{B'} \mathrm{Vol}(\Omega_\lambda \cap B') \geq C_5 R\sqrt{\lambda/\varepsilon}\mathrm{Vol}(B')/(\sqrt{\lambda}\ \log\lambda)^{n-1} \ .$$

The last inequality gives

$$\frac{\mathrm{Vol}(\Omega_\lambda)}{\mathrm{Vol}(B)} \geq \frac{C_6}{(\sqrt{R\lambda})^{2n-2}(\log\lambda)^{n-1}} \ ,$$

which completes the proof of Theorem 1.3. ∎

7 Examples

7.1 An Example on $\mathbb{S}^n$

In this section we show that Theorem 1.3 is sharp up to the $(\log\lambda)^{n-1}$ factor. The example we give will be a sequence of spherical harmonics on the standard sphere $\mathbb{S}^n$. Let us denote by $\mathcal{H}_k^n$ the space of spherical harmonics on $\mathbb{S}^n$ of degree k.

Proposition 7.1. *There exists a sequence $(Y_k^n)_{k\geq 1} \in \mathcal{H}_k^n$ with the following properties:*

1. *The number of nodal domains of Y_k^n is $\geq c_{1,n}k^n$.*
2. *There exist $\geq c_{2,n}k^{n-1}$ nodal domains of Y_k which have the north pole on their boundary.*

Corollary 7.2. *For every eigenvalue λ and $r < 1/\sqrt{\lambda}$ there exists an eigenfunction φ_λ, a nodal domain $\mathcal{A}_\lambda$ and a ball B of radius r such that*

$$\frac{\mathrm{Vol}(\mathcal{A}_\lambda \cap B)}{\mathrm{Vol}(B)} \leq \frac{C_3(n)}{(\sqrt{\lambda})^{n-1}} ,$$

and $\mathcal{A}_\lambda \cap \frac{1}{2}B \neq \emptyset$.

Remark. The last corollary shows that the exponent in Theorem 1.3 cannot be improved even if we fix the radii of the balls considered.

Proof. $\lambda = k(k+n-1)$ for some integer $k \geq 0$. Let Y_k^n be as in Proposition 7.1. Let B be a ball of radius $r < 1/k$ centered at the north pole. By Proposition 7.1 that there exists a nodal domain $\mathcal{A}_\lambda$ for which

$$\frac{\mathrm{Vol}(\mathcal{A}_\lambda \cap B)}{\mathrm{Vol}(B)} \leq \frac{C_4(n)}{k^{n-1}} .$$

The result follows since $\lambda \sim k^2$. $\square$

We now prove Proposition 7.1. First, we introduce spherical coordinates and we review elementary facts about spherical harmonics.

Lemma 7.3. *A point on the sphere $\mathbb{S}^n$ is parametrized by $(\theta_1, \ldots, \theta_{n-1}, \varphi)$, where $0 < \theta_l < \pi$, $0 \le \varphi \le 2\pi$, and*

$$\begin{aligned} x_1 &= \cos\theta_1 \ , \\ &\vdots \\ x_{n-1} &= \sin\theta_1 \ldots \sin\theta_{n-2}\cos\theta_{n-1} \ , \\ x_n &= \sin\theta_1 \ldots \sin\theta_{n-1}\cos\varphi \ , \\ x_{n+1} &= \sin\theta_1 \ldots \sin\theta_{n-1}\sin\varphi \ . \end{aligned}$$

We recall the definition of the zonal spherical harmonics and Legendre Polynomials. Details can be found in chapter 3 of [Gro96]. Consider the natural action of the orthogonal group $O(n+1)$ on $\mathbb{S}^n$. It induces a representation of $O(n+1)$ on $\mathcal{H}^n_k$. The *zonal* spherical harmonic $Z^n_{k,p}$ of degree k with pole $p \in \mathbb{S}^n$ is defined as the unique spherical harmonic in $\mathcal{H}^n_k$, which is fixed by the stabilizer of the point p in $O(n+1)$, and admits the value 1 at p. The Legendre polynomial $P^{n+1}_k(t)$ is defined to be the polynomial on $[-1, 1]$, for which

$$Z^n_{k,p_0}(\theta_1, \ldots, \theta_{n-1}, \phi) = P^{n+1}_k(\cos\theta_1) \ ,$$

where p_0 is the north pole. It is easy to see that for any $p \in \mathbb{S}^n$

$$Z^n_{k,p}(x) = P^{n+1}_k(\langle p, x\rangle) \ .$$

Lemma 7.4 ([Gro96, Proposition 3.3.7])**.** *P^n_k is given by*

$$P^n_k(t) = \alpha_n(-1)^k(1-t^2)^{-(n-3)/2}\frac{\partial^k}{\partial t^k}(1-t^2)^{k+\frac{n-3}{2}} \ ,$$

where α_n are some constants which depend on n.

We define also the associated Legendre functions:

$$E^n_{k,j}(t) = (1-t^2)^{j/2}(\partial^j_t P^n_k)(t) \ .$$

The next lemma is an inductive construction of spherical harmonics:

Lemma 7.5 ([Gro96, Lemma 3.5.3])**.** *Given $G \in \mathcal{H}^{n-1}_j$, let*

$$H(\theta_1, \ldots \theta_{n-1}, \varphi) := E^{n+1}_{k,j}(\cos\theta_1)G(\theta_2, \ldots \theta_{n-1}, \varphi) \ .$$

Then, $H \in \mathcal{H}^n_k$.

Proof of Proposition 7.1. We prove it by induction on n. For $n = 1$, we take $Y_k^1(\varphi) = \sin k\varphi$. Suppose the result is true for $n-1$. Set

$$H_{k,j}^n(\theta_1, \ldots, \theta_{n-1}, \varphi) := E_{k,j}^{n+1}(\cos\theta_1) Y_j^{n-1}(\theta_2, \ldots, \theta_{n-1}, \varphi) \ .$$

By Lemma 7.5 $H_{k,j}^n \in \mathcal{H}_k^n$. From Lemma 7.4 one can see that $E_{k,j}^{n+1}$ has exactly $k-j$ distinct zeroes in the interval $(-1,1)$ it follows that the number of nodal domains of H_n^k is $\geq c_{1,n-1}(k-j)j^{n-1}$, of which $c_{1,n-1}j^{n-1}$ touch the north pole. We define $Y_k^n := H_{k,\lfloor k/2 \rfloor}^n$. □

7.2 An Example on $\mathbb{T}^n$

The following example shows that the exponent in Theorem 1.3 is $\geq n$. We do not know whether this is a sharp example.

Let $\mathbb{T}^n$ be a flat torus parametrized by $(x_1, x_2, \ldots x_n)$, where $0 \leq x_k < 2\pi$. Let $\varphi = \Pi_{j=1}^n \sin kx_j$. φ is an eigenfunction corresponding to eigenvalue $\lambda = nk^2$. Each nodal domain is of diameter $< c/k$ and has area $\leq c/k^2$. Hence, if we take a ball $B \subseteq \mathbb{T}^n$ of radius 1 and we let $\mathcal{A}_\lambda$ be a nodal domain close to the center of the ball, we have

$$\frac{\mathrm{Vol}(\mathcal{A}_\lambda \cap B)}{\mathrm{Vol}(B)} \leq \frac{C(n)}{(\sqrt{\lambda})^n} \ .$$

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