Explicit classification of formal groups over local fields

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Introduction

The goal of this work is to explicitly classify commutative formal groups defined over the ring of integers of a local field with finite residue field in terms of their logarithm both up to isogeny and up to isomorphism.

We introduce two new invariants of formal groups. The first one classifies formal groups up to isogeny of some special sort. The second invariant takes finitely many values on finite height formal groups of fixed dimension over a fixed field. The two invariants together fix a formal group up to strict isomorphism. Both behave well if one extends the base field or applies to it some automorphisms. This is an important advantage of our classification when compared with the classification of Breuil (cf. [1]).

For simplicity in this introduction we state most results for one-dimensional formal groups over a totally ramified extension of $\mathbb{Q}_p$. The reason for this is that in the case of residue fields of cardinality more than p one must use some non-commutative objects. Almost all statements to follow can be extended to the general situation (and even to totally ramified extensions of unramified fields with arbitrary residue field). We will formulate most of this generalizations in the main text. Since this text is rather preliminary, a few of the proofs will be given for the most simple situation.

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We use the following notation throughout the paper.

We denote by K a finite totally ramified (in the introduction) extension of $\mathbb{Q}_p$, by e its absolute ramification index, by π a uniformizing element of K, by $\mathfrak{O}_K$ its ring of integers, and by k its residue field. By F we denote a commutative m-dimensional formal group law over $\mathfrak{O}_K$ (m will be often equal to 1). In the present paper, we consider only commutative finite-dimensional formal group laws.

It is well known that any formal group is strictly isomorphic to a p-typical one. Thus we will assume that the logarithm λ of the formal group F has the form $\Lambda(\Delta)(x)$, where $\Lambda(\Delta) \in K[[\Delta]]$ and $\Delta(ax^b) = ax^{pb}$.

We introduce the following two rings:

$$R = \mathfrak{O}_K[[\Delta]] \otimes K = \mathfrak{O}_K[[\Delta]] \otimes \mathbb{Q}_p \subset K[[\Delta]], \quad R_0 = \mathbb{Z}_p[[\Delta]] \otimes \mathbb{Q}_p = R \cap \mathbb{Q}_p[[\Delta]].$$

Note that f belongs to R if and only if the valuations of its coefficients are bounded from below.

Proposition (A). *A1. Λ can be presented in the form v/u, where $v \in R$ and $u = p - \sum u_i \Delta^i$; here $u_i \in \mathbb{Z}_p$ and the first invertible u_i is u_h, where h is the height of F.*

A2. If F is isogenious to F', then $u(F') = u\varepsilon(\Delta)$ for some $\varepsilon \in 1 + \Delta\mathbb{Z}_p[[\Delta]]$.

This statement can be regarded as an extension of Honda's classification of formal groups over an unramified field. It is also valid in the most general case if one considers u lying in a certain non-commutative ring. One can make a canonical choice in the equivalence class of u by having all the u_i (or their coefficients in the m-dimensional case, in which they are $m \times m$ matrices) to be Teichmüller representatives of F_p (or F_q in the general case). In particular, using this method, we obtain a canonical representative in each strict isomorphism class of formal groups over unramified fields.

We put

$$r(F) = v(\Delta) \mod u(\Delta) \in R \mod u(\Delta) = K \otimes \mathbb{Z}_p[[\Delta]]/(u(\Delta)).$$

Theorem (B). *For finite height formal groups with fixed u, the following statements are valid:*

B1. F and F' are isogenious if and only if there exists an $a \in K$ and $\varepsilon \in R_0^*$ such that $av \equiv \varepsilon(\Delta)v' \mod u$

B2. F and F' are isogenious and there exists an isogeny $f(x) \equiv ap^s x \mod \deg 2$ for $a \in K$ and some $s \in \mathbb{Z}$ if and only if $av \equiv \varepsilon(\Delta)v' \mod u$ for some $\varepsilon \in R_0^*$.

B3. For a fixed $a \in K$ and $m \in \mathbb{Z}$, we have $av \equiv \varepsilon(\Delta)v'\Delta^m \mod u$ for some $\varepsilon \in 1+\Delta\mathbb{Z}_p[[\Delta]]$ if and only if there is an isogeny $f = \sum a_i x^i \equiv ap^s x \mod \deg 2$ from F into F' for some $s \in \mathbb{Z}$ such that the height of f is equal to $sh+m$ and $a_{p^{sh+m}} \equiv u_h^s \mod \pi$.

B4. $r = r'$ if and only if there is an isogeny $f = \sum a_i x^i \equiv p^s x \mod \deg 2$ from F into F' for some $s \in \mathbb{Z}$ such that $f(x) \equiv [p^s]_F(x) \mod \pi\mathfrak{O}_K[[x]]$.

B5. A finite height formal group F is isogenious to a formal group defined over some $K' \subset K$ if and only if $v \equiv av' \mod u$ for some $a \in K^*$ and v' over K'.

Remark (C). C1. Thus, the invariant $I(F) = (u(F), r(F))$ classifies formal groups up to isogeny as described in the fourth part ("strict isogeny"). This statement is also valid in the most general case. One also can write down the generalizations of all parts, though the third part will become rather long.

C2. One can replace the invariant $I(F)$ by a certain element $V(F)$ of $K\{\{\Delta\}\} \mod R$ (i.e., modulo positive degrees). One defines $V(F)$ as the residue modulo R of the fraction v/u considered in the two-dimensional field $K\{\{\Delta\}\}$.

C3. All the statements can also be extended to not necessarily finite height formal groups and morphisms between them.

C4. One can also prove that, for a given u, every $r(F)$ can be realized modulo K^* (modulo $\mathrm{Gl}_m(K)$ in the general case). Thus all isogeny classes of formal groups can be described explicitly.

C5. Parts 1 and 2 describe Fontaine's classification results (see the book [3]) explicitly. The invariant $I(F)$ is a modification of Fontaine's functor. Our definition is much more explicit. The main disadvantage of Fontaine's functor is that it is formulated in terms of modules, so it gives $r(F)$ only modulo a $\mathbb{Z}_p[[\Delta]]^*$ multiplier instead of $1+\Delta\mathbb{Z}_p[[\Delta]]$ (in the totally ramified case). Thus one cannot recover $a_{p^{sh+m}} \mod \pi$ from Fontaine's invariant. One can construct non-isomorphic (even non-strictly) formal groups that are glued together by Fontaine's functor but have distinct I-invariants.

We note also that fixing a base for the formal group law one can formulate substantially more precise results than in the base-independent approach.

Parts B3 - B5 are completely new.

We put

$$D_F = \{f \in K[[\Delta]] : \exp_F(f(\Delta)(x)) \in \mathfrak{O}_K[[x]]\}.$$

This module will be called the invariant (p-typical) Cartier–Dieudonne module for F. In particular, one can recover the theory of Cartier–Dieudonne over local fields by means of the methods of this paper.

Proposition (D). *D1. F is strictly isomorphic to F' if and only if $D_F = D_{F'}$.*

D2. More generally, there is a homomorphism $f(x) \equiv ax \mod \deg 2$ if and only if $aD_F \subset D_{F'}$.

D3. D_F is a free $\mathbb{Z}_p[[\Delta]]$-module, and for every base w_i of $\mathfrak{O}_K$ over $\mathbb{Z}_p$ elements $\lambda_i(\Delta): \ \lambda_i(\Delta)(x) = \lambda(w_i(x))$ form its $\mathbb{Z}_p[[\Delta]]$-base.

D4. $D \subset K[[\Delta]]$ is equal to D_F for some F if and only if it is $\mathbb{Z}_p[[\Delta]]$-free of dimension e, $D \mod \Delta = \mathfrak{O}_K$, $p\mathfrak{O}_K \subset \Delta$, and D is $\langle\pi\rangle$-stable, where $\langle\pi\rangle(f(\Delta))(x) = f(\Delta(\pi x))$.

D5. For $f \in K[[\Delta]]$, the series $f(\Delta)(x)$ is a logarithm of a formal group strictly isomorphic to F if and only if $f \equiv 1 \mod \Delta$ and $f \in D_F$.

Part D4 is a version of the Cartier classification theorem. One can also easily prove an exact analog of Cartier's theorem in this case.

The behavior of modules D with respect to extensions of K and the modules corresponding to formal groups defined over some $K' \subset K$ can be indicated explicitly.

We define

$$M(F) = D_F\mathbb{Z}_p\{\{\Delta\}\} \subset K\{\{\Delta\}\}.$$

Theorem (E). *E1. If F and F' are as in the second part of Theorem B, then there is an isogeny $ap^s x + \dots$ if and only if $ap^s M_F \subset M_{F'}$. In particular, $M(F)$, $u(F)$ and $r(F)$ taken modulo $\mathbb{Q}_p^*$, and R_0^* determine F up to isomorphism.*

E2. For an infinite height (additive reduction in the general case) formal group one has $D_F = M_F \cap R$.

E3. Denote by F_π the formal group law

$$F_\pi(X,Y) = \frac{F(\pi X, \pi Y)}{\pi}.$$

If F is a formal group of finite height, then $M_F = \pi M_{F_\pi}$.

E4. For a fixed K and a fixed dimension of the group, the number of different πM_{F_π} is finite.

Remark (F). F1. We have the following algorithm for classifying all formal groups over a fixed field K.

First one calculates (using the universal p-typical formal group law, for example) all possible πD_{F_π}. Next for each of them and each u one checks what residues modulo uR have elements $v \in \pi M_{F_\pi}$, $v \equiv p \mod \Delta$ such that

$$\langle\pi\rangle(v/u) \in \pi D_{F_\pi}. \qquad (*)$$

After this one recovers $V(F)$, and then, knowing M_F, classifies all formal groups. Note that it is sufficient to check (*) modulo a small power of Δ (Δ^2 for $e \le p^2/2$) and that it is "continuous", so the calculation terminates in finitely many steps. This is fairly simple to see for dimension one and $e \le p^2/2$. In particular, $\Lambda(\Delta)(x)$ is a logarithm of a formal group of height > 1 in this case if and only if $((u(F)/p)\Lambda)(\Delta)(x)$ is a logarithm of a formal group of infinite height.

F2. The M-invariant reflects the obstacle for two (strictly) isogenious formal groups to be isomorphic, i.e., it 'corrects' Fontaine's functor. The number of different M is finite for a fixed K but increases with extension of K. In order to handle that one can consider the filtration

$$\pi D_{F_\pi} \supset \pi^2 D_{F_{\pi^2}} \supset \pi^3 D_{F_{\pi^3}} \supset \dots.$$

For $s \ge \frac{e}{p-1}$ (often for much smaller s), one has for $D_s = \pi^s D_{F_{\pi^s}}$ the equality $D_s = \pi^s \mathfrak{O}_K[[\Delta]]$. In order to recover F it is sufficient to know $V(F)$ and some D_s that contains a v with a required residue modulo uR. Note that for an extension L/K with ramification index r one can easily recover $D^L_{sr}(F)$ from $D^K_s(F)$. Thus, for any group it is sufficient to know some finite module information in order to recover it from the V-invariant in any local field.

F3 Using a more general form of E1, one can easily find some bounds for the minimum possible s in Theorem B.

F4. Using the invariant Cartier–Dieudonne modules, one can easily define the (usual) Cartier–Dieudonne module structure on the kernels of isogenies of formal groups (cf. [6]). Thus, one can use this theory to classify finite (smooth) flat group schemes. If one knows $\mathrm{Ker}[p^s]_F$, then one can calculate the coefficient at Δ^i of a (possible) logarithm modulo p^{s-i-1}. Hence, in order to recover M_F it is sufficient to know $\mathrm{Ker}[p^s]_F$ for any $s \ge 2\log_p(2e/(p-1))$ as a group scheme (usually much smaller is sufficient). Note that if we fix F and change K, then we do not have to increase the exponent. Therefore, one can fix the (strict) isomorphism class of a formal group by fixing the V-invariant and the group scheme corresponding to some (explicitly bounded from above) level of torsion. This statement can be used for proving theorems about good reduction of Abelian varieties. It will be possible to give an explicit answer to the question of Nicholas Katz: whether one can be sure that an Abelian variety that acquires good reduction over K and is defined over $K' \subset K$ has good reduction over K' knowing that some level of p-torsion for this variety (depending only on e) gives a flat group scheme over $\mathfrak{O}_{K'}$.

Remark (General remarks). G1. One can extend all statements formulated above to formal A-modules, where A is a ring of integers of a local field, by making very natural changes.

G2. It also is almost certain that this theory can be extended to general p-divisible groups. Probably, one can calculate the V-invariant for a dual formal group, thus answering the question of Conrad (cf. [2]).

In the one-dimensional case one can prove a more precise statement about classes of isogenious formal groups. It will not be proved in this work.

Theorem (H). *H1. Every one-dimensional formal group is isogenious to a group with logarithm $(v(\Delta)/u(\Delta))(x)$, where $v = p + \sum_{i=1}^{h-1} v_i\Delta^i$ and $v_K(v_i) \geq e(1 - \frac{i}{h})$.*

H2. Every $(v(\Delta)/u(\Delta))(x)$ for v as in H1, is a logarithm of a formal group over $\mathfrak{O}_K$ (we call it a "nice" group) .

Remark (I). I1. One can also show that if $v' \mod u$ has the form described in the theorem, then the group F' is strictly isomorphic to F. Moreover, all homomorphisms between "nice" formal groups can be described in terms of $V(F)$. The Newton polygon of a "nice" group and the "residues" of the torsion elements of $F(\mathfrak{O}_K^{\mathrm{alg}})$ can be calculated. Using this, one can calculate the Newton polygons and some invariants of the Tate module $T(F)$ for all groups isogenious to a given "nice" group. Unfortunately, it is quite difficult to determine which of them are defined over $\mathfrak{O}_K$.

I2. Theorem 2 can be extended to the case of an arbitrary residue field, thus giving a generalization of the classical result of Laffaille.

I3. Probably these results can be extended to multidimensional formal groups. An analog of the first part is not completely clear yet.

We begin section 1 by recalling the notion of a p-typical formal group. For simplicity, we deal only with formal groups of this type, though our methods work for general formal groups. Next we recall the results of Honda on classification of formal groups over rings of integers of unramified local fields. Their generalization (by Hazewinkel) to a wider class of rings will be applied in this paper. We also describe a canonical representative in each strict isomorphness class of formal group laws. This result seems to be new. We end the section by proving the first part of proposition A in the general case.

In the section 2, we describe the restriction of scalars for formal groups over complete discrete valuation fields.

The goal of the third section is application of Honda's classification of formal groups over unramified fields to classification of formal groups over general local fields. The main idea is to replace an m-dimensional group over $\mathfrak{O}_K$ by a me-dimensional one over an unramified ring W by means of restriction of scalars. We illustrate this method by classifying formal groups over the rings of ramification index $< p$. For simplicity, we describe the proof only in the one-dimensional case. Making slight changes in the formulas, one can obtain the general proof. In fact, authors were able to carry out the corresponding calculations for any $e \leq 2p-2$ (for one-dimensional groups, at least).

In section 4 we define the fractional part invariant. Next we state and prove a generalization of the first four parts of Theorem B.

In section 5, we define the invariant Cartier–Dieudonne modules for formal groups. The difference from the Cartier definition is that we take logarithms of p-typical curves. That gives us a canonical imbedding of our module in $K[[\Delta]]^m$. In order to illustrate usefulness of such modification we prove B5.

In section 6, we define the module invariant M_F. Next we prove that along with the I-invariant it classifies formal groups up to an isomorphism. We prove some properties of M_F. In the end of the section we apply our methods to classification of one-dimensional formal groups of height > 1 for $e \leq p^2/2$.

Notation. Throughout the paper $M_m(\mathfrak{A})$ will denote the ring of $m \times m$ matrices over a (possibly, non-commutative) ring $\mathfrak{A}$, I_m will denote the unit matrix of size m; e_i will denote the m-vector $(0, 0, \dots, 1, 0, \dots, 0)$ (1 is on the i-th place).

v_K is the normalized valuation on K, $e = v_K(p)$ is the absolute ramification index of K, $\mathfrak{M}$ is the maxiaml ideal of K.

We will also use the notation from the introduction.

m will usually mean the dimension of a formal group F, $X = (X_i) = X_1, \dots X_m$ will denote formal variables, as well as x (one variable).

1 Classification over unramified fields and related topics

In the book [4] the results of Honda on classification of formal groups over rings of integers unramified local fields. were generalized to a wider class of rings. In one of the proofs we will apply this classification to formal group laws over $\mathfrak{R}[[t]]$, where $\mathfrak{R}$ is an unramified ring.

1.1 p-typical groups

Since throughout this paper we will work with commutative formal group laws over characteristic zero rings, our formal group laws will have logarithms. The following statement is well known.

Proposition 1.1.1. *Let $\mathfrak{A}$ be a commutative $\mathbb{Z}_p$-algebra. Then a formal group law with logarithm $\lambda = (\lambda_i)$, $1 \le i \le m$, where $\lambda_i = \sum a^i_{i_1 \dots i_m} X_1^{i_1} \dots X_m^{i_m}$ is strictly isomorphic to a formal group law, whose logarithm is equal to (λ'_i), for $\lambda'_i = \sum a'^i_{i_1 \dots i_m} X_1^{i_1} \dots X_m^{i_m}$ for $a'^i_{i_1 \dots i_m} X_1^{i_1} = a^i_{i_1 \dots i_m} X_1^{i_1} \dots X_m^{i_m}$ if and only if $i_s = 0$ for all s except one, the remaining i_s is a power of p, all other $a'^i_{i_1 \dots i_m}$ are 0.*

Thus we can assume that the logarithm λ of the formal group F has the form $\Lambda(\Delta)(X)$, where $\Lambda(\Delta) = (\Lambda_i)$ is in the matrix ring $M_m(K[[\Delta]])$ and $\Delta(ax_i^b) = ax_i^{pb}$ for any i, $1 \le i \le m$.

1.2 Classification over σ-rings. The results of Honda

We consider a torsion free commutative $\mathbb{Z}_p$-algebra $\mathfrak{A}$ with an operator σ satisfying $\sigma(x) - x^p \in p\mathfrak{A}$ for any $x \in \mathfrak{A}$.

For any such pair $\mathfrak{A}, \sigma$ we introduce a non-commutative ring $\mathfrak{A}[[\Delta]]'$, which is equal to the ring $\mathfrak{A}[[\Delta]]$ as a right $\mathfrak{A}$-module and where the the relation $\Delta a = \sigma(a)\Delta$ is fulfilled for any $a \in \mathfrak{A}$.

We will often identify $\mathfrak{A}[[\Delta]]'$ and $\mathfrak{A}[[\Delta]]$ as sets and $\mathfrak{A}$-modules.

Theorem 1.2.1. *1. $\Lambda(\Delta)(X)$, $\Lambda(\Delta) \in \mathbb{Q}_p\mathfrak{A}[[de]]$, is a logarithm of a formal group over $\mathfrak{A}$ if and only if $\Lambda = pU^{-1}$ for some $U \in M_m(\mathfrak{A}[[\Delta]]')$, $U \equiv pI_m \mod (\Delta)$.*

2. Λ and Λ' give strictly isomorphic formal group laws if and only if $U' = \mathfrak{E}U$ for some $\mathfrak{E} \in M_m(\mathfrak{A}[[\Delta]]')$, $\varepsilon \equiv I_m \mod (\Delta)$.

3. More generally, there exists a homomorphism f from F to F' of dimensions m and m', $f \equiv AX \mod \deg 2$, A is some $m \times m'$ matrix over $\mathfrak{A}$, if and only if there exist $C \in M_{m \times m'}(\mathfrak{A}[[\Delta]]')$ such that $CU = U'A$.

4. If $\Lambda \equiv \Lambda' \mod p$, then $\Lambda'(X)$ is the logarithm of a formal group F' over $\mathfrak{A}$ if and only if $\Lambda(X)$ satisfies the same property. In this case F' is strictly isomorphic to F.

Following Honda, we will call matrices, satisfying the conditions for U in the theorem, **special** elements of $M_m(\mathfrak{A}[[\Delta]]')$.

This result can naturally be extended to formal modules. See [4] for details.

1.3 Canonical representatives

The previous theorem gives a classification of formal groups over $\mathfrak{A}$ in terms of equivalence classes of certain matrices U. In the one-dimensional case one can choose in any class a polynomial of the minimal possible degree, thus obtaining a canonical representatives. This method cannot be extended to multi-dimensional groups. However, canonical representatives can be chosen in a completely different way.

We fix a system of representatives $\theta: \mathfrak{A}/p\mathfrak{A} \to \mathfrak{A}$.

Proposition 1.3.1. *Each equivalence class of 'special' matrices contains a unique representative U such that $(U - pI_m) = r\Delta$ where the coefficients of r belong to $\theta(\mathfrak{A}/p\mathfrak{A})$.*

Proof. 1. We prove the existence of U by induction on the powers of Δ. Thus we assume that the coefficients of r_s, where $r_s\Delta = (U_s - pI_m)$, at the powers of Δ less than s for some $s \geq 1$ belong to $\theta(\mathfrak{A}/p\mathfrak{A})$. We put $U = \sum V_l\Delta^l$ and take

$$U_{s+1} = I_m + \theta(\frac{\theta(V_s) - V_s}{p})\Delta^s,$$

θ of a matrix V denotes a matrix whose coefficients are the representatives of coefficients of V. Obviously, the coefficients of $(U_s - p)I_m\Delta$ at powers of Δ less than $s + 2$, belong to $\theta(\mathfrak{A}/p\mathfrak{A})$. The limit of U_i will be the desired U, it will lie in the same equivalence class as all U_i.

2. Suppose that U and U' are equivalent, distinct, and satisfy the condition of the theorem. Let s be the smallest number such that $U' \not\equiv U \mod \Delta^s$. Then $U' = \mathfrak{E}U$, $\mathfrak{E} \equiv I_m \mod \Delta^s$. Since $U' - U = (\mathfrak{E} - I_m)U$, we have $U \equiv U' \mod (\Delta^s, p)$. Since the coefficients of U and U' are representatives, $U \equiv U \mod \Delta^s$. We obtain a contradiction. $\square$

A natural choice of θ is the Teichmüller representative map. This choice is very convenient when one studies reduction of formal group laws.

1.4 General form of Proposition A

Let N be the inertia subfield of a complete discrete valuation field K, and let $\mathfrak{O}_N$ be its ring of integers. The ring $\mathfrak{O}_N$ has a natural Frobenius σ. We have $e = [K : N]$.

Let W denote the ring $\mathfrak{O}_N[[\Delta]]'$ defined as in subsection 1.2, and let W' denote $N[[\Delta]]'$.

This notation will be used throughout the paper except the next subsection (where any other N will be possible).

The ring $K[[\Delta]]$ has a natural structure of a left W-module. Note that in order to define it one does not need to extend σ to K.

We consider the ring $\mathfrak{A} = \mathfrak{O}[[t]]$ with $\sigma(t) = t^p$ and $\mathfrak{A}[[\Delta]]'$. Obviously, $\mathfrak{A}$ satisfies the conditions of Theorem 1.2.1.

Proposition 1.4.1. *Suppose that $\Lambda \in M_m(K[[\Delta]])$ comes from a formal group F (i.e., $\Lambda(\Delta)(X)$ is the logarithm of a p-typical group F). Then Λ can be presented in the form v/u where $vp^l\pi^{-p^l} \in M_m(\mathfrak{O}_K[[\Delta]])$, $l = [\log_p(e/(p-1))]$, and u is a special element of $M_m(W)$.*

Proof. We fix an epimorphism $g : A \to \mathfrak{O}_K$ by assuming $g(t) = \pi$. The existence of a universal p-typical formal group law and its properties (freeness of the corresponding algebra) imply that there exists a p-typical formal group law G over A such that $g_*(G) = F$.

Theorem 1.2.1 implies that there exists a special matrix U over $\mathfrak{A}[[\Delta]]'$ such that the logarithm of G is equal to $pU^{-1}(X)$.

We present U as $u - w$, where $u \in M_m(W)$, $u \equiv pI_m \mod \Delta$, $w \in tM_m(\mathfrak{A}[[\Delta]]')\Delta$.

Then we have

$$U^{-1} = (u-w)^{-1} = (I_m - u^{-1}w)^{-1}u^{-1} = (\sum_i (u^{-1}w)^i)u^{-1}.$$

It is easily seen that $u^{-1} = \sum_{i=0}^{\infty} p^{-i-1}\Delta^i u_i$ for a certain $u_i \in M_m(\mathfrak{O})$ (for example, it follows from the fact that $pu^{-1}(X)$ is a logarithm of a Honda formal group over $\mathfrak{O}$). Applying the equality $\Delta t = t^p\Delta$, we obtain that $u^{-1}w$ can be presented in the form $\sum_{i=0}^{\infty} p^{-i-1}t^{p^i}u_{1i}\Delta^{i+1}$ for some $u_{1i} \in M_m(\mathfrak{A}[[\Delta]]')$. One can also easily prove by induction that $(u^{-1}w)^j$ can be written in the form

$$\sum_{i=0}^{\infty} p^{-i-j}t^{p^i+\dots+p^{i+j-1}}u_{ji}\Delta^{i+j}$$

for some $u_{ji} \in M_m(\mathfrak{A}[[\Delta]]')$. It follows that

$$U^{-1} = \sum_{i=0} p^{-i-1}t^{p^i}u^{-1}w_i \tag{1}$$

for some $w_i \in M_m(\mathfrak{A}[[\Delta]]')$ (since p^i is the smallest possible degree of t at p^{-i-1}).

We checks that $p^{-l}\pi^{p^l} \mid p^{-i}\pi^{p^i}$ for any i. Indeed, the valuation v_i of $\pi^{p^i}p^{-i}$ is equal to $p^i - ei$. The difference $v_i - v_{i+1}$ is equal to $(p-1)p^i - e$, hence v_l is the minimum of v_i. We obtain $v_i \geq p^l - el$.

Since w is obtained by substitution of π into (1), we obtain the claim. $\square$

2 Restriction of scalars and other auxiliary results

2.1 Restriction of scalars

In order to apply the results of the preceding section to ramified local fields we will replace an m-dimensional group over $\mathfrak{O}_K$ by an me-dimensional one over W. In order to do this we need some properties of the Weil restriction of scalars. Some of them are quite general and well known, so we will not prove them. Other ones are special for the setting of formal group laws. We formulate the results for an arbitrary extension of complete discrete valuation fields. That will allow us to apply them both to formal modules and to formal group laws over multi-dimensional rings (in future).

Let K/N be an extension of complete discrete valuation fields of degree s, and let $\mathfrak{O}$ be the ring of integers of N. As always in this paper, the characteristic of K is 0.

We consider the categories C_N and C'_N of formal groups over $\mathfrak{O}$, C_K and C'_K are the same for $\mathfrak{O}_K$. The objects of C_N and C'_N are commutative finite-dimensional formal group laws, the morphisms of C_N are homomorphisms of formal group laws over $\mathfrak{O}$, the morphisms of C'_N are homomorphisms over N. The categories C_K and C'_K are defined similarly. We have $C_N \subset C'_N$ and $C_K \subset C'_K$.

We fix a basis $w = (w_1, \ldots, w_s)$ of $\mathfrak{O}_K$ over $\mathfrak{O}$.

Consider an operator $S = S_w : N^s \to K$ that maps (n_i) into $\sum n_i w_i$ and extend it to various polynomial rings over N.

Our goal is to construct a functor ϕ from C'_K to C'_N that maps C_K to C_N.

Suppose that we have an m-dimensional formal group law $F = (F_l)$ over $\mathfrak{O}_K$, i.e., an m-tuple of series in variables $X_1, \ldots, X_m, Y_1, \ldots, Y_m$. We introduce the variables X_l^r, Y_l^r for $1 \le l \le m$ and $1 \le r \le s$. Assume that the new variables 'take their values in N' and satisfy the relations $X_l = S((X_l^r))$, $Y_l = S((Y_l^r))$. Then standard properties of Weil's restriction of scalars imply that there exists a unique ms-tuple (F_l^r) of formal power series in $X_1, \ldots, X_m, Y_1, \ldots, Y_m$ over $\mathfrak{O}$ satisfying

$$F_i(S((X_l^r)), S((X_l^r))) = S((w_r F_i^r(X_1, \ldots, X_m, Y_1, \ldots, Y_m)) \qquad (2)$$

for all i, $1 \le i \le m$. Moreover, $F_w = (F_l^r)$ is a formal group law. Applying similar process to homomorphisms of formal group laws, we obtain the desired functor.

Remark 2.1.1. 1. A logarithm of a formal group law is an isomorphism with G_a^m (the m-th power of the additive group law), hence $\phi(\lambda)$ is the logarithm of F.

2. An $m \times n$ matrix A can be regarded as a homomorphism from G_a^m into G_a^n. Thus it corresponds to a unique $ns \times ms$ matrix A' over N and this correspondence is functorial and compatible with ϕ. A' is integral if and only if so is A (cf. the reasoning below). We will denote A' by $\phi(A)$.

3. If C is a matrix of the base change from (w) into (w'), i.e., $C(w) = (w')$, then $\phi'(F(X,Y)) = C\phi F(C^{-1}X, C^{-1}Y)$ and $\phi'(f(X)) = C\phi F(C^{-1}X)$ for F being a formal group, where f is a homomorphism of formal groups.

We formulate the main properties of this functor.

Proposition 2.1.2. *1. Let $f \in C'_K(F_1, F_2)$. Then $\phi(f) \in C_N(\phi(F_1), \phi(F_2))$ if and only if $f \in C_K(F_1, F_2)$.*

2. There exists a homomorphism f from F_1 of dimension m_1 into F_2 of dimension n_2, $f \equiv AX \mod \deg 2$, A is an $m_1 \times m_2$ matrix over $\mathfrak{O}_K$, if and only if there exists a homomorphism $f' \in C_N(\phi(F_1), \phi(F_2))$, $f' \equiv A'X \mod \deg 2$.

3. F_1 and F_2 are strictly isomorphic if and only if so are $\phi(F_1)$ and $\phi(F_2)$.

Proof. 1. Suppose that the coefficient $a^l((a_i))$ of f_l at $\prod X_i^{a_i}$ is non-integral. We choose a j such that $w_j \in \mathfrak{O}_K^*$. We denote the coefficients of $\phi(f)$ as b. According to (2)

$$S((b_v^l((b_i^r))) = w_j^{\sum_i a_i} a^l((a_i)),$$

where $b_i^r = 0$ for $r \neq j$ and $b_j^j = a_j$. Thus at least one of $b_v^l((b_i^r))$ is also non-integral.

2. If $f \equiv AX \mod \deg 2$ then $\phi(f) \equiv A'X \mod \deg 2$ according to the definition of ϕ.

The converse implication. It is well known that there exist homomorphisms $g \in C'_K(F_1), F_2)$, $g' \in C'_N(\phi(F_1), \phi(F_2))$ satisfying $g \equiv AX \mod \deg 2$, $g' \equiv A'X \mod \deg 2$ and they are unique. Thus the statement follows from part 1.

3. This is a special case of part 2 with A equal to I_m. □

2.2 Fractional parts

We consider the ring $R_X = \mathfrak{O}_K[[X_i]] \times Q_p$, i.e., the ring of series with bounded denominators. We denote by DR_X the K-module of series whose partial derivatives belong to R_X.

Definition 2.2.1. We denote the residue of an element $f \in K[[X_i]]$ modulo R_X by $\{f\}$ and call it the fractional part of f.

Thus $\{f\} = \{g\}$ if and only if $f - g \in R_X$.

Lemma 2.2.2. *Let $f, g \in DR_X$ and $h_1, h_2 \in \mathfrak{O}_K[[X_i]]_0^m$ (i.e., the constant terms are zero).*

1. If $\{f\} = \{g\}$ and $h_1 \equiv h_2 \mod (\pi)$, then $\{f(h_1)\} = \{g(h_2)\}$.

2. Let λ be a logarithm of a finite height formal group. If $\{\lambda(h_1)\} = \{\lambda(h_2\}$, then $h_1 \equiv h_2 \mod (\pi)$.

3. We introduce the right action of NW on $K[[X]]$ in the obvious way. Then $\{fr\} = \{f\}r$ for any $r \in NW,\ f \in K[[X]]$.

Proof. 1. The equality $\{f(h_1)\} = \{g(h_1)\}$ is obvious. Let now $h_2 = (h_{2i})$ be equal to $(h_{1i}) + \pi(r_i)$, $r_i \in \mathfrak{O}_K[[X_i]]^m$.

We decompose

$$g(h_2) = g(h_1) + \sum_{(a_i)} g_{(a_i)} C_{(a_i)} \pi^{\sum_i a_i} r_1^{a_1} \dots r_m^{a_m},$$

where (a_i) are multiindices, $g_{(a_i)}$ are coefficients of g, $C_{(a_i)}$ are polynomial coefficients. One can easily check that there exists a constant c (depending on the ramification index of K) such that

$$\pi^c C_{(a_i)} \pi^{\sum_i a_i} / \gcd(a_i) \in \mathfrak{O}_K. \tag{3}$$

Since $g \in DR_X$, the sequence $g_{(a_i)} \gcd(a_i)$ has bounded denominators. Applying (3), we obtain $\{g(h_1)\} = \{g(h_2)\}$.

2. We have $\lambda(h_1) - \lambda(h_2) = \lambda(h_1 \underset{F}{-} h_2)$. Thus it is sufficient to prove $\lambda(h) \in R_X^m \implies \pi \mid h$. It is obvious for $m = 1$. The proof of the general case using $\lambda(X) = \lim_i p^{-i}[p^i](X)$ will be given elsewhere.

3. This is obvious since $R_X \cdot r \subset R_X$ for any $r \in NW$. $\square$

We introduce the valuation v_X on R_X as the minimum of the valuations of coefficients.

Remark 2.2.3. It is easily seen that (3) also implies that $v_X(g(h_1) - g(h_2)) \geq \min v_X(\frac{dg}{dX_i}) - c$.

3 Logarithmic matrix; classification for small ramification index

The goal of the this section is to apply Honda's classification of formal groups over unramified fields to the classification of formal groups over general local fields. We apply the functor ϕ and consider a matrix that corresponds to the logarithm of $\phi(F)$.

3.1 Construction of the matrix

We define an operator $\langle\alpha\rangle$ on $K[[\Delta]]$: $\langle\alpha\rangle(\sum c_i\Delta^i) = \sum c_i\alpha^{p^i}\Delta^i$.

Thus for any $h \in K[[\Delta]]$ we have

$$\langle\alpha\rangle(h)(x) = h(\alpha x).$$

In what follows, we always assume that N is the inertia subfield of K and w is a $\mathfrak{O}$-base of $\mathfrak{O}_K$. Sometimes we will impose some extra restrictions on w.

Definition 3.1.1. For an m-tuple $f = (f_i)$ of p-typical series in X_i over K we define the matrix $T_w(f)$ consisting of the columns (t_j^i), the coefficients of t_j^i are $t_{jk}^{il} \in W'$, by the equality

$$f(w_i x e_j)_k = (\sum_l w_l t_{jk}^{il})(xe_j).$$

Thus $S_l(t_{jk}^{il}) = \langle w_i\rangle\Lambda_j$.

The following statement is obvious.

Lemma 3.1.2. *Let A be an $n \times m$ matrix over K, and let $A' = \phi(A)$ be the $en \times em$ matrix defined in Remark 2.1.1. Then $T(Af) = a'T(F)$.*

If $\Lambda(X)$ is a logarithm of a formal group F, we call $T_w(\Lambda)$ the logarithmic matrix for F.

3.2 Main properties of the logarithmic matrix

We study the connection of $\phi(F)$ with the formal group obtained from $T(\Lambda)$.

Proposition 3.2.1. *1. Let $\Lambda \in K[[\Delta]]^m$. Then $\lambda = \Lambda(X)$ is a logarithm of a formal group F over $\mathfrak{O}_K$ if and only if $\lambda_e = T(\Lambda)((X_j^i))$ is a logarithm of a formal group over $\mathfrak{O}$.*

2. If the conditions of the first part are fulfilled, then $T(\Lambda)((X_j^i))$ is a logarithm of a formal group law that is strictly isomorphic to $\phi(F)$.

Proof. Obviously, $T(\Lambda)((X_j^i))$ is the p-typical part of $\phi\lambda$. Thus all statements follow form Proposition 1.1.1, except the fact that λ is a logarithm of a formal group over $\mathfrak{O}_K$ if λ_e is a logarithm of a formal group over $\mathfrak{O}$.

Suppose that $\lambda \equiv \lambda' \mod \deg p^s$, where λ' is a logarithm of some p-typical formal group over $\mathfrak{O}_K$. Note that this is always true for $s = 1$. Then λ'_e (i.e., the p-typical part of $\phi(\lambda)$) is a logarithm of a formal group over

$\mathfrak{O}$, besides $\lambda'_e \equiv \lambda_e \mod \deg p^s$. Using the form of the logarithm of universal p-typical group law (see [4]), we obtain $\lambda'_e \equiv \lambda_e \mod (\deg p^{s+1}, 1/p)$. The same reasoning, as in the proof of Proposition 2.1.2 shows that $\lambda \equiv \lambda' \mod (\deg p^s, 1/p)$. Applying the form of the logarithm of universal p-typical group law again, we obtain that there exists a logarithm λ'' of some p-typical formal group over $\mathfrak{O}_K$ such that $\lambda \equiv \lambda'' \mod \deg p^s$. Hence for any l we have $\lambda \equiv \lambda_l \mod \deg p^l$, where λ_l is some 'integral' logarithm. Thus λ gives an integral group law (since the group law is integral modulo any degree). $\square$

We also express the condition for the existence of homomorphisms in terms of logarithmic matrix.

Proposition 3.2.2. *There exists a homomorphism f from F into F' of dimensions m and m' respectively, $f \equiv AX \mod \deg 2$, A is some $m \times m'$ matrix over $\mathfrak{O}_K$, if and only if there exist $C \in M_{me\times m'e}(W)$ such that $\phi(A)T(F) = T(F')C$*

Proof. According to Proposition 2.1.2 we should find out when there exists a homomorphism g from $\phi(F)$ to $\phi(F')$, $g \equiv \phi(A)X \mod \deg 2$. Besides, we can replace $\phi(F)$ and $\phi(F')$ by F_1 and F'_1, whose logarithms are $T(F)(X_i^j)$ and $T(F')(X_i^j)$.

We should check the condition of Theorem 1.2.1. The corresponding special elements are equal to $T(F)^{-1}$ and $T(F')^{-1}$. Multiplying the equality $CT(F)^{-1} = T(F')^{-1}\phi(A)$ by $T(F)$ from the right and by $T(F')$ from the left, we obtain the claim. $\square$

Remark 3.2.3. We can apply elementary transformations over W to the columns of the matrix T without changing the strict isomorphism class of the corresponding me-dimensional group.

3.3 Classification for small ramification

We classify formal groups for $e < p$. In order to make the proof perfectly clear for the reader we consider the one-dimensional case. It can easily be seen that a very similar reasoning works for an arbitrary dimension.

Theorem 3.3.1. *1. $\lambda = \Lambda(\Delta)(X)$ is a logarithm of a p-typical formal group if and only if $\Lambda = vu^{-1}$ for some $u \in M_m(W)$, $u \equiv pI_m \mod \Delta$ and $v \in M_m(\mathfrak{O}_K[\Delta])$, $v \equiv pI_m \mod \pi\Delta$.*

2. F corresponding to vu^{-1} is strictly isomorphic to F_1 corresponding to $v_1u_1^{-1}$ if and only if $u_1 = \varepsilon u$, $v_1 = v + gu$, where $\varepsilon \in I_m + \Delta M_m(W)$, $g \in \pi M_m(\mathfrak{O}_K[\Delta])\Delta$.

Proof. We assume that $m = 1$.

We choose $w_i = \pi^{i-1}$, $1 \le i \le e$.

1. A p-typical series $\lambda = \Lambda(x)$ can be a logarithm of a formal group only if it is equal to $1 + \sum_{i>0} \frac{c_i}{p^i}\Delta^i$, $c_i \in \mathfrak{O}_K$. Let $c_r = \sum c_{ri} w_i$ be the decomposition of c_r in w_i, and let $T(\Delta) = (T_1, T_2, \dots, T_e)$ be the matrix corresponding to λ. Then from the definition of this matrix we obtain

$$\begin{pmatrix} \Lambda_1 \\ \vdots \\ \Lambda_e \end{pmatrix},$$

where $\Lambda_r = \sum \frac{c_{ir}}{p^i}\Delta^r$ for $r > 1$, $\Lambda_1 = 1 + \sum \frac{c_{i1}}{p^i}\Delta^r$. Since $p\pi \mid \pi^p$, the r-th column for $r > 1$ has the form

$$\begin{pmatrix} p\alpha_{1,r} \\ \vdots \\ p\alpha_{r-1,r}\Delta \\ \varepsilon_r \\ \alpha_{r+1,r}\Delta \\ \vdots \\ \alpha_{e,r}\Delta \end{pmatrix},$$

where $a_{i,r} \in W$, $\varepsilon_r \in 1 + W\Delta$.

We replace $T_2, \dots, T_e$ by

$$T'_r = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varepsilon_r \\ \alpha_{r+1,r}\Delta \\ \vdots \\ \alpha_{e,r}\Delta \end{pmatrix}.$$

This change adds to T some $M \in p\Delta M_e[W]$. Hence, according to Theorem 1.2.1, the new matrix gives an 'integral' logarithm if and only if T does, and the new logarithm will give a strictly isomorphic formal group.

Applying elementary transformations to $T'_2, \dots, T'_e$ (cf. Remark 3.2.3), we eventually obtain the matrix

$$T' = \begin{pmatrix} \Lambda_1 & 0 & \dots & 0 \\ \Lambda_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Lambda_e & 0 & \dots & 1 \end{pmatrix}.$$

According to Theorem 1.2.1, T' gives an integral logarithm if and only if there exist a special matrix U over W, $UT' = pI_e$. Clearly, U should have the form

$$\begin{pmatrix} u(\Delta) & 0 & \dots & 0 \\ r_1 & p & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r_{e-1} & 0 & \dots & p \end{pmatrix},$$

where $u \in p + W\Delta$, $r_i \in W\Delta$. Matrix U of this form is the inverse to T if and only if $\Lambda_1 = p$, $r_i\Lambda_1 + p\Lambda_i = 0$ for $i > 1$. Hence $\Lambda_1 = p/u$, $\Lambda_i = -r_{i+1}/u$. We obtain the only if part of the claim.

Now we prove the converse statement. We represent v in the form $\sum \Lambda_i \pi^{i-1}$. Since $\pi \mid v$, we have $p \mid \Lambda_1$. We take $u' = pu/\Lambda_1$. For this u' (with the same Λ) we have $\Lambda_1 = p$. Therefore we can construct U and T', then T and Λ. Hence the if part is also valid.

2. We construct the matrices T' and T'_1 in the same way as above, U and U_1 are their inverses of the form described above.

We want to check when $U' = CU$, where $C \in M_e[[W]]$, $C \equiv I_m \mod \Delta$. Obviously, this can happen only for

$$C = \begin{pmatrix} \varepsilon & 0 & \dots & 0 \\ g_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ g_{e-1} & 0 & \dots & 1 \end{pmatrix},$$

where $\varepsilon \in 1 + W\Delta$, $g_i \in W\Delta$. Performing the matrix mutliplication, we obtain the claim. □

In a similar way one can describe formal groups of height > 1 for $e = p$. It is possible since in this case we also have $a_{12} \in W$. It is possible also to classify formal groups for $e \leq 2p - 2$. To this end one should take into account the second column of T. The method of the last section seems to be more useful, as it easily gives classification of formal groups of height > 1 for $e \leq p^2/2$.

Remark 3.3.2. 1. One can also prove a result about homomorphisms between formal groups similar to part 3 of 1.2.1. One should take v into account in the same way as in the results of the next section.

2. One can choose a canonical representative in each class of strict isomorphism by demanding the coefficients of u at positive degrees of Δ to be Teicmüller representatives (see Proposition 1.3.1), the coefficients of v at positive degrees of Δ to have zero trace into $\mathfrak{O}$.

3. One can easily prove that in the one-dimensional case the height of F is equal to the index of the first invertible coefficient of u.

3.4 Main matrix lemma

We formulate a lemma that will allow us to check the conditions of Theorem 1.2.1. We will use it (later) for different rings R.

Lemma 3.4.1. *I Let $\mathfrak{R}, \mathfrak{L}$ be integral domains (not necessarily commutative), $gr \subset \mathfrak{L}$, T_1, T_2 be be matrices over $\mathfrak{L}$, T_1 is non-degenerate.*

1. $T_1^{-1}T_2 \in M_m(\mathfrak{R})$ if and only if $T_2(\mathfrak{R}^m) \subset T_1(\mathfrak{R}^m)$.

2. $T_1^{-1}T_2 \in \mathrm{Gl}_m(\mathfrak{R})$ if and only if $T_2(\mathfrak{R}^m) = T_1(\mathfrak{R}^m)$.

II For a matrix $T \in \mathrm{Gl}_m(\mathfrak{L})$ we have $pT^{-1} \in M_m(\mathfrak{R})$ if and only if $p\mathfrak{R}^m \subset T(\mathfrak{R}^m)$.

Proof. I 1. If $T_1^{-1}T_2 \in M_m(\mathfrak{R})$, then $T_2(\mathfrak{R}^m) = T_1(T_1^{-1}T_2\mathfrak{R}^m) \subset T_1(\mathfrak{R}^m)$.

The inverse implication: for any i there exist $v_i \in \mathfrak{R}^m$ such that $T_2e_i = Tv_i$. We have $T_2 = tI_m = T(e_i)$, where (e_i) is the matrix consisting of all e_i. Hence $T_1(v_i) = T_2$, where (v_i) is the matrix consisting of all v_i, we obtain the claim.

2. The second claim follows immediately from the first one applied to the pairs T_1, T_2 and T_2, T_1.

II This follows immediately from I1 if we take $T_1 = T$ and $T_2 = pI_m$. ☐

Now we combine this lemma with Propositions 3.2.1 and 3.2.2.

Proposition 3.4.2. *1. A p-typical m-tuple $\lambda = \Lambda(X)$, $\Lambda \equiv I_m \mod \Delta$, is a logarithm of a formal group law over $\mathfrak{O}_K$ if and only if $pW^m \in T(\Lambda)W^m$.*

2. There exists a homomorphism f from F to F' of dimensions m and m' respectively, $f \equiv AX \mod \deg 2$, A is some $m \times m'$ matrix over $\mathfrak{O}_K$, if and only if $\phi(A)T(F)(W^m) \subset T(F')(W^m)$.

Proof. Since $T(F) \equiv I_{me} \mod \Delta$ and $T(F') \equiv I_{m'e} \mod \Delta$, the matrices $T(F)$ and$T(F')$ are non-degenerate.

1. According to Proposition 3.2.1, we must check that the matrix $pT(\lambda)$ is integral (see theorem 1.2.1). Applying part II of Lemma 3.4.1 for $\mathfrak{R} = W$, $\mathfrak{L} = W'$, we obtain the claim.

2. We apply Proposition 2.1.2. The conditions of Lemma 3.4.1, part I1, are obviously fulfilled for $\mathfrak{R} = W$, $\mathfrak{L} = W'$, the matrix $T_1 = T(F')$, and $T_2 = \phi(A)T(F)$. Hence we obtain the claim. ☐

4 Fractional part invariant: classification up to a strict isogeny

In this section we prove that the fractional part of a logarithm of a formal group classifies formal groups up to isogeny of an explicitly described sort.

We will consider the algebra $NW = N \times W$, which is equal to $\cup_s p^{-s} W$.

4.1 Classification up to rational isogeny; homomorphisms

For a formal group law F we define $r(F)$ to be the residue of $\Lambda \mod M_m(R)$. We have $r(F) \in M_m(R)u^{-1}/M_m(R)$, where $\Lambda = vu^{-1}$. We always assume that u has the minimum possible rank over W.

First we formulate how can $r(F)$ of two strictly isomorphic formal groups differ.

Proposition 4.1.1. *If F is strictly isomorphic to F', then*

1. $u' = u\eta$ for $\eta \in I_m + M_m(W)\Delta$ (i.e., the ideals, generated by u and u' coincide)

2. $r' = r\varepsilon$ for $\varepsilon \in I_m + M_m(W)\Delta$.

Proof. 1. For totally ramified one-dimensional case the statement follows from Proposition 3.2.2 combined with the fact that $\det(T(F)) = \frac{p}{u}s$, $s \in 1 + \Delta\mathbb{Z}_p[[\Delta]]$.

The general proof is quite technical and will be given elsewhere.

2. According to Proposition 3.2.2, $T(F') = T(F)\mathfrak{E}$ for some $\mathfrak{E} \in I_{me} + M_{me}(W)$.

Then we have $T(F') \equiv T(F)\mathfrak{E} \mod NW$. We assume that $w_1 = 1$, $\pi \mid w_j$ for $j > 1$. Then we obtain $S(T(F)(e_i^1)) = \Lambda_i$ and $S(T(F)(e_i^j) \in NW^m$ for $j > 1$. Thus $vu^{-1} = \varepsilon v' u'^{-1}$ for some $\varepsilon \in I_m + M_m(W)$, ε is the corresponding block of $\mathfrak{E}$. $\square$

Thus one can also define the invariant $I(F)$ as in the introduction. It will be well defined if we take for $u(F)$ the canonical representative. Otherwise one should consider u up to a $I_m + M_m(W)\Delta$ multiplier.

One can easily show that for each F any $\varepsilon \in I_m + M_m(W)\Delta$ is possible, i.e., the claim of Proposition is the best possible.

We prove the following necessary and sufficient condition for the existence of homomorphisms for $\varepsilon \in \mathrm{Gl}_{m'}(NW)$.

Theorem 4.1.2. *1. There exists a homomorphism f from F into F', the dimensions of F, F' are m, m', $f(X) \equiv Ap^s x \mod \deg 2$ for a fixed $A \in$*

$M_{m\times m'}(K)$ and some $s\in\mathbb{Z}$ if and only if $Ar = r'\varepsilon$ for some $\varepsilon\in M_{m'}(NW)$, where $r = r9F)$, $r' = r(F')$.

2. F and F' of dimension m and finite height are isogenious if and only if there exists an $A\in \mathrm{Gl}_m(K)$ and $\varepsilon\in M_m(NW)$ such that $Ar = r'\varepsilon$.

Proof. 1. Again we take $w_1 = 1$, $\pi \mid w_j$ for $j > 1$. Then we obtain $S(T(F)(NW^m)) = vu^{-1}(NW^m) + R^m$. Thus the condition on the fractional parts is equivalent to $AT(F)(NW^{me}) \subset T(F')(NW^{m'e})$. According to Proposition 3.4.2 for $R = NW$ this is equivalent to the existence of $B \in M_{m'e}(NW)$, $AT(F) = T(F')B$. Since $C\in M_m(NW)$ if and only if $p^sC\in M_m(W)$ for some $s\in\mathbb{Z}$, Theorem 1.2.1 gives us the claim.

2. For finite height formal groups, f is isogeny if and only if $f \mod \deg 2$ is an invertible operator. Hence part 2 follows from part 1 immediately.

□

We call two formal groups **rationally isogenious** if they satisfy the conditions of the theorem for $A = I_m$.

4.2 Strict isogeny: classification

Now we describe what does it mean that two logarithms have equal fractional parts. Note that this gives us an answer to the question which formal groups are glued together by the r-invariant (we call them **strictly isogenious**).

Theorem 4.2.1. *For finite height formal groups $r = r'$ if and only if there is an homomorphism $f\equiv p^sI_mX \mod \deg 2$ from F into F' for some $s\in\mathbb{Z}$ such that $f(x)\equiv [p^s]_F(X) \mod \pi\mathfrak{O}_K[[X]]^m$.*

Proof. Suppose that $r = r'$. As is proved above, we have a homomorphism $f\equiv p^sI_mX \mod \deg 2$ from F to F'. We have the equality

$$\lambda'(f(X)) = p^s\lambda(X) = p^s\lambda([p^s]_F(X)). \tag{4}$$

According to the first part of Lemma 2.2.2 we obtain

$$\{\lambda(f(X))\} = \{p^s\lambda([p^s]_F(X))\}.$$

Now the second part of Lemma 2.2.2 immediately gives us the claim.

The converse implication is even easier: one just calculates fractional parts in equality (4). □

Remark 4.2.2. 1. More generally, the same reasoning proves that for fixed r-invariants of F and F' the knowledge of $f \mod \deg 2$ fixes f modulo π.

In particular, we obtain that two strictly isomorphic logarithms have equal fractional parts if and only if the strict isomorphism between them is congruent to X modulo π. Thus our technique can be applied to the study of deformation problems.

2. Since strict isomorphisms multiply r by $\varepsilon \in I_m + M_m(W)\Delta$ and any such ε is possible, it seems natural to describe homomorphisms between F and F' fixing r and r' only modulo $I_m + M_m(W)\Delta$. We will try to do this in the next subsection.

4.3 Strict isogeny: homomorphisms in the one-dimensional totally ramified case

Now we give a more precise version of Theorem 4.1.2 by considering $\varepsilon \in I_m + M_m(W)\Delta$ (instead of $\mathrm{Gl}(NW)$). Unfortunately, in order to describe the result for multi-dimensional case one should introduce many technical definitions and results. This will be done in subsequent papers. For simplicity we also assume $N = \mathbb{Q}_p$. Note that in this case there are no homomorphisms between finite height groups with non-equivalent u. We have $W = \mathbb{Z}_p[[\Delta]]$.

We expand u as $p - \sum u_i\Delta^i$, $u_i \in \mathbb{Z}_p$. We have $u_h \in \mathbb{Z}_p^*$, $u_i \in p\mathbb{Z}_p$ for $i < h$, where h is the height of F (see the remark below).

Theorem 4.3.1. *Suppose that two finite height formal groups have the same u, Then for a fixed $a \in K$ and $m \in \mathbb{Z}$ we have $av \equiv \varepsilon(\Delta)v'\Delta^m \mod u$ for some $\varepsilon \in 1 + \Delta\mathbb{Z}_p[[\Delta]]$ if and only if there is an isogeny $f = \sum a_i x^i \equiv ap^s x \mod \deg 2$ from F into F' for some $s \in \mathbb{Z}$ such that the height of f equals $sh + m$ and $a_{p^{sh+m}} \equiv u_h^s \mod \pi$.*

Proof. 1. Let the logarithms satisfy the condition on fractional parts. Then, according to Theorem 4.1.2, some isogeny $f \equiv ap^s x \mod \deg 2$ does exist. Therefore we must check that f satisfies other assertions of the claim.

We denote by P_f the series $\prod_{t \in \operatorname{Ker} f}(x \underset{F}{-} t)$. According to a result of Lubin (see [4]) f can be presented as a composition of P_f and an isomorphism. Hence we can write

$$f = cx \circ g(x) \circ P_f(x),$$

where $c \in \mathfrak{O}_K^*$, $g(x) \equiv x \mod \deg 2$.

This is equivalent to the relation

$$ap^s\lambda(x) = \lambda'(cx \circ g(x) \circ P_f(x)).$$

Now we calculate how each of this functions affects fractional parts. We denote the height of f by H.

I. We have $c^{p^l} \to \theta_c$ for $l \to \infty$, where θ_c is the Teichmüller representative of $\theta \mod \pi$. Thus $\{\lambda'(cx)\} = \theta_c\{\lambda'(x)\}$.

II. The series $c^{-1}\lambda'(cx)$ is a logarithm of a formal group $F_c(x,y) = c^{-1}F(cx,cy)$, which is isomorphic to F, the isomorphism is equal to cx. Hence $\lambda'(cx \circ g(x))$ is a logarithm of some formal group, strictly isomorphic to F_c.

According to Proposition 4.1.1 for strictly isomorphic formal groups the fractional part of logarithms differ by a $\eta' \in 1 + W\Delta$. Hence

$$\lambda'(cx \circ g(x) = \{\lambda'(x)\}\theta_c\eta'.$$

III. Obviously, $P_f \equiv x^{p^H} \mod \pi$. Hence Lemma 2.2.2 implies

$$\{\lambda'(cx \circ g(x) \circ P_f(x))\} = \lambda'(cx \circ g(x))\Delta^H = \{\lambda'(x)\}\theta_c\eta\Delta^H.$$

Thus we obtain the relation

$$ap^s r(F) = r(F')\eta\theta_c\Delta^H.$$

On the other hand, since $P_f \equiv x^{p^H} \mod \pi$, we have $c \equiv a_{p^H} \mod \pi$.

We can replace u by its canonical representative (with respect to Teichmüller representatives). Then in the ring $R_u = R_0/u$ (note that we are in the commutative situation!) we have $p = \Delta^h \sum_{i\geq 0} u_{h+i}\Delta^i$. Therefore, in R_u we have $p^s = u_h^s\eta(\Delta)\Delta^{sh}$ for some $\eta \in 1 + W\Delta$. Hence we have the congruence

$$av'\Delta^{hs+m}u_h^s\eta \equiv \eta'\theta_{a_{p^H}}\Delta^H v'\Delta^m \mod u.$$

If $H \neq hs + m$ or $a_{p^{sh+m}} \not\equiv u_h^s \mod \pi$, then the valuation of the element $d = \Delta^{hs+m}u_h^s\eta - \theta_{a_{p^H}}\Delta^H$ in the field $\mathbb{Q}_pW/(u)$ is equal to $\min(H, hs+m)$. Therefore, d is invertible which is impossible since $dr = 0$ and $r(F) \neq 0$. Thus we obtain the claim.

2. If f satisfies all conditions of our theorem, then the calculations of fractional parts carried out above show immediately that the relation $av \equiv \varepsilon(\Delta)v'\Delta^m \mod u$ for some $\varepsilon \in 1 + \Delta\mathbb{Z}_p[[\Delta]]$ is fulfilled. □

Remark 4.3.2. 1. In particular, applying the proposition to $f = [p]_F$, $F' = F$, we obtain that h defined in terms of u coincides with the height of F (we did not use that in the proof).

2. In all cases one can replace the invariant $r(F)$ by a certain element $V(F)$ of $K\{\{\Delta\}\} \mod R$ (i.e., modulo positive degrees). One defines $V(F)$ to be the residue modulo R of the fraction vu^{-1} considered in the matrix ring over the two-dimensional field $K\{\{\Delta\}\}$.

5 Invariant Cartier–Dieudonne modules

5.1 Category of D-modules

Definition 5.1.1. We denote by $\mathfrak{D}$ the category of (left) W-submodules D of $K[[\Delta]]^m$, satisfying the following conditions.

1. D is a free W-module of rank em.
2. $D \mod \Delta = \mathfrak{O}_K^m$.
3. $p\mathfrak{O}_K^m \subset D$.

Lemma 5.1.2. *The elements $v_1, \ldots, v_{me} \in D$ form a W-base of $D \in \mathfrak{D}$ if and only if $v_i \mod \Delta$ form a $\mathfrak{O}$-base of $\mathfrak{O}_K^m$.*

Proof. The only if part is obvious.

Now we prove that v_i satisfying the conditions of Lemma form a base of D.

Suppose that v_i' is a W-base of D, $1 \le i \le me$. Then we have $(v_i) = P(v_i')$ for some $P \in M_m(W)$. Since $(v_i) \mod \Delta = (v_i') \mod \Delta$ and the dimension of $D \mod \Delta$ over $W \mod \Delta$ is equal to the number of v_i', $P \mod \Delta$ is invertible. Thus P is invertible, hence v_i also form a base of D. □

Lemma 5.1.3. *If $D \subset D'$ for $D, D' \in \mathfrak{D}$, then $D = D'$*

Proof. Choose a basis v_i of D. Since $v_i \mod \Delta$ generate $\mathfrak{O}_K^m$, v_i is also a W-base of D'. □

Sometimes it can be useful to replace the axioms of $\mathfrak{D}$ (especially the first one) by slightly different ones.

Proposition 5.1.4. *I If D, satisfying conditions 2 and 3 of 5.1.1 satisfies also one of the following conditions:*

1. D is generated by $\le me$ elements.

2. Any $v \in W^m\Delta \cap D$ also belongs to $D\Delta$,

then it belongs to $\mathfrak{D}$.

II Vice versa, any $D \in \mathfrak{D}$ satisfies 1 and 2.

Proof. I 1. Let $v_i \in D$, $1 \le i \le r$ be generating elements. Since the dimension of $D \mod \Delta$ over $\mathfrak{O} = W \mod \Delta$ is greater that or equal to s, the elements v_i are W-independent. Since this dimension is equal to me, we also obtain $s = me$.

2. We choose v_i, $1 \le i \le me$ such that $v_i \mod \Delta$ generate $D \mod \Delta$. For $v \in D$ we can choose $c_i \in \mathfrak{O}$ such that $v \equiv \sum v_i c_i \mod \Delta$. Hence $v = \sum v_i c_i + v_1\Delta$ for some $v_1 \in D$. Similarly $v_1 \equiv \sum v_i c_{i1} \mod \Delta$. Applying

this procedure successively and passing to the limit, we obtain that v lies in the W-envelope of v_i. Thus D satisfies condition 1 in this lemma.

II The first condition is obvious.

Let now v belong to $W^m\Delta \cap D$. We choose a base v_i for D; $v_i \mod \Delta$ are $\mathfrak{O}$-independent. We have $v = \sum v_i w_i$. Taking this equality modulo Δ, we obtain that $w_i \in W\Delta$, hence $v \in D\Delta$. $\square$

5.2 Two definitions of D_F

Definition 5.2.1. For a formal group F with logarithm $\lambda = \Lambda(X)$ one defines

$$D_F = S(T_F(W^m)) = \langle\langle w_i\rangle\Lambda_j\rangle.$$

Proposition 5.2.2. *For formal groups F_1 and F_2 of dimensions m_1 and m_2, whose D-modules are equal to D_1 and D_2, respectively, the following statements are valid.*

1. Let A be a $m_1 \times m_2$ matrix over $\mathfrak{O}_K$. There exists a homomorphism f from F_1 into F_2, $F(X) \equiv AX \mod \deg 2$ if and only if $AD_1 \subset D_2$.

2. For $m_1 = m_2$ the groups F_1 and F_2 are strictly isomorphic if and only if $D_1 = D_2$.

3. For any F $D_F \in \mathfrak{D}$.

Proof. 1. According to part 2 of Proposition 3.4.2 it is sufficient to check $A'(T_1(W^{m_1})) \subset T_2(W^{m_2})$. This is equivalent to the statement of the proposition since S is a bijection.

2. This follows immediately from part 1 for $A = I_{m_1}$.

3. Properties 1 and 2 are obvious (cf. Proposition 5.1.4). Property 3 follows from part 1 of Proposition 3.4.2. $\square$

Now we prove that D_F is the logarithm of the module of p-typical curves (i.e., of the classical Cartier–Dieudonne module).

Proposition 5.2.3. $D_F = D'_F$, *where*

$$D'_F = \{f \in K[[\Delta]]^m : \ \exp_F(f(x)) \in \mathfrak{O}_K[[x]],$$

here $\exp_F$ *is the composition inverse to* λ_F.

Proof. First we check $D_F \subset D'_F$.

We have $\exp_F(\langle w_i\rangle\Lambda(xe_i)) = w_i x e_i$, hence the generators of D_F belong to D'_F.

Next we check that D'_F is an W-module. For $c_i \in \mathbb{Z}_p$ and $h \in K[[\Delta]]^m$ one has

$$\exp_F(h \sum c_i \Delta_i)(x) = \sum_{(F)} [c_i]_F \exp_F(h(X^{p^i})),$$

therefore D'_F is an $\mathbb{Z}_p\{\{\Delta\}\}$-module. It remains to check that for any Teichmüller representative θ we have $D'F\theta \subset D'F$. This assertion follows immediately from the relation $\exp_F(h\theta(x)) = \theta \exp_F(x)$.

Now, according to Lemma 5.1.3, it is sufficient to prove $D'_F \in \mathfrak{D}$. We have $\exp_F(h(x)) \equiv h(x) \mod \deg 2$, hence $D'_F \mod \Delta \subset \mathfrak{O}_K^m$. Thus properties 2 and 3 of the definition of $\mathfrak{D}$ follow from the fact $D_F \subset D'_F$. Obviously, D'_F also satisfies condition 2 of Proposition 5.1.4. Therefore $D'_F \in \mathfrak{D}$ and the proposition is proved. □

Remark 5.2.4. Combining the two descriptions of D_F, one easily sees that for any base t_i of $\mathfrak{O}_K$ over $\mathbb{Z}_p$ the elements $v_{ijl} = \langle w_i t_j \rangle \Lambda_l$ form a $\mathbb{Z}_p[\Delta]$-base of D_F.

Indeed, v_i belong to D_F and $v_{ijl} \mod \Delta$ are a $\mathbb{Z}_p$-base of $D_f \mod \Delta$.

5.3 Basic properties of D_F

We prove that a module $D \in \mathfrak{D}$ corresponds to a formal group over $\mathfrak{O}_K$ if and only if it is $\langle \pi \rangle$-stable.

Proposition 5.3.1. *For $D \in \mathfrak{D}$ the following conditions are equivalent*

1. *$D = D_F$ for some F.*
2. *For any $a \in \mathfrak{O}_K$ we have $\langle a \rangle D \subset D$*
3. *$\langle \pi \rangle D \subset D$*
4. *There exist $\Lambda_i \in D$, $\Lambda_i \equiv e_i \mod \Delta$ such that $\langle w_j \rangle \Lambda_i \in D$, where w_i is some $\mathfrak{O}$-base of $\mathfrak{O}_K$.*

Proof. 1 $\Longrightarrow$ 2: We have $\exp_F(\langle a \rangle (f(\Delta))(x)) = \exp_F(f(\Delta)(ax))$. It remains to apply Proposition 5.2.3.

2 $\Longrightarrow$ 3 is obvious.

3 $\Longrightarrow$ 4: We choose $\Lambda_i \in D : \Lambda_i \equiv e_i \mod \Delta$. We have $\langle \pi^s \rangle \Lambda_i \in D$. Thus we can take $w_j = \pi^{j-1}$.

4 $\Longrightarrow$ 1: We have $\langle w_j \rangle \Lambda_i \equiv w_j e_i \mod \Delta$. Hence, according to Lemma 5.1.2, the elements $\langle w_j \rangle \Lambda_i$ generate D.

We take a formal group F over K whose logarithm is equal to $(\Lambda_i(e_i X_i))$. The module D satisfies property 3 of the definition, therefore, according to Proposition 3.4.2, F is a p-typical formal group over $\mathfrak{O}_K$. We have $D \subset D_F$, thus $D = D_F$ □

Instead of π we could take any other polynomial generator of $\mathfrak{O}_K$ over $\mathfrak{O}$, for example, $1+\pi$.

Next we prove that the logarithms of formal groups isomorphic to F can be described in terms of module D_F.

Proposition 5.3.2. *For $f \in M_m K[[\Delta]]$ the series $f(\Delta)(X)$ is a logarithm of a formal group strictly isomorphic to F if and only if $f \equiv I_m \mod \Delta$ and $f(\Delta)(xe_i) \in D_F$.*

Proof. If F' is strictly isomorphic to F, we have $D_{F'} = D_F$ and everything is clear.

Vice versa, for a sequence of $\Lambda_i \in d : \Lambda_i \equiv e_i \mod \Delta$ we can construct a formal group F' as is done in the previous proof. Again we have $D = D_F \subset D_{F'}$.

□

Thus we can choose any set of $\Lambda_i \equiv e_i \mod \Delta$ in D.

We also note that the intersection of D_F with K^m always contains a module that is slightly larger than $p\mathfrak{O}_K^m$.

Remark 5.3.3. 1. For $s = -[e/(1-p)]$ we have $\pi^s \mathfrak{O}_K W^m \subset D_F$.

2. Let $a \in \mathfrak{O}_K$, $v_K(a) = l$, $l \le s$. Then for a formal group law $F_a = a^{-1}F(aX, aY)$ one has $\pi^{s-l}\mathfrak{O}_K W^m \subset D_{F_a}$.

Proof. 1. This is a special case of the second part for $a = 1$.

2. We have

$$D_{F_a} = \{f \in K[[\Delta]]^m : \exp_F(af(x)) \in a\mathfrak{O}_K[[x]].$$

Hence it is sufficient to check that $\pi^l \mid \exp_F(aX)$.

Since all partial derivarives of λ are integral, the coefficients of λ at X^a are divisible by $\frac{1}{a}$. Hence one can easily proof by induction on the degree of monomials that the coefficients of $\exp_F$ at $\prod X_i^{a_i}$ are divisible by $1/\prod a_i!$. Moreover, $\exp_F(X) \equiv X \mod \deg p$. It follows that for $(a_i) \ne e_i$ the valutation of the coefficients of $\exp_F$ at $\prod X_i^{a_i}$ are $> -\frac{\sum a_i}{p-1}$. Thus for $(x_i) \in \pi^s \mathfrak{O}_K^m$ one has $\min v(x_i) < \min v(\exp_F((x_i X_i))_i - x_i X_i)$. □

5.4 The change of the base field

We describe how the module D_F behaves if we replace K by $L \supset K$.

We denote the ring of integers of L by $\mathfrak{O}_L$.

Proposition 5.4.1. *Let F be a formal group over $\mathfrak{O}_K$, s_i be an $\mathfrak{O}_K$-base of $\mathfrak{O}_L$. Then $D_L(F) = \langle\langle s_i\rangle D_K(F)\rangle$ (the envelope can be understood in the Abelian group sense).*

Proof. From the exponential definition of D_F (see Proposition 5.2.3) it is obvious that $\langle s_i\rangle D_K(F) \subset D_L(F)$ for all i.

On the other hand, $D' = \langle s_i\rangle D_K(F)$ contains $\langle s_i\rangle \Lambda_j w$ for all i and j and all $w \in W$. Applying Remark 5.2.4 for $t_i = s_j w_l \theta_r$, where θ_r is a $\mathbb{Z}_p$-base of $\mathfrak{O}$ consisting of Teichmüller representatives, we obtain that D' contains a $\mathbb{Z}_p$-base of $D_L(F)$. □

Now suppose that K' is a subfield of K and π' is a uniformizing element of K'.

Proposition 5.4.2. *A finite height formal group F is isogenious to a formal group defined over K' if and only if $r = Ar'\varepsilon$ for some r' defined over K', $A \in \mathrm{Gl}_m(K)$, and $\varepsilon \in \mathrm{Gl}_m(W)$.*

Proof. The only if part follows immediately from Theorem 4.1.2.

Now suppose that $r = Ar'\varepsilon$. We consider the module $D_1 = A^{-1}D \cap K'[[\Delta]]$. We have $D_1 \mod \Delta \subset A^{-1}\mathfrak{O}_K^m \cap K'^m$ and $pA^{-1}\mathfrak{O}_K^m \cap K'^m \subset D'$. Hence if we choose $A' \in \mathrm{Gl}_m(K')$ so that $A'D_1 \mod \Delta = {\mathfrak{O}'_K}^m$, then the module $A'D_1$ satisfies properties 2 and 3 of the definition of $\mathfrak{D}$ (over K'). Obviously, D' satisfies propery 2 of Proposition 5.1.4. Thus $D \in \mathfrak{D}'$. Since for any $a \in \mathfrak{O}'_K$ we have $\langle a\rangle D' \subset D'$, it corresponds to some formal group F' over $\mathfrak{O}'_K$.

It remains to prove that F' is isogenious to F. Using Proposition 5.4.1, we obtain $AA'^{-1}D_{F'} \subset D_F$. Hence there exist a homomorphism $f \equiv AA'^{-1}X \mod \deg 2$ from F' into F.

On the other hand, the condition on the fractional parts implies that there exists an $s \in \mathbb{Z}$ such that for all i, $1 \le i \le m$, $p^s\Lambda_i \in AA'^{-1}D_{F'}$. Hence the inverse homomorphism also exists.

□

Note that when the conditions of the proposition are not fulfilled one still obtains a formal group F' over K' and a canonical (up to an isomorphism) homomorphism from F into F'.

6 The module invariant

The goal of this section is an explicit description of a module invariant M_F. It completes the V-invariant to a classification of formal groups up to a strict isomorphism; furthermore, the set of possible M_F for F of finite height formal groups of fixed dimension over a fixed field K is finite.

6.1 Definition and main properties

As proved above, the module D_F belongs to $u^{-1}(R^m)$. We can consider u^{-1} as a matrix over $K\{\{\Delta\}\}$. Therefore we can embed D_F in $K\{\{\Delta\}\}$.

Note that we can define on $K\{\{\Delta\}\}$ the natural right action of

$$\Omega = \{\sum_{i\in\mathbb{Z}} a_i\Delta^i,\ a_i \in \mathfrak{O},\ a_i \to 0 \text{ for } i \to -\infty\},$$

the multiplication, as in W, is defined by the relation $\Delta a = \sigma(a)\Delta$ for any $a \in \mathfrak{O}$. We have $w \subset \Omega$.

Definition 6.1.1. For a formal group F we define M_F as $D_F\Omega \subset K\{\{\Delta\}\}$.

Note that M_F is a Ω-module.

We describe the main properties of M_F that allows to use it in the classification questions as a complement of the invariant $r(F)$.

Theorem 6.1.2. *1. Suppose that F and F' satisfy the conditions of part 1 of Theorem 4.1.2, i.e., that for some $A \in M_{m\times m'}(K)$ and $\varepsilon \in M_{m'}(NW)$ we have $Ar = r'\varepsilon$. Then there exists a homomorphism f from F to F', $f(X) \equiv Ap^s x \mod \deg 2$ for a fixed $s \in \mathbb{Z}$ if and only if $Ap^sM_F \subset M_{F'}$.*

2. Suppose that F is a finite height group and for some $A \in M_{m\times m'}(K)$ and $\varepsilon \in M_{m'}(NW)$ we have $Ar = r'\varepsilon$. If $p^sA \in M_{m\times m'}(\mathfrak{M}^l)$, where $l = p^{[\log_p(e/(p-1))]} - e[\log_p(e/(p-1))] - [e/(1-p)]$, $\varepsilon \in M_{m'}(NW)$, then there exists a homomorphism f from F to F', $f(X) \equiv Ap^s x \mod \deg 2$.

3. F and F' are strictly isomorphic if and only if they are rationally isogenious and $M_F = M_{F'}$.

Proof. 1. If such an f exists, then we have $AD_F \subset D_{F'}$. It follows that $AD_FN \subset D_{F'}N$, which is equivalent to the condition on the fractional parts. The condition on M_F follows immediately from the definition of M_F.

Vice versa, let the conditions on F and F' be fulfilled. Then for the W-modules $Y = AD_F$ and $Z = D_{F'}$ we have $YN = ZN$ and $Y\Omega = Z\Omega$. Let z_i be a W-base of Z (recall that $D_{F'}$ is W-free). For any $r \in Z\Omega N$ the coefficients $r_i \in \Omega N$ such that $r = \sum z_i r_i$ are unique. for $y \in Y$, $y = \sum z_i y_i$ we have $y_i \in NW \cap \Omega = W$. Hence $Y \subset Z$ and Proposition 5.2.2 gives us the claim.

2. It is sufficient to check that $p^sAM_F \subset M_{F'}$. According to Remark 5.3.3, we have $(\mathfrak{M}^{-[e/(1-p)]}\Omega)^{m'} \subset M'_F$. It remains to check that $M_F \subset (\mathfrak{M}^r\Omega)^m$ for $r = p^{[\log_p(e/(p-1))]} - e[\log_p(e/(p-1))]$. It is sufficient to check that $\langle w_i\rangle\Lambda_i \in (\mathfrak{M}^r\Omega)^m$. We can take the base w_i such that $w_1 = 1$ and $w_i \in \mathfrak{M}$ for $i > 1$. Since for finite height groups $u \in \mathrm{Gl}_m(\Omega)$, Proposition 1.4.1 implies

that $\Lambda_i \in (\mathfrak{M}^r\Omega)^m$. For $s \in \mathfrak{M}$ (in particular, $s = w_i$, $i > 1$) we obtain $\lambda(sxe_i) \in \mathfrak{M}^r\mathfrak{O}_K[[X]]$ directly from the fact that the partial derivatives of λ are integral, thus $\Lambda_{ij} = \sum w_r\Delta^r$ for $w_r \in p^{-r}W$.

3. This is a special case of part 1 for $A = I_m$. $\square$

6.2 M_F for finite height groups. Algorithm for classification of formal groups

Proposition 6.2.1. *1. If the logarithm of a formal group law F belongs to R_X^m, then $D_F = M_F \cap R^m$.*

2. If F is a finite height formal group, then $M_F = \pi M_{F_\pi}$, where F_π is the formal group law $\pi^{-1}F(\pi X, \pi Y)$.

3. If F is a finite height formal group, then $D_F \cap R^m = \pi D_{F\pi}$.

Proof. 1. We have $D_F = \langle\langle w_i\rangle\Lambda_i\rangle$. Hence $M_F = \bigoplus v_i\Omega$, where $v_i \in R^m$, $1 \le i \le me$. Thus $D_F \subset M_F \cap R^m$.

On the other hand, $p\mathfrak{O}_K^m \subset D_f \mod \Delta$ implies that v_i form an NW-base of R. Hence $\sum v_ir_i \in R^m$ for $r_i \in \Omega$ if and only if $r_i \in NW$. Thus for $v \in M_F \cap R^m$ we have $v = \sum v_iu_i$ for $u_i1\Omega \cap NW = W$.

2. The map πX is a homomorphism from F_π into F. Thus $\pi D_{F_\pi} \subset D_F$ and we obtain $\pi M_{F_\pi} \subset M_F$.

We present Λ as vu^{-1}. Since F is a finite height group, u is invertible in $M_m(\Omega)$. Therefore $M_F = \langle v(e_i)\rangle$. Since $v(e_i) \in D_F$, we have $\exp_F(v(xe_i)) \in \mathfrak{O}[[x]]^m$. As $v(e_i) \in R^m$, we also obtain $\exp_F(v(xe_i)) \in \pi\mathfrak{O}[[x]]^m$. Thus, according to Proposition 5.2.3 for the group F_π, $\pi^{-1}v(e_i) \in D_{F\pi}$. Hence $M_F \subset \pi M_{F_\pi}$.

3. Since $D_{F\pi} \in R^m$, we have $\pi D_{F\pi} \subset D_F \cap R^m$. On the other hand, $D_F \cap R^m \subset M_F \cap R^m$ and it remains to apply part 2.

$\square$

One can easily check that $\lambda \in R_X$ if the reduction of the formal group law F modulo π is equal to G_a^m.

Now we prove a result that is crucial for the construction and classification of formal group laws.

Theorem 6.2.2. *1. $\lambda = \Lambda(X)$ is a logarithm of a formal group law F over $\mathfrak{O}_K$ if and only if $\langle\pi\rangle\lambda_i \in \pi D_{F'}$ for $1 \le i \le m$ and $(\Lambda u)_i \in \pi D_{F'}$ for some special u and a formal group F' over $\mathfrak{O}_K$ satisfying $\pi^{s-1}\mathfrak{O}_K W^m \subset D_{F'}$.*

If this happens then F' is strictly isomorphic to F_π.

2. Let $s = -[e/(1-p)]$, $a \in \mathfrak{O}_K$, and $v_K(a) = l \le s$. Suppose that for some Λ, a special u, and a formal group law F' over $\mathfrak{O}_K$ satisfying $a^{-1}\pi^s\mathfrak{O}_K W^m \subset D_{F'}$ we have $\langle a\rangle\Lambda_i \in aD_{F'}$ and $(\Lambda u)_i \in \pi^s D_{F'}$. Then Λ

is a logarithm of a formal group law F over $\mathfrak{O}_K$ and F' is strictly isomorphic to F_a.

Proof. 1. According to the previous statement, we have $\langle\pi\rangle\lambda_i \in \pi D_{F_\pi}$ and $(\Lambda u(F))_i \in \pi D_{F_\pi}$. On the other hand, suppose that the conditions on Λ are fulfilled for some F' and u. The elements $v_{ij} = \langle\pi^j\rangle\lambda_i$ for $1 \le j \le e$ form a W-base of πD_{F_π}. Since v_{i1} belong to $\pi D_{F'}$, v_{ij} belong to $\pi D_{F'}$ for any $j > 1$. Thus F' is strictly isomorphic to F_π.

Now it remains to prove assertion 3 for $a = \pi$.

2. According to Remark 5.3.3, we have $a^{-1}\pi^s\mathfrak{O}_K W^m \subset D_{F'}$. Next, for any $b \in \mathfrak{O}_K^*$ from Proposition 5.2.3 one immediately obtains the equality $D_{F_b} = b^{-1}D_F$. Thus we can assume $a = \pi^s$.

We prove by induction on $r \ge 0$ that the module $D_r = \pi^{-r}\langle D, \langle\pi^i\rangle\rangle W$, $i \ge r$ corresponds to some formal group.

Suppose that this statement is fulfilled for $r = n+1$, $n \ge 0$. We have $\Lambda u = v \in D'^m$, where $u = pI_m - \sum u_i\Delta^i$. If $n > 0$, we have

$$p\langle\pi^n\rangle\Lambda = \sum\langle\pi^{p^i n}\rangle(\Lambda)u_i + \langle\pi^n\rangle v. \tag{5}$$

Hence for any $n \ge 0$ we have $\langle\pi^n\rangle\Lambda u' \in \pi^{n+1}D_{n+1}^m$ for a special u'.

According to Lemma 5.1.2 we can complete the elements $r_i = (\langle\pi^n\rangle\Lambda u')_i$, $1 \le i \le m$ to a W-base r_j, $1 \le j \le me$, of $\pi^{n+1}D_{n+1}$. It follows that $\pi^n D_n$ is generated over W by $(\pi^n\Lambda)_i$ and r_j for $j > m$. Hence $D_n \in D$, since the other properties are obvious. Since D_n contains $\pi^{-n}\langle\pi^j\rangle\Lambda_i$ for $j \ge n$, we apply Proposition 5.3.1 for $w_j = \pi^{j-1}$ and obtain that D_n is equal to D_{F_n} for some F_n over $\mathfrak{O}_K$.

Now suppose that the conditions on Λ are fulfilled for some F' and u.

As in the proof of the first part, we see that $v_{ij} = \langle\pi^j\rangle\lambda_i \in \pi^s D_{F'}$ for $l \le j < l+e$ form a W-base of πD_{F_π}. Hence F' is strictly isomorphic to F_{π^s}. □

Remark 6.2.3. In particular, the conditions of assertion 3 are fulfilled if $\langle a\rangle\Lambda_i \in a\mathfrak{O}_K^m W$ and $\Lambda u_i \in a\mathfrak{O}_K^m W$, since in this case one can take F' equal to the m-th power of the additive formal group law.

In particular, one can obtain all Honda formal group laws (Lubin–Tate ones, for example) by means of this statement. The canonical representatives of the isogeny classes of formal groups presented in [5] for the case of algebraically closed residue field can also be constructed (see also Theorem H in the introduction).

Now we describe a general algorithm for classifying formal group laws of fixed dimension m over a fixed field K.

First one describes all possible πD_{F_π}. To this end one can can use the universal p-typical formal group law to construct the logarithms of the form $\pi^{-1}\lambda(\pi X)$. Since D_F depends only on the residues of the coefficients of Λ modulo π^l, where $l = -[e/1-p]$ and only the first few coefficients of $\langle\pi\rangle\Lambda$ may not divide π^l, the number of different πD_{F_π} is finite.

Next one fixes πD_{F_π}, and for each u describes Λ satisfying the conditions of the previous proposition. Thus one obtains the description of all p-typical logarithms of formal group laws.

In order to check which of them give strictly isomorphic formal group laws, one can check what residues modulo $R^m u$ have elements $(v_i) \in \pi M_{F_\pi}^m$, $v \equiv pI_m \mod \Delta$ such that

$$\langle\pi\rangle(v/u) \in \pi D_{F_\pi}^m. \tag{$*$}$$

After doing this one recovers $r(F)$. Next it is possible to calculate M_F (using Proposition 6.2.1 for finite height groups). Hence one obtains the pair $r(F), M(F)$ and is able to apply Theorem 6.2.2.

Note that the condition (*) depends only on the first few coefficients of u and v. Thus if we take u equal to a canonical representative, calculation will terminate in finitely many steps.

6.3 Classification of formal groups for $e \le p^2/2$

We illustrate the method described in the previous subsection by classifying one-dimensional formal groups of height > 1 for $e \le p^2/2$. This result can easily be extended to multidimensional groups if one defines the height as a certain vector.

6.3.1 Calculation of πD_{F_π}

According to Remark 5.3.3, it is sufficient to calculate $\lambda_\pi = \pi^{-1}\lambda(\pi x) = \Lambda_\pi(\Delta)(x)$ modulo $\pi^{1-[e/(1-p)]}$.

The universal p-typical group law is obtained by inverting the power series $u_c = 1 - \sum c_i \Delta^i p^{-i}$ in $\mathfrak{A}[[\Delta]]'$, where $\mathfrak{A} = \mathbb{Z}_p[c_i]$, $i > 0$, $\sigma(c_i) = c_i^p$, and applying it to x.

We have

$$u_c^{-1} = 1 + \frac{c_1}{p}\Delta + \left(\frac{c_2}{p} + \frac{c_1^{p+1}}{p^2}\right)\Delta^2 + \dots. \tag{6}$$

For the formal group λ_π we have $\pi^{p^i-1} \mid c_i$. Hence for $e \le \frac{p^2}{2}$ we obtain

$\Lambda_\pi \equiv 1 + \frac{c}{p}\Delta \mod \pi^{1-[e/(1-p)]}$, $\pi^{p-1} \mid c = c_1$. Therefore,

$$\pi D_{F\pi} = \langle \pi^i + \frac{\pi^{(i-1)p+1}c}{p} \rangle,\ i \geq 1.$$

We denote $v_K(c)$ by d.

From this one easily obtains that $a \in \pi D_{F_\pi}$ (which is equivalent to $a\Delta \in \pi D_{F_\pi}$) for $a \in \mathfrak{M}$ if and only if $v_K(a) \geq 1 + \frac{e-d}{p-1}$.

6.3.2 Checking (*)

If $h > 1$ then, according to Proposition 1.3.1, one can assume $u \equiv p \mod \Delta^2$. For such u and $e \leq \frac{p^2}{2}$ one easily sees

$$\langle \pi \rangle (v/u) \equiv \langle \pi \rangle (v/p) \mod \pi^{-[e/(p-1)]}.$$

Next, the module πD_{F_π} was constructed for

$$\Lambda = \sum_{i \geq 0} \frac{c}{\pi^{p-1}}^{1+\dots+p^{i-1}} \left(\frac{\Delta}{p}\right)^i.$$

Since $h > 1$, we have $\frac{c}{\pi^{p-1}} \in \mathfrak{M}$, hence $v' = p + c\frac{\Delta}{p\pi^{p-1}} \in \pi D_{F_\pi}$ and v' satisfies (*) for $u = p$. Thus we obtain that v satisfies (*) if and only if $\langle \pi \rangle \frac{v-v'}{p} \in D_F$. On the other hand, we have $\langle \pi \rangle \frac{\Delta^2}{p} D_{F_\pi} \in D_{F_\pi}$. Therefore, v satisfies (*) for $v \in \pi D_{F_\pi}$ if and only if $v \equiv v' + v_1\Delta \mod \deg 2$, where $v_1 \in \mathfrak{M}$, $v_K(v_1) \geq e - p + 1 + \frac{e-d}{p-1}$. Furthermore, we obtain

$$v_1\Delta + \frac{v_1^p c\Delta^2}{p\pi^{p-1}} = \langle v/\pi \rangle v'\Delta \in \pi D_{F\pi}.$$

Hence the final answer for $e > p$ is $d \geq p$,

$$v = p + c\frac{\Delta}{p\pi} + v_1\Delta + \frac{v_1^p c\Delta^2}{p\pi^{p-1}} + v_2\Delta^2$$

for any $v_2 \in \pi D_{F\pi}$ and $v_1 \in \mathfrak{M}$ satisfying $v_K(v_1) \geq e - p + 1 + \frac{e-d}{p-1}$. One can easily calculate $r(F)$.

For a finite height group one has $M_F = \pi D_{F_\pi}\Omega$. Since πD_{F_π} is generated over $W[\langle \pi \rangle]$ by $\pi + \frac{\pi c}{p}$, we obtain that $\pi D_{F_\pi}\Omega$ corresponding to c' coincides with $\pi D_{F\pi}$ for c if and only if $v(c-c') \geq \max(e+1, e+1+\frac{e-d}{p-1})$ (see subsection 6.3.1). For an infinite height group law one can calculates M_F directly using $D_F \subset R$.

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