On p-adic Uniformization of Fake Projective Planes [1]

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ABSTRACT: In this paper we prove that fake projective planes (Mumford's one and those considered by Ishida-Kato) are (connected) Shimura varieties. We also describe the level structures and minimal fields of definition explicitly.

1. INTRODUCTION

The aim of this paper is to exhibit some explicit examples of p-adic uniformization of Shimura surfaces of PEL-type. Each of our examples gives, on the complex analytic side, a surface of general type having complex unit-ball as the topological universal covering; more specifically, they are smooth surfaces having the same topological Betti numbers as $\mathbb{CP}^2$, so-called "*fake projective planes*", which are among the most interesting classes of ball-quotient surfaces in algebraic geometry.

One of the fake projective planes discussed in this paper was discovered by Mumford [Mu79]; p-adic uniformization was already the crucial strategy of construction there. Later in part of [IK98] Masanori Ishida and the author observed other possible fake projective planes according to Mumford's idea, and proved there exist (at least) two more different fake projective planes, collaborated with the group-theorists' classification of discrete groups acting on Bruhat-Tits buildings in [CMSZ93a] and [CMSZ93b]. As far as the author knows, these three are the only known fake projective planes.

However, these surfaces are, a priori, constructed as surfaces over the field of 2-adic numbers $\mathbb{Q}_2$, not over $\mathbb{C}$. (A fake projective plane over an arbitrary field may be defined to be a smooth surface enjoying: $\mathrm{c}_1^2 = 3\mathrm{c}_2 = 9$, $\mathrm{p_g} = \mathrm{q} = 0$, and the canonical class is ample; if one considers this condition in the complex analytic category, it is equivalent to that the surface in question is a fake projective plane in the origial sense.) In this sense, although those constructions as above prove the "existence" of such surfaces, they are not really concrete as complex surfaces. For instance, topological or even some algebro-geometric properties of them (e.g. fundamental groups, field of definition, etc.) are quite obscure; even the arithmeticity of the fundamental groups seems unknown.

The theory of p-adic uniformization, on the other hand, involves surprizingly rich algebro-geometric and arithmetic aspects concerning with Shimura varieties. This was first observed by Čerednik [Če76] and Drinfeld [Dr76] for Shimura curves. The generalizations to higher dimensions have been developed over the past two decades; recently, Boutot-Zink [BZ95] and Rapoport-Zink [RZ96] gave powerfully extended generalities. These theories give a nice symmetric picture involving p-adic and complex uniformizations of certain Shimura varieties of PEL-type; viz. such a Shimura variety has both p-adic (for very special p) and complex analytic uniformizations, and they can be viewed symmetrically by means of "changing-invariants" trick.

Shimura varieties of this kind are, therefore, much expected to possess rich structures involving the interplay between p-adic and complex analysis. However, as far as the author knows, very few such examples are known in dimension two or high, in contract with numerous explicit examples of Shimura curves of this type (see, e.g. [Ku79][Ku94][An98]).

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Sketch of Results. In this paper, we will see that the p-adic constructions of the three fake projective planes as above actually fit in with the picture of the theory by Boutot-Zink [BZ95] and Rapoport-Zink [RZ96], and give very explicit examples in dimension two. This implies, in particular, the following:

(1) These fake projective planes are realized as connected components of certain Shimura varieties, and hence, are solutions of certain moduli problems parametrizing suitable abelian varieties with additional structures.

(2) In particular, they are, as complex surfaces, *arithmetic* quotient of the complex unit-ball; moreover, we can describe the fundamental groups fairly explicitly in terms of congruence conditions.

(3) Besides, describing connected components of the Shimura varieties by means of the theory of complex multiplication, we can find out the minimal fields of definition for those surfaces. In fact, we will prove that they are defined over the absolute class fields of suitable imaginary quadratic fields.

The structure of this paper is as follows: In the next section, we will survey very briefly the theory of p-adic uniformizations of Shimura varieties, basically after Boutot-Zink [BZ95] and Rapoport-Zink [RZ96], in order to show how it works to come up with the results as above. The subsequent sections are devoted to present examples; in §3 we will discuss Mumford's fake projective plane, and the other two fake projective planes in §4.

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Notation. The notation $A \otimes B$ (absence of the base ring) only occurs when A and B are $\mathbb{Q}$-algebra, and tensors are taken over $\mathbb{Q}$. For a field extension F/E, we denote by $\mathrm{Res}_{F/E}$ the Weil restriction. If E is a p-adic field, E^{ur} denotes the maximal unramified extension in an algebraic closure.

2. p-adic uniformization of Shimura varieties of PEL-type

This section surveys the generalities of p-adic uniformization of Shimura varieties of PEL-type. Our references on this subject are Rapoport [Ra90], Boutot-Zink [BZ95], and Rapoport-Zink [RZ82, §4][RZ96], through which the reader can always justify the sometimes sketchy statements in this section. Since the purpose of this section is to provide language to describe the examples in the subsequent sections, we do not really need the full generality, and will be fairly specific. For instance, knowledge of the theory over general CM fields is not needed, and so we will limit ourselves to the case only over imaginary quadratic fields, which will make the situation simpler and explicit.

A. *The fundamental data.* Throughout this section we fix an integer $d > 2$ and a prime number p. Let K be an imaginary quadratic extension of $\mathbb{Q}$ such that the prime p decomposes to the product $\mathfrak{p}\overline{\mathfrak{p}}$ of two distinct prime ideals in K. Let D be a central division algebra over K of dimension d^2. For any place ν of K we write $D_\nu := D \otimes_K K_\nu$. We choose once for all an infinite place $\varepsilon\colon K \hookrightarrow \mathbb{C}$, i.e. a CM-type, of K. We assume: zw

(a) D_ε is isomorphic to the matrix algebra $\mathrm{M}_d(\mathbb{C})$.
(b) The Hasse invariant of $D_\mathfrak{p}$ (resp. $D_{\overline{\mathfrak{p}}}$) is $\frac{1}{d}$ (resp. $-\frac{1}{d}$).

A positive involution on D is a $\mathbb{Q}$-linear endomorphism $x \mapsto x^*$ of D such that $(xy)^* = y^*x^*$, $x^{**} = x$ for all $x, y \in D$, and that $\mathrm{tr}_{D/\mathbb{Q}}(xx^*) > 0$ for all non-zero $x \in D$, where $\mathrm{tr}_{D/\mathbb{Q}}$ denotes the reduced trace. Fix a positive involution $*$ on D such that its restriction to K coincides with the complex conjugation. The assumption (a) yields an isomorphism

$$\Phi\colon D \otimes \mathbb{R} \xrightarrow{\sim} \mathrm{M}_d(\mathbb{C})$$

of $\mathbb{R}$-algebras. We may assmue that Φ gives an isomorphism between $\mathbb{R}$-algebras with involution, where $\mathrm{M}_d(\mathbb{C})$ is equipped with the standard involution $A \mapsto {}^t\overline{A}$. Indeed, any involution of $\mathrm{M}_d(\mathbb{C})$ is a standard one up to inner automorphisms. We fix a $*$-stable order $\mathcal{O}_D$ of D such that $\mathcal{O}_{D_\mathfrak{p}} := \mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\mathfrak{p}}$ (resp. $\mathcal{O}_{D_{\overline{\mathfrak{p}}}} := \mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{\overline{\mathfrak{p}}}}$) is a maximal order of $D_\mathfrak{p}$ (resp. $D_{\overline{\mathfrak{p}}}$). For any finite place ν, $\mathcal{O}_{D_\nu}$ denotes $\mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\nu}$.

We shall further introduce the following data: Let V be D viewed as a left D-module. Let $b \in D$ be a non-zero element which is anti-symmetric, i.e. $b^* = -b$. We define the anti-symmetric $\mathbb{Q}$-bilinear form $\psi = \psi_b\colon V \times V \to \mathbb{Q}$ by $\psi(x, y) := \mathrm{tr}_{D/\mathbb{Q}}(ybx^*)$ for $x, y \in V$. The form ψ clealy satisfies $\psi(\alpha x, y) = \psi(x, \alpha^* y)$ for $x, y \in V$ and $\alpha \in D$. It is also easily verified that ψ is non-degenerate if b is invertible. By Γ we denote $\mathcal{O}_D$ viewed as a left $\mathcal{O}_D$-module, and define $\Gamma_p := \Gamma \otimes_\mathbb{Z} \mathbb{Z}_p$. Then we require: zw

(c) The form ψ induces a perfect paring $\psi_p\colon \Gamma_p \otimes \Gamma_p \to \mathbb{Z}_p$.

Via the isomorphism Φ chosen as above, the element b can be represented by an anti-Hermitian matrix, written simply as $\Phi(b)$. We assume: zw

(d) There exists $P \in \mathrm{M}_d(\mathbb{C})$ such that ${}^t\overline{P}\Phi(b)P = \mathrm{diag}(-\sqrt{-1}, \ldots, -\sqrt{-1}, \sqrt{-1})$.

B. *The Shimura variety.* Define an algebraic group G over $\mathbb{Q}$ as follows: Given a $\mathbb{Q}$-algebra R, we set

$$G(R) := \{g \in ((D \otimes R)^{\mathrm{op}})^\times \mid \psi(xg, yg) = c(g)\psi(x, y),\ c(g) \in R^\times,\ x, y \in V \otimes R\}.$$

Note that we have an equality $\mathrm{Nm}\, g \cdot \overline{\mathrm{Nm}\, g} = c(g)^d$, where Nm denotes the reduced norm of $(D \otimes R)^{\mathrm{op}}/K \otimes R$.

To see that the algebraic group $G_\mathbb{R}$ is isomorphic to a group of unitary similitudes, we consider, as in [RZ96, 6.37], a decomposition

$$V \otimes \mathbb{R} \xrightarrow{\sim} \mathbb{C}^d \otimes_\mathbb{C} W$$

so that $D \otimes \mathbb{R} \cong \mathrm{M}_d(\mathbb{C})$ acts from the left on the right hand side by the usual matrix multiplication on the first factor. Since $D \otimes \mathbb{R}$ is simple, we have $\mathrm{End}_{D\otimes\mathbb{R}}(V \otimes \mathbb{R}) = (D \otimes \mathbb{R})^{\mathrm{op}}$, and the right action of $(D \otimes \mathbb{R})^{\mathrm{op}} \cong \mathrm{M}_d(\mathbb{C})^{\mathrm{op}}$ on $V \otimes \mathbb{R}$ can be seen as the usual matrix multiplications on W from the right. Then $G_\mathbb{R}$ is the group of unitary similitudes with respect to the anti-Hermitian form H on W, where H is induced by ψ by the formula

$$\psi(z_1 \otimes w_1, z_2 \otimes w_2) = \mathrm{tr}_{\mathbb{C}/\mathbb{R}}({}^t\overline{z}_1 z_2 H(w_1, w_2))$$

for $w_1, w_2 \in W$ and $z_1, z_2 \in \mathbb{C}^d$. By a suitable choice of basis, the form H is represented by the matrix $J := \mathrm{diag}(-\sqrt{-1}, \ldots, -\sqrt{-1}, \sqrt{-1})$, and then we deduce that $G_\mathbb{R}$ is isomorphic to $\mathrm{GU}(d-1, 1)$, the group of unitary similitudes of (W, J).

We define a homomorphism $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_\mathrm{m} \to G_\mathbb{R}$ so that $h(r)$, $r \in \mathbb{R}^\times$, acts on W as the multiplication by r and $h(\sqrt{-1})$ acts as the multiplication by $-J$. By [Ko92, 4.1] the pair $(G_\mathbb{R}, h)$ satisfies the conditions (1.5.1), (1.5.2), and (1.5.3) in [De71], so that we get the associated Shimura variety; for a sufficiently small level structure C, i.e. an open compact subgroup of $G(\mathbb{A}_\mathrm{f})$, the corresponding Shimura variety will be denoted by $\mathcal{S}_C$:

$$\mathcal{S}_C\colon = G(\mathbb{Q}) \setminus X_\infty \times G(\mathbb{A}_\mathrm{f})/C,$$

where X_∞ is the set of all conjugates of h under the action of $G(\mathbb{R})$, that is, in our case, the open unit-ball in $\mathbb{C}^{d-1}$. The Shimura variety $\mathcal{S}_C$ is a finite disjoint union of quasi-projective manifolds, which are arithmetic quotients of the complex unit-ball. The Shimura field will be denoted by E, which is $\varepsilon(K)$ in our case. Given $\mathcal{S}_C$, Sh_C stands for its canonical model, a quasi-projective variety over E.

A small remark on the structure of G: Consider a subgroup G_1 of G defined by

$$G_1(R)\colon = \{u \in G(R) \mid c(u) = 1\}$$

for a $\mathbb{Q}$-algebra R. Then, in case d is odd, there exists an epimorphism

$$G_1 \times \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m},K} \longrightarrow G$$

by $(u,k) \mapsto uk$. Indeed, for any $g \in G(R)$, we can take $k = \mathrm{Nm}\, g \cdot c(g)^m$ so that $gk^{-1} \in G_1(R)$, where $d = 2m+1$.

C. *The moduli problem over E.* The Shimura variety Sh_C with respect to a level structure $C \subset G(\mathbb{A}_\mathrm{f})$ represents the moduli functor $\underline{\mathcal{S}h}_C\colon \mathbf{Sch}_E \to \mathbf{Set}$ defined as follows: The Shimura data (G,h) gives rise to a Hodge decomposition $V_\mathbb{R} = V_{0,\mathbb{C}} \oplus V_{1,\mathbb{C}}$, where $V_{0,\mathbb{C}}$ (resp. $V_{1,\mathbb{C}}$) is the subspace of $V_\mathbb{R}$ on which $h(z)$ acts by z (resp. by $\overline{z}$). As the conjugacy class of h is defined over E, this decomposition comes from a decomposition $V = V_0 \oplus V_1$ defined over E. Then, for an E-scheme T, $\underline{\mathcal{S}h}_C(T)$ is the set of all isomorphism classes of quadruples $(A, \iota, \overline{\Lambda}, \overline{\eta})$ consisting of the following data: zw

(i) An abelian scheme A over T considered up to isogeny.

(ii) A homomorphism $\iota\colon D \to \mathrm{End}^\circ(A)$ such that, for any $\delta \in D$,

$$\mathrm{tr}(\iota(\delta)|\mathrm{Lie}(A)) = \mathrm{tr}(\delta|V_0),$$

where $\mathrm{End}^\circ(A)\colon = \mathrm{End}(A) \otimes \mathbb{Q}$.

(iii) A $\mathbb{Q}$-homogeneous polarization $\overline{\Lambda}$ of A such that ι gives a homomorphism of $\mathbb{Q}$-algebras respecting the involutions $*$ on D and the Rosatti involution on $\mathrm{End}^\circ(A)$.

(iv) A modulo C-class $\overline{\eta}$ of $D \otimes \mathbb{A}_\mathrm{f}$-linear symplectic similitudes

$$\eta\colon (V(A), \text{Riemann form}) \xrightarrow{\sim} (V \otimes \mathbb{A}_\mathrm{f}, \psi \otimes \mathbb{A}_\mathrm{f}),$$

where $V(A)$ denotes the rational Tate module of A.

D. *Local model.* ¿From now on, the level structure C will be always assumed to be of form $C = C_pC^p$, where $C^p \subset G(\mathbb{A}_\mathrm{f}^p)$ and: zw

(e) C_p is the maximal open compact subgroup of $G(\mathbb{Q}_p)$ consisting of those elements of $G(\mathbb{Q}_p)$ which map Γ_p onto itself.

Localizing the above moduli problem to the infinite place $E \hookrightarrow \mathbb{C}$, one gets the moduli problem of which the solution is given by the complex Shimura manifold $\mathcal{S}_C$, which has the complex unit-ball uniformization. The theory of p-adic uniformization of Shimura varieties, which is the main subject of this section, asserts that, localizing Sh_C to the special place

$$\nu\colon E \longhookrightarrow \overline{\mathbb{Q}}_p$$

defined by the prime $\varepsilon(\mathfrak{p})$, again one gets an interesting analytic variety which has a uniformization of a certain kind. In order to see this, it will be more convenient to extend the localized moduli problem over E_ν to that over the integer ring $\mathcal{O}_{E_\nu}$, which amounts to consider a model of $\mathrm{Sh}_C \otimes_E E_\nu$.

Let $\underline{\mathcal{A}}_C\colon \mathbf{Sch}_{\mathcal{O}_{E_\nu}} \to \mathbf{Set}$ be the moduli functor defined as follows: For an $\mathcal{O}_{E_\nu}$-scheme T the set $\underline{\mathcal{A}}_C(T)$ consists of the isomorphism classes of the quadruple $(A, \iota, \overline{\Lambda}, \overline{\eta}^p)$ of the following data: zw

(i)$'$ An abelian scheme A over T considered up to isogeny of degree prime to p.

(ii)$'$ An injective homomorphism $\iota\colon \mathcal{O}_D \hookrightarrow \mathrm{End}^{(p)}(A)$ such that, for $\delta \in \mathcal{O}_D$,

$$\det{}_{\mathcal{O}_T}(\iota(\delta)|\mathrm{Lie}(A)) = \det{}_{\overline{\mathbb{Q}}_p}(\delta|V_0),$$

where $\mathrm{End}^{(p)}(A) := \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$.

(iii)$'$ A $\mathbb{Q}$-homogeneous polarization $\overline{\Lambda}$ of A, which contains a polarization of degree prime to p, such that ι gives a homomorphism between $\mathbb{Z}_{(p)}$-algebras with involution with $\mathcal{O}_D$ (resp. $\mathrm{End}^{(p)}(A)$) considered together with $*$ (resp. the Rosatti involution).

(iv)$'$ A modulo C^p-class $\overline{\eta}^p$ of $D \otimes \mathbb{A}^p_{\mathrm{f}}$-linear symplectic similitudes

$$\eta\colon (V^p(A), \text{Riemann form}) \xrightarrow{\sim} (V \otimes \mathbb{A}^p_{\mathrm{f}}, \psi \otimes \mathbb{A}^p_{\mathrm{f}}),$$

where $V^p(A)$ denotes the prime-to-p part of the rational Tate module of A.

The condition in (ii)$'$ calls for an explanation. This condition has been introduced in [Ko92, §5], and, as explained there, the equality should be read as an identity of polynomial functions on $\delta \in \mathcal{O}_D$. Over schemes of characteristic zero, it can be replaced by the usual "trace" condition as in (ii), while it is crucial over positive characteristics.

By the standard argument as in [Ko92, §5], one sees that the étale sheafification of the functor $\underline{\mathcal{A}}_C$ can be represented by a quasi-projective scheme $\mathcal{A}_C$ over $\mathcal{O}_{E_\nu}$ (even projective due to [BZ95, 1.1]).

We need to relate $\mathcal{A}_C$ to the Shimura variety Sh_C; to do this, one has to take a bit care of the difference between the data (iv) and (iv)$'$. Let $(A, \iota, \overline{\Lambda}, \overline{\eta}^p)$ be as above. By [RZ96, 6.39] the Tate module of A together with the pairing induced by the polarization is isomorphic to (Γ_p, ψ_p) as a symplectic $\mathcal{O}_{D_p}$-module (cf. [RZ96, 6.38]). This gives a similitude between $(V_p(A), \text{Riemann form})$ and $(V \otimes \mathbb{Q}_p, \psi \otimes \mathbb{Q}_p)$. So, if our group G satisfies the Hasse principle, then, combining with the datum (iv)$'$, we get a similitude as in (iv), and thus we get an isomorphism

$$\mathcal{A}_C \otimes_{\mathcal{O}_{E_\nu}} E_\nu \xrightarrow{\sim} \mathrm{Sh}_C \otimes_E E_\nu,$$

i.e. the scheme $\mathcal{A}_C$ is giving a local model of Sh_C.

For verifying the Hasse principle for G, we refer the argument by Kottwitz [Ko92, §7]; our situation is of the case (A) in his sense, and we find that G satisfies the Hasse principle, for the center of G is an imaginary quadratic extension of $\mathbb{Q}$.

E. *p-adic uniformization.* Now let κ be the residue field of $\overline{E}_\nu(\cong \overline{\mathbb{Q}}_p)$, and take a point $[A_t, \iota_t, \overline{\Lambda}_t, \overline{\eta}^p_t] \in \underline{\mathcal{A}}_C(\kappa)$. Let us denote by X_t the p-divisible group corresponding to the abelian variety A_s. The p-divisible group X_t has an $\mathcal{O}_{D_p}$-action so that it is a formal $\mathcal{O}_{D_p}$-module, where $\mathcal{O}_{D_p}:=\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}_{D_\mathfrak{p}} \times \mathcal{O}_{D_{\overline{\mathfrak{p}}}}$. By this action one has the decomposition

$$X_t = X_1 \times X_2,$$

where X_1 (resp. X_2) is a formal $\mathcal{O}_{D_\mathfrak{p}}$ (resp. $\mathcal{O}_{D_{\overline{\mathfrak{p}}}}$)-module. The "determinant" condition in (ii)$'$ is equivalent with that X_1 is a special formal $\mathcal{O}_{D_\mathfrak{p}}$-module in the sense of Drinfeld [Dr76]. By [Dr76] it is known that there is only one isogeny class of special formal $\mathcal{O}_{D_p}$-module over κ.

Due to [RZ96, 6.39], the Tate module of X_t together with the pairing induced by the polarization is isomorphic to (Γ_p, ψ_p) as a symplectic $\mathcal{O}_{D_p}$-module (cf. [RZ96, 6.38]). Since the involution $*$ gives an isomorphism $\mathcal{O}_{D_{\overline{\mathfrak{p}}}} \cong \mathcal{O}^{\mathrm{op}}_{D_\mathfrak{p}}$ and $\mathcal{O}^{\mathrm{op}}_{D_\mathfrak{p}} \cong \mathrm{End}_{\mathcal{O}_{D_\mathfrak{p}}}(\Gamma_\mathfrak{p})$, there exists an isomorphism $\Gamma_p \cong \Gamma_\mathfrak{p} \times \Gamma_\mathfrak{p}^\vee$, where $\Gamma_\mathfrak{p}:=\Gamma \otimes_{\mathcal{O}_K} K_\mathfrak{p}$. Hence we have $X_2 \cong {X_1}^\vee$, and the polarization on $X_t \cong X_1 \times {X_1}^\vee$ has to be the tautological one (the Poincaré bundle), for it has been required to be compatible with the $\mathcal{O}_{D_p}$-action. The significance of this fact is that, when we take the p-divisible group associated to the data as above, we can throw away the polarization!

Summarizingly, once we consider the associated p-divisible groups, we have only to concern with a single isogeny class. This enables us to deal with the moduli data on an equal footing, and leads us to the context of p-adic periods. So, fixing one point $[A_s, \iota_s, \overline{\Lambda}_s, \overline{\eta}^p_s] \in \underline{\mathcal{A}}_C(\kappa)$, we shall be interested in classifying quasi-isogenies $X_s \to X_t$; because of the rigidity property of such quasi-isogenies (cf. [RZ96, 3.62]), they form a p-adic formal scheme over $\mathbb{Z}_p^{\mathrm{ur}}$. In fact, Drinfeld [Dr76] proved that they are parametrized by a formal scheme $\widehat{\Omega}^{\mathrm{ur}}$, which is base extension of the formal scheme Ω over $\mathbb{Z}_p$ of which the Raynaud generic fiber is known to be the *Drinfeld symmetric space.*

Then, pushing out the fixed data $[A_s, \iota_s, \overline{\Lambda}_s, \overline{\eta}^p_s]$ by quasi-isogenies, and applying the theory of Serre-Tate local moduli, we get an epimorphism of formal schemes (*the uniformization morphism*)

$$\widehat{\Omega}^{\mathrm{ur}} \times G(\mathbb{A}^p_\mathrm{f})/C^p \longrightarrow \widehat{\mathcal{A}}_C \otimes \mathcal{O}^{\mathrm{ur}}_{E_\nu},$$

where $\widehat{\mathcal{A}}_C$ denotes the ν-adic completion of $\mathcal{A}_C$; the second factor on the left hand side serves for the level structure. In order to understand the fibers of this morphism, we introduce an algebraic group I over $\mathbb{Q}$ by setting, for a $\mathbb{Q}$-algebra R,

$$I(R):=\{\varphi \in (\mathrm{End}^\circ_D A_s) \otimes R \mid \varphi\varphi' \in R^\times\},$$

where $\varphi \mapsto \varphi'$ denotes the involution induced by the Rosatti involution. This is the algebraic group of quasi-isogenies of (A_s, ι_s), and so we can consider the natural action of $I(\mathbb{Q})$ on $\widehat{\Omega}^{\mathrm{ur}} \times G(\mathbb{A}_\mathrm{f})/C$ (cf. [BZ95, §1]). The algebraic group I is, in fact, an inner form of G such that $I(\mathbb{A}^p_\mathrm{f}) \cong G(\mathbb{A}^p_\mathrm{f})$ and $I(\mathbb{Q}_p)$ is isomorphic to the group, denoted by $J(\mathbb{Q}_p)$, of all quasi-isogenies of the formal $\mathcal{O}_{D_p}$-module X_s. Now the main theorem of the theory of p-adic uniformization of Shimura varieties asserts:

Theorem 2.1 ([BZ95, 1.12], [RZ96, 6.51]). *We have an isomorphism of formal schemes*

$$I(\mathbb{Q}) \setminus \widehat{\Omega}^{\mathrm{ur}} \times G(\mathbb{A}_{\mathrm{f}})/C \xrightarrow{\sim} \widehat{\mathcal{A}}_C \otimes \mathcal{O}_{E_\nu}^{\mathrm{ur}}.$$

Furthermore, this isomorphism is equivariant with respect to the Galois action of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) \cong \mathrm{Gal}(E_\nu^{\mathrm{ur}}/E_\nu)$ in a suitable sense (see the references cited above for the precise meaning). The left hand side is a finite disjoint union of quotients of $\widehat{\Omega}^{\mathrm{ur}}$ by the groups

$$\Gamma_g = I(\mathbb{Q}) \cap (J(\mathbb{Q}_p) \times g^{-1}C^p g) \subset J(\mathbb{Q}_p)$$

with $g \in G(\mathbb{A}_{\mathrm{f}}^p)$, where $I(\mathbb{Q})$ is now considered as a subgroup of $I(\mathbb{A}_{\mathrm{f}}) \cong J(\mathbb{Q}_p) \times G(\mathbb{A}_{\mathrm{f}}^p)$.

F. *Calculation of the algebraic group I.* It is important to describe the algebraic group I more explicitly. We should do this in the case that the following two conditions are satisfied: zw

(f) For any finite place ℓ which does not divide p, D_ℓ is isomorphic to the matrix algebra $\mathrm{M}_d(K_\ell)$.

(g) For any finite prime ℓ which ramifies or is prime in K, there exists $c \in K_{\tilde{\ell}}$ with $\overline{c} = -c$ such that $\mathrm{Nm}_{D\otimes_K K_{\tilde{\ell}}/\mathbb{Q}_\ell}(cb)$ is in the image of $\mathrm{Nm}_{K_{\tilde{\ell}}/\mathbb{Q}_\ell}$, where $\widetilde{\ell}$ is the prime of K lying above ℓ.

Note that the condition (g) is always satisfied if d is odd; indeed, setting $f = \mathrm{Nm}_{D\otimes_K K_{\tilde{\ell}}/\mathbb{Q}_\ell}(cb)$ for some c, the norm of $f \cdot cb$ becomes a square number.

Let A_s be the abelian variety over κ fixed above. First we need to describe $\mathrm{End}_D^\circ A_s$ with the Rosatti involution $\varphi \mapsto \varphi'$. We claim:

Proposition 2.2. *There exists an isomorphism*

$$(\mathrm{End}_D^\circ A_s, \varphi \mapsto \varphi') \xrightarrow{\sim} (\mathrm{M}_d(K), \tau\colon M \mapsto {}^t\overline{M}).$$

Proof. Let R be a commutative algebra over $\mathbb{Q}$, and suppose we have an isomorphism $(\mathrm{End}_D^\circ A_s) \otimes R \cong \mathrm{M}_d(K) \otimes R$ of algebras over R. Let us denote by ρ the involution on $\mathrm{M}_d(K) \otimes R$ induced by the Rosatti involution on $\mathrm{End}_D^\circ A_s$. By Skolem-Noether theorem, there exists $U \in \mathrm{M}_d(K) \otimes R$ such that $M^\tau = UM^\rho U^{-1}$ for any $M \in \mathrm{M}_d(K) \otimes R$. If we require $U^\tau = U$, then U is uniquely determined up to constants in $R^\times$. Then $(\mathrm{M}_d(K) \otimes R, \rho)$ and $(\mathrm{M}_d(K) \otimes R, \tau)$ are isomorphic if and only if there exist $f \in R^\times$ and $W \in \mathrm{M}_d(K) \otimes R$ such that $fU = WW^\tau$. This observation shows that $\mathrm{Isom}((\mathrm{End}_D^\circ A_s, '), (\mathrm{M}_d(K), \tau))$ is a torsor over the $\mathbb{Q}$-group scheme T defined by

$$T(R) := \{W \in \mathrm{M}_d(K) \otimes R \mid WW^\tau \in R^\times\}$$

for $\mathbb{Q}$-algebra R. The group scheme T satisfies the Hasse principle, as is shown by an arguement similar to that in the end of D. So it suffices to find the desired isomorphism only locally at each place of $\mathbb{Q}$.

At a finite place $\ell \neq p$, by Tate's theorem and the level structure η_s, we have

$$(\mathrm{End}_D^\circ A_s) \otimes \mathbb{Q}_\ell \cong \mathrm{End}_{D\otimes\mathbb{Q}_\ell}(V \otimes \mathbb{Q}_\ell) \cong (D \otimes \mathbb{Q}_\ell)^{\mathrm{op}} \cong \mathrm{M}_d(K) \otimes \mathbb{Q}_\ell.$$

At the infinite place, we similarly have $(\mathrm{End}_D^\circ A_s) \otimes \mathbb{R} \cong \mathrm{M}_d(K) \otimes \mathbb{R}$ by the assumption (a). So we treat these two cases at the same time. First, let ℓ be a place of $\mathbb{Q}$ which decomposes in K as $\ell = \nu\overline{\nu}$. We look at the isomorphism $(\mathrm{End}_D^\circ A_s) \otimes \mathbb{Q}_\ell \cong \mathrm{M}_d(\mathbb{Q}_\nu) \times \mathrm{M}_d(\mathbb{Q}_\nu)$; the involution τ on the right hand side is now the switch of factors. By looking at the action of Rosatti involution on the rational Tate module, which is isomorphic to $V \otimes \mathbb{Q}_\ell$ by the level structure, we find

$$((\mathrm{End}^\circ_D A_s) \otimes \mathbb{Q}_\ell, \mathrm{Rosatti}) \cong (D^{\mathrm{op}}_\ell, \delta \mapsto b\delta^* b^{-1}).$$

Hence, the action of ρ on $\mathrm{M}_d(\mathbb{Q}_\nu) \times \mathrm{M}_d(\mathbb{Q}_\nu)$ is described by $\gamma \mapsto (b,b)\gamma^*(b^{-1}, b^{-1})$, where $*$ is now the switch. So in this case one can choose $U = (b^{-1}, b^{-1})$, and then, $W = (b^{-1}, 1)$ gives a solution. Also in the case that ℓ ramifies or remains prime in K, we can find f and W as above due to the assumption (g), for isomorphism classes of Hermitian forms of fixed rank over a local field are characterized by their determinant modulo the image of the norm mapping.

At p, we shall consider the Dieudonné isocrystal $(N, \mathbb{F})$ associated to A_s; we have $(\mathrm{End}^\circ_D A_s) \otimes \mathbb{Q}_p \cong \mathrm{End}_{D \otimes \mathbb{Q}_p^{\mathrm{ur}}}(N, \mathbb{F})$ (cf. [RZ96, 6.29]). Since the associated p-divisible group splits like $X_s \cong X_1 \times X_1{}^\vee$, N must also split as

$$N \cong V \otimes \mathbb{Q}_p^{\mathrm{ur}} \cong (V \otimes_K K_{\mathfrak{p}}^{\mathrm{ur}}) \oplus (V \otimes_K K_{\overline{\mathfrak{p}}}^{\mathrm{ur}}).$$

Due to [RZ96, 3.59, 3.60], the isocrystal $V \otimes_K K_{\mathfrak{p}}^{\mathrm{ur}}$, together with the action of $D_{\mathfrak{p}} \otimes_{K_{\mathfrak{p}}} K_{\mathfrak{p}}^{\mathrm{ur}}$, is characterized by an isocrystal which is isotypic of slope zero. So the endomorphism algebra of it is isomorphic to $\mathrm{M}_d(K_{\mathfrak{p}})$. We can do the same at $\overline{\mathfrak{p}}$, and thus we have

$$(\mathrm{End}^\circ_D A_s) \otimes \mathbb{Q}_p \cong \mathrm{End}_{D \otimes \mathbb{Q}_p^{\mathrm{ur}}}(N, \mathbb{F}) \cong \mathrm{M}_d(K) \otimes \mathbb{Q}_p \cong \mathrm{M}_d(\mathbb{Q}_p) \times \mathrm{M}_d(\mathbb{Q}_p).$$

In this case, the Rosatti involution is just the switch of factors, for the polarization on X_s is tautological, as we have seen before. So we find that in this case the solution W trivially exists, and hence the proposition has been proved. □

Corollary 2.3. *The algebraic group I is isomorphic to a group of unitary similitudes with respect to the positive definite Hermitian form; more specifically, we have a functorial isomorphism*

$$I(R) \cong \{g \in (\mathrm{M}_d(K) \otimes R)^\times \mid {}^t\overline{g}g \in R^\times\}$$

for any $\mathbb{Q}$-algebra R.

Note that, since the prime p splits on K, $I(\mathbb{Q}_p)$ is isomorphic to $\mathrm{GL}_d(\mathbb{Q}_p)$.

G. *Connected components.* Let us consider the torus T over $\mathbb{Q}$ defined by

$$T := \{(k, f) \in \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_{\mathrm{m},K} \times \mathbb{G}_{\mathrm{m},\mathbb{Q}} \mid k\overline{k} = f^d\},$$

and define an epimorphism $\vartheta: G \to T$ by sending $g \mapsto (\mathrm{Nm}_{D^{\mathrm{op}}/K}\, g, c(g))$, where $\mathrm{Nm}_{D^{\mathrm{op}}/K}$ is the reduced norm on D^{op}. The kernel of ϑ is the derived group G^{der} of G. By [De71, §2] we know that the morphism ϑ induces the canonical morphism of Shimura varieties

$$\mathcal{S}_C = G(\mathbb{Q}) \setminus X_\infty \times G(\mathbb{A}_{\mathrm{f}})/C \longrightarrow T(\mathbb{Q})_+ \setminus T(\mathbb{A}_{\mathrm{f}})/\vartheta(C)$$

which induces the Galois-equivariant bijection between the set of all connected components of both sides.

The description of the torus differs according to the parity of d; viz., those when d is even or odd (cf. [Ko92, §7]). When d is even, say $d = 2m$, then $(k, f) \mapsto (kf^{-m}, f)$ gives an isomorphism between T and $U_1 \times \mathbb{G}_{\mathrm{m},\mathbb{Q}}$, where U_1 is the kernel of the norm map from $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m},K}$ to $\mathbb{G}_{\mathrm{m},\mathbb{Q}}$. When d is odd, say $d = 2m+1$, then $(k,f) \mapsto kf^{-m}$ gives an isomorphism between T and $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m},K}$.

The Shimura variety $T(\mathbb{Q})_+ \setminus T(\mathbb{A}_\mathrm{f})/\vartheta(C)$ in case d is odd can be described fairly explicitly in terms of the theory of complex multiplication. More precisely, the Shimura variety $T(\mathbb{Q})_+ \setminus T(\mathbb{A}_\mathrm{f})/\vartheta(C)$ has the canonical model defined over the Shimura field $E = \varepsilon(K)$, and the Galois group $\mathrm{Gal}(F/E)$ acts simply transitively, where F is the ray class field of E corresponding to the level structure $\vartheta(C)$. Since the canonical morphism as above induces the morphism between the canonical models which is equivariant with the Galois action, we know that the number of connected components of $\mathcal{S}_C$ is equal to the order of $\mathrm{Gal}(F/E)$, and that each connected component of $\mathcal{S}_C$ has F as the field of definition.

3. The first example: Mumford's fake projective plane

In this and following sections we will present explicit examples of the p-adic uniformization of Shimura varieties discussed in the previous section with $d = 3$, $p = 2$. Each Shimura variety in our examples has connected components isomorphic to algebraic surfaces of a special kind, so-called, *fake projective planes*, i.e. compact complex surfaces of general type which have the same topological Betti numbers as $\mathbb{CP}^2$. The first example is related to the *Mumford's fake projective plane*, which was discovered by Mumford [Mu79].

Let us first fix the fundamental data as in A of §2: Let $\zeta = \zeta_7$ be a primitive 7th-root of unity, and set $L := \mathbb{Q}(\zeta)$; L is a cyclic extension of $\mathbb{Q}$ of degree 6 having an intermediate quadratic extension $K := \mathbb{Q}(\lambda)$ $(\cong \mathbb{Q}(\sqrt{-7}))$, where $\lambda = \zeta + \zeta^2 + \zeta^4$. The Galois group of L/K is generated by the Frobenius map $\sigma\colon \zeta \mapsto \zeta^2$, and that of $K/\mathbb{Q}$ is generated by the complex conjugation $z \mapsto \overline{z}$. Note that the prime 2 decomposes on K, more explicitly, $2 = \lambda\overline{\lambda}$ gives the prime factorization, and the prime 7 ramifies. We fix an infinite place (CM-type) $\varepsilon\colon K \hookrightarrow \mathbb{C}$ by $\lambda \mapsto \frac{-1+\sqrt{-7}}{2}$. Note that the class number of K is 1.

Construction of D. We set $\mu := \lambda/\overline{\lambda}$, and define the central division algebra D over K of dimension 9 by

$$D := \bigoplus_{i=0}^{2} L\Pi^i$$

with $\Pi^3 = \mu$ and $\Pi z = z^\sigma \Pi$ for $z \in L$; D satisfies the conditions (a), (b), and (f) in §2. The involution $*$ on D is defined by $z^* = \overline{z}$ for $z \in L$ and $\Pi^* = \overline{\mu}\Pi^2$. It is easily verified that $*$ is positive. As for the order, we set

$$\mathcal{O}_D := \mathcal{O}_L \oplus \mathcal{O}_L\overline{\lambda}\Pi \oplus \mathcal{O}_L\overline{\lambda}\Pi^2,$$

which is clealy stable under $*$. Moreover, one can easily verify that $\mathcal{O}_{D_\lambda}$ and $\mathcal{O}_{D_{\overline{\lambda}}}$ are maximal.

Finally, we introduce a non-degenerate anti-symmetric $\mathbb{Q}$-bilinear form $\psi = \psi_b$ on $V = D$ by putting $b = (\lambda - \overline{\lambda}) - \overline{\lambda}\Pi + \overline{\lambda}\Pi^2$. In order to check the conditions (c) and (d) it is convenient to have an explicit description of the isomorphism $\Phi\colon D_\varepsilon \xrightarrow{\sim} \mathrm{M}_3(\mathbb{C})$ of algebras with involutions. Let $\theta\colon L \hookrightarrow \mathbb{C}$ be given by $\zeta \mapsto \exp\frac{2i\pi}{7}$, and $\theta_i := \theta \circ \sigma^i$ for $i = 0, 1, 2$. Then we have

$$V \otimes_{K,\varepsilon} \mathbb{C} \xrightarrow{\sim} \bigoplus_{i=0}^{2} V \otimes_{L,\theta_i} \mathbb{C}.$$

We look at the action of D on the factor $V\otimes_{L,\theta_0}\mathbb{C}$; subject to the base $\{1\otimes 1, \Pi\otimes 1, \Pi^2\otimes 1\}$ one obtains a matrix presentation of elements in D determined by

$$z\mapsto\begin{bmatrix} z & & \\ & z^{\sigma^2} & \\ & & z^{\sigma}\end{bmatrix} \quad \text{and} \quad \Pi\mapsto\begin{bmatrix} & & \mu \\ 1 & & \\ & 1 & \end{bmatrix},$$

which induces the desired isomorphism Φ. (Here we dropped θ_0 and ε for simplicity.) Then we have

$$\Phi(b)=\begin{bmatrix} \lambda-\overline{\lambda} & \lambda & -\lambda \\ -\overline{\lambda} & \lambda-\overline{\lambda} & \lambda \\ \overline{\lambda} & -\overline{\lambda} & \lambda-\overline{\lambda}\end{bmatrix}.$$

(Here $\varepsilon(\lambda)$ is abbreviated simply to λ.) The characteristic polynomial is $t^3-3\sqrt{-7}t^2-15t-\sqrt{-7}$. In particular, the condition (d) is satisfied; as $\mathrm{Nm}_{D/K}\, b=2\lambda+1$ is both λ and $\overline{\lambda}$-adic unit, we also verify (c).

Level structure. Let us first notice that, in the isomorphy in Corollary 2.3, the involution $g\mapsto {}^t\overline{g}$ on the right hand side can be replaced by the one defined by $g\mapsto g' := H^{-1}\,{}^t\overline{g}H$, where $H := \Phi(h)$ with $h=3+\overline{\lambda}\Pi+\overline{\lambda}\Pi^2\in D$; viz.

$$H=\begin{bmatrix} 3 & \overline{\lambda} & \overline{\lambda} \\ \lambda & 3 & \overline{\lambda} \\ \lambda & \lambda & 3\end{bmatrix};$$

this is allowed since there exists an decomposition $H={}^t\overline{W}W$ over $\mathbb{Q}$ according as $3+\overline{\lambda}\Pi+\overline{\lambda}\Pi^2=(\lambda+\Pi)^*(\lambda+\Pi)$.

To define the level structure, it suffices to define the prime-2-part C^2. But we need to have a non-trivial component only at 7; viz. we define C^2 in the form $C^2=C^{2,7}C_7$ with $C_7\in G(\mathbb{Q}_7)$ and $C^{2,7}$ being the maximal one:

$$C^{2,7} := \{g\in G(\mathbb{A}_{\mathrm{f}}^{2,7}) \mid \Gamma^{2,7}g\subseteq\Gamma^{2,7}\},$$

with $\Gamma^{2,7}=\Gamma\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}^{2,7}$, where $\widehat{\mathbb{Z}}^{2,7}=\lim\mathbb{Z}/m\mathbb{Z}$ with the projective limit taken over m such that $(2,m)=(7,m)=1$.

Now the component C_7 is defined as follows: By Corollary 2.3, the fact stated below Theorem 2.1, and what we have just done above, we have an isomorphism

$$G(\mathbb{Q}_7)\xrightarrow{\sim}\{g\in\mathrm{GL}_3(\widetilde{\mathbb{Q}}_7) \mid g^*Hg=c(g)H,\ c(g)\in\mathbb{Q}_7^\times\},$$

where $\widetilde{\mathbb{Q}}_7$ is the ramified quadratic extension of $\mathbb{Q}_7$, and $*$ is the matrix transposition followed by the Galois action. Let $\widetilde{C}_7$ be the maximal open compact subgroup of $G(\mathbb{Q}_7)$ consisiting of g which maps $\Gamma\otimes_{\mathbb{Z}}\mathbb{Z}_7$ to itself. We consider modulo $\sqrt{-7}$ reduction of $\widetilde{C}_7$, which is a subgroup G_0 in $\mathrm{GL}_3(\mathbb{F}_7)$. The matrix H mod $\sqrt{-7}$ is of rank 1, and has 2 dimensional null space N_0. Restricting elements in G_0 to N_0, we obtain a homomorphism

$$\pi\colon\widetilde{C}_7\longrightarrow\mathrm{GL}_2(\mathbb{F}_7).$$

The last group has $2^6 3^2 7$ elements. Let S be a 2-Sylow subgroup of $\mathrm{GL}_2(\mathbb{F}_7)\cap\{g \mid \det g=\pm 1\}$. Now we set $C_7 := \pi^{-1}(S)$, and thus, the definition of the level structure is finished.

Theorem 3.1. *The canonical model Sh_C of the Shimura variety obtained by the data as above has three geometrically connected non-singular projective surface over $\mathbb{Q}(\zeta_7)$ such that the base change over $\mathbb{Q}_2(\zeta_7)$ are isomorphic to the Mumford's fake projective plane* [Mu79].

Corollary 3.2. *On the complex analytic side, the Mumford's fake projective plane is an arithmetic quotient of the 2-dimensional complex unit-ball having the minimal field of definition $E = \mathbb{Q}(\zeta_7)$.*

More explicitly, indeed, it is the quotient of the 2-dimensional unit-ball by the arithmetic group $\Gamma_\infty := G(\mathbb{Q}) \cap C \subset G(\mathbb{R}) \cong \mathrm{GU}(2,1)$.

Proof of Theorem 3.1. First, we notice that we constructed the level structure $C = C_2C^2$ precisely so that the associated discrete group $\Gamma_1 = I(\mathbb{Q}) \cap (J(\mathbb{Q}_2) \times C^2)$ modulo the center of $J(\mathbb{Q}_2) \cong I(\mathbb{Q}_2) \cong \mathrm{GL}_3(\mathbb{Q}_2)$ coincides with the one constructed by Mumford in [Mu79]. Indeed, Mumford's Γ was a subgroup of the unitary group with respect to H characterized by the same congruence condition as above. Also by [RZ96, 6.51] the isomorphism in Theorem 2.1 descends to $\mathbb{Q}_2 \cong E_\nu$. So it suffice to see that every element g in $I(\mathbb{Q})$ can be written as $g = uk$ with $u \in I(\mathbb{Q})$ and $k \in \mathbb{Q}^\times$ such that ${}^t\overline{u}Hu = H$. But this follows from an argument similar to that we have remarked at the end of B of §2. Hence we deduce that one, and therefore, every connected component of the associated Shimura variety $\mathcal{S}_C$ is isomorphic to Mumford's fake projective plane.

Next we look at the morphism $\vartheta : G \to T$ as in G of §2 in order to observe connected components. To calculate $\vartheta(C)$ we have only to concern with the component at 7. We claim the following: Under the canonical identification $T(\mathbb{Q}_7) \cong \widetilde{\mathbb{Q}}_7^\times$, where $\widetilde{\mathbb{Q}}_7^\times$ is the ramified quadratic extension of $\mathbb{Q}_7$, we have $\vartheta(C_7) \cong \pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times$, where $\widetilde{\mathbb{Z}}_7^\times$ is the integer ring of $\widetilde{\mathbb{Q}}_7^\times$. First we note that the map ϑ is described by

$$\vartheta : C_7 \ni \gamma \mapsto ({}^t\overline{\gamma}H\gamma H^{-1})^{-1} \det \gamma \in \widetilde{\mathbb{Z}}_7^\times.$$

Since $\pm 1 + 7\,\mathbb{Z}_7^\times \subset C_7$ we immediately deduce that $\vartheta(C_7)$ contains $\pm 1 + 7\,\mathbb{Z}_7^\times$. Moreover, since $\pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times \subset C_7$, $\vartheta(C_7)$ contains the element $k^2\overline{k}^{-1}$ for any $k \in \pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times \subset C_7$; but, since $k\overline{k} \in \vartheta(C_7)$, we deduce that

$$(\pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times)^3 = \pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times \subseteq \vartheta(C_7).$$

Hence the claim follows if we check that $(\vartheta(C_7) \bmod \sqrt{-7}) = \{\pm 1\}$. As we have seen above any $\gamma \in C_7$ has a decomposition $\gamma = k\theta$ in C_7 with ${}^t\overline{\theta}H\theta H^{-1} = 1$. Since $\det {}^t\overline{\theta} \det H\theta H^{-1} \equiv (\det\theta)^2 \equiv 1 \bmod \sqrt{-7}$ we have $\vartheta(\gamma) \equiv \pm k$. But, since $k = \gamma\theta^{-1} \bmod \sqrt{-7}$ belongs to P, the order of k is either one or two; hence $k \equiv \pm 1 \bmod \sqrt{-7}$. We therefore get $\vartheta(\gamma) \equiv \pm 1 \bmod \sqrt{-7}$, which proves the claim.

Now in view of the theory of complex multiplication we find that the Shimura variety $T(\mathbb{Q})_+ \backslash T(\mathbb{A}_\mathrm{f})/\vartheta(C)$ has three connected components having field of definition $L = \mathbb{Q}(\zeta)$ which are permuted by $\mathrm{Gal}(L/K)$, since the ray class field corresponding to C is easily checked to be L. We therefore proved the all statements in the theorem. □

4. The second and third examples

The second and third examples contain fake projective planes as connected components, which are the ones constructued in [IK98].

In this section we work over the 45th-cyclotomic field $\mathbb{Q}(\zeta_{45})$; inside it are the subfields $\mathbb{Q}(\zeta_9)$ and $\mathbb{Q}(\zeta_{15})$. We set $\lambda := \zeta_{15} + \zeta_{15}^2 + \zeta_{15}^4 + \zeta_{15}^8$, of which the minimal polynomial is given by $x^2 - x + 4$. We set $K := \mathbb{Q}(\lambda)$ ($\cong \mathbb{Q}(\sqrt{-15})$); the class number of K is 2. The prime 2 decomposes on K; more explicitly, $2\mathcal{O}_K = \Lambda\overline{\Lambda}$ gives the prime factoization, where $\Lambda = 2\mathbb{Z} + (\lambda+1)\mathbb{Z}$. The primes 3 and 5 ramify on K; viz. $3\mathcal{O}_K = \mathfrak{P}_3^2$ with $\mathfrak{P}_3 = 3\mathbb{Z} + (\lambda+1)\mathbb{Z}$, and $5\mathcal{O}_K = \mathfrak{P}_5^2$ with $\mathfrak{P}_5 = 5\mathbb{Z} + (\lambda+2)\mathbb{Z}$. A CM-type embedding $\varepsilon: K \hookrightarrow \mathbb{C}$ is fixed by $\lambda \mapsto \frac{1+\sqrt{-15}}{2}$. We define L to be the composite field of K and the maximal real subfield of $\mathbb{Q}(\zeta_9)$; L is generated over K by $\zeta_9 + \zeta_9^{-1}$ of which the minimal polynomial over K is $x^3 - 3x + 1$. The Galois group of L/K is generated by the Frobenius $\sigma: \zeta_9 + \zeta_9^{-1} \mapsto \zeta_9^2 + \zeta_9^{-2}$, while that of $K/\mathbb{Q}$ is generated by the complex conjugation.

Putting $\mu := \overline{\lambda}/\lambda$, we can define D and $*$ by the same recipe as in §3; the conditions (a), (b), and (f) can be easily verified. The order $\mathcal{O}_D$ should be defined by

$$\mathcal{O}_D := \mathcal{O}_L \oplus \mathcal{O}_L\lambda\Pi \oplus \mathcal{O}_L\lambda\Pi^2.$$

The $\mathbb{Q}$-bilinear form $\psi = \psi_b$ on $V = D$ is given by the element $b = (\lambda - \overline{\lambda}) - \lambda\Pi + \lambda\Pi^2$, which guarantees the conditions (c) and (d). Indeed, one expresses elements in D by matrices in a similar way as in §3.

Level structures. In order to define level structures, we replace the involution $g \mapsto {}^t\overline{g}$ on the right hand side of the isomorphy in Corollary 2.3 by $g \mapsto g' := H^{-1}\,{}^t\overline{g}H$ by

$$H = \begin{bmatrix} 10 & 2\lambda - 6 & -\lambda + 3 \\ 2\overline{\lambda} - 6 & 10 & 2\lambda - 6 \\ -\overline{\lambda} + 3 & 2\overline{\lambda} - 6 & 10 \end{bmatrix}.$$

This replacement is possible since $H = {}^t\overline{A}A/3$ with

$$A = \begin{bmatrix} 0 & 0 & 2\lambda - 1 \\ \lambda - 2 & -2\lambda - 2 & 3 \\ -2\lambda + 4 & \lambda - 2 & -\lambda + 2 \end{bmatrix}.$$

We define two distinct level structures ${}^{\mathrm{I}}C$ and ${}^{\mathrm{II}}C$ which define two Shimura varieties. The both are expressed in the form $C_2C^{2,3}C_3$, with C_2 and $C^{2,3}$ being maximal, given by a formula similar to those in §3. the components at 3 are defined in terms of congruence conditions modulo $3\mathcal{O}_K = \mathfrak{P}_3^2$. Put $t := 2\lambda - 1 \bmod 3$. Then the modulo 3 reduction of the maximal open subgroup $\widetilde{C}_3$ of $I(\mathbb{Q}_3) \cong G(\mathbb{Q}_3)$ is a subgroup of $\mathrm{GL}_3(\mathbb{F}_3[t]/(t^2))$. We define its subgroups ${}^{\mathrm{I}}U$ and ${}^{\mathrm{II}}U$: ${}^{\mathrm{I}}U$ is generated by the invertible scalar matrices and the following seven matrices:

$${}^{\mathrm{I}}h_0 = \begin{bmatrix} t+2 & 2t+1 & 2t \\ t+2 & 0 & 2t \\ 0 & 0 & t+1 \end{bmatrix}, \quad {}^{\mathrm{I}}h_1 = \begin{bmatrix} 2 & 0 & 1 \\ 2t+2 & t+1 & 2t+1 \\ 0 & 0 & 2t+2 \end{bmatrix},$$

$${}^{\mathrm{I}}h_2 = \begin{bmatrix} 0 & t+1 & 2 \\ 1 & t & 2t \\ 0 & 0 & 2t+2 \end{bmatrix}, \quad {}^{\mathrm{I}}h_3 = \begin{bmatrix} 0 & 0 & 2t+1 \\ 1 & 0 & 1 \\ 0 & 1 & t+2 \end{bmatrix},$$

$${}^{\mathrm{I}}h_4 = \begin{bmatrix} t+1 & 2 & 0 \\ 0 & 2 & 2t+2 \\ 0 & 0 & 2t+2 \end{bmatrix}, \quad {}^{\mathrm{I}}h_5 = \begin{bmatrix} 2 & 0 & 0 \\ 2t+1 & t+2 & 0 \\ 1 & 2 & 1 \end{bmatrix},$$

$$ {}^{\mathrm{I}}h_6 = \begin{bmatrix} t & 2 & 0 \\ t+2 & 2 & 2t+1 \\ 2 & 0 & 0 \end{bmatrix}. $$

${}^{\mathrm{II}}U$ is generated by the invertible scalar matrices and the following seven matrices:

$$ {}^{\mathrm{II}}h_0 = \begin{bmatrix} t+1 & t+1 & 2t \\ 0 & 1 & t \\ 0 & 0 & t+1 \end{bmatrix}, \quad {}^{\mathrm{II}}h_1 = \begin{bmatrix} t+2 & t+2 & 2t+1 \\ t & 2 & 2 \\ t & 0 & 1 \end{bmatrix}, $$

$$ {}^{\mathrm{II}}h_2 = \begin{bmatrix} 2 & 2t+2 & t+2 \\ t & t+2 & 2 \\ t & t & 1 \end{bmatrix}, \quad {}^{\mathrm{II}}h_3 = \begin{bmatrix} 0 & 2t+2 & t+1 \\ 2t+1 & 1 & 2 \\ 0 & 0 & 2t+1 \end{bmatrix}, $$

$$ {}^{\mathrm{II}}h_4 = \begin{bmatrix} 2t+2 & t+2 & 0 \\ t & 2t+2 & 2 \\ t & 2t & 1 \end{bmatrix}, \quad {}^{\mathrm{II}}h_5 = \begin{bmatrix} t+2 & 0 & 0 \\ 1 & 2t+2 & 0 \\ t & t & 2t+1 \end{bmatrix}, $$

$$ {}^{\mathrm{II}}h_6 = \begin{bmatrix} 1 & t+2 & 0 \\ t+1 & t & t+1 \\ 2t & t & t+1 \end{bmatrix}. $$

It is easily verified that the groups ${}^{\mathrm{I}}U$ and ${}^{\mathrm{II}}U$ are contained in the image of $\widetilde{C}_3$ under the reduction map. Now we define ${}^{\mathrm{I}}C_3$ and ${}^{\mathrm{II}}C_3$ to be the inverse image of ${}^{\mathrm{I}}U$ and ${}^{\mathrm{II}}U$ under the reduction map, respectively. The level structures ${}^{\mathrm{I}}C$ and ${}^{\mathrm{II}}C$ are defined to be $C_2C^{2,3}\,{}^{\mathrm{I}}C_3$ and $C_2C^{2,3}\,{}^{\mathrm{II}}C_3$, respectively.

We denote the associated Shimura varieties by the above data with the level structure ${}^{\mathrm{I}}C$ (resp. ${}^{\mathrm{II}}C$) by $\mathcal{S}_{{}^{\mathrm{I}}C}$ (resp. $\mathcal{S}_{{}^{\mathrm{II}}C}$). We claim:

Theorem 4.1. *The canonical model* $\mathrm{Sh}_{{}^{\mathrm{I}}C}$ *(resp.* $\mathrm{Sh}_{{}^{\mathrm{II}}C}$*) of the Shimura variety* $\mathcal{S}_{{}^{\mathrm{I}}C}$ *(resp.* $\mathcal{S}_{{}^{\mathrm{II}}C}$*) consists of two connected components, each of which is a geometrically connected non-singular projective surface over* $E' = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$*, such that the base change of each connected component at the prime* Λ *is isomorphic over* $E'_\Lambda \cong \mathbb{Q}_{2^2}$ *to the base change of the surface* $X_{\Gamma_{\mathrm{B}_1},\eta}$ *(resp.* $X_{\Gamma_{\mathrm{B}_2},\eta}$*) obtained in* [IK98].

Corollary 4.2. *The two complex fake projective planes considered in* [IK98]*, other than the Mumford's one, are arithmetic quotients of the* 2*-dimensional complex unit-ball having the minimal field of definition* $E' = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$*.*

In order to prove the theorem, first we show that the discrete group

$$ {}^{\mathrm{I}}\Gamma = I(\mathbb{Q}) \cap (J(\mathbb{Q}_2) \times {}^{\mathrm{I}}C^2) \quad \text{and} \quad {}^{\mathrm{II}}\Gamma = I(\mathbb{Q}) \cap (J(\mathbb{Q}_2) \times {}^{\mathrm{II}}C^2) $$

coincide modulo center with those discovered by Cartwright, Mantero, Steger, and Zappa in [CMSZ93a] and [CMSZ93b]. More precisely, $\Gamma_{\mathrm{B}.1}$ and $\Gamma_{\mathrm{B}.2}$, respectively, in their notation (see [CMSZ93b, §5]). These groups act on vertices of Bruhat-Tits building related to the Drinfeld Symmetric space Ω simply transitively, and, as was shown in [IK98], they are torsion-free. So, by Mumford's criterion [Mu79], one deduces that the quotients of Ω by these groups are fake projective planes. Moreover, it was proved in [IK98] that these fake projective planes, including the Mumford's one, are pairwise non-isomorphic.

Proposition 4.3. *We have equalities* ${}^{\mathrm{I}}\Gamma = \Gamma_{\mathrm{B}.1}$ *and* ${}^{\mathrm{II}}\Gamma = \Gamma_{\mathrm{B}.2}$ *modulo scalar matrices.*

Proof. The inclusions $\Gamma_{\mathrm{B}.1} \subseteq {}^{\mathrm{I}}\Gamma$ and $\Gamma_{\mathrm{B}.2} \subseteq {}^{\mathrm{II}}\Gamma$ can be seen without so much pain; actually, the matrices ${}^{\mathrm{I}}h_i$ and ${}^{\mathrm{II}}h_i$ are precisely the modulo 3 reductions of those g_i of the corresponding groups described in [CMSZ93b, pp. 182]. In order to show the other inclusions, we should calculate the stabilizer of one vertex of the Bruhat-Tits building. Let $[M_0]$ be a vertex, i.e. the similarity class of a lattice, where M_0 is the standard lattice $\mathbb{Z}_2^3$. Elements in ${}^{\mathrm{I}}\Gamma$ or ${}^{\mathrm{II}}\Gamma$ has entries belonging to $\mathbb{Z}[\lambda, \frac{1}{2}]$. Let γ be such a matrix in $I(\mathbb{Q})$, and assume it fixes $[M_0]$. We can assume that γ is H-unitary due to the similar argument as in the end of B of §2; taking inverse if necessary, we may moreover assume $\gamma M_0 \subseteq M_0$. Then γ has entries in $\mathbb{Z}[\lambda, \frac{1}{2}] \cap \mathbb{Z}_2 = \mathbb{Z}[\lambda, \frac{1}{\bar{\lambda}}]$. Since γ is H-unitary, we moreover deduce that they are actually belonging to $\mathbb{Z}[\lambda, \frac{1}{\bar{\lambda}}] \cap \frac{1}{2\cdot 3\cdot 5}\mathbb{Z}[\frac{1}{\lambda}] \subseteq \mathbb{Z}[\frac{\lambda}{2}]$, where 2, 3, and 5 are the primes which appear in denominators of the entries of H^{-1}.

Lemma 4.4. *The matrix γ must be of form $\pm S^i$ $(i = 0, 1, 2)$, where*

$$S = \begin{bmatrix} 0 & -1 & 1-\lambda \\ 1 & -1 & 3-\lambda \\ 0 & 0 & 1 \end{bmatrix}.$$

(*Note that S is of order* 3.)

Proof of Lemma 4.4. Since γ is H-unitary, each column of γ has H-norm 10. There exist only finitely many such vectors having entries in $\mathbb{Z}[\frac{\lambda}{2}]$. So if we have a complete list of such vectors, then all the possible γ can be determined. This procedure is elementary, but it would be too painful to perform by hand. The author carried it out by computer search collaborated with MAPLE V. Actually there are precisely 42 solutions for the vectors of norm 10. By checking the H-unitarity of all combinations of such vectors, the author got precisely 6 matrices as in the lemma. Note that the matrix S is precisely the one which appeared in [CMSZ93b, pp. 182] (denoted by s there). □

We continue the proof of Proposition 4.3. By the above observations, we only need to show that the similarity class of S does not belong to each of ${}^{\mathrm{I}}\Gamma$ and ${}^{\mathrm{II}}\Gamma$ modulo scalar matrices. Here we prove the proposition for ${}^{\mathrm{II}}\Gamma$, but exactly the same argument runs for the other. In the following we drop the superscript ${}^{\mathrm{II}}-$.

Since the modulo $\mathfrak{P}_3$ reduction of the Hermitian form H is of rank 1, modulo $\mathfrak{P}_3$ reduction of every element of Γ leaves invariant the 2-dimensional null space of H mod $\mathfrak{P}_3$. More explicitly, looking at

$$H \bmod \mathfrak{P}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

one considers the upper left 2×2-minor of each $P^{-1}\gamma P$ mod $\mathfrak{P}_3$ for $\gamma \in \Gamma$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

This gives a mapping

$$\rho \colon \Gamma \longrightarrow \mathrm{GL}_2(\mathbb{F}_3).$$

By definition of Γ, the image of ρ is generated by the upper left 2×2-minor of $P^{-1}h_iP$ for $i = 0, \ldots, 6$. Let us denote the resulting matrices by $\overline{h}_i$ for $i = 0, \ldots, 6$. Then one calculates

$$\overline{h}_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \overline{h}_1 = \overline{h}_2 = \overline{h}_4 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad \overline{h}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\overline{h}_5 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad \overline{h}_6 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

By this, it is easy to see that the image of ρ is exactly $\mathrm{GL}_2(\mathbb{F}_3)$. Let $\overline{\rho}\colon U \to \mathrm{GL}_2(\mathbb{F}_3)$ be the reduction map which factors ρ, i.e. $\overline{\rho}(h_i) = \overline{h}_i$. By observing the fundamental relations among $\overline{h}_i$'s, one easily deduce that $\operatorname{Ker}\overline{\rho}$ is generated by scalar matrices with determinant 1 and some matrices of form $h_{i_1}\cdots h_{i_N}$ with N being multiples of 3. In particular, one sees that every element in $\operatorname{Ker}\overline{\rho}$ has determinant 1, for every h_i has determinant $2t+1$.

Now assume that a scalar times S were belonging to Γ; then U contains the matrix

$$s = \begin{bmatrix} 2t+1 & 1 & t+1 \\ 0 & t+1 & t \\ 0 & 2t & 1 \end{bmatrix},$$

the modulo 3 reduction of S. Since $\overline{\rho}(s) = \overline{\rho}(h_0)$, we know $s \in h_0 \cdot \operatorname{Ker}\overline{\rho}$; but this violates with the fact $\det s = 1$. Thus we conclude the proof of the proposition. □

Proof of Theorem 4.1. It remains only to prove the statements on connected components and the fields of definition. As in the first example, we shall look at the morphism $\overline{\vartheta}\colon {}^{\mathrm{I}}C_3, {}^{\mathrm{II}}C_3 \to (\mathbb{F}_3[t]/(t^2))^\times$ given by $uk \mapsto (\det u)k\overline{k}$, where the conjugation $\bar{\cdot}$ is now given by $t \mapsto -t$. The $\overline{\vartheta}$ factors through ${}^{\mathrm{I}}U$ or ${}^{\mathrm{II}}U$, and in both cases, one easily sees that $\overline{\vartheta}$ is surjective. So, similarly as in the first example, our the level structures $\vartheta({}^{\mathrm{I}}C)$ and $\vartheta({}^{\mathrm{II}}C)$ of the torus T are the maximal one. So each of the Shimura varieties $\mathcal{S}_{{}^{\mathrm{I}}C}$ and $\mathcal{S}_{{}^{\mathrm{II}}C}$ has exactly two connected components defined over the absolute class field $E' \cong \mathbb{Q}(\sqrt{-3}, \sqrt{5})$ of K. Since $E'_\Lambda \cong \mathbb{Q}_{2^2}$ (the unramified quadratic extension of $\mathbb{Q}_2$), and the 2-adic uniformizations by $\Gamma_{\mathrm{B}.1}$ and $\Gamma_{\mathrm{B}.2}$ were carried out over $\mathbb{Q}_2$, the isomorphism in Theorem 2.1 descends onto $\mathbb{Q}_{2^2}$. □

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