DERIVED FUNCTORS OF GRADED THEORIES

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We introduce the notion of a graded theory $\mathbf{T}$ which corresponds to an algebraic theory as studied by Lawvere [13]; see (8.1) below. The models of a graded theory form a category $model(\mathbf{T})$. Many graded algebraic objects in algebra and topology are examples of such models. We show that the category $model(\mathbf{T})$ can be equivalently described by an iterated Grothendieck construction $\Gamma = (\Gamma^1, \Gamma^2, \dots)$. Here $\Gamma^1 : \mathbf{A}_0 \to \mathbf{A}_1$ is a functor and $\Gamma^{n+1} : \Gamma^n \mathbf{A}_n \to \mathbf{A}_{n+1}$ is a functor defined on the Grothendieck construction of $\Gamma^n, n \geq 1$. The categories $\mathbf{A}_i, i \geq 0$, are abelian categories. We study the derived functors $L_i\Gamma^{n+1}$ and we obtain certain vanishing lines for such functors. In section § 10 we consider the derived functors of a functor

$$F : model(\mathbf{T}) \to \mathbf{B}$$

where $\mathbf{B}$ is an abelian category. This yields in particular corollary (10.6) which is a generalized version of the main result 3.1 of Blanc [4]. We introduce new transparent methods to prove the result.

There are many fields of application like

- the Bousfield-Kan spectral sequence [6], [7],
- the Quillen homology of algebras [16], and
- the Blanc homology spectral sequence [3], [4],
- the homotopy operation spectral sequence [18].

We refer the reader to [4] and [18] for the details on these examples. Our motivating example is the spherical graded theory (8.1") obtained by the homotopy category of finite one point unions of spheres S^n with $n \geq 2$. The homotopy groups $\{\pi_n X\}_{n\geq 2}$ of a space describe a model of the spherical theory. See also [9], [10], [17] where such models are called π -algebras. Applications on π -algebras of the theory in this paper are given in [18]. In particular the associated iterated Grothendieck-construction $\Gamma = (\Gamma_1, \Gamma_2, \dots)$ of a graded theory $\mathbf{T}$ and the reduction functors of Γ below turn out to be of crucial importance for the computation of homotopy groups of a space; see [18]. Dual results are also available for the graded theory of Eilenberg-Mac Lane spaces; see $(8.1''')$.

§ 1 Grothendieck construction

We study partial resolutions in a Grothendieck construction $\Gamma\mathbf{A}$ of a functor $\Gamma : \mathbf{C} \to \mathbf{A}$. The results of this section will be used inductively in section § 2 where we consider iterated Grothendieck constructions. This leads to a vanishing line of derived functors in § 3.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let $\mathbf{A}$ be an abelian category, for example $\mathbf{A} = \mathbf{Ab}$ the category of abelian groups. We say that an epimorphism in $\mathbf{A}$ is a surjective map in $\mathbf{A}$ and similarly we say that a monomorphism in $\mathbf{A}$ is an injective map in $\mathbf{A}$.

(1.1) Definition. Let $\mathbf{C}$ be a category and let $\Gamma : \mathbf{C} \to \mathbf{A}$ be a functor. Then we obtain the category $\Gamma\mathbf{A}$ as follows. Objects are triple (X, A, η) where X is an object in $\mathbf{C}$ and where $\eta : \Gamma(X) \to A$ is a morphism in $\mathbf{A}$. Morphisms $(X, A, \eta) \to (Y, B, \lambda)$ in $\Gamma\mathbf{A}$ are pairs (f, g) where $f : X \to Y$ is a morphism in $\mathbf{C}$ such that the diagram

$$\begin{tikzcd} \Gamma(X) \arrow[r, "\Gamma(f)"] \arrow[d, "\eta"] & \Gamma(Y) \arrow[d, "\lambda"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

commutes in $\mathbf{A}$. We call $\Gamma\mathbf{A}$ the *Grothendieck construction* of Γ. An object (X, A, η) in $\Gamma\mathbf{A}$ is also denoted by $\eta = (X, A, \eta)$.

Recall that a *coequalizer* of a pair $f, g : X \to Y$ of morphisms in a category $\mathbf{C}$ is a morphism $u : Y \to U$ such that $uf = ug$ and for all $\alpha : Y \to Z$ in $\mathbf{C}$ with $\alpha f = \alpha g$ there is a unique $\bar{\alpha} : U \to Z$ with $\alpha = \bar{\alpha} u$. A functor $\Gamma : \mathbf{C} \to \mathbf{C}'$ *preserves* a coequalizer diagram

(1.2) $$X \rightrightarrows Y \to U$$

in $\mathbf{C}$ if the induced diagram

$$\Gamma X \rightrightarrows \Gamma Y \to \Gamma U$$

is a coequalizer diagram in $\mathbf{C}'$. One readily checks that coequalizers exist in an abelian category $\mathbf{A}$ and that an exact sequence

(1.3) $$A_1 \xrightarrow{i} A_0 \xrightarrow{q} A \to o$$

in $\mathbf{A}$ corresponds to a coequalizer diagram

(2) $$A_1 \overset{f}{\underset{g}{\rightrightarrows}} A_0 \xrightarrow{q} A$$

where $f = 0$ and $g = i$.

We have canonical functors

(1.4) $$\begin{tikzcd} & \mathbf{C} \times \mathbf{A} & \\ \mathbf{C} \arrow[r, "I_1"] & \Gamma\mathbf{A} \arrow[u, "\tilde{Q}"] \arrow[d, "Q"] & \mathbf{C} \times \mathbf{A} \arrow[l, "I_o"'] \\ & \mathbf{A} & \end{tikzcd}$$

Here Q carries the object η in $\Gamma\mathbf{A}$ to the cokernel

$$Q(\eta) = \mathrm{cok}(\eta : \Gamma(X) \to A) \tag{1}$$

of η in the abelian category $\mathbf{A}$ and $\tilde{Q}$ carries (X, A, η) to the pair $(X, Q\eta)$. Moreover I_0 and I_1 are full inclusions of categories. The functor I_1 carries X to the object

$$I_1(X) = (X, \Gamma X, 1_{\Gamma X}) \tag{2}$$

given by the identity $1_{\Gamma X}$ of ΓX. The functor I_0 carries the pair (X, A) in $\mathbf{C} \times \mathbf{A}$ to the object

$$I_0(X, A) = (X, A, 0) \tag{3}$$

given by the zero homomorphism $0 : \Gamma(X) \to A$. Clearly $\tilde{Q}I_0$ is the identical functor. We also write

$$\begin{aligned} I_0(X) = I_0(X, 0) &= (X, 0, 0) \quad \text{for} \quad X \in \mathbf{C}, \\ I_0(A) = I_0(*, A) &= (*, A, 0) \quad \text{for} \quad A \in \mathbf{A}. \end{aligned}$$

For each object (X, A, η) in $\Gamma\mathbf{A}$ we have the commutative diagram

$$\begin{tikzcd} \Gamma(X) \arrow[r, equal] \arrow[d, "1"'] & \Gamma(X) \arrow[r, equal] \arrow[d, "\eta"] & \Gamma(X) \arrow[d, "0"] & \\ \Gamma(X) \arrow[r, "\eta"] & A \arrow[r, "q"] & Q(\eta) \arrow[r] & 0 \end{tikzcd} \tag{4}$$

in which the bottom row is an exact sequence in $\mathbf{A}$. This gives us the canonical morphisms

$$I_1(X) \xrightarrow{(1,\eta)} (X, A, \eta) \xrightarrow{(1,q)} I_0(X, Q(\eta)) \tag{1.5}$$

in $\Gamma\mathbf{A}$.

We now consider chain complexes and simplicial objects. Let *chain* $(\mathbf{A})$ be the category of chain complexes A_* in $\mathbf{A}$ with $A_i = 0$ for $i < 0$. Moreover for a category $\mathbf{C}$ let $s\mathbf{C}$ be the category of simplicial objects $X.$ in $\mathbf{C}$. By the Dold-Kan theorem we have isomorphisms of categories

$$\mathit{chain}\,(\mathbf{A}) \underset{N}{\overset{K}{\rightleftarrows}} s\mathbf{A} \tag{1.6}$$

where N is the 'normalization functor' with $NK \cong \mathrm{id}$ and $KN \cong \mathrm{id}$. Here id is the identical functor. Compare [8]. For a chain complex A_* the *homology* H_iA_* is defined. We also write for a simplicial object $A.$ in $s\mathbf{A}$

(1.7) $$\pi_i(A.) = H_i(NA.)$$

with $i \geq 0$. These are the *homotopy groups* of $A.$. For a chain complex A_* the corresponding object in degree n is denoted by A_n. For a simplicial object $A.$, however, the object in degree n is denoted by $A_{(n)}$. We write

(1) $$length(A.) \leq n$$

if the chain complex $NA.$ satisfies $(NA.)_k = 0$ for $k > n$. That is, $NA.$ is a chain complex concentrated in degree $0, \dots, n$. Clearly for an exact sequence

$$A. \to B. \to C. \to 0$$

in $s\mathbf{A}$ we have

(2) $$length(B.) \leq \mathrm{Max}\{length(A.), length(C.)\}$$

This is a consequence of the exactness of the functor N_n, see [8].

(1.8) Definition. A *resolution* of an object A in $\mathbf{A}$ is a chain complex A_* in *chain* $(\mathbf{A})$ with $H_0A_* = A$ and $H_i(A_*) = 0$ for $i > 0$. This is a *projective resolution* if all A_i in A_* are projective in $\mathbf{A}$. We say that $\mathbf{A}$ has *enough projectives* if each object A in $\mathbf{A}$ has a projective resolution. Moreover we say that $\mathbf{A}$ has *global dimension* $gd(\mathbf{A}) \leq n$ if each object A in $\mathbf{A}$ has a projective resolution A_* with $A_i = 0$ for $i > n$. Hence $gd(\mathbf{Ab}) = 1$. More generally for any subring R of the rationals $\mathbb{Q}$ the category of R-modules has global dimension ≤ 1. A category of vector spaces has global dimension $= 0$.

(1.9) Definition. Let $\Gamma\mathbf{A}$ be the Grothendieck construction of $\Gamma : \mathbf{C} \to \mathbf{A}$ as in (1.1). A simplicial object

$$\eta. = (X., A., \eta.)$$

in $s\Gamma\mathbf{A}$ is given by a simplicial map

$$\eta. : \Gamma(X.) \to A.$$

in $s\mathbf{A}$ with $X. \in s\mathbf{C}$ and $A. \in s\mathbf{A}$. We say that $\eta.$ is a *partial resolution* if the map

$$N\eta. : N\Gamma(X.) \to NA. \in chain\,(\mathbf{A})$$

induced by the normalization functor N has the following properties (a), (b) and (c):

(a) $N\eta.$ is injective,
(b) the cokernel of $N\eta.$ is projective in each degree,
(c) the chain complex $NA.$ is a resolution of $A = H_0NA.$.

Now assume that the coequalizer diagram

$$X_{(1)} \rightrightarrows X_{(0)} \to X$$

exists in the category $\mathbf{C}$ where $X_{(1)} \rightrightarrows X_{(0)}$ is given by the simplicial object $X.$. Then we write

$$X = \pi_0(X.)$$

We have a simplicial map $X. \to X$ (where X is the constant simplicial object given by X) which induces the map

$$q : \pi_0(\Gamma X.) \to \pi_0 \Gamma X = \Gamma X$$

We say that $\eta.$ above is an $X.$ *-partial resolution of* $\eta : \Gamma X \to A \in \Gamma\mathbf{A}$ if the diagram

$$\begin{tikzcd} \pi_0\Gamma X. \arrow[r, "(\eta.)_*"] \arrow[dd, "q"'] & \pi_0 A. \arrow[d, equal] \\ & H_0 N A. \arrow[d, equal] \\ \Gamma X \arrow[r, "\eta"] & A \end{tikzcd}$$

commutes in $\mathbf{A}$.

(1.10) Proposition. *Assume* $\mathbf{A}$ *has enough projectives. Then for each simplicial object* $X.$ *in* $s\mathbf{C}$ *and for each* $\eta : \Gamma(\pi_0 X.) \to A$ *in* $\mathbf{A}$ *there exists an* $X.$ *-partial resolution* $\eta. : \Gamma X. \to A.$ *of* η*. Moreover the exact sequence in* $s\mathbf{A}$

$$0 \to \Gamma X. \xrightarrow{\eta.} A. \xrightarrow{q.} Q\eta. \to 0$$

induces an isomorphism

$$\partial : \pi_{i+1}(Q\eta.) \cong \pi_i(\Gamma X.) \quad \textit{for} \quad i \geq 1$$

and an exact sequence

$$0 \to \pi_1(Q\eta.) \to \pi_0\Gamma(X.) \xrightarrow{\eta q} A \to \pi_0(Q\eta.) \to 0$$

Proof. Since $\mathbf{A}$ has enough projectives we know by the arguments in (I.6.12) in [2] that $chain(\mathbf{A})$ is a cofibration category. See also [15]. Weak equivalences are homology isomorphisms and cofibrations are injective maps $X_* \to A_*$ for which the cokernel is projective. Now consider the chain map

$$\tilde{\eta} : N(\Gamma X.) \to A$$

Here A is the chain complex concentrated in degree 0 given by A and $\tilde{\eta}$ is the chain map for which the following diagram commutes

$$\begin{array}{ccc} H_0(N\Gamma X.) & \xrightarrow{(\tilde{\eta})_*} & H_0 A \\ \| & & \\ \pi_0 \Gamma X. & & \| \\ q\downarrow & & \\ \Gamma \pi_0 X. & \xrightarrow{\eta} & A \end{array}$$

Then the factorization axiom in a cofibration category shows that there exists a factorization

$$N(\Gamma X.) \overset{i}{\rightarrowtail} C_* \xrightarrow{q} A$$

where i is a cofibration and q is a weak equivalence and $qi = \tilde{\eta}$. Now let

$$\eta. : \Gamma X. \cong KN\Gamma X. \xrightarrow{Ki} KC_* = A.$$

be given by the functor K in (1.6). q.e.d.

Addendum. *We can choose the $X.$ -partial resolution $\eta. : \Gamma X. \to A.$ of η in such a way that*

$$\mathit{length}(Q\eta.) \leq \mathit{length}(\Gamma X.) + gd(\mathbf{A}) + 1$$

Each partial resolution $\eta. : \Gamma X. \to A. \in s\Gamma\mathbf{A}$ yields a short exact sequence

(1.11) $$0 \to \Gamma(X.) \xrightarrow{\eta.} A. \xrightarrow{q.} Q\eta. \to 0$$

in $s\mathbf{A}$ defined as in (1.9) (4). Moreover this corresponds to a morphism

(*) $$I_1(X.) \xrightarrow{(1,\eta.)} (X., A., \eta.)$$

in $s\Gamma\mathbf{A}$ as in (1.4) (5).

(1.12) Lemma. *Let $\Lambda : \Gamma\mathbf{A} \to \mathbf{A}'$ be a functor to the abelian category $\mathbf{A}'$. Then a partial resolution $\eta. : IX. \to A. \in s\Gamma\mathbf{A}$ yields by (*) an exact sequence in $s\mathbf{A}'$:*

$$0 \to \Lambda I_1(X.) \to \Lambda\eta.$$

Proof. We show that the map $(1, \eta_{(n)}) : I_1 X_{(n)} \to \eta_{(n)} = (X_{(n)}, A_{(n)}, \eta_{(n)})$ is split injective in each degree $n \geq 0$. In fact, since $(Q\eta.)_{(n)} = Q\eta_{(n)}$ is projective we see by (1.11) that $\eta_{(n)} : \Gamma(X_{(n)}) \to A_{(n)}$ admits a retraction r. Such a retraction defines also a retraction $(1, r)$ of $(1, \eta_{(n)})$ in (*). q.e.d.

§ 2 Polynomial and reducible functors

Using cross effects we define polynomial functors and the notion of a reducible functor $\Lambda : \Gamma\mathbf{A} \to \mathbf{A}$ where $\Gamma\mathbf{A}$ is the Grothendieck construction of a functor Γ. Such reducible functors are crucial for the results on vanishing lines of derived functors below.

A category $\mathbf{C}$ *with sums and zero object* is a category in which finite sums exist and for which the initial object $*$ (which is the empty sum) is also a final object. We therefore get for objects X, Y in $\mathbf{C}$ a *zero map* $0 : X \to * \to Y$ and for a sum $X \vee Y$ in $\mathbf{C}$ we obtain *retractions*

$$\begin{aligned} r_1 &= (1,0) : X \vee Y \to X \\ r_2 &= (0,1) : X \vee Y \to Y \end{aligned}$$

For example an abelian category $\mathbf{A}$ is a category with sums $A \oplus B$ and zero object 0. If $\Gamma : \mathbf{C} \to \mathbf{A}$ is a functor with $\Gamma(*) = 0$ then we define the *cross effect*

$$\Gamma(X \mid Y) = \text{kernel}\,\{(r_1)_*, (r_2)_* : \Gamma(X \vee Y) \to \Gamma(X) \oplus \Gamma(Y)\} \tag{2.1}$$

Hence we have a natural direct sum decomposition

$$\Gamma(X \vee Y) = \Gamma(X) \oplus \Gamma(Y) \oplus \Gamma(X \mid Y)$$

and we have the map

$$P : \Gamma(X \mid X) \subset \Gamma(X \vee X) \xrightarrow{(1,1)_*} \Gamma(X)$$

which is natural in X.

(2.2) Lemma. *If $\mathbf{C}$ is a category with sums and zero object and if $\Gamma : \mathbf{C} \to \mathbf{A}$ satisfies $\Gamma(*) = 0$ then also $\Gamma\mathbf{C}$ is a category with sums and zero object.*

Proof. The sum of $\eta : \Gamma(X) \to A$ and $\lambda : \Gamma(Y) \to B$ in $\Gamma\mathbf{A}$ is the map

$$\eta \vee \lambda : \Gamma(X \vee Y) \to A \oplus \Gamma(X \mid Y) \oplus B \tag{1}$$

which is $(\eta(r_1)_*, r_{12}, \lambda(r_2)_*)$ where r_{12} is the projection given by (4.1). The zero object $*$ in $\Gamma\mathbf{C}$ is the identity $\Gamma(*) = 0$ of the zero object in $\mathbf{A}$. q.e.d.

We point out that the functor $I_1 : \mathbf{C} \to \Gamma\mathbf{A}$ carries sums to sums, that is $I_1(X \vee Y) \cong I_1(X) \vee I_1(Y)$; see (1.4). The functor I_0, however, has not this property. The sum of $(X, A), (Y, B) \in \mathbf{C} \times \mathbf{A}$ is $(X, A) \vee (Y, B) = (X \vee Y, A \oplus B)$. We have the commutative diagram in $\mathbf{A}$

$$\begin{array}{ccccc} \Gamma(X \vee Y) & = & \Gamma(X \vee Y) & = & \Gamma(X \vee Y) \\ \downarrow{\scriptstyle r_{12}} & & \downarrow{\scriptstyle (0, r_{12}, 0)} & & \downarrow{\scriptstyle (0,0)} \\ \Gamma(X \mid Y) & \xrightarrow{i} & A \oplus \Gamma(X \mid Y) \oplus B & \longrightarrow & A \oplus B \end{array} \tag{2}$$

in which the bottom row is short exact. The inclusion i admits a retraction p which is the projection satisfying $p(o, r_{12}, 0) = r_{12}$. Hence we obtain a "split short exact sequence" in $\Gamma\mathbf{A}$ $r_{12} \rightarrowtail I_0(X, A) \vee I_0(Y, B) \twoheadrightarrow I_0((X, A) \vee (Y, B))$.

(2.3) Definition. Let $\mathbf{A}$ be an abelian category and let $\mathbf{C}$ be a category with sums and zero object. Let $\Gamma : \mathbf{C} \to \mathbf{A}$ be a functor with $\Gamma(*) = 0$. Then for $X_1, \dots, X_q \in \mathbf{C}$ the $q - th$ cross effect $\Gamma(X_1 \mid \dots \mid X_q), q \geq 1$, is the kernel of the map

$$\Gamma(X_1 \vee \ldots \vee X_q) \to \bigoplus_{i=1.}^{q} \Gamma(\partial_i)$$

for which the $i - th$ coordinate is induced by

$$\begin{cases} 1 \vee 0 \vee 1 : X_1 \vee \ldots \vee X_q \to \partial_i \quad \text{with} \\ \partial_i = X_1 \vee \ldots \vee X_{i-1} \vee * \vee X_{i+1} \vee \ldots \vee X_q \end{cases}$$

We say that Γ has *degree* q if $\Gamma(X_1 \mid \dots \mid X_{q+1}) = 0$ for all $X_1, \dots, X_{q+1} \in \mathbf{C}$. The functor Γ is *linear*, resp. *quadratic* if Γ has degree 1, resp. 2. Moreover Γ is *polynomial* if there is $1 \leq q < \infty$ such that Γ has degree q. We also call q the *polynomial degree* of Γ. See [11].

(2.4) Definition. An *epimorphism* in $\Gamma\mathbf{A}$ is a morphism $(1, f) : \eta \to \lambda$ of the form

(1)
$$\begin{tikzcd} \Gamma X \arrow[r, equal] \arrow[d, "\eta"'] & \Gamma X \arrow[d, "\lambda"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

where f is an epimorphism in $\mathbf{A}$. Let $\mathbf{A}'$ be an abelian category. We say that a functor

(2)
$$\Lambda : \Gamma\mathbf{A} \to \mathbf{A}'$$

preserves epimorphisms if Λ carries an epimorphism (1) to an epimorphism. Similarly we say that a functor $\lambda : \mathbf{C} \times \mathbf{A} \to \mathbf{A}'$ preserves epimorphisms if $\Lambda(1, f)$ is an epimorphism for each morphism $(1, f) : (X, A) \to (X, B)$ where f is an epimorphism. Moreover we say that Λ is I_1 *-reduced* if $\Lambda I_1(X) = 0$ for $X \in \mathbf{C}$.

Now let $\mathbf{C}$ be a category with sums and zero object $*$ and let $\Gamma : \mathbf{C} \to \mathbf{A}$ be a functor into the abelian category $\mathbf{A}$ satisfying $\Gamma(*) = 0$. Moreover let

$$\Lambda : \Gamma\mathbf{A} \to \mathbf{A}'$$

be a functor into the abelian category $\mathbf{A}'$. Then we define a new functor

(2.3)
$$\Lambda' : \Gamma\mathbf{A} \to \mathbf{A}'$$

by the following natural exact sequence with $\eta : \Gamma X \to A \in \Gamma\mathbf{A}$,

(1)
$$\Lambda(I_1 X \mid I_0 \Gamma X) \to \Lambda(\eta \mid I_0 \Gamma X) \to \Lambda'(\eta) \to 0$$

Here the left hand side is induced by $(1,\eta) : I_1 X \to \eta$, see (1.5), and we use the notation on $I_0 A$ with $A = \Gamma X$ in (1.4) (3). It is clear that Λ' is a well defined functor. Moreover if Λ is I_1 -reduced we obtain a natural map

$$(2) \qquad \Lambda'(\eta) \xrightarrow{P'} \Lambda(\eta)$$

as follows. Consider the commutative diagram in which the top row is exact.

$$\begin{tikzcd}
\Lambda(I_1 X \mid I_0 \Gamma X) \arrow[r] \arrow[d] & \Lambda(\eta \mid I_0 \Gamma X) \arrow[r] \arrow[d] & \Lambda'(\eta) \arrow[r] \arrow[dd, "P'"] & 0 \\
\Lambda(I_1 X \mid I_1 X) \arrow[r] \arrow[d, equal] & \Lambda(\eta \mid I_1 \Gamma X) \arrow[d] & & \\
\Lambda(I_1 X \mid I_1 X) \arrow[r] & \Lambda(\eta \mid \eta) \arrow[r, "P"] & \Lambda(\eta) &
\end{tikzcd}$$

Here we use the canonical maps $(0,1) : I_0 \Gamma X \to I_1 \Gamma X$ and $(1,\eta) : I_1 X \to \eta$. Since Λ is I_1 reduced we see that the composite of the maps of the bottom row is zero. Hence the factorization P' is well defined.

(2.4) Proposition. *Assume $\Lambda : \Gamma\mathbf{A} \to \mathbf{A}'$ is I_1 -reduced and preserves epimorphisms. Then there is a functor $\Lambda'' : \mathbf{C} \times \mathbf{A} \to \mathbf{A}'$ together with an exact sequence*

$$\Lambda'(\eta) \xrightarrow{P'} \Lambda(\pi) \xrightarrow{\pi} \Lambda''(X, Q\eta) \to 0$$

which is natural in $(\eta : \Gamma X \to A) \in \Gamma\mathbf{A}$. Moreover Λ' is I_1 -reduced and Λ' and Λ'' preserve epimorphisms.

We call Λ'' a *reduction functor* of Λ since the domain category of Λ'' does not involve the functor Γ. We point out that the functor Λ'' and the projection π is completely determined by P' since (2.4) implies that we have for $\eta = I_0(X, A)$ the exact sequence

$$(2.5) \qquad \Lambda'(I_0(X,A)) \xrightarrow{P'} \Lambda(I_0(X,A)) \xrightarrow{\pi_0} \Lambda''(X,A) \to 0$$

Moreover for $(1,q) : \eta \to I_0(X, Q\eta)$ the diagram

$$\begin{tikzcd}
\Lambda(\eta) \arrow[r, "\pi"] \arrow[d, "(1,q)_*"'] & \Lambda''(X, Q\eta) \arrow[d, equal] \\
\Lambda(I_0(X, Q\eta)) \arrow[r, "\pi_0"] & \Lambda''(X, Q\eta)
\end{tikzcd}$$

so that π in (2.4) is given by $\pi_0(1,q)_* = \pi$.

Proof of (2.4). Consider the diagram

$$
\begin{tikzcd}
\Gamma X \arrow[r, equal] \arrow[d, "\eta"'] & \Gamma X \arrow[d, "\lambda"] \\
A \arrow[r, "f", shift left] \arrow[r, "g"', shift right] \arrow[d, "q"'] & B \arrow[d, "q"] \\
Q\eta \arrow[r, "\alpha"] & Q\lambda
\end{tikzcd} \tag{1}
$$

where $(1,f)$ and $(1,g)$ are both maps $\eta \to \lambda$ in $\Gamma\mathbf{A}$ which induce $\alpha = Q(1,f) = Q(1,g)$. Hence there exists

$$Q\eta \xrightarrow{\delta} L \overset{i}{\subset} B \tag{2}$$

with $f + i\delta q = g$. Here L is the image of λ which is the kernel of q. We obtain g as the composite of maps in $\Gamma\mathbf{A}$

$$
\begin{tikzcd}
\Gamma X \arrow[r, equal] \arrow[d, "\eta"'] & \Gamma X \arrow[r, equal] \arrow[d, "(\eta,0)"] & \Gamma X \arrow[r, equal] \arrow[d, "(\lambda,0)"] & \Gamma X \arrow[d, "\lambda"] \\
A \arrow[r, "(1,q)"] & A \oplus Q\eta \arrow[r, "f\oplus\delta"] & B \oplus L \arrow[r, "(1,i)"] & B
\end{tikzcd} \tag{3}
$$

The morphism $(\eta,0),(\lambda,0)$ in the diagram can be expressed as sums in $\Gamma\mathbf{A}$ of the form

$$
\begin{cases} (\eta,0) = \eta \vee I_0(Q\eta) \\ (\lambda,0) = \lambda \vee I_0(L) \end{cases} \tag{4}
$$

and hence using cross effects of Λ we get by (3) the maps

$$
\begin{tikzcd}
\Lambda(\eta) \arrow[r, "\bar{H}"] \arrow[d, "(1,g)_*"'] & \Lambda(\eta) \oplus \Lambda(\eta \mid I_0 Q\eta) \oplus \Lambda I_0(Q\eta) \arrow[d, "\Lambda(f)\oplus\Lambda(f|\delta)\oplus\Lambda I_0(\delta)"] \\
\Lambda(\lambda) & \Lambda(\lambda) \oplus \Lambda(\lambda \mid I_0 L) \oplus \Lambda I_0(L) \arrow[l, "\bar{P}"']
\end{tikzcd} \tag{5}
$$

Here the maps $\bar{H}$ and $\bar{P}$ have the coordinates

$$
\begin{cases} \bar{H} = (1, H, (1,q)_*) \\ \bar{P} = (1, P'', 0) \end{cases} \tag{6}
$$

The third coordinate of $\bar{P}$ is actually trivial by the following argument: Let $\lambda_0 : \Gamma X \to L$ be the map with $i\lambda_0 = \lambda$. Then we have the composite

$$0 = \Lambda I_1 X \overset{(1,\lambda_0)_*}{\twoheadrightarrow} \Lambda(\lambda_0) \xrightarrow{(1,i)_*} \Lambda(\lambda)$$

where $(1,\lambda_0)_*$ is an epimorphism since Λ preserves epimorphisms and where $(1,i)_* = 0$ since Λ is I_1 -reduced. Hence we see that also

(7)
$$\begin{tikzcd}
\Gamma(*) \arrow[r, "0"] \arrow[d, "0"] & \Gamma X \arrow[r, equal] \arrow[d, "\lambda_0"'] & \Gamma X \arrow[d, "\lambda"'] \\
L \arrow[r, "1"] & L \arrow[r, "i"] & B
\end{tikzcd}$$

induces the trivial map $\Lambda I_0 L \to \Lambda(\lambda)$. But this map is the third coordinate of $\bar{P}$ in (6). We now consider the composite

(8)
$$P''' : \Lambda(\lambda \mid I_0 \Gamma X) \xrightarrow{(1|I_0\lambda_0)_*} \Lambda(\lambda \mid I_0 L) \xrightarrow{P''} \Lambda(\lambda)$$

Here $(1 \mid I_0\lambda_0)_*$ is an epimorphism since Λ preserves epimorphisms. Hence we have

(9)
$$image(P'') = image(P''') = image(P')$$

where $P'' = P(1 \mid (0,i))$ is given by (7) and where P' is given by (2.3) (2). Hence if we devide out $image(P')$ we obtain a natural exact sequence

(10)
$$\Lambda(\lambda, I_0 L) \to \Lambda(\lambda) \to \tilde{\Lambda}(\lambda) \to 0$$

where $\tilde{\Lambda} : \Gamma\mathbf{A} \to \mathbf{A}'$ is a well defined functor. By (5) we see that $(1,f)$ and $(1,g)$ in (1) induce the same map

(11)
$$(1,f)_* = (1,g)_* : \tilde{\Lambda}(\eta) \to \tilde{\Lambda}(\lambda)$$

Hence there is a unique functor Λ'' as in (2.4) satisfying $\tilde{\Lambda}(\eta) = \Lambda''(X, Q\eta)$.

Now it is clear by (2.3) (1) that Λ' preserves epimorphisms since λ has this property. Moreover by (2.3) (1) it is clear that λ' is I_1-reduced. q.e.d.

(2.6) Definition. Let $\Lambda : \Gamma\mathbf{A} \to \mathbf{A}'$ be a functor which preserves epimorphisms. Then we define inductively functors

$$\begin{aligned}
&\Lambda_0 : \mathbf{C} \to \mathbf{A}' \\
&\Lambda_i : \mathbf{C} \times \mathbf{A} \to \mathbf{A}', i \geq 1, \\
&\Lambda_{(i)} : \Gamma\mathbf{A} \to \mathbf{A}'
\end{aligned}$$

as follows. Let $\Lambda_0(X) = \Lambda(I_1 X)$ and let $\Lambda_{(1)}$ be defined by the exact sequence

(1)
$$\Lambda(I_1 X) \to \Lambda(\eta) \to \Lambda_{(1)}(\eta) \to 0$$

which is natural in $(\eta : \Gamma X \to A) \in \Gamma\mathbf{A}$. Then $\Lambda_{(1)}$ is I_1 -reduced and preserves epimorphisms. Hence we obtain by (2.4) for $i \geq 1$ the exact sequence

$$(2) \qquad \Lambda_{(i+1)}(\eta) \to \Lambda_{(i)}(\eta) \to \Lambda_i(X, Q\eta) \to 0$$

where $\Lambda_{(i+1)} = \Lambda'_{(i)}$ and $\Lambda_i = \Lambda''_{(i)}$. We call the sequence of functors $\Lambda_i, i \geq 0$, the *reduction functors* of Λ. Moreover we say that Λ is *reducible* if there is $N < \infty$ such that for all split injective $(\eta : \Gamma X \to A) \in \Gamma\mathbf{A}$ we have

$$(3) \qquad \Lambda_{(i)}(\eta) = 0 \quad \text{for} \quad i > N$$

In this case we say that Λ has *reduction degree* $\leq N$. By (1) and (2) and by the definition in (2.3) (1), (2.5) we obtain inductively the natural epimorphisms $(k \geq 0)$

$$(4) \qquad \Lambda(I_0(X, A) \mid I_0\Gamma X \mid \ldots \mid I_0\Gamma X) \xrightarrow{q_k} \Lambda_{k+1}(X, A) \to 0$$

where the left hand side is the $(k+1)$ -cross effect of Λ with k-times $I_0\Gamma X$.

(2.7) Theorem. *Let $\mathbf{C}$ be a category with sums and zero object $*$ and let $\Gamma : \mathbf{C} \to \mathbf{A}$ be a functor with $\Gamma(*) = 0$ which is polynomial of degree $\leq k$. Moreover let $\Lambda : \Gamma\mathbf{A} \to \mathbf{A}'$ be a polynomial functor which preserves epimorphisms. Then Λ is reducible of reduction degree $\leq k$ and all reduction functors $\Lambda_i, i \geq 0$, are polynomial and preserve epimorphisms for $i \geq 1$.*

Proof. We have surjections

$$\begin{aligned}
(1) \qquad & \Lambda(\eta) \to \Lambda_{(1)}(\eta) \to 0 \\
& \Lambda_{(1)}(\eta \mid I_0\Gamma X) \to \Lambda_{(2)}(\eta) \to 0 \\
(2) \qquad & \Lambda_{(i)}(\eta \mid I_0\Gamma X) \to \Lambda_{(i+1)}(\eta) \to 0, i \geq 1,
\end{aligned}$$

by (2.3) (1). Hence since Λ is polynomial we see that also $\Lambda_{(1)}$ is and therefore also $\Lambda_{(2)}$. In fact if $\Lambda_{(i)}$ is polynomial then by (2) also $\Lambda_{(i+1)}$ is polynomial, $i \geq 1$. Also it is clear by (1) and (2) that all $\Lambda_{(i)}$ preserve epimorphisms for $i \geq 1$ and therefore by (2.4) also all $\Lambda_i, i \geq 1$, preserve epimorphisms in $\mathbf{A}$. In order to prove that Λ is reducible we have to consider $\Lambda_{(i)}(\eta)$ where η is split injective. In this case we have an isomorphism $\eta = I_1(X) \vee I_0(A)$ where $A = Q\eta$ and we consider the functor $\Omega_{(i)} : \mathbf{A} \to \mathbf{A}'$ given by

$$(3) \qquad \Omega_{(i)}(A) = \Lambda_{(i)}(I_1(X) \vee I_0(A))$$

It is clear by (1) that $\Omega_{(1)}$ is polynomial of degree $\leq k$ since Λ has degree $\leq k$. Moreover by (2) we see that *degree* $(\Omega_{(i+1)}) \leq$ *degree*$(\Omega_{(i)}) - 1$. Hence $\Omega_{(k+1)}$ has degree 0 and hence $\Omega_{(k+1)}(A) = \Omega_{(k+1)}(0)$. This shows that $\Lambda_{(k+1)}(\eta) = 0$ since $\Lambda_{(k+1)}$ is I_1-reduced. q.e.d.

(2.8) Remark. The results and the notation above generalize in an obvious way for a functor

$$\Lambda : (\Gamma \mathbf{A}) \times \mathbf{B} \to \mathbf{A}'$$

where $\mathbf{B}$ is an abelian category. In fact, we can consider the functor

$$\Gamma' : \mathbf{C} \xrightarrow{\Gamma} \mathbf{A} \xrightarrow{i} \mathbf{A} \times \mathbf{B}$$

where i is the inclusion which carries A to $(A, 0)$. Then

$$\Gamma'(\mathbf{A} \times \mathbf{B}) = (\Gamma \mathbf{A}) \times \mathbf{B}$$

In the following we have $\mathbf{B} = \mathbf{A}^r$ where $\mathbf{A}^r = \mathbf{A} \times \ldots \times \mathbf{A}$ is the r-fold product category of $\mathbf{A}, r \geq 0$.

§ 3 Iterated Grothendieck constructions and derived functors

Let $\mathbf{A}$ be an abelian category. We consider a sequence

$$\Gamma = (\Gamma^1, \Gamma^2, \ldots, \Gamma^m)$$

of functors $\Gamma^i, 1 \leq i \leq m$. This is an infinite sequence $(\Gamma^1, \Gamma^2, \ldots)$ if $m = \infty$.

(3.1) Definition. The sequence Γ of functors is an *iterated Grothendieck construction* on $\mathbf{A}$ if the following property (1) holds:

$$(1) \qquad \begin{cases} \Gamma^1 : \mathbf{A} \to \mathbf{A} \quad \text{and} \\ \Gamma^k : \Gamma^{k-1}\mathbf{A} \to \mathbf{A} \quad \text{for} \quad 2 \leq k \leq m \end{cases}$$

Here $\Gamma^{k-1}\mathbf{A}$ is the Grothendieck construction of Γ^{k-1} in (1.1). There is a more general form of an iterated Grothendieck construction Γ on a sequence $(\mathbf{A}^0, \mathbf{A}^1, \ldots, \mathbf{A}^m)$ of abelian categories $\mathbf{A}^i$ satisfying

$$(1') \qquad \begin{cases} \Gamma^1 : \mathbf{A}^0 \to \mathbf{A}^1 \quad \text{and} \\ \Gamma^k : \Gamma^{k-1}\mathbf{A}^{k-1} \to \mathbf{A}^k \quad \text{and} \quad 2 \leq k \leq m \end{cases}$$

The notation and results considered below for the case (1) have an obvious generalization for the case (1').

An object $X = (A^{\leq m}, \eta^{\leq m}) \in \Gamma^m \mathbf{A}$ is given by an $(m+1)$ -tuple

$$A^{\leq m} = (A^0, A^1, \ldots, A^m)$$

of objects in $\mathbf{A}$ and by an m-tuple

$$\eta^{\leq m} = (\eta^1, \eta^2, \ldots, \eta^m)$$

of morphisms in $\mathbf{A}$. Here X defines the *restriction* $X^k = (A^{\leq k}, \eta^{\leq k})$ for $k = 0, \ldots, m$ where we set $X^0 = A^0$ and the morphisms η^k are morphisms

$$\eta^k : \Gamma^k(X^{k-1}) \to A^k \quad (1 \le k \le m)$$

We also write $X^{k-1} = (\eta^1, \dots, \eta^{k-1})$ for $k \ge 2$ so that morphisms

$$(2) \qquad \begin{cases} \eta^1 : \Gamma^1(A^0) \to A^1 \\ \eta^k : \Gamma^k(\eta^1, \dots, \eta^{k-1}) \to A^k \quad (2 \le k \le m) \end{cases}$$

describe the object $X = (A^{\le m}, \eta^{\le m})$. A morphism

$$(3) \qquad f^{\le m} : (A^{\le m}, \eta^{\le m}) \to (B^{\le m}, \lambda^{\le m}) \quad \text{in} \quad \Gamma^m \mathbf{A}$$

is a tuple $f^{\le m} = (f^0, \dots, f^m)$ of morphisms $f^i : A^i \to B^i$ with $0 \le i \le m$ such that all f^i are compatible with the η^i, that is, for $0 < i < m$ the diagram

$$\begin{array}{ccc} \Gamma^i(A^{\le i-1}, \eta^{\le i-1}) & \xrightarrow{(f^{\le i-1})_*} & \Gamma^i(B^{\le i-1}, \lambda^{\le i-1}) \\ \downarrow \eta^i & & \lambda^i \downarrow \\ A^i & \xrightarrow{f^i} & B^i \end{array}$$

commutes. For $1 \le k < m$ we have full inclusions

$$(4) \qquad \mathbf{A} = \Gamma^0\mathbf{A} \xrightarrow{I_1} \Gamma^1\mathbf{A} \xrightarrow{I_1} \Gamma^2\mathbf{A} \xrightarrow{I_1} \dots \xrightarrow{I_1} \Gamma^k\mathbf{A} \xrightarrow{I_1} \Gamma^{k+1}\mathbf{A}$$

given by I_1 in (1.4). For $m = \infty$ let $\Gamma^\infty\mathbf{A}$ be the limit category of the sequence (4). Objects in $\Gamma^\infty\mathbf{A}$ are of the form $(A^\infty, \eta^1, \eta^2, \dots)$ where $A^\infty = (A^0, A^1, \dots)$ is a sequence of objects in $\mathbf{A}$ and where $\eta^1, \eta^2, \dots$ is a sequence of morphisms η^k in $\mathbf{A}$ with the property in (2) for $k \ge 1$.

We obtain the canonical functors

$$(3.2) \qquad \bar{Q}, \bar{K} : \Gamma^m\mathbf{A} \to \mathbf{A}^m \quad (m \le \infty)$$

Here $\bar{Q}$ carries the object $(A^{\le m}, \eta^{\le m})$ to the object $(Q\eta^1, \dots, Q\eta^m) \in \mathbf{A}^m$ where $Q\eta^i$ = cokernel (η^i) as in (1.4) (1). We call $\bar{Q}$ the functor of *indecomposables* on $\Gamma^m\mathbf{A}$. Moreover we define accordingly the functor $\bar{K}$ by kernels; that is, $\bar{K}$ carries $(A^{\le m}, \eta^{\le m})$ to the object $(K\eta^1, \dots, K\eta^m) \in \mathbf{A}^m$ where $K\eta^i$ = kernel (η^i). We call $\bar{K}$ the *kernel functor* on $\Gamma^m\mathbf{A}$.

Partial resolutions (1.9) lead to the following definition.

(3.3) Definition. Let $X = (A^{\le n}, \eta^{\le n})$ be an object in $\Gamma^n\mathbf{A}$. A *resolution of* X in $\Gamma^n\mathbf{A}$ is a simplicial object

$$X_\cdot = (A_\cdot^{\le n}, \eta_\cdot^{\le n}) \in s\Gamma^n\mathbf{A}$$

with the properties (a) and (b).

(a) $N(A_\cdot^0)$ is a projective resolution of A^0.
(b) For $1 \le k < n$ the simplicial object $\eta_\cdot^{k+1}$ given by $X_\cdot$ is an $X_\cdot^k$ -partial resolution of η^{k+1} where $X_\cdot^k$ is the restriction of $X_\cdot$.

(3.4) Lemma. *Assume* $\mathbf{A}$ *has enough projectives. Then resolutions exist in* $\Gamma^n\mathbf{A}$ *for all* $n \le m$.

This is an inductive application of (1.10).

(3.5) Definition. Let $\mathbf{B}$ be an abelian category, let Γ be an iterated Grothendieck construction and let

$$\Lambda : \Gamma^m\mathbf{A} \to \mathbf{B}$$

be a functor. If $\mathbf{A}$ has enough projectives we define the *derived functors*

$$L_i\Lambda : \Gamma^m\mathbf{A} \to \mathbf{B} \quad \text{for} \quad i \ge 0$$

by the homotopy groups

$$(L_i\Lambda)(X) = \pi_i(\Lambda X.)$$

where $X.$ is a resolution of $X \in \Gamma^m\mathbf{A}$ as in (3.3). We have the natural map

$$(L_0\Lambda)(X) = \pi_0(\Lambda X.) \to \Lambda(\pi_0 X.) = \Lambda(X).$$

We write $L_0\Lambda = \Lambda$ if this map is an isomorphism for all X. As usual, different resolutions yield canonically isomorphic derived functors. Hence the derived functors $L_i\Lambda$ are well defined up to canonical isomorhism since resolutions exist by (3.4). Using the functor of indecomposables $\bar{Q}$ in (3.2) we get the following result by the second part of (1.10).

(3.6) Proposition. *The derived functors* $L_i\bar{Q}$ *of* $\bar{Q} : \Gamma^m\mathbf{A} \to \mathbf{A}^m$ *satisfy in degree* k *with* $1 \le k \le m$ *the formula*

$$(L_{i+1}\bar{Q})_k = L_i\Gamma^k \quad \textit{for} \quad i \ge 1$$

and if $L_0\Gamma^k = \Gamma^k$ *one has the exact sequence*

$$0 \to (L_1\bar{Q})_k(X) \to \Gamma^k X^{k-1} \xrightarrow{\eta^k} A^k \to (L_0\bar{Q})_k(X) \to 0$$

Here $X = (A^{\le m}, \eta^{\le m}) \in \Gamma^m\mathbf{A}$ *yields by restriction* $X^k = (A^{\le k}, \eta^{\le k}) \in \Gamma^k\mathbf{A}$. *This implies* $L_0\bar{Q} = \bar{Q}$ *and* $L_1\bar{Q} = \bar{K}$ *is the kernel functor* $\bar{K}$ *in (3.2).*

§ 4 Vanishing line for derived functors

Let $\mathbf{A}$ be an abelian category which has enough projectives and let $\Gamma = (\Gamma^1, \dots)$ be an iterated Grothendieck construction on $\mathbf{A}$. Then the derived functors

$$L_i\Gamma^{n+1} : \Gamma^n\mathbf{A} \to \mathbf{A} \tag{4.1}$$

are defined by (3.5). We say that these functors have a *vanishing line* if for all n there exists $N_{n+1} < \infty$ such that $L_i\Gamma^{n+1} = 0$ for $i > N_{n+1}$. We now describe conditions on Γ which imply that a vanishing line exists. We say that Γ *preserves zero objects* if $\Gamma^n(*) = 0$ for all $n \ge 1$. In this case (2.2) shows that all categories $\Gamma^n\mathbf{A}$ have sums and a zero object. Hence cross effects $\Gamma^{n+1}(X \mid Y)$ are defined for $X, Y \in \Gamma^n\mathbf{A}$. Moreover we say that $\Gamma = (\Gamma^1, \dots)$ *preserves multi-epimorphisms* if for all $n \geq 0$ the functor Γ^{n+1} carries a map $f^{\leq n}$ as in (3.1) (3) for which all $f^i, 0 \leq i \leq n$, are epimorphisms to an epimorphism. Hence in this case the reduction functors of $\Gamma^{n+1}, n \geq 0$, are defined by (2.4). But in addition also all reduction functors of reduction functors are inductively defined by (2.4) since all these functors preserve again multi-epimorphisms.

(4.2) Definition. Assume $\Gamma = (\Gamma^1, \dots)$ preserves zero objects and multi-epimorphisms. Then Γ is said to be *reducible* if all $\Gamma^{n+1}, n \geq 0$, are reducible and if the same holds for all reduction functors. Moreover also the reduction functors of reduction functors (obtained inductively) again are reducible.

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, \dots\}$ and let

$$\varepsilon = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^r) \in (\mathbb{N}_0)^r \tag{4.3}$$

be a tuple of elements $\varepsilon^i \in \mathbb{N}_0$ with $1 \leq i \leq r$. Let $\ell(\varepsilon) = r$ be the *length* of ε and let

$$\begin{cases} \varepsilon^{-1}(\mathbb{N}) = \{i; 1 \leq i \leq r, \varepsilon^i \geq 1\} \\ \varepsilon^{-1}(0) = \{i; 1 \leq i \leq r, \varepsilon^i = 0\} \end{cases} \tag{4.4}$$

Clearly $\{1, \dots, r\} = \varepsilon^{-1}(\mathbb{N}) \cup \varepsilon^{-1}(0)$. Moreover let $\mid \varepsilon \mid$ be the number of elements in $\varepsilon^{-1}(\mathbb{N})$. We also write for $k \in \mathbb{N}_0$ and ε as above

$$(\varepsilon, k) = (\varepsilon^1, \varepsilon^2, \dots, \varepsilon^r, k)$$

so that $\ell(\varepsilon, k) = \ell(\varepsilon) + 1$. If the iterated Grothendieck construction $\Gamma = (\Gamma_1, \dots)$ is reducible we obtain for $n \geq 0$ functors

$$\Gamma^{n+1}_{\varepsilon} : (\Gamma^n \mathbf{A}) \times \mathbf{A}^{|\varepsilon|} \to \mathbf{A} \tag{4.5}$$

where ε is an element as in (4.3) with $\ell(\varepsilon) \geq 0$. Here (4.5) coincides with Γ^{n+1} if $\ell(\varepsilon) = 0$. If (4.5) is defined for $\ell(\varepsilon) \leq r$ and $n \geq 0$ then we define $\Gamma^{n+1}_{(\varepsilon,k)}$ by the reduction functor

$$\Gamma^{n+1}_{(\varepsilon,k)} = (\Gamma^{n+2}_{\varepsilon})_k \quad \text{of} \quad \Gamma^{n+2}_{\varepsilon}$$

Hence the functors (4.5) applied to split injective objects are trivial if all $\varepsilon^i, 1 \leq i \leq \ell(\varepsilon)$, are sufficiently large. We call (4.5) the *reduction functors associated* to the reducible Grothendieck construction Γ.

(4.6) Definition. We say that a functor $\Lambda : \mathbf{A}^k \to \mathbf{A}$ is *bounded* by a bounding function $\lambda : \mathbb{N}_0^k \to \mathbb{N}_0$ if for all simplicial objects $A^i_\cdot \in s\mathbf{A}$ $(1 \leq i \leq k)$ with *length* $(A^i_\cdot) \leq a^i < \infty$ one has

$$length \Lambda(A^1_\cdot, \dots, A^k_\cdot) \leq \lambda(a^1, \dots, a^k).$$

Accordingly we say that the reducible iterated Grothendieck construction Γ is bounded if all functors

$$\Gamma^1_\varepsilon : \mathbf{A} \times \mathbf{A}^{|\varepsilon|} \to A$$

associated to Γ are bounded by some bounding function γ_ε with $\gamma_\varepsilon = 0$ if $\Gamma^1_\varepsilon = 0$. In fact, since all objects of $\mathbf{A} \times \mathbf{A}^{|\varepsilon|}$ are split injective we see that $\Gamma^1_\varepsilon = 0$ if all $\varepsilon^i, i \in \{1, \dots, l(\varepsilon)\}$, are sufficiently large.

(4.7) Theorem. *Let Γ be an iterated Grothendieck construction on $\mathbf{A}$ which preserves zero objects and multi-epimorphisms and let Γ be reducible and bounded. Moreover assume that the global dimension of $\mathbf{A}$ is finite. Then the derived functors $L_i\Gamma^{n+1}$ in (4.1) have a vanishing line $N_{n+1}, n \geq 0$.*

Let $d < \infty$ be the global dimension of $\mathbf{A}$. Then we define inductively numbers $a^n, 0 \leq n$, as follows

$$\begin{aligned} a^0 &= d \\ a^{n+1} &= 1 + d + N_{n+1} \quad \text{for} \quad n \geq 0, \quad \text{where} \\ N_{n+1} &= \operatorname{Max}\{\gamma_\varepsilon(a^0, a^{i_1}, \dots, a^{i_{|\varepsilon|}}), \varepsilon \in (\mathbb{N}_0)^n\} \end{aligned}$$

and $\varepsilon^{-1}(\mathbb{N}) = \{i_1 < \dots < i_{|\varepsilon|}\}$; see (4.4).

(4.8) Addendum. $L_i\Gamma^{n+1} = 0$ *for* $i > N_{n+1}$.

Proof. Let $X_\cdot = (A_\cdot^{\leq m}, \eta_\cdot^{\leq m})$ be a resolution of $X \in \Gamma^m\mathbf{A}$. We can choose $X_\cdot$ inductively as in the addendum (A) of (1.10). Hence we have for $1 \leq n < m$

$$b^n \leq \mathit{length}(A_\cdot^n) \leq c^n + d + 1 \tag{1}$$

where

$$\begin{cases} b^n = \mathit{length}\, Q(\eta_\cdot^n) \quad \text{and} \\ c^n = \mathit{length}\,(\Gamma^n X_\cdot^{n-1}) \end{cases}$$

Moreover we have by the inductive application of (2.7) the following diagram in which the row and the columns are exact sequences in $s\mathbf{A}$ given by (2.4). In the diagram we assume for simplicity that the reduction degree is always ≤ 1. It is easy to extend the diagram to the general case.

$$\begin{array}{ccccc} \Gamma^{n-1}_{(0,0)}(X_\cdot^{n-2}) & & & & \Gamma^{n-1}_{(1,0)}(X_\cdot^{n-2}, Q\eta_\cdot^n) \\ \downarrow & & & & \downarrow \\ \Gamma^n_{(0)}(X_\cdot^{n-1}). & \longrightarrow & \Gamma^{n+1}(X_\cdot^n) & \longrightarrow & \Gamma^n_{(1)}(X_\cdot^{n-1}, Q\eta_\cdot^n) \\ \downarrow & & & & \downarrow \\ \Gamma^{n-1}_{(1,0)}(X_\cdot^{n-2}, Q\eta_\cdot^{n-1}, Q\eta_\cdot^{n-1}) & & & & \Gamma^{n-1}_{(1,1)}(X_\cdot^{n-2}, Q\eta_\cdot^{n-1}, Q\eta_\cdot^n) \end{array}$$

The 4 corners of this diagram are given by the functors Γ^{n-1}_ε with $\ell(\varepsilon) = 2$. We can extend the diagram in such a way that we obtain a large diagram in which the corners are given by the functors Γ_ε with $\ell(\varepsilon) = n$. This shows by (1.7) (2) that

(2) $$c^{n+1} = \mathit{length}\,\Gamma^{n+1}(X^n_{\cdot}) \leq \mathrm{Max}\{c^1_\varepsilon, \varepsilon \in (\mathbb{N}_0)^n\}$$

where

$$c^1_\varepsilon = \mathit{length}\,\Gamma^1_\varepsilon(A^0_{\cdot}, Q\eta^{i_1}_{\cdot}, \dots, Q\eta^{i_{|\varepsilon|}}_{\cdot})$$

with $\varepsilon^{-1}(\mathbb{N}) = \{i_1 < \ldots < i_{|\varepsilon|}\}$. Since γ_ε is a bounding function for Γ^1_ε we get

(3) $$c^1_\varepsilon \leq \gamma_\varepsilon(\mathit{length}\,(A^0_{\cdot}), b^{i_1}, \dots, b^{i_{|\varepsilon|}})$$

We have to show $c^{n+1} \leq N_{n+1}$. By (1.8) we can choose $\mathit{lenght}(A^0_{\cdot}) = d = a^0$. Now assume $b^k \leq a^k$ for $k \leq n$. Then we get by (3)

(3) $$c^1_\varepsilon \leq \gamma_\varepsilon(a^0, a^{i_1}, \dots, a^{i_{|\varepsilon|}}), \varepsilon \in (\mathbb{N}_0)^n,$$

and hence by (2)

(5) $$c^{n+1} \leq N_{n+1} \quad \text{hence}$$

(6) $$b^{n+1} \leq 1 + d + c^{n+1} \leq 1 + d + N_{n+1} = a^{n+1} \quad \text{by (1) for} \quad n+1 < m.$$

This yields the result. q.e.d.

§ 5 Polynomial iterated Grothendieck constructions

We consider an iterated Grothendieck construction $\Gamma = (\Gamma^1, \dots)$ which preserves zero ojects so that all categories $\Gamma^n\mathbf{A}$ have sums. We say that Γ is polynomial if all $\Gamma^{n+1}, n \geq 0$, are polynomial functors; see (2.3).

(5.1) Theorem. *Let $\Gamma = (\Gamma^1, \dots)$ be an iterated Grothendieck construction which preserves zero obejcts and multi-epimorphisms and which is polynomial. Then Γ is reducible and bounded.*

Proof. This is an inductive application of (2.7) which shows that all reduction functors exist and are polynomial. Hence also all functors Γ^1_ε are polynomial and thus they are bounded by the following lemma. q.e.d.

(5.2) Lemma. *Let $\Lambda : \mathbf{A} \to \mathbf{B}$ be a functor between abelian categories. If $\Lambda(0) = 0$ and if Λ has degree k then one has for any simplicial object $A_{\cdot} \in s\mathbf{A}$ the formula*

$$\mathit{length}\,\Lambda(A_{\cdot}) \leq k \cdot \mathit{length}\,(A_{\cdot})$$

This is proved in 4.23 of [8]. See also [14]. Combining (5.1) and (3.7) we thus get:

(5.3) Theorem. *Let Γ be an iterated Grothendieck construction on* $\mathbf{A}$ *which preserves zero objects and multi-epimorphisms. Assume that Γ is polynomial and assume that the global dimension of* $\mathbf{A}$ *is finite. Then the derived functors $L_i\Gamma^{n+1}$ have a vanishing line $N_{n+1}, n \geq 0$, as in (4.7).*

As a special case of (5.3) we obtain the following results.

(5.4) Corollary. *Let $\Gamma = (\Gamma^1, \dots)$ be an iterated Grothendieck construction on* $\mathbf{A}$ *which preserves zero objects and multi-epimorphisms. If all Γ^{n+1} are linear, $n \geq 0$, and if d is the global dimension of* $\mathbf{A}$ *with $d < \infty$ then*

$$L_i\Gamma^{n+1} = 0 \quad \textit{for} \quad i > d + n(d+1) = N_{n+1}.$$

Proof. All Γ^1_ε are linear and hence by (5.2) we have

$$\gamma_\varepsilon(a^0, a^{i_1}, \dots, a^{i_{|\varepsilon|}}) = \mathrm{Max}\,\{a^0, a^{i_1}, \dots, a^{i_{|\varepsilon|}}\}$$

. Hence (5.4) is a consequence of (4.7). q.e.d.

The next definition and corollary is motivated by a result of Blanc [4]; see (10.7) below.

(5.5) Definition. Let $\Gamma = (\Gamma^1, \dots)$ be an iterated Grothendieck construction on $\mathbf{A}$ which preserves zero objects. We say that an object $X = (A^{\leq n}, \eta^{\leq n}) \in \Gamma^n\mathbf{A}, n \geq 0$, has *connectivity degree* $\mid X \mid$ if $A^k = 0$ for $k <\mid X \mid$ and $A^k \neq 0$ for $k = \mid X \mid$. Given a natural number $r \geq 1$ we say that $\Gamma = (\Gamma^1, \dots)$ is r *-polynomial* if for $n \geq 0$ and $q \geq 2$ and any set of objects $X_1, \dots, X_q \in \Gamma^n\mathbf{A}$ with

$$\sum_{i=1}^{q}(r + \mid X_i \mid) > r + n + 1$$

we have $\Gamma^{n+1}(X_1 \mid \dots \mid X_q) = 0$. Since $r \geq 1$ this implies that Γ^{n+1} is polynomial.

(5.6) Corollary. *Let $\Gamma = (\Gamma^1, \dots)$ be an iterated Grothendieck construction on* $\mathbf{A}$ *which preserves zero objects and multi-epimorphisms. Moreover assume that Γ is r-polynomial with $r > d \geq 0$. Here d is the global dimension of* $\mathbf{A}$. *Then*

$$L_i\Gamma^{n+1} = 0 \quad \textit{for} \quad i > d + n(d+1).$$

If $r > n$ then all Γ^{i+1} with $0 \leq i \leq n$ in (5.5) are linear and in this case (5.6) corresponds to (5.4). Moreover if Γ is $(r+1)$ -polynomial then Γ is also r-polynomial and hence it suffices to prove the corollary for $r = d+1$.

Proof of (5.6). Using (5.3) and (4.7) the corollary is a consequence of the following fact which we shall prove below. Let $A^i_\cdot \in s\mathbf{A}$ be a simplicial object for $0 \leq i \leq n$ and assume

$$\text{(1)} \qquad \textit{length}\,(A^i_\cdot) \leq d + i(d+1).$$

Then all functors Γ^1_ε with $\varepsilon \in (\mathbb{N}_0)^n$ satisfy

$$\text{(2)} \qquad \textit{lenght}\,(\Gamma^1_\varepsilon(A^0_\cdot, A^{i_1}_\cdot, \dots, A^{i_{|\varepsilon|}}_\cdot) \leq d + n(d+1)$$

where $\varepsilon^{-1}(\mathbb{N}) = \{i_1 < \ldots < i_{|\varepsilon|}\}$. For the proof of (2) we first observe that the functor I_0 in (1.4) satisfies

$$I_0(X, A) = I_0(X) \vee I_0(A) \tag{3}$$

For simplicity we assume now that the reduction degree is always ≤ 1. The general case with reduction degree > 1 is proved by a similar argument where the inductive application of the surjection (2.6) (4) is needed for the description of Γ^1_ε as a quotient of cross effects involving the functors $\Gamma^i, i \geq 1$.

Since we assume reduction degree ≤ 1 the functor Γ^1_ε is only needed for $\varepsilon^i \in \{0, 1\}$. Hence we obtain by (2.6) (4) a surjection

$$\Gamma^{n+e(\varepsilon)+1} J_\varepsilon(B^0, B^{i_1}, \ldots, B^{i_r}) \to \Gamma^1_\varepsilon(B^0, B^{i_1}, \ldots, B^{i_r})$$

where $\varepsilon^{-1}(\mathbb{N}) = \{i_1 < \ldots < i_r\}$ and $B^i \in \mathbf{A}$. Here we define the full inclusion

$$J_\varepsilon : \mathbf{A} \times \mathbf{A}^r \to \Gamma^n \mathbf{A} \tag{4}$$

as follows. Let $J_\tau = I_1$ for $\tau = 0$ and let $J_\tau = I_0$ for $\tau = 1$. Then J_ε is the composite $J_\varepsilon = J_{\varepsilon^1} \circ \ldots \circ J_{\varepsilon^r}$. Now (3) implies that the functor J_ε in (4) satisfies

$$J_\varepsilon(B^0, B^1, \ldots B^{|\varepsilon|}) = \bigvee_{j=0}^{|\varepsilon|} J^j_\varepsilon(B^j) \tag{5}$$

Here $J^j_\varepsilon : \mathbf{A} \to \Gamma^n \mathbf{A}$ is a functor for which the connectivity degree $\mid J^j_\varepsilon(B) \mid$ with $B \in \mathbf{A}$ (see (5.5)) satisfies

$$\mid J^j_\varepsilon(B) \mid = \begin{cases} 0 & \text{if} \quad j = 0 \\ i_j & \text{if} \quad 1 \leq j \leq \mid \varepsilon \mid \end{cases}$$

with $\{i_1 < \ldots < i_j\} = \varepsilon^{-1}(\mathbb{N})$. Now (5) and the definition of $\Gamma^{n+1} J_\varepsilon$ shows that we have the cross effect decomposition

$$\begin{cases} \Gamma^{n+1} J_\varepsilon(B^0, B^1, \ldots, B^{|\varepsilon|}) = \bigoplus\limits_{t \geq 1} \; \bigoplus\limits_{0 \leq a_1 < \ldots < a_t \leq |\varepsilon|} \Lambda_{\varepsilon,a}(B^{a_1}, \ldots, B^{a_t}) \\ \Lambda_{\varepsilon,a}(B^{a_1}, \ldots B^{a_t}) = \Gamma^{n+1}(J^{a_1}_\varepsilon(B^{a_1}) \mid \ldots \mid J^{a_t}_\varepsilon(B^{a_t})) \end{cases} \tag{6}$$

Here the assumption that Γ^{n+1} is r-polynomial implies that $\Lambda = \Lambda_{\varepsilon,a}$ is a functor as in the following lemma. Hence this lemma implies (2) by use of the decomposition (6). For this we set $C^\tau = B^{a_\tau} = A^{j_\tau} \in \mathbf{A}$ with $j_\tau = i_{a_\tau}$ for $a_\tau \geq 1$ and $j_\tau = 0$ for $a_\tau = 0$. Moreover we set $r = d + 1$. q.e.d.

(5.7) Lemma. *Let $n \geq 0, r \geq 1, t \geq 1$ and $0 \leq j_1 < \ldots < j_t \leq n$ be given. Let $\Lambda : \mathbf{A}^t \to \mathbf{A}$ be a functor with the property that*

$$\Lambda(C_1^1 \mid \ldots \mid C_{s_1}^1 \,,\, C_1^2 \mid \ldots \mid C_{s_2}^2 \,,\, \ldots \,,\, C_1^t \mid \ldots \mid C_{s_t}^t) = 0$$

for all $s_1, s_2, \ldots, s_t \geq 1$ satisfying

$$\sum_{\tau=1}^{t} s_\tau (j_\tau + r) > r + n + 1.$$

Then for $C_\cdot^\tau \in s\mathbf{A}$ with

$$\textit{length}\,(C_\cdot^\tau) \leq r - 1 + j_\tau r \quad \textit{for} \quad \tau = 1, \ldots, t$$

one has

$$\textit{length}\,\Lambda(C_\cdot^1, \ldots, C_\cdot^t) \leq r - 1 + nr.$$

For $t = 1$ the lemma is a consequence of (5.2). In fact, for $t = 1$ we have $s_1 \geq \deg(\Lambda) + 1$ and hence we get

$$\begin{aligned} \textit{lenght}\Lambda(C_\cdot^1) &\leq \deg(\Lambda)(r - 1 + j_1 r) \\ &\leq (s_1 - 1)(r - 1 + j_1 r) \\ &< \left(\tfrac{r+n+1}{j_1+r} - 1 \right) (r - 1 + j_1 r) \overset{(*)}{\leq} r + nr \end{aligned}$$

Here the proof of (*) for $0 \leq j_1 \leq n$ and $n \geq 0, r \geq 1$ is a nice exercise.

Proof of (5.7). The normalization of $C_\cdot^\tau$ yields the objects

$$K_i^\tau = (NC_\cdot^\tau)_i \tag{1}$$

with $K_i^\tau = 0$ for $i < 0$ and $i > r - 1 + j_\tau r$. Moreover we have by the Dold-Kan theorem

$$C_{(m)}^\tau = \bigoplus_{0 \leq \lambda^\tau \leq m} \bigoplus_{I \in F_{\lambda,m}} K_{m-\lambda^\tau}^\tau \tag{2}$$

where $F_{\lambda,m}$ is the set of all sequences $I = \{0 \leq u_1 < \ldots < u_\lambda < m\}$. We allow $\lambda = 0$. We have to show that

$$\Lambda(C_{(m)}^1, \ldots, C_{(m)}^t) \quad \text{is degenerate} \tag{3}$$

for $m > r - 1 + nr$ or equivalently for $m \geq r + nr$ where $r \geq 1$. Using (2) and (3) we get a decomposition of (3) by terms of the form

$$\Lambda(K_{m-\lambda_1^1}^1 \mid \ldots \mid K_{m-\lambda_{s_1}^1}^1, \ldots, K_{m-\lambda_1^t}^t \mid \ldots \mid K_{m-\lambda_{s_t}^t}^t) \tag{4}$$

with

$$(5) \qquad \begin{cases} m - \lambda_\alpha^\tau \leq r - 1 + j_\tau r \quad \text{for} \quad 1 \leq \alpha \leq s_\tau \\ \sum_{\tau=1}^{t} s_\tau (j_\tau + r) \leq n + r + 1 \end{cases}$$

Hence (5) implies

$$(6) \qquad (m - \lambda_\alpha^\tau)/r < j_\tau \quad \text{for} \quad 1 \leq \alpha \leq s_\tau$$

and

$$(7) \qquad \sum_{\tau=1}^{t} \sum_{\alpha=1}^{s_\tau} (r + (m - \lambda_\alpha^\tau)/r) < n + r + 1$$

Using (7) we get

$$(8) \qquad \sum_{\tau=1}^{t} \sum_{\alpha=1}^{s_\tau} (m - \lambda_\alpha^\tau) < rn + r^2 + r - \sum_{\tau=1}^{t} s_\tau r^2 \leq rn + r^2 + r - tr^2 \leq rn + r$$

Hence for $rn + r \leq m$ the left hand side of (8) is smaller than m and this implies by Lemma 5.2.2 in Blanc [4] that (4) and hence (3) is degenerate. q.e.d.

§ 6 Models of theories

A *theory* $\mathbf{T}$ is a category with initial object $*$ and with finite sums denoted by $X \vee Y$ for $X, Y \in \mathbf{T}$. We consider $*$ as the empty sum. A map between theories $\mathbf{T}$ and $\mathbf{T}'$ is a functor $F : \mathbf{T} \to \mathbf{T}'$ which preserves sums. Let $\mathbf{Set}$ be the category of sets and let $\mathbf{T}^{op}$ be the opposite category of $\mathbf{T}$. A *model* of the theory $\mathbf{T}$ is a functor

$$(6.1) \qquad M : \mathbf{T}^{op} \to \mathbf{Set}$$

(i.e. a contravariant functor $\mathbf{T} \to \mathbf{Set}$) such that M carries sums in $\mathbf{T}$ to products of sets in $\mathbf{Set}$. This means that M carries the inclusions $i_X : X \to X \vee Y$, $i_Y : Y \to X \vee Y$ to the projections $p_X = i_X^* : M(X \vee Y) \to M(X)$ and $p_Y = i_Y^* : M(X \vee Y) \to M(Y)$ of the product set

$$M(X \vee Y) = M(X) \times M(Y)$$

In particular M carries the empty sum $*$ in $\mathbf{T}$ to the final object $*$ in $\mathbf{Set}$ consisting of a single point. See Lawvere [13]. For example given an object X in $\mathbf{T}$ then we obtain the model

$$Mor_X = \mathbf{T}(-, X) : \mathbf{T}^{op} \to \mathbf{Set}$$

which carries Y to the set $\mathbf{T}(Y,X)$ of morphisms in $\mathbf{T}$. Then Mor_X is called a *free model*. Let $model(\mathbf{T})$ be the *category of models*; objects are models M, M' as above and morphisms are natural transformations $M \to M'$; for this we assume that the category $\mathbf{T}$ is small so that $model(\mathbf{T})$ is a well defined category. We have the Yoneda embedding of categories

$$\mathbf{T} \subset model(\mathbf{T}) \tag{6.2}$$

which carries X to Mor_X. A map $F : \mathbf{T}' \to \mathbf{T}$ between theories induces a functor

$$F^* : model(\mathbf{T}) \to model(\mathbf{T}')$$

which carries M to $M \circ F$. Here F^* is also called an "algebraic functor". By 3.7.7, 3.7.5 of Borceux [5] we know that any algebraic functor F^* has a left adjoint.

A cogroup $X = (X, 0, \mu, \nu)$ in a theory $\mathbf{T}$ is an object X together with maps

$$0 : X \to *, \mu : X \to X \vee X, \nu : X \to X \tag{6.3}$$

satisfying the usual identities (see Baues [1]) such that $(M(X), 0^*, \mu^*, \nu^*)$ is a group. The cogroup X is *abelian* if the interchange map $T : X \vee X \to X \vee X$ with $Ti_1 = i_2, Ti_2 = i_1$ satisfies $T\mu = \mu$. Then $M(X)$ is an abelian group. A map $f : X \to Y$ between cogroups is *linear* if $\mu f = (f \vee f)\mu$. If X and Y are cogroups then $X \vee Y$ is in the canonical way also a cogroup.

We call $\mathbf{T}$ a *theory of cogroups* if each object X of $\mathbf{T}$ has the structure of a cogroup such that this structure is compatible with sums in $\mathbf{C}$; compare Baues [1].

(6.4) Example. A *ringoid* (or a pre-additive category) is a category $\mathbf{R}$ with the property that all morphism sets $\mathbf{R}(A, B)$ with $A; B \in \mathbf{R}$ are abelian groups and composition in $\mathbf{R}$ is bilinear. A ringoid with only one object is a ring. An *additive category* $\mathbf{A}$ is a ringoid in which sums $A \vee B = A \oplus B$ exist. Then each object in $\mathbf{A}$ is an abelian cogroup with the comultiplication

$$\mu_A = i_1 + i_2 : A \to A \vee A \tag{1}$$

In fact, an additive category is the same as a theory $\mathbf{A}$ in which each object is an abelian cogroup and in which all morphisms are linear.

Each ringoid $\mathbf{R}$ yields in a canonical way the additive category $Add(\mathbf{R})$ which is the *additive completion* of $\mathbf{R}$. Objects in $Add(\mathbf{R})$ are tuple $(X_0, \dots, X_r), r \geq 0$, of objects in $\mathbf{R}$ and morphisms $(X_0, \dots, X_r) \to (Y_0, \dots, Y_k)$ are matrices of morphisms in $\mathbf{R}$. An *additive functor* $F : \mathbf{R} \to \mathbf{R}'$ between ringoids is a functor which induces for $X, Y \in \mathbf{R}$ a homomorphism $F : \mathbf{R}(X, Y) \to \mathbf{R}'(FX, FY)$ between abelian groups. A (right) $\mathbf{R}$ *-module* M is an additive functor

$$M : \mathbf{R}^{op} \to \mathbf{Ab} \tag{2}$$

Let $mod(\mathbf{R})$ be the category of (right) $\mathbf{R}$ -modules; morphisms are natural transformations. Then one readily checks that

$$mod(\mathbf{R}) = model(Add(\mathbf{R})) \tag{3}$$

where the right hand side is the category of models of the theory $Add(\mathbf{R})$. We point out that $mod(\mathbf{R})$ is an abelian category.

(6.5) Lemma. *Let* $\mathbf{C}$ *be a theory of cogroups and let*

$$X \xrightarrow{f,g} Y \xrightarrow{s} X$$

be a diagram in $model(\mathbf{C})$ *with* $fs = gs = 1_Y$. *Then* $X \rightrightarrows Y \to N$ *is a coequalizer in* $model(\mathbf{C})$ *if and only if for all* $C \in \mathbf{C}$ *the induced sequence*

$$kernel\{f : X(C) \to Y(C)\} \xrightarrow{g'} Y(C) \to N(C) \to 0$$

is an exact sequence in the category of groups. Here g' *is the restriction of* g.

Proof. Assume the sequence is exact for each C and let $h : Y \to U$ be given with $hf = hg$ then h yields a unique quotient map $\bar{h} : N(C) \to U(C)$ since $hg(x) = 0$ for $f(x) = 0$. Here $\bar{h}$ is natural in C. Hence $X \rightrightarrows Y \to N$ is a coequalizer. On the other hand if $Y \to N$ is the coequalizer of (f, g) then we get a map $\alpha : cokernel(g') \to N$ and also a map $\beta : N \to cokernel(g')$ where β is the inverse of α. q.e.d.

(6.6) Definition. Let $\mathbf{C}$ be a theory in which the initial object $*$ is a zero object. Then we have the inclusion

$$\mathbf{C} \to model(\mathbf{C})$$

which carries X to $Mor_X = \mathbf{C}(-, X)$. We call sums

$$X = \bigvee_{i \in J} Mor_{X_i} \quad \text{in} \quad model(\mathbf{C})$$

with $X_i \in \mathbf{C}$ *free objects* in $model(\mathbf{C})$. Here J is *finitely generated* if J is finite. Such sums exist for all index sets J. A *free presentation* $g : Y \to X$ of an object $N \in model(\mathbf{C})$ is a coequalizer diagram

$$Y \vee X \underset{(g,1)}{\overset{(0,1)}{\rightrightarrows}} X \to N$$

in $model(\mathbf{C})$ where X and Y are free objects. Here $0 : Y \to * \to X$ is defined by the zero object $*$ in $\mathbf{C}$. We say that two free presentations $g : Y \to X$ and $g' : Y' \to X'$ of N are *strongly equivalent* if $X = X'$ and if there is a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] & X \arrow[r] \arrow[d, equal] & N \arrow[d, equal] \\
Y' \arrow[r, "g'"] & X' \arrow[r] & N
\end{tikzcd}$$

(6.7) Lemma. *Let* $\mathbf{C}$ *be a theory of cogroups. Then all objects* N *in* $model(\mathbf{C})$ *have a free presentation*

Proof. We can choose a free X together with a surjective map $q : X \to N$. For example let

$$X = \bigvee_{C \in \mathbf{C}} \bigvee_{\alpha \in N(C)} \mathbf{C}(-, C_\alpha)$$

where $C_\alpha = C$ for all $\alpha \in N(C)$. Then q is defined by $q(g) = g^*(\alpha)$ for $g \in \mathbf{C}(C', C_\alpha), C' \in \mathbf{C}$. This is a map between models for which

$$q : X(X') \to N(X')$$

is surjective for all $X' \in \mathbf{C}$. Since $N(*) = 0$ we see that $kernel(q)$ is again a model of $\mathbf{C}$ and hence there exists a free object Y and a surjective map $\bar{g}$ such that g is the composite

$$g : Y \xrightarrow{\bar{g}} kernel(q) \subset X.$$

Now it is easy to see that $(0,1)$ and $(g,1)$ form a coequalizer as in (6.6). q.e.d.

§ 7 Split extension of theories

We here describe certain split extensions of theories and we show that the category of models of such an extension is canonically the Grothendieck construction of a functor Γ.

(7.1) Definition. Let $\mathbf{C}$ be a theory and let $\mathbf{R}$ be an additive category and let

$$D : \mathbf{R}^{op} \times \mathbf{C} \to \mathbf{Ab} \tag{1}$$

be a bifunctor which is additive in $\mathbf{R}$ and which satisfy $D(U,*) = 0$ for $U \in \mathbf{R}$. Then we define a category $\mathbf{T}$ as follows. Objects are pairs (U,X) with $U \in \mathbf{R}$ and $X \in \mathbf{C}$ and morphisms

$$(f,g,\delta) : (U,X) \to (V,Y) \tag{2}$$

in $\mathbf{T}$ are given by morphisms $f : U \to V \in \mathbf{R}$, $g : X \to Y \in \mathbf{C}$ and elements $\delta \in D(U,Y)$. Composition of such morphisms is defined by

$$(f,g,\delta)(f_1,g_1,\delta_1) = (ff_1, gg_1, f_1^*\delta + g_*\delta_1) \tag{3}$$

We have the *projection functor*

$$q : \mathbf{T} \to \mathbf{R} \times \mathbf{C}$$

which is the identity on objects and which carries (f,d,δ) to (f,g). We call $\mathbf{T}$ together with q the *split linear extension* of $\mathbf{R} \times \mathbf{C}$ by D; see Baues [2]. By the properties of D in (1) we see that sums in $\mathbf{R} \times \mathbf{C}$ denoted by

$$(U,X) \vee (V,Y) = (U \oplus V, X \vee Y) \tag{4}$$

are also sums in $\mathbf{T}$.

Given a split linear extension $\mathbf{T}$ of $\mathbf{R} \times \mathbf{C}$ we have full inclusions of theories

$$\mathbf{C} \xrightarrow{\lambda} \mathbf{T} \xleftarrow{\rho} \mathbf{R} \tag{7.2}$$

with $\lambda(X) = (0, X)$ and $\rho(U) = (U, *)$. We clearly have the equation of abelian groups

$$\mathbf{T}(\rho U, (V, Y)) = \mathbf{R}(U, V) \oplus D(U, Y)$$

by use of (7.1) (2). Here ρU is an abelian cogroup in $\mathbf{T}$.

Let Δ be the *left adjoint* of

$$\lambda^* : model(\mathbf{T}) \to model(\mathbf{C})$$

with adjunction maps

$$\begin{cases} \varepsilon : N \to \lambda^* \Delta(N) & \text{for} \quad N \in model(\mathbf{C}) \\ \tau : \Delta \lambda^*(M) \to M & \text{and} \quad M \in model(\mathbf{T}) \end{cases}$$

The functor Δ carries the free model $\mathbf{C}(-, X)$ with $X \in \mathbf{C}$ to the free model $\mathbf{T}(-, \lambda X) \in model(\mathbf{T})$. In fact we have

$$model(\mathbf{T})(\mathbf{T}(-, \lambda X), M) = M(\lambda X) = model(\mathbf{C})(\mathbf{C}(-, \lambda X), M\lambda)$$

and this shows $\Delta(\mathbf{C}(-, X)) = \mathbf{T}(-, \lambda X)$.

Using Δ and ρ we obtain the composite functor

(7.3) $$\rho^* \Delta : model(\mathbf{C}) \xrightarrow{\Delta} model(\mathbf{T}) \xrightarrow{\rho^*} model(\mathbf{R}) = \mathbf{A}$$

Here $model(\mathbf{R}) = mod(\mathbf{R}) = \mathbf{A}$ is an abelian category since $\mathbf{R}$ is an additive category. For a set K let $\mathbb{Z}[K]$ be the free abelian group generated by K; we have the inclusion $K \subset \mathbb{Z}[K]$ denoted by $x \mapsto [x] \in \mathbb{Z}[K]$ for $x \in K$. We also obtain the functor

(7.4) $$\Gamma : model(\mathbf{C}) \to model(\mathbf{R}) = \mathbf{A}$$

which carries $N \in model(\mathbf{C})$ to $\Gamma(N) \in \mathbf{A}$ with

$$\Gamma(N)(U) = D(U, -) \otimes_{\mathbf{C}} \mathbb{Z}[N(-)]$$

Here the right hand side is the abelian group obtained as a quotient of

$$\bigoplus_{Y \in \mathbf{C}} D(U, Y) \otimes \mathbb{Z}[N(Y)]$$

by the relations

$$\delta \otimes [f^* y] \sim (f_* \delta) \otimes [y]$$

for $f : Y \to Z \in \mathbf{C}, \delta \in D(U, Y), y \in N(Z)$. It is clear that $\Gamma(N)(U)$ is a well defined contravariant functor in $U \in \mathbf{R}$ and that $\Gamma(N)$ is natural in N so that the functor Γ in (7.4) is well defined.

(7.5) Lemma. *The functors* Γ *and* $\rho^*\Delta$ *in (7.4) and (7.3) respectively coincide, that is* $\Gamma = \rho^*\Delta$.

Proof. We define a functor

$$\Delta' : model(\mathbf{C}) \to model(\mathbf{T})$$

by

$$\Delta'(N)(U,X) = \Gamma(N)(U) \times N(X) \tag{1}$$

for $(U,X) \in \mathbf{T}$ and by

$$(f,g,\delta)^* : \Gamma(N)(V) \times N(Y) \to \Gamma(N)(U) \times N(X) \tag{2}$$

$$\begin{cases} (f,g,\delta)^*(w,z) = (f^*w + \delta \otimes [z], g^*z) \\ w \in \Gamma(N)(V), z \in N(Y) \\ f : U \to V \in \mathbf{R} \\ g : V \to Y \in \mathbf{C} \\ \delta \in D(U,Y) \end{cases}$$

One readily checks that Δ' is a well defined functor; compare (7.1) (3). Now we claim that $\Delta = \Delta'$ is the left adjoint of λ^*. In fact since $\lambda^*(\Delta')(N) = N$ we have a natural map $\Delta \to \Delta'$ given by the identity of N. On the other hand we have for each $M \in model(\mathbf{T})$ and $\alpha : N \to M\lambda$ a unique map

$$\bar{\alpha} : \Delta'(N) \to M \in model(\mathbf{T})$$

with $\bar{\alpha}\lambda^* = \alpha$. In fact, since $(U,X) = \rho U \vee \lambda X$ is a sum we have for the inclusions i_1, i_2 the commutative diagram

$$\begin{array}{ccccc} \Gamma(N)(U) & \xleftarrow{(i_1)^*} & \Delta'(N)(U,X) & \xrightarrow{(i_2)^*} & N(X) \\ \downarrow \bar{\alpha} & & \downarrow \bar{\alpha} & & \downarrow \alpha \\ M(\rho U) & \xleftarrow{(i_1)^*} & M(U,X) & \xrightarrow{(i_2)^*} & M(\lambda X) \end{array}$$

Here $\bar{\alpha}$ on the left hand side satisfies for $\delta \in D(U,Y), y \in N(Y)$ the equation

$$\begin{aligned} \bar{\alpha}(\delta \otimes [y]) &= \bar{\alpha}(f,g,\delta)^*(0,y); \quad \text{see (2)}, \\ &= (f,g,\delta)\bar{\alpha}(0,y) \\ &= (f,g,\delta)^*(0,\alpha y) \end{aligned}$$

with $(f,g,\delta) : (U,*) \to (U,Y)$ given by $f = 1 =$ identity of U, $g = 0 : * \to Y$ and $\delta \in D(U,Y)$. Hence the extension $\bar{\alpha}$ is unique and therefore we see that $\Delta' \to \Delta \to \Delta'$ is the identity. Clearly also $\Delta \to \Delta' \to \Delta$ is the identity. q.e.d.

(7.6) Corollary. *The adjunction map ε in (7.2) is an isomorphism. Moreover the adjunction map τ is given for $N = M\lambda$ by the commutative diagram*

$$\begin{array}{ccc} \Delta(N)(U,X) & = & \Gamma(N)(U) \times N(X) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau'} \\ M(U,X) & = & M(\rho U) \times M(\lambda X) \end{array}$$

Here τ' carries the pair $(\delta \otimes [y], z)$ to $((1,0,\delta)^*(0,y), z)$ with $z \in N(X)$ and $(0,y) \in M(U,Y) = M(\rho U) \times M(\lambda Y)$ and $(1,0,\delta) : (U,*) \to (U,Y) \in \mathbf{T}$.

(7.7) Proposition. *Let $\mathbf{T}$ be the split linear extension of $\mathbf{R} \times \mathbf{C}$ by D. Then there is a canonical isomorphism of categories*

$$\psi : model(\mathbf{T}) = \Gamma\mathbf{A}$$

where $\Gamma\mathbf{A}$ is the Grothendieck construction of Γ in (7.3).

Proof. Let M be a model of $\mathbf{T}$ and let $N = M\lambda$ be the induced model of $\mathbf{C}$. We have the adjunction map

$$\tau : \Delta(N) = (\Delta\lambda^*)(M) \to M \quad \text{in} \quad model(\mathbf{T})$$

and hence τ induces a map

$$\tau_* : \Gamma(N) = \rho^*\Delta(N) \to \rho^* M = A \quad \text{in} \quad \mathbf{A} \tag{1}$$

Now ψ carries M to the object (N, A, τ_*) in $\Gamma\mathbf{A}$. It is clear that ψ is a well defined functor. Next we define a functor

$$\chi : \Gamma\mathbf{A} \to model(\mathbf{T}) \tag{2}$$

which is the inverse of ψ. Let (N, A, η) be an object in $\Gamma\mathbf{A}$ with $N \in model(\mathbf{C})$ and $\eta : \Gamma N \to A \in \mathbf{A}$. We define $M = \chi(N, A, \eta)$ with $M\lambda = N$ as follows. The contravariant functor

$$M : \mathbf{T} \to Set$$

carries an object (U, X) to the product set

$$M(U,X) = A(U) \times N(X) \tag{3}$$

A morphism $(f, g, \delta) : (U, X) \to (V, Y)$ in $\mathbf{T}$ induces the function

$$(f,g,\delta)^* : M(V,Y) = A(V) \times N(Y) \to A(U) \times N(X) = M(U,X)$$

defined for $v \in A(V), y \in N(Y)$ by

$$(f,g,\delta)^*(v,y) = (f^*v + \partial, g^*(y)) \tag{4}$$

Here $\partial = \eta\delta^*\varepsilon(y)$ is the image of y under the composition

$$y \in N(Y) \xrightarrow{\varepsilon} (\lambda^*\Delta(N))(Y) = \Delta(N)(\lambda Y) \xrightarrow{\delta^*}$$
$$\Delta(N)(\rho U) = (\rho^*\Delta(N))(U) = (\Gamma(N))(U) \xrightarrow{\eta} A(U)$$

Here $\delta : \rho U = (U, *) \to (0, Y) = \lambda Y$ is the morphism in $\mathbf{T}$ given by $(0, 0, \delta)$. Using (6.5) (3) we have to check

$$\begin{aligned}(f_1, &g_1, \delta_1)^*(f, g, \delta)^*(v, g)\\ &= (f_1, g_1, \delta_1)^*(f^*(v) + \eta\delta^*\varepsilon(y), g^*(y))\\ &= (f_1^*(f^*(v) + \eta\delta^*\varepsilon(y)) + \eta\delta_1^*\varepsilon(g^*(y)), g_1^*g^*(y))\\ &= ((ff_1)^*(v) + \eta(\alpha_1^*\delta + \xi_*\delta_1)^*\varepsilon(y), (gg_1)^*(y))\end{aligned}$$

This follows from the equation

$$f_1^*\eta\delta^*\varepsilon(y) + \eta\delta_1^*\varepsilon(g^*y) = \eta(f_1^*\delta + g_*\delta_1)^*\varepsilon(y)$$

where η is a homomorphism. Hence M is a well defined object in $\mathit{model}(\mathbf{T})$ and the functor χ which carries (N, A, η) to M is well defined. Now one can check that χ is the inverse of ψ. For example we obtain $\chi\psi(M) = M$ since

$$M(U, X) = M\rho U \times M\lambda X = (\chi\psi(M))(U, X) \tag{5}$$

Here the left hand side holds since M is a model and since $(U, X) = \rho U \vee \lambda X$ is a sum in $\mathbf{T}$. The right hand side is a consequence of the definitions of χ and ψ. Since (5) is compatible with induced maps we see that $M = \chi\psi(M)$. On the other hand we get for $(N, A, \eta) \in \Gamma\mathbf{A}$ above the natural equation

$$\psi\chi(N, A, \eta) = \psi(M) = (N, A, \eta) \tag{6}$$

where M is defined as in (3). For this we have to understand the adjunction map $\tau : \delta(N) \to M$ described in (7.6). Now (4) shows that (6) holds. q.e.d.

(7.8) Addendum. *For the functors I_0 and I_1 in (1.4) the following diagrams commute.*

$$\begin{tikzcd} \mathit{model}(\mathbf{C}) \arrow[r, "I_1"] \arrow[d, equal] & \Gamma\mathbf{A} \arrow[d, equal, "\psi"] \\ \mathit{model}(\mathbf{C}) \arrow[r, "\Delta"] & \mathit{model}(\mathbf{T}) \end{tikzcd}$$

$$\begin{tikzcd} \mathit{model}(\mathbf{C}) \times \mathbf{A} \arrow[r, "I_0"] \arrow[d, equal] & \Gamma\mathbf{A} \arrow[d, equal, "\psi"] \\ \mathit{model}(\mathbf{R} \times \mathbf{C}) \arrow[r, "q^*"] & \mathit{model}(\mathbf{T}) \end{tikzcd}$$

Proof. For $N \in model(\mathbf{C})$ and $M = \Delta N$ the map τ in (7.7) (1) is the identity. Hence τ_* is the identity and therefore $\psi\Delta(N) = I_1(N)$. Next we have

$$model(\mathbf{R} \times \mathbf{C}) = model(\mathbf{R}) \times model(\mathbf{C})$$

with $model(\mathbf{R}) = \mathbf{A}$ so that the second diagram is well defined. For N and $A \in \mathbf{A}$ we consider $\chi I_0(N, A) = M$ with χ defined in (7.7) (3). Since $\eta = 0$ in (7.7) (4) we see that $(f, g, \delta)^*(v, y) = (f^*v, g^*y)$. But this shows that $M = q^*(N, A)$. q.e.d.

Now let $\mathbf{T}$ be the split linear extension of $\mathbf{R} \times \mathbf{C}$ by $D : R^{op} \times \mathbf{C} \to \mathbf{Ab}$. Let

$$free(\mathbf{C}) \subset model(\mathbf{C})$$

for the full subcategory consisting of free objects. Then we have the canonical functor

$$\bar{D} : \mathbf{R}^{op} \times free(\mathbf{C}) \to \mathbf{Ab} \tag{7.9}$$

which extends D such that the functor $X \mapsto D(U, X)$ preserves filtered colimits for $X \in free(\mathbf{C})$. We also have the equation

$$\bar{D}(U, X) = model(\mathbf{T})(\rho^* Mor_U, \lambda^* X)$$

Using $\bar{D}$ we get the following useful description of the functor $\Gamma = \rho^*\Delta$.

(7.10) Lemma. *Let $\mathbf{C}$ be a theory of cogroups with a zero object $*$. Moreover let a free presentation of $N \in model(\mathbf{C})$ be given as in (6.6). Then the sequence*

$$\bar{D}(-, Y \vee X)_2 \xrightarrow{(g,1)_*} \bar{D}(-, X) \to \Gamma(N) \to 0$$

is an exact sequence in $mod(\mathbf{R})$ where

$$\bar{D}(U, X \vee Y)_2 = kernel\{(0, 1)_* : \bar{D}(U, Y \vee X) \to \bar{D}(U, X)\}$$

for $U \in \mathbf{R}$ and where $(g, 1)_*$ is the restriction of the map $(g, 1)_* : \bar{D}(U, Y \vee X) \to \bar{D}(U, X)$.

Proof. We consider the functor

$$\Delta'' : model(\mathbf{C}) \to model(\mathbf{T})$$

defined as follows. For a finitely generated free object X in $model(\mathbf{C})$ let

$$\Delta''(X) = \mathbf{T}(-, \lambda^* X) \tag{1}$$

Moreover if a finitely generated free presentation of $N \in model(\mathbf{C})$ is given as in (6.6) then let $\Delta''(N)$ be defined by the exact sequence of groups with $Z \in \mathbf{T}$

$$\mathbf{T}(Z, \lambda^*(X \vee Y))_2 \to \mathbf{T}(Z, \lambda^* X) \to \Delta''(N)(Z) \to 0 \tag{2}$$

One can check that the functor Δ'' does not depend on the choice of the free presentation and hence is well defined; if the presentation is not finitely generated we use compatibility with filtered colimits. We claim that $\Delta'' = \Delta$ is the left adjoint Δ of λ^*. In fact Δ, as a left adjoint, preserves coequalizers so that the free presentation in (6.6) is mapped to the coequalizer

$$(3) \qquad \Delta(X \vee Y) \underset{(g,1)_*}{\overset{(0,1)_*}{\rightrightarrows}} \Delta(X) \to \Delta(N) \to 0$$

For a free object X, however, it is clear that $\Delta(X) = \Delta''(X)$ by (1). Hence by (2) and (3) we obtain the canonical surjective map

$$(4) \qquad \Delta''(N)(Z) \to \Delta(N)(Z)$$

which is natural in M and Z. On the other hand we know by (6.6) and (2) that

$$(5) \qquad \lambda^* \Delta''(N) = N$$

Hence we have also a canonical map $\Delta(N) \to \Delta''(N)$ which is the inverse of (4). q.e.d.

(7.11) Corollary. *Let* $\mathbf{C}$ *be a theory of cogroups with a zero object* $*$. *Then* Γ *carries a free presentation in model*$(\mathbf{C})$ *to a coequalizer in* $\mathbf{A}$.

Proof. For a free object X we have by (7.10) $\bar{D}(-, X) = \Gamma(X)$. Hence the corollary is an immediate consequence of (7.10). q.e.d.

The next two lemmas are needed in the proof of theorem (7.14) below.

(7.12) Lemma. *Let* g *be a free presentation of* M *in model*$(\mathbf{T})$. *Then* $g' = \lambda^* g$ *is a free presentation of* $\lambda^* M$ *in model* $(\mathbf{C})$.

Proof. For $(V, Y) \in \mathbf{T}$ and $X \in \mathbf{C}$ we have

$$\begin{aligned}(\lambda^* \mathbf{T}(-, (V, Y)))(X) &= \mathbf{T}(\lambda X, (V, Y)) \\ &= \mathbf{T}((0, X), (V, Y)) \\ &= \mathbf{C}(X, Y) \\ &= \mathbf{C}(-, Y)(X)\end{aligned}$$

Hence λ^* carries a free object to a free object. Using (6.5) we see that $\lambda^* g$ is a free presentation of $\lambda^* M$. q.e.d.

The next lemma uses the functor

$$\tilde{Q} : \Gamma \mathbf{A} \to model(\mathbf{C}) \times \mathbf{A}$$

given by (1.4).

(7.13) Lemma. *Let g' be a free presentation of N in model$(\mathbf{C})$ and let g'' be a free presentation of B in model$(\mathbf{R}) = \mathbf{A}$. If $M \in$ model$(\mathbf{C})$ is given with $\tilde{Q}\psi(M) = (N, B)$ then there exists a free presentation g of M with $\tilde{Q}\psi(g) = (g', g''')$ where g''' is strongly equivalent to g''.*

Proof. Let $\eta : \Gamma(N) \to A$ be given by $\chi(M)$ so that the cokernel of η is $Q\eta = B$. Then $g' : Y \to X$ and $g'' : G \to F$ yield the following commutative diagram with exact rows and columns in $\mathbf{A}$.

$$\begin{tikzcd}
\Gamma(Y \vee X)_2 \arrow[r] \arrow[d, "\eta_1"] & \Gamma(X) \arrow[r] \arrow[d, "\eta_0"] & \Gamma(N) \arrow[r] \arrow[d, "\eta"] & 0 \\
A_1 \arrow[r] \arrow[d] & A_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
G \oplus D \arrow[r, "g'''"] \arrow[d] & F \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Here we set $A_0 = \Gamma(X) \oplus F$ and $A_1 = \Gamma(Y \vee X)_2 \oplus G \oplus D$ where D is free in $\mathbf{A}$. We choose for $F \to B$ a lift $F \to A$ which defines the surjective map $A_0 \to A$. We can choose g''' in such a way that $g'''(D) = 0$ and $g''' \mid G = g''$. Now η_0 and η_1 correspond to free objects in *model*$(\mathbf{C})$ which yield via the diagram above the free presentation g of M with the properties in (7.13). q.e.d.

In the following result we describe crucial properties of the reduction functors Λ_i in (2.6).

(7.14) Theorem. *Let $\mathbf{C}$ be a theory of cogroups with zero object $*$ and let $\mathbf{T}$ be a split linear extension of $\mathbf{R} \times \mathbf{C}$ as in (7.1). Moreover let*

$$\Lambda : model(\mathbf{T}) = \Gamma\mathbf{A} \to \mathbf{A}'$$

be a functor which carries a free presentation to a coequalizer. Then Λ preserves epimorphisms and the reduction functors

$$\begin{aligned}
&\Lambda_0 : model(\mathbf{C}) \to \mathbf{A}' \\
&\Lambda_i : model(\mathbf{C}) \times \mathbf{A} \to \mathbf{A}', i \geq 1,
\end{aligned}$$

carry a free presentation to a coequalizer.

Here a free presentation in the product category *model*$(\mathbf{C}) \times \mathbf{A}$ is a pair of free presentations in *model*$(\mathbf{C})$ and in $\mathbf{A} =$ *model*$(\mathbf{R})$ respectively. In addition to (7.14) we get the following result. We say that a functor $f : \mathbf{A} \to \mathbf{A}'$ between abelian categories is *weakly right exact* if each exact sequence $A \xrightarrow{g} B \to C \to 0$ in $\mathbf{A}$ yields an exact sequence

$$F(A \oplus B)_2 \xrightarrow{(g,1)_*} FB \to FC \to 0$$

in $\mathbf{A}'$.

(7.15) Addendum. *Given $A \in \mathbf{A}$ the reduction functors*

$$\Lambda_0, \Lambda_i(-, A) : model(\mathbf{C}) \to \mathbf{A}'$$

with $i \geq 1$ carry a free presentation in $model(\mathbf{C})$ to a coequalizer in $\mathbf{A}'$. Moreover for $X \in model(\mathbf{C})$ the reduction functor

$$\Lambda_i(X, -) : \mathbf{A} \to \mathbf{A}'$$

is weakly right exact.

Proof of (7.14). Let $\eta : \Gamma N \to A \in \Gamma\mathbf{A}$ and let

(1)
$$\begin{tikzcd} \Gamma Y \arrow[r, "\Gamma g"] \arrow[d, "\eta_Y"] & \Gamma X \arrow[r] \arrow[d, "\eta_X"] & \Gamma N \arrow[d, "\eta"] \\ A_1 \arrow[r, "g'"] & A_0 \arrow[r] & A \end{tikzcd}$$

be morphisms in $\Gamma\mathbf{A}$ such that $(g, g') : \eta_Y \to \eta_X$ is a free presentation of η. Here we identify $model(\mathbf{T}) = \Gamma\mathbf{A}$ via ψ in (7.7). Let $(1, f) : \eta \to \lambda$ be an $\mathbf{A}$ -epimorphism. Then we can choose as in (7.13) a free presentation $\lambda'' \to \lambda'$ of λ for which the following diagram commutes; in particular we may choose $\lambda' = \eta_X$.

$$\begin{tikzcd} \eta_Y \arrow[r] & \eta_X \arrow[r, "p"] \arrow[d, equal] & \eta \arrow[d, "{(1,f)}"] \\ \lambda'' \arrow[r] & \lambda' \arrow[r, "q"] & \lambda \end{tikzcd}$$

Hence since Λ carries p and q to epimorphisms also $\Lambda(1, f)$ is an epimorphism. Next we observe that the inclusion (see (7.8))

(2)
$$\Delta : model(\mathbf{C}) \xrightarrow{I_1} \Gamma\mathbf{A} = model(\mathbf{T})$$

carries a free model to a free model; in fact for $X \in \mathbf{C}$ the functor Δ carries $\mathbf{C}(-, X)$ to $\mathbf{T}(-, \lambda X)$. Moreover Δ preserves coequalizers. Hence we see that $\Lambda_0 = \Lambda I_1$ carries free presentations to coequalizers. Next we prove that the functors

(3)
$$model(\mathbf{T}) = \Gamma\mathbf{A} \xrightarrow{\Lambda_{(i)}} \mathbf{A}', i \geq 1,$$

carry a free presentation to a coequalizer. In fact, since λ^* in (7.12) carries a free presentation to a free presentation we see that the top row and the row in the middle of the next diagram are exact.

$$
\begin{array}{lcccccccc}
 & \Lambda I_1(Y\vee X)_2 & \longrightarrow & \Lambda I_1 X & \longrightarrow & \Lambda I_1(N) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & \Lambda(\eta_Y\vee\eta_X)_2 & \longrightarrow & \Lambda(\eta_X) & \longrightarrow & \Lambda(\eta) & \longrightarrow & 0 \\
(4) & \downarrow & & \downarrow & & \downarrow & & \\
 & \Lambda_{(1)}(\eta_Y\vee\eta_X)_2 & \longrightarrow & \Lambda_{(1)}(\eta_X) & \longrightarrow & \Lambda_{(1)}(\eta) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & & 0 & & 0 & &
\end{array}
$$

Moreover all columns are exact by (2.6) (1). This implies that also the bottom row is exact. Therefore $\Lambda_{(1)}$ has the property in (3). Now assume $\Lambda_{(i)} = \Omega, i \geq 1$, has the property in (3). Then we have by (2.3) (1) the exact sequence

$$
(5) \qquad \Omega(I_1 N \mid I_0\Gamma N) \to \Omega(\eta \mid I_0\Gamma N) \to \Lambda_{(i+1)}(\eta) \to 0
$$

for $\eta \in \Gamma\mathbf{A}$.

Let $C \to B \to A \to 0$ be an exact sequence in $\mathbf{A}$ which for example coincides with the exact sequence

$$
(6) \qquad \Gamma(Y\vee X)_2 \to \Gamma(X) \to \Gamma(N) \to 0
$$

given by (1) and (7.11). Then we can choose as in (7.13) a diagram in $\Gamma\mathbf{A}$

$$
\begin{array}{ccccc}
 & & \gamma'' & & \\
 & & \downarrow & & \\
\gamma' & \longrightarrow & \gamma & \longrightarrow & \eta\vee I_0 A \\
\downarrow & & \downarrow & & \| \\
\eta_Y\vee I_0 C & \longrightarrow & \eta_X\vee I_0 B & \longrightarrow & \eta\vee I_0 A
\end{array}
$$

in which the column in the middle and the row in the middle are free presentations. Since Ω has property (3) we thus see that the induced sequence

$$
\Omega((\eta_Y\vee I_0 C)\vee(\eta_X\vee I_0 B))_2 \to \Omega(\eta_X\vee I_0 B) \to \Omega(\eta\vee I_0 A) \to 0
$$

is exact and therefore also the sequence

$$
(7) \qquad \Omega(\eta_Y\vee\eta_X \mid I_0 C\vee I_0 B)_2 \to \Omega(\eta_X \mid I_0 B) \to \Omega(\eta \mid I_0 A) \to 0
$$

is exact. Using (5) we get the following commutative diagram where $A \to B \to C \to 0$ is given by (6).

$$\begin{array}{ccccccc}
\Omega(I_1Y \vee I_1X \mid I_0C \vee I_0B)_2 & \longrightarrow & \Omega(I_1X \mid I_0B) & \longrightarrow & \Omega(I_1N \mid I_0A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Omega(\eta_Y \vee \eta_X \mid I_0C \vee I_0B)_2 & \longrightarrow & \Omega(\eta_X \mid I_0B) & \longrightarrow & \Omega(\eta \mid I_0A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Lambda_{(i+1)}(\eta_Y \vee \eta_X)_2 & \longrightarrow & \Lambda_{(i+1)}(\eta_X) & \longrightarrow & \Lambda_{(i+1)}(\eta) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Here the columns are exact by (5) and the top row and the row in the middle are exact by (7). Hence the bottom row is exact. This proves (3) for $\Lambda_{(i+1)}$. Hence all $\Lambda_{(i)}, i \geq 1$, carry a free presentation to a coequalizer.

Now we can use the definition of Λ_i by the exact sequence

$$\Lambda_{(i+1)}(I_0(X,A)) \xrightarrow{P'} \Lambda_{(i)}(I_0(X,A)) \to \Lambda_i(X,A) \to 0 \tag{8}$$

in (2.5) and (2.6) (2). Using (7.13) we obtain for a free presentation (g', g'') of (X, A) in $model(\mathbf{C}) \times \mathbf{A}$ a free presentation g of $I_0(X,A)$ in $model(\mathbf{T})$ such that $\tilde{Q}(g) = (g', g''')$ where g''' is strongly equivalent to g''. Since $\Lambda_{(i)}$ and $\Lambda_{(i+1)}$ satisfy (3) we thus see by (8) that Λ_i carries the free presentation (g', g'') to a coequalizer. q.e.d.

Proof of (7.15). We choose a commutative diagram in $\mathbf{A}$

$$\begin{array}{ccccccc}
G' & \longrightarrow & F' & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
G & \longrightarrow & F & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \\
C & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

The rows and the columns are exact in $\mathbf{A}$ and F, F', G, G' are free in $\mathbf{A}$. For $\eta \in \Gamma\mathbf{A}$ we thus get the diagram

$$\begin{array}{ccccc}
(\eta_Y, G') & \longrightarrow & (\eta_Y, F') & \longrightarrow & (*, 0) \\
\downarrow & & \downarrow & & \downarrow \\
(\eta_Y, G) & \longrightarrow & (\eta_X, F) & \longrightarrow & (\eta, A) \\
\downarrow & & \downarrow & & \| \\
(*, C) & \longrightarrow & (\eta, B) & \longrightarrow & (\eta, A)
\end{array}$$

Here the top row and the row in the middle and also the left hand column in the middle are free presentations in $model(\mathbf{C}) \times \mathbf{A}$. Hence they yield by Λ_i coequalizers and this shows that

$$\begin{array}{ccccccc}
\Lambda_i((*, C) \vee (\eta, B))_2 & \longrightarrow & \Lambda_i(\eta, B) & \longrightarrow & \Lambda_i(\eta, A) & \longrightarrow & 0 \\
\| & & & & & & \\
\Lambda_i(\eta, C \oplus B)_2 & & & & & &
\end{array}$$

is exact. Hence the functor $\Lambda_i(\eta, -)$ is weakly right exact for all $\eta \in \Gamma\mathbf{A}$.

Next we consider the diagram

$$\begin{array}{ccccc}
 & & (*, G) & & \\
 & & \downarrow & & \\
(\eta_Y, G) & \longrightarrow & (\eta_X, F) & \longrightarrow & (\eta, A) \\
\downarrow & & \downarrow & & \| \\
(\eta_Y, 0) & \longrightarrow & (\eta_X, A) & \longrightarrow & (\eta, A)
\end{array}$$

here the row in the middle and also the column in the middle are free presentations in $model(\mathbf{C}) \times \mathbf{A}$. Hence they are mapped to coequalizers by Λ_i. This shows that the sequence

$$\begin{array}{ccccccc}
\Lambda_i((\eta_Y, 0) \vee (\eta_X, A))_2 & \longrightarrow & \Lambda_i(\eta_X, A) & \longrightarrow & \Lambda_i(\eta, A) & \longrightarrow & 0 \\
\| & & & & & & \\
\Lambda_i(\eta_Y \vee \eta_X, A)_2 & & & & & &
\end{array}$$

is exact. Hence for each A the functor $\Lambda_i(-, A)$ carries a free presentation to a coequalizer. q.e.d.

§ 8 Graded theories

We introduce the notion of a graded theory and we show that the category of models of a graded theory is an iterated Grothendieck construction.

(8.1) Definition. A *graded theory* $\mathbf{T}$ is a theory $\mathbf{T}$ with a zero object $*$ together with a set $\mathcal{S}$ of objects in $\mathbf{T}$ and a function

$$| \quad | : \mathcal{S} \to \mathbb{Z}$$

which carries $S \in \mathcal{S}$ to the *degree* $| S |$ with $| S | \geq 0$. Moreover the following properties (a), (b), (c) are satisfied.

(a) The objects X of $\mathbf{T}$ are exactly the finite sums of objects in $\mathcal{S}$, that is

$$X = S_1 \vee \ldots \vee S_r$$

with $S_1, \ldots, S_r \in \mathcal{S}$ and $0 \leq r < \infty$. Let $| X |= Min\{| S_i |; 1 \leq i \leq r\}$ be the *degree of* X and let $\dim(X) = Max\{| S_i |; 1 \leq i \leq r\}$ be the (formal) *dimension of* X. For $n \geq 0$ we define the *n-skeleton* of X by

$$X^n = \bigvee_{\substack{i \in \{1,\dots,r\} \\ |S_i| \leq n}} S_i$$

We have the canonical inclusion $X^n \subset X$ and $\dim(X) \leq n$ if and only if $X^n = X$ and $| X | \geq k$ if and only if $X^k = *$.

(b) Each object S of $\mathcal{S}$ is an abelian cogroup and thus by (a) also each object X of $\mathbf{T}$ is an abelian cogroup. Moreover each map $f : S \to X$ in $\mathbf{T}$ with $| X | \geq | S |$ is linear and $\mathbf{T}(S, X) = 0$ for $| X | > | S |$.

(c) Each map $f : S \to U \vee Y$ in $\mathbf{T}$ where $| U | \geq | S |$ and $\dim(Y) < | S |$ splits as a sum $f = i_1 f_1 + i_2 f_2$ where $f_1 : S \to U$ and $f_2 : S \to Y$.

As a special case of graded theories $\mathbf{T}$ we can consider such $\mathbf{T}$ for which all ringoids $\mathbf{R}_n, n \geq 0$, coincide so that the associated abelian categories $\mathbf{A}_n, n \geq 0$, satisfy $\mathbf{A}_0 = \mathbf{A}_1 = \dots$. In particular, one has the following special case of such a graded theory.

(8.1') Definition. Let R be a ring. A *graded R-theory* is a graded theory $\mathbf{T}$ with the following properties. For $n \geq 0$ there is at most one object $S = S_n \in \mathcal{S}$ with $| S | = n$ and the endomorphism ring $\mathbf{R}_n$ of S_n coincides with the ring R.

(8.1") Example. Let $r \geq 2$ and let

$$\mathcal{S}_r = \{S^r, S^{r+1}, \dots\}$$

be the set of all spheres S^n with $n \geq r$. We define the degree function

$$\begin{cases} | \quad |: \mathcal{S}_r \to \mathbb{N}_0 \quad \text{by} \\ | S^{n+r} | = n \quad \text{for} \quad n \geq 0 \end{cases}$$

Hence the degree of the sphere does not coincide with the dimension of the sphere. Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed spaces and basepoint preserving maps. We associate with $\mathcal{S}_r$ the full subcategory

$$\mathbf{T}(\mathcal{S}_r) \subset \mathbf{Top}^*/\simeq$$

consisting of finite one point unions $S_1 \vee \dots \vee S_k$ with $S_i \in \mathcal{S}_r$ for $1 \leq i \leq k, k \geq 0$. Then it is easily seen that $\mathbf{T}(\mathcal{S}_r)$ is a graded $\mathbb{Z}$ -theory. Moreover $\mathbf{T}(\mathcal{S}_r)$ is $(r-1)$ -polynomial in the sense of (9.4) below as follows from the Hilton-Milnor theorem [12]. Compare [17], [9], [10].

(8.1"') Example. Let $r \geq 1$ and let $\mathcal{A}$ be a set of abelian groups. Moreover let

$$\mathcal{K}_r(\mathcal{A}) = \{K(A, n+r); n \geq 0, A \in \mathcal{A}\}$$

be the corresponding set of *Eilenberg-Mac Lane spaces* $K(A, n+r)$. We define the degree function

$$\begin{aligned} | \quad |&: \mathcal{K}_r(\mathcal{A}) \to \mathbb{N}_0 \quad \text{by} \\ | K(A, n+r) | &= n \quad \text{for} \quad n \geq 0 \end{aligned}$$

We associate with $\mathcal{K}_r(\mathcal{A})$ the full subcategory

$$\mathbf{K}_r(\mathcal{A}) \subset \mathbf{Top}^*/\simeq$$

consisting of products $K_1 \times \ldots \times K_k$ with $K_i \in \mathcal{K}_r(\mathcal{A})$ for $1 \leq i \leq k, k \geq 0$. Then it is easily seen that the opposite category $\mathbf{K}_r(\mathcal{A})^{op}$ is a graded theory. Moreover $\mathbf{K}_r(\mathcal{A})^{op}$ is r-polynomial in the sense of (9.4) below. The system of *cohomology groups* $M = \{H^{n+r}(X;A), n \geq 0, A \in \mathcal{A}\}$ together with all cohomology operations is a model of $\mathbf{K}_r(\mathcal{A})^{op}$.

Let $\mathbf{R}(\mathcal{A})$ be the full subcategory of $\mathbf{Ab}$ consisting of all abelian groups $A \in \mathcal{A}$. Then $\mathbf{R}(\mathcal{A})$ is a ringoid and all abelian categories $\mathbf{A}_i$ associated with the theory $\mathbf{K}_r(\mathcal{A})$ coincide with the category of modules

$$\mathbf{A}_i = mod\,(\mathbf{R}(\mathcal{A})), i \geq 0.$$

If $Add\,(\mathbf{R}(\mathcal{A}))$ is an abelian category then the global dimension of $mod\,(\mathbf{R}(\mathcal{A}))$ is at most 2. For example this holds if $\mathcal{A} = Cyc$ consists of all cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/p^i$ where p is a prime and $i \geq 1$. This also holds if $\mathcal{A} = Cyc_p$ consists of all $\mathbb{Z}/p^i, i \geq 1$, where p is a fixed prime. Hence for these cases we can apply the results on the vanishing line below. This enriches considerably the classical theory of modules are the Steenrod-algebra which corresponds to the case $\mathcal{A} = \{\mathbb{Z}/p\}$.

Remark. Blanc [4] defines for a ring R a *category of regular graded algebras* CRGA. It is easy to check that for each CRGA there exists a graded R^{op} -theory $\mathbf{T}$ such that

$$CRGA = model(\mathbf{T}).$$

In fact, $\mathbf{T}$ is the category of finitely generated free RGA's; see Blanc [4] 2.1.4. The category $model(\mathbf{T})$ given by graded R-theories, however, are considerably more general than CRGA's in the sense of Blanc [4]. Blanc [4] 2.4.2, 5.1.1 uses free simplicial resolutions with a CW-basis of an object in the category CRGA. Such resolutions are special cases of the resolutions in $\Gamma^\infty \mathbf{A}$ defined in § 2. Hence derived functors in this paper yield the derived functors considered by Blanc [4] as a special case.

The definition of a graded theory $\mathbf{T}$ implies the following properties of $\mathbf{T}$. Let $\mathbf{T}_n, n \geq 0$, be the full subcategory of $\mathbf{T}$ consisting of objects X with $\dim(X) \leq n$. Moreover let $\mathbf{R}_n$ be the full subcategory of $\mathbf{T}$ consisting of objects $S \in \mathcal{S}$ with $\mid S \mid = n$. Then (8.1) (b) shows that $\mathbf{R}_n$ is a ringoid with $\mathbf{R}_0 = \mathbf{T}_0$. Let

(8.2) $$\mathbf{A}_n = mod(\mathbf{R}_n) = model(Add(\mathbf{R}_n))$$

be the abelian category of right $\mathbf{R}_n$ -modules; see (6.4). These are the abelian categories associated to the graded theory $\mathbf{T}$. We have full inclusions of theories

(8.3) $$\mathbf{T}_n \xrightarrow{\lambda} \mathbf{T}_{n+1} \xleftarrow{\rho} Add(\mathbf{R}_{n+1})$$

Here λ is the inclusion of the subcategory and ρ carries $U = (S_1, \ldots, S_r) \in Add(\mathbf{R}_{n+1})$ with $S_i \in \mathcal{S}$ and $\mid S_i \mid = n+1$ for $1 \leq i \leq r$ to the sum

(1) $$\rho(U) = S_1 \vee \ldots \vee S_r$$

in $\mathbf{T}$. Moreover we define a bifunctor

(2) $$\begin{aligned} &D_{n+1} : Add(\mathbf{R}_{n+1})^{op} \times \mathbf{T}_n \to \mathbf{Ab} \\ &D_{n+1}(U, Y) = \mathbf{T}(\rho U, Y) \end{aligned}$$

Here D_{n+1} has the properties of the bifunctor D in (7.1), that is D_{n+1} is additive in U and $D_{n+1}(U, *) = 0$.

(8.4) Lemma. *Let $\mathbf{T}$ be a graded theory. Then the theory $\mathbf{T}_{n+1}$ is a split extension of $Add(\mathbf{R}_{n+1}) \times \mathbf{T}_n$ by D_{n+1}.*

Proof. Each object X of $\mathbf{T}_{n+1}$ has the unique decomposition

$$X = \rho U \vee \lambda Y$$

where $\dim(Y) \leq n$ and $U \in Add(\mathbf{R}_{n+1})$. A morphism h in $\mathbf{T}_{n+1}$ is given by

$$h : X = \rho U \vee \lambda Y \to \rho U' \vee \lambda Y' = X'$$

Using (8.1) (b), (c) we see that h carries λY to $\lambda Y'$, that is, there is $g : Y \to Y' \in \mathbf{T}_n$ with $hi_2 = i_2\lambda(g)$. Moreover hi_1 splits by (8.1) (c), that is, there are $f : U \to U'$ and $\delta \in D_{n+1}(U, Y')$ such that $hi_1 = i_1 f + i_2\delta$. This shows that $\mathbf{T}_{n+1}$ is canonically isomorphic to the split extension described in (7.1). q.e.d.

The lemma shows that we can apply (7.7) inductively. This leads to the following crucial result.

(8.5) Theorem. *Let $\mathbf{T}$ be a graded theory and let $\mathbf{A} = (\mathbf{A}_0, \mathbf{A}_1, \ldots)$ be the sequence of abelian categories associated to $\mathbf{T}$ by $\mathbf{A}_n = mod(\mathbf{R}_n)$ for $n \geq 0$. Then there exists a sequence of functors $\Gamma = (\Gamma^1, \Gamma^2, \ldots)$ with*

$$\begin{cases} \Gamma^1 : \mathbf{A}_0 \to \mathbf{A}_1 \\ \Gamma^{n+1} : \Gamma^n \mathbf{A}_n \to \mathbf{A}_{n+1}, n \geq 0 \end{cases}$$

such that the category of models of $\mathbf{T}$ is the iterated Grothendieck construction

$$model(\mathbf{T}) = \Gamma^\infty \mathbf{A}.$$

If there is $m \geq 0$ with $| S | \leq m$ for all $S \in \mathcal{S}$ then we get

$$model(\mathbf{T}) = \Gamma^m \mathbf{A}_m.$$

Using (7.7) we obtain inductively the isomorphism of categories, $n \geq 0$,

(8.6) $$\psi : model(\mathbf{T}_n) = \Gamma^n \mathbf{A}_n$$

and the functors

$$\Gamma^{n+1} : \Gamma^n \mathbf{A}_n = model(\mathbf{T}_n) \xrightarrow{\Delta} model(\mathbf{T}_{n+1}) \xrightarrow{\rho^*} mod(\mathbf{R}_{n+1}) = \mathbf{A}_{n+1}$$

Here Δ is the left adjoint of the functor λ^* induced by $\lambda : \mathbf{T}_n \to \mathbf{T}_{n+1}$ and ρ^* is induced by $\rho : Add(\mathbf{R}_{n+1}) \to \mathbf{T}_{n+1}$; compare (8.3). Now (8.5) is a direct consequence of (7.7). It is clear that $mod(\mathbf{R}_n)$ has enough projectives for $n \geq 0$. This shows by (2.7) that *resolutions* exist in $\Gamma^n \mathbf{A}_n$ for all $n \geq 0$. Moreover one readily checks that all Γ^n preserve zero objects so that *sums* in $\Gamma^n \mathbf{A}$ can be described by cross effects as in (4.2). The multi-epimorphism in $\Gamma^n \mathbf{A}_n$ correspond via ψ in (8.6) exactly to surjective maps in $model(\mathbf{T}_n)$; here we say that a map $f : M \to N$ in $model(\mathbf{T}_n)$ is *surjective* if for all objects $X \in \mathbf{T}_n$ the induced map $f : M(X) \to N(X)$ is a surjective map between sets.

Using (7.14) inductively we get the following result; compare (4.2).

(8.7) Theorem. *Let $\mathbf{T}$ be a graded theory and let $\Gamma = (\Gamma^1, \dots)$ be the associated iterated Grothendieck construction. Then Γ^{n+1} carries a multi-epimorphism in $\Gamma^n \mathbf{A}_n$ to an epimorphism in $\mathbf{A}_{n+1}, n \geq 0$. Hence all reduction functors*

$$\Gamma_\varepsilon^{n+1} : (\Gamma^n \mathbf{A}_n) \times \mathbf{A}_{i_1} \times \dots \times \mathbf{A}_{i_{|\varepsilon|}} \to \mathbf{A}_{n+\ell(\varepsilon)+1}$$

are defined for all $\varepsilon \in (\mathbb{N}_0)^{\ell(\varepsilon)}$ with $\ell(\varepsilon) \geq 0$ and $n + \varepsilon^{-1}(\mathbb{N}) = \{i_1 < \dots < i_{|\varepsilon|}\}$. These reduction functors carry a multi-epimorphism to an epimorphism and they carry a free presentation to a coequalizer.

The theorem shows that the reduction functor Γ_ε^{n+1} is determined by its restriction to finitely generated free objects. This is very helpful for the computation of the reduction functors. As in (7.15) we get the following additional property of the reduction functors Γ_ε^{n+1} in (8.7).

(8.8) Addendum. *For $X \in \Gamma^n \mathbf{A}_n$ and $A_i \in \mathbf{A}_i$ the functor $\Gamma_\varepsilon^{n+1}(-, A_{i_1}, \dots, A_{i_{|\varepsilon|}})$ carries a free presentation in $\Gamma^n \mathbf{A}_n$ to a coequalizer and the functor*

$$\Gamma_\varepsilon^{n+1}(X, A_{i_1}, \dots, A_{i_t}, -, A_{i_{t+2}}, \dots, A_{i_{|\varepsilon|}})$$

with $0 \leq t <| \varepsilon |$ is weakly right exact.

Proof of (8.7) and (8.8). It suffices to show that Γ^{n+1} carries a multi-epimorphism $f^{\leq n} : \eta^{\leq n} \to \lambda^{\leq n}$ to an epimorphism. For this we observe that a free presentation $Y \to X \to \eta^{\leq n}$ can be used to obtain a free presentation $Y' \to X \to \lambda^{\leq n}$ where $X \to \lambda^{\leq n}$ is the composite of $X \to \eta^{\leq n}$ and $f^{\leq n}$. Hence since Γ^{n+1} carries a free presentation to an epimorphism we see that Γ^{n+1} carries $f^{\leq n}$ to an epimorphism. Now we can apply (7.14) and (7.15) inductively. This yields the results. q.e.d.

§ 9 Derived functors for graded theories

Let $\mathbf{T}$ be a graded theory. We have seen in (8.6) that resolutions exist in $model(\mathbf{T}_n)$. Hence we obtain for each object $X \in model(\mathbf{T}_n)$ the derived functors $(n \geq 0)$

(9.1) $$L_i \Gamma^{n+1}(X) = \pi_i(\Gamma^{n+1} X.)$$

where $X.$ is a resolution of X in $s\,model(\mathbf{T}_n)$.

(9.2) Proposition. $L_0\Gamma^{n+1} = \Gamma^{n+1}$ for $n \geq 0$.

Proof. Since free presentations exist we see that there exists a resolution $X.$ for which

$$X_{(1)} \rightrightarrows X_{(0)} \to X$$

is a free presentation of X as in (6.6). Now (7.11) shows that Γ^{n+1} carries a free presentation to a coequalizer. This implies $L_0\Gamma^{n+1} = \Gamma^{n+1}$. q.e.d.

The next result is an immediate corollary of (8.7) and (4.7).

(9.3) Theorem. *Let* $\mathbf{T}$ *be a graded theory and let* $\Gamma = (\Gamma^1, \dots)$ *be the associated iterated Grothendieck construction with* $\Gamma^{n+1} : \Gamma^n\mathbf{A}_n \to \mathbf{A}_{n+1}$*. Assume the abelian category* $\mathbf{A}_n$ *has finite global dimension* $d_n < \infty$ *for* $n \geq 0$*. Moreover assume that all reduction functors* Γ^{n+1}_ε *are reducible and that all reduction functors* Γ^1_ε *are bounded by* γ_ε *with* $\gamma_\varepsilon = 0$ *if* $\Gamma^1_\varepsilon = 0$*. Then the derived functors* $L_i\Gamma^{n+1}$ *have a vanishing line* $N_{n+1} < \infty$*, that is*

$$L_i\Gamma^{n+1} = 0 \quad \textit{for} \quad i > N_{n+1}.$$

Here N_{n+1} *is inductively defined by*

$$\begin{cases} a^0 = d_0 \\ a^{n+1} = 1 + d_{n+1} + N_{n+1} \quad \textit{for} \quad n \geq 0 \\ N_{n+1} = Max\{\gamma_\varepsilon(a^0, a^{i_1}, \dots, a^{i_{|\varepsilon|}}), \varepsilon \in (\mathbb{N}_0)^n\} \end{cases}$$

The maximum is taken over all $\varepsilon \in (\mathbb{N}_0)^n$*. Such an* ε *defines* $i_1, \dots, i_{|\varepsilon|}$ *by* $\varepsilon^{-1}(\mathbb{N}) = \{i_0 < \dots < i_{|\varepsilon|}\}$.

A graded theory $\mathbf{T}$ is *polynomial* if for all $S \in \mathcal{S}$ the functors

$$\pi^S : \mathbf{T} \to \mathbf{Ab} \quad \text{with} \quad \pi^S(X) = \mathbf{T}(S, X) \tag{9.4}$$

are polynomial. Moreover $\mathbf{T}$ is polynomial of *degree* $\leq k$ if all these functors are of degree $\leq k$. In particular $\mathbf{T}$ is *linear*, resp. *quadratic* if $\mathbf{T}$ is polynomial of degree 1, resp. 2. One readily checks that $\mathbf{T}$ is linear if and only if all morphisms in $\mathbf{T}$ are linear; see (6.3). For $r \geq 1$ we say that $\mathbf{T}$ is *r-polynomial* if for $n \geq 0$ and $q \geq 2$ and any set of objects $X_1, \dots, X_q$ in $\mathbf{T}$ with

$$\sum_{i=1}^{q}(r + \mid X_i \mid) > r + n + 1$$

and $S \in \mathcal{S}$ with $\mid S \mid = n + 1$ we have

$$\pi^S(X_1 \mid \dots \mid X_q) = 0.$$

Since $r \geq 1$ this shows that $\mathbf{T}$ is polynomial.

(9.5) Lemma. *Let* $\mathbf{T}$ *be a graded theory with associated iterated Grothendieck construction* $\Gamma = (\Gamma^1, \dots)$. *Then* $\mathbf{T}$ *is polynomial (resp. polynomial of degree* k*, resp.* r*-polynomial) if and only if* Γ *is polynomial (resp. polynomial of degree* k*, resp.* r*-polynomial).*

Proof. Since cross effects behave well with respect to coequalizers we see that (9.5) is a consequence of (7.10) and (8.4). q.e.d.

(9.6) Theorem. *Let* $\mathbf{T}$ *be a graded theory which is polynomial and let* $\Gamma = (\Gamma^1, \dots)$ *be the associated iterated Grothendieck construction. Assume that all associated abelian categories* $\mathbf{A}_n$ *have finite global dimension* $d_n < \infty$ *for* $n \geq 0$. *Then the derived functors* $L_i\Gamma^{n+1}, n \geq 0, i \geq 0$, *have a vanishing line* $N_{n+1} < \infty$ *with*

$$L_i\Gamma^{n+1} = 0 \quad \textit{for} \quad i > N_{n+1}$$

where N_{n+1} *can be computed as in (9.3).*

Proof. This is a consequence of (9.5), (9.3) and (5.3). q.e.d.

(9.7) Corollary. *Let* $\mathbf{T}$ *be a theory as in (9.6). If* $\mathbf{T}$ *is linear then*

$$L_i\Gamma^{n+1} = 0 \quad \textit{for} \quad i > d_0 + \sum_{k=1}^{n}(d_k + 1)$$

(9.8) Corollary. *Let* $\mathbf{T}$ *be a theory as in (9.6). If* $\mathbf{T}$ *is* r*-polynomial with* $r > d = Max\{d_0, d_1, \dots, d_n\}$. *Then*

$$L_i\Gamma^{n+1} = 0 \quad \textit{for} \quad i > d + n(d+1).$$

Here (9.7) corresponds to (5.4) and (9.8) corresponds to (5.6).

§ 10 Functors on categories of models

Let $\mathbf{T}$ be a graded theory and let $model(\mathbf{T})$ be the category of models of $\mathbf{T}$. We here consider functors

$$F : model(\mathbf{T}) \to \mathbf{B} \tag{10.1}$$

where $\mathbf{B}$ is an abelian category. The derived functors L_iF are defined for $i \geq 0$ by

$$(L_iF)(M) = \pi_i(FM.)$$

where $M.$ is a resolution of M; such resolutions exist by (3.4) and (8.5).

(10.2) Definition. For each model M of $\mathbf{T}$ and $k \geq 0$ we define a new model M^k and a surjective map

$$q : M \to M^k \quad \text{in} \quad model(\mathbf{T}) \tag{1}$$

by setting $M^k(S) = M(S)$ for $\mid S \mid \leq k$ and $M^k(S) = 0$ for $\mid S \mid > k$. Then q is the identity on $M(S)$ for $\mid S \mid \leq k$ and is trivial otherwise. For $Y \in \mathbf{T}$ we have

$$M^k(Y) = M(Y^k) \tag{2}$$

where Y^k is the k-skeleton of Y. Hence M^k is also defined by the composite

$$M^k : \mathbf{T} \xrightarrow{j} \mathbf{T}_k \xrightarrow{i} \mathbf{T} \xrightarrow{M} \mathbf{Set} \tag{3}$$

Here j carries Y to Y^k and i is the inclusion functor. Hence M^k is a well defined model of $\mathbf{T}$. We say that the functor F in (10.1) has *height*$(F) \leq k$ if for all $M \in model(\mathbf{T})$ the induced map

$$q_* : F(M) \to F(M^k) \tag{4}$$

is an isomorphism.

(10.3) Lemma. *A functor F has height$(F) \leq k$ if and only if F is naturally isomorphic to a composite functor*

$$model(\mathbf{T}) \xrightarrow{i^*} model(\mathbf{T}_k) \xrightarrow{G} \mathbf{B}$$

Proof. If F is given then we have $G = \Gamma j^*$ with j defined as in (10.2) (3). q.e.d.

(10.4) Theorem. *Let $\mathbf{T}$ be a graded theory which is polynomial and for which the associated abelian categories $\mathbf{A}_i$ have global dimension $d_i < \infty$ for $i \geq 0$. Let F be a functor as in (10.1) which preserves zero objects and which carries surjective maps to epimorphisms. Moreover assume that F is polynomial and of finite height. Then there exists $N < \infty$ with $L_iF = 0$ for $i > N$.*

(10.5) Corollary. *Assume $\mathbf{T}$ and F have the properties in (10.4). If $\mathbf{T}$ is linear and if F is linear with height $(F) = k$. Then*

$$L_iF = 0 \quad \textit{for} \quad i > d_0 + \sum_{i=1}^{k}(d_i + 1)$$

For $M \in model(\mathbf{T})$ we write $\mid M \mid \geq m$ if $M(S) = 0$ for $S \in \mathcal{S}$ with $\mid S \mid < m$. The functor F in (10.1) with *height*$(F) = k < \infty$ is called *r-polynomial* ($r \geq 1$) if for $q \geq 2$ and any set of objects $M_1, \dots, M_q \in model(\mathbf{T})$ with

$$\sum_{i=1}^{q}(r + \mid M_i \mid) > r + k + 1$$

we have $F(M_1 \mid \dots \mid M_q) = 0$. Since $r \geq 1$ this implies that F is polynomial.

(10.6) Corollary. *Assume $\mathbf{T}$ and F have the properties in (10.4). Assume $\mathbf{T}$ is r-polynomial and assume F has finite height$(F) = k < \infty$ and F is r-polynomial. If $r > d = Max\{d_0, \dots, d_k\}$ then*

$$L_iF = 0 \quad \textit{for} \quad i > d + k(d+1)$$

Proof of (10.4). Let $\Gamma = (\Gamma^1, \dots)$ be the iterated Grothendieck construction associated to $model(\mathbf{T})$. Then the assumptions in (10.4) imply that

$$\Gamma_G = (\Gamma^1, \Gamma^2, \dots, \Gamma^k, G)$$

is an iterated Grothendieck construction with G given by F as in (10.3). Here we use (8.6). Moreover Γ_G satisfies assumptions as in (5.3). This shows that (10.4) is a consequence of the generalized form of (5.3). Similarly (10.5) and (10.6) are consequences of the generalized form of (5.4) and (5.6) respectively. q.e.d.

(10.7) Remark. Theorem (10.4) and corollary (10.6) yield the correct list of assumptions needed for the result 3.1 of Blanc [4].

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