QUADRATIC FUNCTORS AND METASTABLE HOMOTOPY I
(HOMOLOGICAL ALGEBRA OF QUADRATIC FUNCTORS)

by

Hans–Joachim Baues

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

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For the sixtieth birthday of Keith Hardie

Abstract: Quadratic functors lead naturally to the notion of a "quadratic module" M for which tensor products $A \otimes M$ and Hom–groups Hom(A,M) are defined. Moreover there are derived functors of $\otimes$ and Hom which generalize the classical Tor and Ext groups respectively. For example the functors Ω and R of Eilenberg-Mac Lane turn out to be such derived functors.
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Classical homological algebra is concerned with modules, additive functors, like the tensor product and the Hom functor, and their derived functors, like Tor and Ext.
In this paper we describe and exploit a "quadratic extension" of homological algebra. Indeed quadratic functors lead to the notion of a quadratic module M and there is canonically a quadratic tensor product $A \otimes M$ and a quadratic Hom–group Hom(A,M). The elements of Hom(A,M) are quadratic forms on the module A with values in M. We introduce and study quadratic derived functors which in particular yield the groups $\mathrm{Tor}_n(A,M)$ and $\mathrm{Ext}^n(A,M)$ respectively. These groups are embedded in long exact sequences as in the classical case, see §9. The functors given by $A \otimes M$, Hom(A,M) and the derived functors Tor_n, Ext^n are quadratic in A and additive in M.

Of special interest are quadratic $\mathbb{Z}$–modules M which are the quadratic analogues of abelian groups. They appear frequently in the meta stable range of algebraic topology; for example homotopy groups of spheres and the (co) homology of Eilenberg-

Mac Lane spaces yield in a natural way quadratic $\mathbb{Z}$–modules, see [4]. A quadratic $\mathbb{Z}$–module M is just a pair of homomorphisms

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

between abelian groups satisfying $HPH = 2H$, $PHP = 2P$. In case M_e, M_{ee} are homotopy groups of spheres these are the operators of the EHP–sequence where H is the Hopf invariant and where P is induced by the Whitehead product, see [4], [11], [16]. For example

$$\mathbb{Z}^\Gamma = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}),\ \mathbb{Z}^S = (\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}),\text{ and } \mathbb{Z}^\Lambda = (0 \longrightarrow \mathbb{Z} \longrightarrow 0)$$

are quadratic $\mathbb{Z}$–modules. They lead to the following quadratic tensor products and torsion products respectively which turned out to be well known quadratic functors from abelian groups A to abelian groups. In fact, there are natural isomorphisms

$A \otimes \mathbb{Z}^S \cong SP^2(A)$ (symmetric square)

$A \otimes \mathbb{Z}^\Lambda \cong \Lambda^2(A)$ (exterior square)

$A \otimes \mathbb{Z}^\Gamma \cong \Gamma(A)$ (Whitehead's functor)

$A *' \mathbb{Z}^\Gamma \cong R(A)$ and $A \otimes' \mathbb{Z}^\Gamma \cong \Omega(A)$.

Here R and Ω are the functors of Eilenberg–Mac Lane [9] with $R(A) = H_5K(A,2)$ and $\Omega(A) \oplus A \otimes \mathbb{Z}/3 = H_7K(A,3)$. These examples illustrate that simple algebraic data like $\mathbb{Z}^\Gamma, \mathbb{Z}^S$ or $\mathbb{Z}^\Lambda$ above are the crucial ingredients of fairly intricate quadratic functors. Further examples of this kind are given in the paper. In [4] we describe exact sequences for homotopy groups of Moore spaces and (co)–homology groups of Eilenberg–Mac Lane spaces in terms of quadratic derived functors. These results made it necessary to develop a quadratic extension of homological algebra as considered in this paper.

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§1 *Modules*

We here fix some basic notations on categories, ringoids, rings and modules respectively, compare also [14]. In particular, we describe the tensor product for modules and ringoids. A bold face letter like $\underline{C}$ denotes a category, $Ob(\underline{C})$ and $Mor(\underline{C})$ are the classes of objects and morphisms respectively. We identify an object A with its identity $1_A = 1 = A$ so that $Ob(\underline{C}) \subset Mor(\underline{C})$. We also write $f \in \underline{C}$ if f is a morphism or an object in $\underline{C}$. The set of morphisms $A \longrightarrow B$ is $\underline{C}(A,B)$. Surjective maps and injective maps are indicated by arrows $\twoheadrightarrow$ and $\rightarrowtail$ respectively.

A <u>ringoid</u> $\underline{R}$ is a category for which all morphism sets are abelian groups and for which composition is bilinear, (equivalently a ringoid is a category enriched over the monoidal category of abelian groups). A ringoid is also called a 'pre additive category', or an $\underline{Ab}$–category, see [13]. We prefer the notion 'ringoid' since in this paper a ringoid will play the role of a ring. In fact, a ringoid $\underline{R}$ with a single object e will be identified with the ring R given by the morphism set $R = \underline{R}(e,e)$. Recall that a <u>biproduct</u> (or a direct sum) in a ringoid $\underline{R}$ is a diagram

(1.1) $$X \underset{r_1}{\overset{i_1}{\rightleftarrows}} X \vee Y \underset{r_2}{\overset{i_2}{\leftrightarrows}} Y$$

which satisfies $r_1 i_1 = 1$, $r_2 i_2 = 1$ and $i_1 r_1 + i_2 r_2 = 1$. Sums and products in a ringoid are as well biproducts, see [13]. An <u>additive category</u> is a ringoid in which biproducts exist. Clearly the category $\underline{Ab}$ of abelian groups is an additive category with biproducts denoted by $X \oplus Y$. A functor $F : \underline{R} \longrightarrow \underline{S}$ between ringoids is <u>additive</u> if

(1.2) $$F(f+g) = F(f) + F(g)$$

for morphisms $f,g \in \underline{R}(X,Y)$. Moreover, we say that F is <u>quadratic</u> if Δ, with

(1.3) $$\Delta(f,g) = F(f+g) - F(f) - F(g),$$

is a bilinear function. A module with coefficients in a ringoid $\underline{R}$ or equivalently an $\underline{R}$–<u>module</u> is an additive functor

(1.4) $$M : \underline{R} \longrightarrow \underline{Ab}\,.$$

In case $\underline{R}$ has only one object e we identify M with the $\underline{R}(e,e)$–module $M(e)$, which is a module over a ring in the usual sense, compare also [14]. An $\underline{R}$–module is also called a <u>left</u> $\underline{R}$–module. A <u>right</u> $\underline{R}$–module N is a contravariant additive functor $N : \underline{R} \longrightarrow \underline{Ab}$. For $f \in \underline{R}(X,Y)$ we use the notation

$$(1.5)\qquad \begin{cases} M(f)(x) = f_*(x) = f\cdot x & x\in M(X),\\ N(f)(y) = f^*(y) = y\cdot f & y\in N(Y).\end{cases}$$

A right $\underline{\underline{R}}$–module is the same as an $\underline{\underline{R}}^{op}$–module where $\underline{\underline{R}}^{op}$ is the opposite category which is a ringoid. In case $\underline{\underline{R}}$ is small (that is, if the class of objects in $\underline{\underline{R}}$ is a set) let $\underline{\underline{M}}(\underline{\underline{R}})$ be the category of $\underline{\underline{R}}$–modules. Morphisms in $\underline{\underline{M}}(\underline{\underline{R}})$ are natural transformations. The category $\underline{\underline{M}}(\underline{\underline{R}})$ is an abelian category; as an example one has $\underline{\underline{M}}(\mathbb{Z}) = \underline{\underline{Ab}}$. We now recall the definition of tensor products of modules.

(1.6) Definition: Let $\underline{\underline{R}}$ be a small ringoid, let A be an $\underline{\underline{R}}^{op}$–module and let B be an $\underline{\underline{R}}$–module. The tensor product $A \otimes_{\underline{\underline{R}}} B$ is the abelian group generated by the elements $a\otimes b$, $a \in A(X)$, $b \in B(X)$ where X is any object in $\underline{\underline{R}}$. The relations are

$$\begin{cases} (a+a')\otimes b = a\otimes b + a'\otimes b\\ a\otimes(b+b') = a\otimes b + a\otimes b'\\ (a''\cdot\varphi)\otimes b = a''\otimes(\varphi\cdot b)\end{cases}$$

for $a,a' \in A(X)$, $b,b' \in B(X)$, $\varphi: X \longrightarrow Y \in \underline{\underline{R}}$, $a'' \in A(Y)$. For maps $f: A \longrightarrow A' \in \underline{\underline{M}}(\underline{\underline{R}})$ and $g: B \longrightarrow B' \in \underline{\underline{M}}(\underline{\underline{R}}^{op})$ we have the induced homomorphisms $f\otimes g: A\otimes_{\underline{\underline{R}}}B \longrightarrow A'\otimes_{\underline{\underline{R}}}B'$ by $(f\otimes g)(a\otimes b) = (fa)\otimes(gb)$. Whence the tensor product is a biadditive functor $\otimes_{\underline{\underline{R}}}: \underline{\underline{M}}(\underline{\underline{R}}^{op}) \times \underline{\underline{M}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$.

If $\underline{\underline{R}} = R$ is a ring then $A\otimes_{\underline{\underline{R}}}B$ above is the usual tensor product of modules over R. We also need tensor products of ringoids:

(1.7) Definition: The tensor product $\underline{\underline{R}}\otimes\underline{\underline{S}}$ of ringoids $\underline{\underline{R}},\underline{\underline{S}}$ is the following ringoid. Objects are pairs (X,Y) with $X \in Ob(\underline{\underline{R}})$, $Y \in Ob(\underline{\underline{S}})$ and the morphisms $(X,Y) \longrightarrow (X',Y')$ are the elements of the tensor product of abelian groups $\underline{\underline{R}}(X,Y) \otimes_{\mathbb{Z}} \underline{\underline{S}}(X,Y)$. Composition is defined by $(f\otimes g)(f'\otimes g') = (ff')\otimes(gg')$. Any biadditive functor $F: \underline{\underline{R}} \times \underline{\underline{S}} \longrightarrow \underline{\underline{Ab}}$ has a unique additive factorization (as well denoted by F) $F: \underline{\underline{R}}\otimes\underline{\underline{S}} \longrightarrow \underline{\underline{Ab}}$ with $F(f\otimes g) = F(f,g)$. For example an $\underline{\underline{R}}$–module A and on $\underline{\underline{S}}$–module $\underline{\underline{B}}$ yield the $\underline{\underline{R}}\otimes\underline{\underline{S}}$–module $A\otimes B$ given by $(A\otimes B)(f\otimes g) = A(f)\otimes_{\mathbb{Z}}B(g)$.

§2 *Quadratic $\mathbb{Z}$–modules*

Let Add($\mathbb{Z}$) be the category of finitely generated free abelian groups. The additive functors $F : \underline{Add}(\mathbb{Z}) \longrightarrow \underline{Ab}$ are in one one correspondence with abelian groups, the correspondence is given by $F \longmapsto F(\mathbb{Z})$. In this section we introduce quadratic $\mathbb{Z}$–modules which are in one one correspondence with quadratic functors $\underline{Add}(\mathbb{Z}) \longrightarrow \underline{Ab}$. In this sense a quadratic $\mathbb{Z}$–module is just the "quadratic analogue" of an abelian group.

(2.1) Definition. A quadratic $\mathbb{Z}$–module

$$M = (M_e \xrightarrow{H} M_{ee} \xrightarrow{P} M_e)$$

is a pair of abelian groups M_e, M_{ee} together with homomorphisms H, P which satisfy $PHP = 2P$ and $HPH = 2H$. A morphism $f : M \longrightarrow N$ between quadratic $\mathbb{Z}$–modules is a pair of homomorphisms $f : M_e \longrightarrow N_e$, $f : M_{ee} \longrightarrow N_{ee}$ $(i = 1,2)$ which commute with H and P respectively. Let $\underline{QM}(\mathbb{Z})$ be the category of quadratic $\mathbb{Z}$–modules. For a quadratic $\mathbb{Z}$–module M we define the involution $T = HP - 1 : M_{ee} \longrightarrow M_{ee}$. Then the equations for H and P are equivalent to $PT = P$ and $TH = H$. Moreover we get $TT = 1$ since $1+T = HP = HPT = T+T^2$. We define for $n \in \mathbb{Z}$ the function

$$\begin{cases} n_* : M_e \longrightarrow M_e \\ n_*(x) = nx + (n(n-1)/2)PH(x),\ x \in M_e \end{cases}$$

One can check that $(n \cdot m)_* = n_* m_*$ and that $(n+m)_* = n_* + m_* + nmPH$. Let $\mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$, $n \geq 0$, be the cyclic group of order n. We call M a quadratic $\mathbb{Z}/n$–module if $n \cdot M_{ee} = 0$ and $n_* M_e = 0$. We identify a quadratic $\mathbb{Z}$–module M satisfying $M_{ee} = 0$ with the abelian group M_e, this yields the full inclusion $\underline{Ab} = \underline{M}(\mathbb{Z}) \subset \underline{QM}(\mathbb{Z})$. Next we observe that there is a duality functor $D : \underline{QM}(\mathbb{Z}) \longrightarrow \underline{QM}(\mathbb{Z})$ with $D(M)$ given by the interchange of the roles of H and P respectively, that is $D(M) = ((DM)_e \xrightarrow{H^D} (DM)_{ee} \xrightarrow{P^D} (DM)_e)$ with $(DM)_e = M_{ee}$ and $(DM)_{ee} = M_e$, $H^D = P$ and $P^D = H$. Clearly $DD(M) = M$. Moreover an additive functor $A : \underline{Ab} \longrightarrow \underline{Ab}$ induces a functor $A[\_] : \underline{QM}(\mathbb{Z}) \longrightarrow \underline{QM}(\mathbb{Z})$. Here we define the quadratic $\mathbb{Z}$–module $A[M]$ by $A[M]_e = A(M_e)$ and $A[M]_{ee} = A(M_{ee})$ with H and P given by $A(H)$ and $A(P)$ respectively. For example the functor $A = \_\otimes_{\mathbb{Z}} C$, $C \in \underline{Ab}$, carries M to $[M] \otimes_{\mathbb{Z}} C$.

(2.2) Proposition: There is a ring Q together with an isomorphism $\chi : \underline{QM}(\mathbb{Z}) \cong \underline{M}(Q)$ of categories where $\underline{M}(Q)$ is the category of Q–modules.

Proof: For $M \in \underline{QM}(\mathbb{Z})$ we have inclusions and projections $(\tau = e,ee)$

(1) $$M_\tau \xrightarrow{i_\tau} M_e \oplus M_{ee} \xrightarrow{r_\tau} M_\tau$$

They yield the following endomorphisms of the abelian group $M_e \oplus M_{ee}$

(2) $$a = i_e r_e,\ b = i_{ee} r_{ee},\ h = i_{ee} H r_e,\ p = i_e P r_{ee}$$

which satisfy the relations

(3) $$\begin{cases} a^2 = a,\ b^2 = b,\ ab = ba = 0, \\ a + b = 1, \\ ha = 0,\ bh = 0,\ pa = 0,\ bp = 0, \\ php = 2p,\ hph = 2h. \end{cases}$$

Let Q be the ring generated by a,b,h,p such that the relations are satisfied. Then χ in (2.2) carries M to the Q–module $M_e \oplus M_{ee}$ defined by (2). As a $\mathbb{Z}$–module Q is given by $Q = \mathbb{Z}^6$ with basis (a,b,h,p,ph,hp). Moreover the quadratic $\mathbb{Z}$–module $\chi^{-1}(Q)$, as well denoted by Q, is given by

(4) $$\begin{cases} Q_e = a \cdot Q = \mathbb{Z}^3 \text{ with basis } (a,ap,aph), \\ Q_{ee} = b \cdot Q = \mathbb{Z}^3 \text{ with basis } (b,bh,bhp), \end{cases}$$

and by

(5) $$H = P = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix},\ T = HP - 1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(2.3) Corollary: The category $\underline{QH}(\mathbb{Z})$ is an abelian category.

Recall that an object X in an additive category is indecomposable if X admits no isomorphism $X \cong A \oplus B$ with $A \neq 0$ and $B \neq 0$. It is an interesting problem to classify all finitely generated indecomposable quadratic $\mathbb{Z}$–modules up to isomorphism. This leads to the following examples. We say that a quadratic $\mathbb{Z}$–module M is of cyclic type if M_e and M_{ee} are cyclic groups. Let $1_n \in \mathbb{Z}/n$ be the generator and let $k : \mathbb{Z}/n \longrightarrow \mathbb{Z}/m$ be the homomorphism with $k(1_n) = k \cdot 1_m$, $k \in \mathbb{Z}$, $m \mid k \cdot n$.

Then we obtain the following list where $C = \mathbb{Z}$ or $C = \mathbb{Z}/p^i$, p = prime, $s,t \geq 1$.

(2.4)

M	M_e	M_{ee}	H	P
C	C	0	0	0
C^Λ	0	C	0	0
C^Γ	C	C	1	2
C^S	C	C	2	1
$H(t)$	$\mathbb{Z}$	$\mathbb{Z}/2^t$	2^{t-1}	0
$P(s)$	$\mathbb{Z}/s^s$	$\mathbb{Z}$	0	2^{s-1}
$s+t>1, H(s,t)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^t$	2^{t-1}	0
$s+t>1, P(s,t)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^t$	0	2^{s-1}
$s+t>3, M(s,t)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^t$	2^{t-1}	2^{s-1}
$\Gamma(s)$	$\mathbb{Z}/2^{s+1}$	$\mathbb{Z}/2^s$	1	2
$S(s)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^{s+1}$	2	1
$s>1,\ \Gamma'(s)$	$\mathbb{Z}/2^{s+1}$	$\mathbb{Z}/2^s$	$2^{s-1}+1$	2
$s>1,\ S'(s)$	$\mathbb{Z}/2^s$	$\mathbb{Z}/2^{s+1}$	2	$2^{s-1}+1$

The isomorphic objects in the list are given by $C^\Gamma \cong C^S$ if $C = \mathbb{Z}/q^i$ (q odd). With the notations in (2.1) we clearly have $C^\Gamma = [\mathbb{Z}^\Gamma] \otimes_{\mathbb{Z}} C$, $C^S = [\mathbb{Z}^S] \otimes_{\mathbb{Z}} C$ and $C^\Lambda = [\mathbb{Z}^\Lambda] \otimes_{\mathbb{Z}} C$. We leave it to the reader to describe the dualities in the list. An elementary but somewhat elaborate proof shows:

(2.5) Proposition: The quadratic $\mathbb{Z}$–modules in (2.4) furnish a complete list of indecomposable quadratic $\mathbb{Z}$–modules of cyclic type.

(2.6) Definition: Let $F : \underline{R} \longrightarrow \underline{Ab}$ be a quadratic functor and let $X \vee Y$ be a biproduct in $\underline{R}$. The quadratic cross effect $F(X\,|\,Y)$ is defined by the image group

(1) $$F(X\,|\,Y) = \mathrm{im}\{\Delta(i_1 r_1, i_2 r_2) : F(X \vee Y) \longrightarrow F(X \vee Y)\}$$

see (1.3) and (1.1). If $\underline{\underline{R}}$ is an additive category we get by (1) the biadditive functor

(2) $$F(\_|\_) : \underline{\underline{R}} \times \underline{\underline{R}} \longrightarrow \underline{\underline{Ab}}.$$

Moreover we have the isomorphism

(3) $$\Psi : F(C) \oplus F(Y) \oplus F(X|Y) \cong F(X \vee Y)$$

which is given by $F(i_1)$, $F(i_2)$ and the <u>inclusion</u> $i_{12} : F(X|Y) \subset F(X \vee Y)$. Let r_{12} be the <u>retraction</u> of i_{12} obtained by Ψ^{-1} and by the projection to $F(X|Y)$. For the biproduct $X \vee Y$ one has the maps $\mu = i_1 + i_2 : X \longrightarrow X \vee Y$ and $\nabla = r_1 + r_2 : X \vee Y \longrightarrow X$. They yield homomorphisms H and P with

(4) $$F\{X\} = (F(X) \xrightarrow{H} F(X|X) \xrightarrow{P} F(X))$$

by $H = r_{12}F(\mu)$ and $P = F(\nabla)i_{12}$. Moreover we derive from $f+g = \nabla(f \vee g)\mu$ the formula

(5) $$F(f+g) = F(f) + F(g) + PF(f|g)H$$

or equivalently $\Delta(f,g) = PF(f|g)H$, see (1.3).

(2.7) <u>Proposition</u>: Let $F : \underline{\underline{R}} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor and assume $\underline{\underline{R}}$ is an additive category. Then $F\{X\}$ is a quadratic $\mathbb{Z}$–module and $X \longmapsto F\{X\}$ defines a functor $\underline{\underline{R}} \longrightarrow \underline{\underline{QM}}(\mathbb{Z})$.

<u>Proof of</u> (2.7): We define the interchange map

(1) $$\begin{cases} T : X \vee X \longrightarrow X \vee X \\ T = i_2 r_1 + i_1 r_2 \end{cases}$$

Then we have $T\mu = \mu$ and $\nabla T = \nabla$. Moreover T induces a map

(2) $$T : F(X|X) \longrightarrow F(X|X)$$

with $F(T)i_{12} = i_{12}T$ and $r_{12}F(T) = Tr_{12}$. Whence we get $TH = H$ and $PT = P$. Moreover we obtain $HP = 1 + T$ by applying F to the commutative diagram in $\underline{\underline{R}}$

(3) $$\begin{array}{ccccc} X \vee X & \xrightarrow{\nabla} & X & \xrightarrow{\mu} & X \vee X \\ \downarrow{\scriptstyle \mu \vee \mu} & & & & \downarrow{\scriptstyle \nabla \vee \nabla} \\ X \vee X \vee X \vee X & & \xrightarrow[1 \vee T \vee 1]{} & & X \vee X \vee X \vee X \end{array} .$$

Here we use the biadditivity of $F(\_|\_)$ in (2.4).

The significance of quadratic $\mathbb{Z}$–modules is described by the next result which is a special case of (3.7) below. Let $\underline{\underline{Add}}(\mathbb{Z}/n)$ be the full subcategory of $\underline{\underline{Ab}}$ consisting of finitely generated free $(\mathbb{Z}/n)$–modules; $n \geq 0$, (for $n = 0$ we set $\mathbb{Z}/0 = \mathbb{Z}$).

(2.8) Theorem: There is a 1–1 correspondence between quadratic functors $F : \underline{\mathrm{Add}}(\mathbb{Z}/n) \longrightarrow \underline{\mathrm{Ab}}$ and quadratic $\mathbb{Z}/n$–modules M, $n \geq 0$. The correspondence carries F to $F\{\mathbb{Z}/n\}$, see (2.6)(4).

Here a '1–1 correspondence' denotes a bijection which maps isomorphism classes to isomorphism classes. Whence any quadratic functor $F : \underline{\mathrm{Add}}(\mathbb{Z}/n) \longrightarrow \underline{\mathrm{Ab}}$ is completely determined (up to isomorphism) by the fairly simple algebraic data of the quadratic $\mathbb{Z}$–module $F\{\mathbb{Z}/n\}$ which is actually a quadratic $\mathbb{Z}/n$–module. In addition to the correspondence in (2.8) we obtain in (3.7) below an equivalence of categories.

Finally we consider some examples of quadratic functors $\underline{\mathrm{Ab}} \longrightarrow \underline{\mathrm{Ab}}$ which appear in the literature, see for example [3]. Let $\otimes^2$ be the functor which carries the abelian group A to $\otimes^2 A = A \otimes A$. Moreover Let $\hat{\otimes}^2, \Lambda^2, SP^2$ be given by the following relations $(a, b \in A)$:

$$(2.9) \qquad \begin{cases} \hat{\otimes}^2(A) = A\otimes A/(a\otimes b + b\otimes a \sim 0) \\ \Lambda^2(A) = A\otimes A/(a\otimes a \sim 0) \\ SP^2(A) = A\otimes A/(a\otimes b - b\otimes a \sim 0) \end{cases}$$

Next let Γ be the quadratic functor of J.H.C. Whitehead [17], see also [9] §13 where $\Gamma = \Gamma_4$ is shown to be part of a free commutative ring with divided powers. We obtain Γ and a weak quadratic functor $\tilde{\Gamma}$ as follows. A function $f : A \longrightarrow B$ between abelian groups is weak quadratic if $[a,b]_f = f(a+b) - f(a) - f(b)$ is bilinear, and f is quadratic if in addition $f(-a) \in f(a)$. Let $\gamma : A \longrightarrow \Gamma(A)$, resp. $\tilde{\gamma} : A \longrightarrow \tilde{\Gamma}(A)$, be the "universal" quadratic, resp. weak quadratic function. The function $\gamma\,(\tilde{\gamma})$ has the property that any (weak) quadratic function f admits a unique factorization $f = g\gamma$ $(f = \tilde{g}\tilde{\gamma})$ where $g\ (\tilde{g})$ is a homomorphism. We write $g = \gamma_\#(f)$ and $\tilde{g} = \tilde{\gamma}_\#(f)$. For the functors Γ and $\tilde{\Gamma}$ one has the following natural commutative diagram with short exact rows (see (8.16)) which is a pull back diagram.

$$(2.10) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & SP^2(A) & \xrightarrow{\tilde{w}} & \tilde{\Gamma}(A) & \xrightarrow{\tilde{\gamma}_\#(1)} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \tilde{\gamma}_\#(\gamma)} & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & SP^2(A) & \xrightarrow{w} & \Gamma(A) & \xrightarrow{\gamma_\#(q)} & A\otimes\mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

Here q is the quotient map and w, ($\tilde{w}$) carries $\{a\otimes b\}$ to $[a,b]_\gamma$ ($[a,b]_{\tilde{\gamma}}$). The functors in (2.9) and (2.10) lead to the following list of quadratic $\mathbb{Z}$–modules $F\{\mathbb{Z}\}$ associated to a quadratic functor F. In (4.9) below we show that the functors $F : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ in this list are actually completely determined by $F\{\mathbb{Z}\}$.

(2.11)

F	$F(\mathbb{Z}) \xrightarrow{H} F(\mathbb{Z}/\mathbb{Z}) \xrightarrow{P} F(\mathbb{Z})$	$F\{\mathbb{Z}\}$
$\otimes^2$	$\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{(1,1)} \mathbb{Z}$	$\mathbb{Z}^\otimes$
$\hat{\otimes}^2$	$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2$	P(1)
Λ^2	$0 \longrightarrow \mathbb{Z} \longrightarrow 0$	$\mathbb{Z}^\Lambda$
SP^2	$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z}$	$\mathbb{Z}^S$
Γ	$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$	$\mathbb{Z}^\Gamma$
$\tilde{\Gamma}$	$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{(2,1)} \mathbb{Z} \xrightarrow{(1,0)} \mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}^{\tilde{\Gamma}}$

The right hand column is compatible with the notation in (2.4). The basis of $\otimes^2(\mathbb{Z}\,|\,\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ is $(e_1\otimes e_2, e_2\otimes e_1)$ where (e_1,e_2) is a basis of $\mathbb{Z}\oplus\mathbb{Z}$. Moreover the basis of $\tilde{\Gamma}(\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$ is $(\tilde{e}_1,\tilde{e}_2)$ with $\tilde{e}_1 = \tilde{w}\{1\otimes 1\} = \tilde{\gamma}(2) - 2\tilde{\gamma}(1)$ and $\tilde{e}_2$ uniquely determined by $\tilde{\gamma}_\#(1)\tilde{e}_2 = 1$, $\tilde{\gamma}_\#(\gamma)\tilde{e}_2 = \gamma(1)$, see (2.10). The surjection $p = (p_1,p_2) : \tilde{\Gamma}\{\mathbb{Z}\} \longrightarrow \Gamma\{\mathbb{Z}\}$ induced by $\tilde{\gamma}_\#(\gamma)$ in (2.10) satisfies $p_1(\tilde{e}_1) = 2$, $p_1(\tilde{e}_2) = 1$, $p_2(1) = 1$.

We point out that the dual $D(\mathbb{Z}^\otimes)$ of the quadratic $\mathbb{Z}$–module $\mathbb{Z}^\otimes$ in (2.11) is isomorphic to the quadratic $\mathbb{Z}$–module $\mathbb{Z}^{\tilde{\Gamma}}$. Up to isomorphism there is actually only one quadratic $\mathbb{Z}$–module M with $M_e \cong \mathbb{Z}\oplus\mathbb{Z}$ and $M_{ee} \cong \mathbb{Z}$, namely $\mathbb{Z}^{\tilde{\Gamma}}$.

Finally we remark that the 'universal' quadratic $\mathbb{Z}$–module Q in (2.2)(4), (5) is decomposable, namely there are isomorphisms

(2.12) $$Q \cong D\mathbb{Z}^\otimes \oplus \mathbb{Z}^\otimes \cong \mathbb{Z}^{\tilde{\Gamma}} \oplus D\mathbb{Z}^{\tilde{\Gamma}} \cong \mathbb{Z}^{\tilde{\Gamma}} \oplus \mathbb{Z}^\otimes$$

in $\underline{QM}(\mathbb{Z})$. Whence the quadratic functor $\underline{Add}(\mathbb{Z}) \longrightarrow \underline{Ab}$ corresponding to Q via (2.8) is the functor $\Gamma \oplus \otimes^2$. The isomorphism $(D\mathbb{Z}^{\otimes})\oplus\mathbb{Z}^{\otimes} \cong Q$ is given by the matrices

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ for } Q_e \text{ and } \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ for } Q_{ee}.$$

§3 *Quadratic* $\underline{R}$*–modules*

For any ringoid $\underline{R}$ one has quadratic $\underline{R}$–modules which are the "quadratic generalization" of $\underline{R}$–modules in §1. For $\underline{R} = \mathbb{Z}$ they are just the quadratic $\mathbb{Z}$–modules discussed in §2 above.

(3.1) <u>Definition</u>: Let $\underline{R}$ be a ringoid. A <u>quadratic $\underline{R}$–module</u> $M = (M_e, M_{ee}, T, H, P)$ is a pair of functors $M_e : \underline{R} \longrightarrow \underline{Ab}$, $M_{ee} : \underline{R} \times \underline{R} \longrightarrow \underline{Ab}$ (both as well denoted by M) together with natural transformations

$$T = T_{X,Y} : M(X,Y) \longrightarrow M(Y,X) \text{ and } M(X) \xrightarrow{H} M(X,X) \xrightarrow{P} M(X)$$

such that the following properties are satisfied

(1) $PT = P$,

(2) $TH = H$,

(3) $T = HP - 1$ on $M_2(X,X)$,

(4) $TT = 1$.

Moreover the functor M_{ee} is biadditive and the functor M_e is quadratic with

(5) $M(f+g) = M(f) + M(g) + PM(f,g)H$

for $f,g : X \longrightarrow Y \in \underline{R}$. We also write $f_* = M(f)$ and $(f,g)_* = M(f,g)$. A morphism $F : M \longrightarrow N$ between quadratic $\underline{R}$–modules is a pair of natural transformations

(6) $F_e : M_e \longrightarrow N_e$, $F_{ee} : M_{ee} \longrightarrow N_{ee}$

which commute with T,H, and P respectively. Let $\underline{QM}(\underline{R})$ be the <u>category of quadratic $\underline{R}$–modules</u> for a small ringoid $\underline{R}$.

We identify a quadratic $\underline{R}$–module, satisfying $M_{ee} = 0$, with an $\underline{R}$–module. This yields the full inclusion of abelian categories $\underline{M}(\underline{R}) \subset \underline{QM}(\underline{R})$, see (2.3). On the other hand a quadratic $\underline{\underline{R}}$–module M with $M_e = 0$ is the same as a pair (M_{ee},T) where M_{ee} a biadditive functor $\underline{\underline{R}}\times\underline{\underline{R}} \longrightarrow \underline{\underline{Ab}}$ and where $T = T_{X,Y}: M_{ee}(X,Y) \cong M_{ee}(Y,X)$ is a natural transformation with $TT = 1$ and $T_{X,X} = -1$, $X,Y \in \mathrm{Ob}\,\underline{\underline{R}}$. The direct sum $M\oplus N$ of quadratic $\underline{\underline{R}}$–modules is given by $(M\oplus N)_e(X) = M_e(X)\oplus N_e(X)$ and $(M\oplus N)_{ee}(X,Y) = M_{ee}(X,Y)\oplus N_{ee}(X,Y)$.

(3.2) Remark: For the ringoid $\underline{\underline{R}} = \mathbb{Z}$ a quadratic $\underline{\underline{R}}$–module M as in (3.1) is the same as a quadratic $\mathbb{Z}$–module with $M_e = M(e)$, $M_{ee} = M(e,e)$. In fact, for $n \in \underline{\underline{R}}(e,e) = \mathbb{Z}$ the induced map $M(n) = n_*$ is defined in (2.1) and $T = T_{e,e}$ in (3.1) is defined by T in (2.1). This also shows that for the ring $\underline{\underline{R}} = \mathbb{Z}/n$ a quadratic $\underline{\underline{R}}$–module is the same as a quadratic $\mathbb{Z}/n$–module defined (2.1).

The equations (4.1)(1),(2),(3) for a quadratic $\underline{\underline{R}}$–module show that for $X \in \mathrm{Ob}(\underline{\underline{R}})$

(3.3) $$M\{X\} = (M(X) \xrightarrow{H} M(X,X) \xrightarrow{P} M(X))$$

is a quadratic $\mathbb{Z}$–module. Whence M yields a functor $M : \underline{\underline{R}} \longrightarrow \underline{\underline{QM}}(\mathbb{Z})$ which carries the object X to $M\{X\}$. The quadratic $\underline{\underline{R}}$–module M, however, is not determined by this functor since for example $T_{X,Y}$ in (3.1) is given for all pairs $(X,Y) \in \underline{\underline{Ob}}(R) \times \mathrm{Ob}(\underline{\underline{R}})$. In case $\underline{\underline{R}}$ has a single object e, that is, if $\underline{\underline{R}} = R$ is a ring, then a quadratic R–module M consists of a quadratic $\mathbb{Z}$–module

(1) $$M(e) \xrightarrow{H} M(e,e) \xrightarrow{P} M(e)$$

where $M(e,e)$ is an $R \otimes_{\mathbb{Z}} R$–module and where the multiplicative monoid of R acts on $M(e)$ such that H and P are equivariant with respect to the diagonal action on $M(e,e)$ and such that

(2) $$(f+g)_*(x) = f_*(x) + g_*(x) + P(f\otimes g)\cdot(Hx).$$

Here $f_*(x)$ denotes the action of $f \in R$ on $x \in M(e)$.

(3) Examples: Let R be a commutative ring. We define quadratic R–modules R^Λ, R^S, and R^Γ as follows.

M	M(e)	M(e,e)	H	P
R^Λ	0	R	0	0
R^S	R	R	2	1
R^Γ	R	R	1	2

Here $f \in R$ acts on $x \in M(e)$ by $f_*(x) = f\cdot f\cdot x$ and $f\otimes g$ acts on $y \in M(e,e)$ by $(f\otimes g)\cdot y = f\cdot g\cdot y$. Compare also (2.4) and (2.11).

Now let M be a quadratic $\underline{\underline{R}}$–module and let $X \vee Y$ be a biproduct in $\underline{\underline{R}}$. Then one gets the natural isomorphism of abelian groups

(3.4) $$\Psi : M(X) \oplus M(Y) \oplus M(X,Y) \cong M(X \vee Y).$$

The coordinates of Ψ are i_{1*}, i_{2*} and $P(i_1,i_2)_*$; the coordinates of the inverse Ψ^{-1} are r_{1*}, r_{2*} and $(r_1,r_2)_*H$. The isomorphism (3.4) corresponds exactly to (2.4)(3). This shows that (3.4) yields for $X \vee Y \in Ob(\underline{\underline{R}})$ the identification

(1) $$M(X,Y) = M(X \mid Y)$$

of quadratic cross effects. We can iterate (3.4) as follows. For an index set I let $\bigvee_{i \in I} X_i$ be an I–fold biproduct in $\underline{\underline{R}}$. If I is finite we obtain by (3.4) the formula

(2) $$M(\bigvee_{i \in I} X_i) = \bigoplus_{I} M(X_i) \oplus \bigoplus_{i<j} M(X_i,X_j)$$

where we choose an ordering $<$ of I. This formula holds as well for infinite index sets I if M_e and M_{ee} commute with direct limits. Many examples of quadratic $\underline{\underline{R}}$–modules arise as follows.

(3.5) Example: Let $\underline{\underline{R}}$ be a ringoid, let $\underline{\underline{A}}$ be an additive category, and let $i : \underline{\underline{R}} \longrightarrow \underline{\underline{A}}$ be an additive functor. Often $\underline{\underline{R}}$ is a subringoid of $\underline{\underline{A}}$ and i is the inclusion, for example $\underline{\underline{R}} = \underline{\underline{A}}$. Then any quadratic functor $F : \underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}$ yields a quadratic $\underline{\underline{R}}$–module

$$F\{\underline{\underline{R}}\} = i^*F = (F_e, F_{ee}, T, H, P)$$

as follows. The functors $F_e = i^*F$ and $F_{ee} = (i \times i)^* F(\_ \mid \_)$ are the restrictions of the functors F and $F(\_ \mid \_)$, see (2.6). Moreover H,P and T are given as in (2.6) and in the proof of (2.7) respectively. In case $\underline{\underline{R}}$ is the subringoid generated by the identity $1_X \in Ob(\underline{\underline{A}})$ than $F\{\underline{\underline{R}}\}$ is the same as the quadratic $\mathbb{Z}$–module $F\{X\}$ in (2.7).

We now are ready to describe the generalization of theorem (2.8) for quadratic $\underline{\underline{R}}$–modules; for this we recall from (VIII, §2) [13] the

(3.6) Definition: Let $\underline{\underline{R}}$ be a ringoid. Then the free additive category

(1) $$i : \underline{\underline{R}} \subset \underline{\underline{Add}}(\underline{\underline{R}})$$

is given as follows. The objects of $\underline{\underline{Add}}(\underline{\underline{R}})$ are the n–tuple $X = (X_1,\dots,X_n)$ of objects X_i in $\underline{\underline{R}}$, $0 \leq n < \infty$. The morphisms are the corresponding matrices of morphisms in $\underline{\underline{R}}$. The inclusion i carries the object X to the corresponding tuple of length 1. Any additive functor $f : \underline{\underline{R}} \longrightarrow \underline{\underline{A}}$ (where $\underline{\underline{A}}$ is an additive category) has a unique extension $\bar{F}: \underline{\underline{Add}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{A}}$ which carries the tuple X to the biproduct $\bar{F}(X) = FX_1 \vee \ldots \vee FX_n$ in $\underline{\underline{A}}$. Let $\underline{\underline{Quad}}(\underline{\underline{R}})$ be the category of quadratic functors

(2) $$F: \underline{\underline{Add}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}},$$

morphisms are natural transformations.

(3.7) Theorem: There is an equivalence of categories $\underline{\underline{Quad}}(\underline{\underline{R}}) \xrightarrow{\sim} \underline{\underline{QM}}(\underline{\underline{R}})$ which carries F to the restriction $F\{\underline{\underline{R}}\}$ in (3.5).

For a ring $\underline{\underline{R}} = R$ the category $\underline{\underline{Add}}(R)$ coincides with the full subcategory of finitely generated free R–modules in $\underline{\underline{M}}(R)$. Therefore (2.8) is readily obtained by (3.7) above. The inverse of the equivalence (3.7) is given by the tensor products defined in the next section; one gets (3.7) as a corollary of (4.4) below.

§4 *The quadratic tensor product*

We introduce the tensor product of an $\underline{\underline{R}}^{op}$–module and a quadratic $\underline{\underline{R}}$–module, this is the quadratic generalization of the tensor product defined in (1.6).

(4.1) Definition: Let $\underline{\underline{R}}$ be a small ringoid. We define the functor

$$\otimes_{\underline{\underline{R}}}: \underline{\underline{M}}(\underline{\underline{R}}^{op}) \times \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$$

which carries the pair (A,M) to the tensor product $A \otimes_{\underline{\underline{R}}} M$. The abelian group $A \otimes_{\underline{\underline{R}}} M$ is generated by the symbols

(1) $$\begin{cases} a \otimes m, \ a \in A(X), \ m \in M(X) \\ [a,b] \otimes n, \ a \in A(X), \ b \in A(Y), \ n \in M(X,Y) \end{cases}$$

where X,Y are objects in $\underline{\underline{R}}$. The relations are

$$(2)\qquad \begin{cases} (a+b)\otimes m = a\otimes m + b\otimes m + [a,b]\otimes H(m), \\ a\otimes(m+m') = a\otimes m + a\otimes m'\ , \\ [a,a]\otimes n = a\otimes P(n), \\ [a,b]\otimes n = [b,a]\otimes T(n), \\ [a,b]\otimes n \text{ is linear in each variable } a,b, \text{ and } n, \\ (\varphi^* a)\otimes m = a\otimes(\varphi_* m), \\ [\varphi^* a, \Psi^* b]\otimes n = [a,b]\otimes(\varphi,\Psi)_*(n) \end{cases}$$

where φ,Ψ are morphisms in $\underline{R}$ and where a,b,m,m',n are appropriate elements as in (1). (We point out that the last two equations of (2) are redundant if $\underline{R} = \mathbb{Z}$.) For morphisms $F : A \longrightarrow A' \in \underline{M}(\underline{R}^{op})$ and $G : M \longrightarrow M' \in \underline{QM}(\underline{R})$ we define the induced homomorphism

$$(3)\qquad F\otimes G : A\otimes_{\underline{R}} M \longrightarrow A'\otimes_{\underline{R}} M'$$

by the formulas

$$(4)\qquad \begin{cases} (F\otimes G)(a\otimes m) = (Fa)\otimes(G_e m) \\ (F\otimes G)([a,b]\otimes n) = [Fa,Fb]\otimes(G_{ee} n) \end{cases}$$

In case $M_{ee} = 0$ we see that $A\otimes_{\underline{R}} M$ coincides with the tensor product (1.6). Whence the functor $\otimes_{\underline{R}}$ in (4.1) extends canonically the functor $\otimes_{\underline{R}}$ in (1.6).

(4.2) Proposition: The tensor product (4.1) yields an additive functor

$$(1)\qquad A\otimes_{\underline{R}}(\_) : \underline{QM}(\underline{R}) \longrightarrow \underline{Ab}$$

for each A in $\underline{M}(\underline{R})$ and a quadratic functor

$$(2)\qquad (\_)\otimes_{\underline{R}} M : \underline{M}(\underline{R}^{op}) \longrightarrow \underline{Ab}$$

for each M in $\underline{QM}(\underline{R})$. The quadratic cross effect of (2) is given by the formula

$$(3)\qquad (A\,|\,B)\otimes_{\underline{R}} M = (A\otimes B)\otimes_{\underline{R}\otimes\underline{R}} M_{ee}.$$

Here A and B are $\underline{R}^{op}$–modules which yield the $(\underline{R}\otimes\underline{R})^{op}$–module $A\otimes B$ by (1.7) and the $\underline{R}\otimes\underline{R}$–module M_{ee} is given by M. The right hand side of (3) is a tensor product in the sense of (1.6). The isomorphism (3) is obtained by the inclusion

$$(4)\qquad i_{12} : (A\otimes B)\otimes_{\underline{R}\otimes\underline{R}} M_{ee} \rightarrowtail (A\oplus B)\otimes_{\underline{R}} M$$

which carries $a \otimes b \otimes n$ to $[i_1 a, i_2 b] \otimes n$ for $a \in A(X)$, $b \in B(Y)$, $n \in M(X,Y)$. By (3.5) the quadratic functor $F = (\_) \otimes_{\underline{\underline{R}}} M$ is as well a quadratic $\underline{\underline{M}}(\underline{\underline{R}})$–module. Here the structure maps T,H,P are given by the natural transformations

$$(5) \qquad (A \otimes B) \otimes_{\underline{\underline{R}} \otimes \underline{\underline{R}}} M_{ee} \xrightarrow{T} (B \otimes A) \otimes_{\underline{\underline{R}} \otimes \underline{\underline{R}}} M_{ee},$$

$$(6) \qquad A \otimes_{\underline{\underline{R}}} M \xrightarrow{H} (A \otimes B) \otimes_{\underline{\underline{R}} \otimes \underline{\underline{R}}} M_{ee} \xrightarrow{P} A \otimes_{\underline{\underline{R}}} M$$

defined by the formulas

$$(7) \qquad \begin{cases} H(a \otimes m) = (a \otimes a) \otimes H(m) \\ H([a,b] \otimes n) = (a \otimes b) \otimes n + (b \otimes a) \otimes T(n), \\ T((a \otimes b) \otimes n) = (b \otimes a) \otimes T(n), \\ P((a \otimes b) \otimes n) = [a,b] \otimes n. \end{cases}$$

We point out that the tensor product (4.1) is compatible with direct limits in $\underline{\underline{M}}(\underline{\underline{R}}^{op})$ and $\underline{\underline{QM}}(\underline{\underline{R}})$ respectively.

Let $\underline{\underline{A}}$ be an additive category and let $F : \underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor. For a small subringoid $\underline{\underline{R}} \subset \underline{\underline{A}}$ the quadratic $\underline{\underline{R}}$–module $F\{\underline{\underline{R}}\}$ is defined by (3.5). On the other hand each object U in $\underline{\underline{A}}$ gives us the $\underline{\underline{R}}^{op}$–module

$$[\underline{\underline{R}}, U] : \underline{\underline{R}}^{op} \longrightarrow \underline{\underline{Ab}}$$

which carries $X \in \underline{\underline{R}}$ to $\underline{\underline{A}}(X,U) = [X,U]$. We now define a map

$$(4.3) \qquad \lambda : [\underline{\underline{R}}, U] \otimes_{\underline{\underline{R}}} F\{\underline{\underline{R}}\} \longrightarrow F(U)$$

by $\lambda(a \otimes m) = F(a)(m)$ for $a \in [X,U]$, $m \in F(X)$ and $\lambda([a,b] \otimes n) = PF(a|b)(n)$ for $b \in \mathrm{Hom}(Y,U)$ and $n \in F(X|Y)$.

(4.4) Proposition: The homomorphism λ in (4.3) is well defined and natural. Moreover λ is an isomorphism if $U = X_1 \vee \ldots \vee X_r$ is a finite biproduct with $X_i \in \underline{\underline{R}}$ for $i = 1,\ldots,r$ and if $\underline{\underline{R}}$ is a full subringoid of $\underline{\underline{A}}$.

This result is a crucial property of the tensorproduct (4.1) which shows that definition (4.1) is naturally derived from the notion of a quadratic functor.

The proposition shows that a quadratic functor $F : \underline{\underline{Add}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$ is completely determined by the quadratic $\underline{\underline{R}}$–module $F\{\underline{\underline{R}}\} = i^{*}F$. This proves theorem (3.7); in fact, the inverse of the functor (3.7) carries $M \in \underline{\underline{QM}}(\underline{\underline{R}})$ to the quadratic functor $[\underline{\underline{R}},\_] \otimes_{\underline{\underline{R}}} M$. The next corollary illustrates proposition (4.4). Let $\underline{\underline{Cyc}}$ be the full subcategory of $\underline{\underline{Ab}}$ consisting of cyclic groups $\mathbb{Z}/n$ where $n = 0$ or where n is a prime power. Then we have the equivalence of categories

$$(4.5) \qquad \underline{\underline{Add}}(\underline{\underline{Cyc}}) \xrightarrow{\sim} \underline{\underline{FAb}}$$

where $\underline{\underline{FAb}}$ is the full subcategory of $\underline{\underline{Ab}}$ consisting of finitely generated abelian groups. Since each abelian group is the limit of its finitely generated subgroups we get the

(4.6) Corollary: Let $F : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor which commutes with direct limits. Then F is completely determined by the quadratic $\underline{\underline{Cyc}}$–module $F\{\underline{\underline{Cyc}}\}$, see (3.5). In fact, we have the natural isomorphism
$[\underline{\underline{Cyc}},A] \otimes_{\underline{\underline{Cyc}}} F\{\underline{\underline{Cyc}}\} \cong F(A)$ for A in $\underline{\underline{Ab}}$.

We now consider examples of the natural transformation λ in (4.3). A commutative ring R satisfies $R^{op} = R$. Therefore we get for any quadratic functor $F : \underline{\underline{M}}(R) \longrightarrow \underline{\underline{Ab}}$ the natural homomorphism $(A \in \mathrm{Ob}\, \underline{\underline{M}}(R))$

$$(4.7) \qquad \lambda : A \otimes_R F\{R\} \longrightarrow F(A).$$

Here the quadratic R–module $F\{R\}$ is essentially given by the homomorphisms in $\underline{\underline{Ab}}$

$$F(R) \xrightarrow{H} F(R\,|\,R) \xrightarrow{P} F(R),$$

see (2.6)(4) and (3.3)(1), and λ is defined as follows. For $a \in A$ let $\bar{a} : R \longrightarrow A$ be the map in $\underline{\underline{M}}(R)$ with $\bar{a}(1) = a$. Then we get for $m \in F(R)$ and $n \in F(R\,|\,R)$ the formulas $\lambda(a \otimes m) = F(\bar{a})(m)$ and $\lambda([a,b] \otimes n) = PF(\bar{a}\,|\,\bar{b})(n)$. By (4.4) the map λ is an isomorphism if A is a finitely generated free R–module. We call λ the tensor approximation of the quadratic functor F. For $R = \mathbb{Z}$ we have the following examples for which the tensor approximation is actually a natural isomorphism.

(4.8) Proposition: The functors $F = \otimes^2, \hat{\otimes}^2, \Lambda^2, SP^2, \Gamma$, and $\hat{\Gamma}$ in the list (2.11) satisfy $A \otimes_{\mathbb{Z}} F\{\mathbb{Z}\} = F(A)$ for all $A \in Ob(\underline{\underline{Ab}})$.

The torsion functor $F : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ with $F(A) = A*A$, however, is a quadratic functor for which the tensor approximation is no isomorphism, in fact, $F\{\mathbb{Z}\} = 0$ in this case. It is easy to check (4.8) by the definition of the relations in (2.9) and (4.1) respectively. Finally we observe the next result where we use the notation $[M] \otimes_{\mathbb{Z}} C$ in (2.1).

(4.10) Proposition: For $M \in \underline{\underline{QM}}(\mathbb{Z})$ and $A,C \in \underline{\underline{Ab}}$ we have the natural isomorphism

$$A \otimes_{\mathbb{Z}} ([M] \otimes_{\mathbb{Z}} C) \cong (A \otimes_{\mathbb{Z}} M) \otimes_{\mathbb{Z}} C .$$

§5 *The quadratic Hom functor*

Let $\underline{\underline{R}}$ be a small ringoid. For $\underline{\underline{R}}$–modules A,B one has the abelian group $Hom_{\underline{\underline{R}}}(A,B)$ which consists of all natural transformations $A \longrightarrow B$. We now extend this Hom functor for the case that B is a quadratic $\underline{\underline{R}}$–module.

(5.1) Definitions: We define the functor

$$Hom_{\underline{\underline{R}}} : \underline{\underline{M}}(\underline{\underline{R}})^{op} \times \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$$

which carries the pair (A,M) to the abelian group $Hom_{\underline{\underline{R}}}(A,M)$, the elements of which are called quadratic forms $A \longrightarrow M$ over $\underline{\underline{R}}$. A quadratic form $\alpha : A \longrightarrow M$ is given by functions $(X,Y \in Ob(\underline{\underline{R}}))$

(1) $\qquad \alpha_X : A(X) \longrightarrow M(X), \quad \alpha_{X,Y} : A(X) \times A(Y) \longrightarrow M(X,Y)$

such that the following properties are satisfied; (they are analogues to the corresponding properties in (4.1)(2) and they as well define the sume $\alpha + \beta$ of quadratic forms).

$$(2)\qquad \begin{cases} \alpha_X(a+b) = \alpha_X(a) + \alpha_X(b) + P\alpha_{X,X}(a,b) \\ (\alpha+\beta)_X = \alpha_X+\beta_X \\ \alpha_{X,X}(a,a) = H\alpha_X(a) \\ \alpha_{X,Y}(a,b) = T\alpha_{Y,X}(b,a) \\ \alpha_{X,Y} \text{ is bilinear and } (\alpha+\beta)_{X,Y} = \alpha_{X,Y} + \beta_{X,Y} \\ M_e(\varphi)\alpha_X = \alpha_{X_1}A(\varphi) \\ M_{ee}(\varphi,\Psi)\alpha_{X,Y} = \alpha_{X_1,Y_1}(A(\varphi) \times A(\Psi)) \end{cases}$$

Here a,b are appropriate elements in A(X) or A(Y) and $\varphi : X \longrightarrow X_1$, $\Psi : Y \longrightarrow Y_1$ are morphisms in $\underline{\underline{R}}$. The last two equations describe the "naturality" of the quadratic form α, (these equations are redundant if $\underline{\underline{R}} = \mathbb{Z}$). For morphisms $F : A' \longrightarrow A$ in $\underline{\underline{M}}(\underline{\underline{R}})$ and $G : M \longrightarrow M' \in \underline{\underline{QM}}(\underline{\underline{R}})$ we define the induced homomorphisms

$$(3)\qquad \mathrm{Hom}(F,G) : \mathrm{Hom}_{\underline{\underline{R}}}(A,M) \longrightarrow \mathrm{Hom}_{\underline{\underline{R}}}(A',M')$$

by the formulas $\mathrm{Hom}(F,G)(\alpha) = \beta$ with

$$(4)\qquad \beta_X = G_e\alpha_X F, \ \beta_{X,Y} = G_{ee}\alpha_{X,Y}(F\times F).$$

In case $M_{ee} = 0$ we see that $\mathrm{Hom}_{\underline{\underline{R}}}(A,M)$ coincides with the usual group of natural transformations $A \longrightarrow M$, whence the functor (5.1) extends canonically the classical functor $\mathrm{Hom}_{\underline{\underline{R}}}$ for $\underline{\underline{R}}$–modules.

(5.2) Proposition: The Hom–functor (5.1) yields an additive functor

$$(1)\qquad \mathrm{Hom}_{\underline{\underline{R}}}(A,\_) : \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$$

for each A in $\underline{\underline{M}}(\underline{\underline{R}})$ and a quadratic functor

$$(2)\qquad \mathrm{Hom}_{\underline{\underline{R}}}(\_,M) : \underline{\underline{M}}(\underline{\underline{R}})^{op} \longrightarrow \underline{\underline{Ab}}$$

for each M in $\underline{\underline{QM}}(\underline{\underline{R}})$. The quadratic cross effect of (2) is given by the formula

$$(3)\qquad \mathrm{Hom}_{\underline{\underline{R}}}(A\,|\,B,M) = \mathrm{Hom}_{\underline{\underline{R}}\otimes\underline{\underline{R}}}(A\otimes B,M_{ee})$$

Compare (4.2) where we describe the corresponding result for quadratic tensor products. The isomorphism in (3) is obtained by the projection

$$(4)\qquad r_{12} : \mathrm{Hom}_{\underline{\underline{R}}}(A\oplus B,M) \longrightarrow\!\!\!\!\rightarrow \mathrm{Hom}_{\underline{\underline{R}}\otimes\underline{\underline{R}}}(A\otimes B,M_{ee})$$

which carries α to the natural transformation $\beta : A(X) \otimes B(Y) \longrightarrow M_{ee}(X,Y)$ with $\beta(a \otimes b) = \alpha_{X,Y}(i_1 a, i_2 b)$. By (3.5) the quadratic functor $F = \mathrm{Hom}_{\underline{\underline{R}}}(\_,M)$ is a quadratic $\underline{\underline{M}}(\underline{\underline{R}})^{op}$–module; the structure maps T,H,P are given by the following natural transformations

$$(5) \qquad \mathrm{Hom}_{\underline{\underline{R}} \otimes \underline{\underline{R}}}(A \otimes B, M_{ee}) \xrightarrow{T} \mathrm{Hom}_{\underline{\underline{R}} \otimes \underline{\underline{R}}}(B \otimes A, M_{ee})$$

$$(6) \qquad \mathrm{Hom}_{\underline{\underline{R}}}(A,M) \xrightarrow{H} \mathrm{Hom}_{\underline{\underline{R}} \otimes \underline{\underline{R}}}(A \otimes B, M_{ee}) \xrightarrow{P} \mathrm{Hom}_{\underline{\underline{R}}}(A,M)$$

defined by

$$(7) \qquad \begin{cases} (T\beta)(a \otimes b) = T\beta(a \otimes b), \\ (H\alpha)(a \otimes b) = \alpha(a,b) + T\alpha(b,a), \\ (P\beta)(a) = H\beta(a \otimes a) \text{ and } (P\beta)(a,b) = \beta(a \otimes b)\,. \end{cases}$$

(5.3) Examples: Let R be a commutative ring and consider the quadratic R–modules R^{Λ}, R^{S} and R^{Γ} defined in (3.3)(3). Moreover let A be an R–module.

(1) A quadratic form $\alpha : A \longrightarrow R^{\Lambda}$ can be identified with an R–bilinear map $\alpha : A \times A \longrightarrow R$ satisfying $\alpha(a,a) = 0$. Whence α is just an alternating bilinear form.

(2) A quadratic form $\alpha : A \longrightarrow R^{S}$ can be identified with a function $\alpha : A \longrightarrow R$ which satisfies $\alpha(\lambda \cdot a) = \lambda^2 \cdot \alpha(a)$ for $\lambda \in R$, $a \in A$ and for which the function

$$\Delta_\alpha : A \times A \longrightarrow R,\ \Delta_\alpha(a,b) = \alpha(a+b) - \alpha(a) - \alpha(b)$$

is R–bilinear. Thus α is the same as a quadratic form on A in the classical sense, see for example [1], [15].

(3) A quadratic form $\alpha : A \longrightarrow R^{\Gamma}$ can be identified with a pair of functions $\alpha : A \longrightarrow R$, $\Delta : A \times A \longrightarrow R$ for which $\alpha : A \longrightarrow R$, $\Delta : A \times A \longrightarrow R$ for which $\alpha(\lambda a) = \lambda^2 \alpha(a)$ and for which Δ is symmetric R–bilinear with $2\Delta(a,b) = \alpha(a+b) - \alpha(a) - \alpha(b)$ and $\Delta(a,a) = \alpha(a)$. If R is uniquely 2–divisible α is a special quadratic form as in (2) since in this case Δ is determined by α.

(5.4) Example: Let X be a connected space and let A be an abelian group. J.H.C. Whitehead [17] considered the Pontrjagin square which is a function (n = even)

$$p : H^n(X,A) \longrightarrow H^{2n}(X,\Gamma A).$$

Here the quadratic $\mathbb{Z}$–module $\Gamma\{A\} = (\Gamma A \longrightarrow A\otimes A \longrightarrow \Gamma A)$, see (2.6)(4), yields the induced quadratic $\mathbb{Z}$–module

$$\mathscr{H} = (H^{2n}(X,\Gamma A) \xrightarrow{H_*} H^{2n}(X,A\otimes A) \xrightarrow{P_*} H^{2n}(X,\Gamma A).$$

Now the Pontrjagin square p together with the <u>cup product</u> is a quadratic form $H^n(X,A) \longrightarrow \mathscr{H}$ over $\mathbb{Z}$, compare [17].

(5.5) <u>Lemma</u>: Let R be a ring and let F be a finitely generated free R–module. Then $Hom_R(F,R)$ is an R^{op}–module such that

$$\chi : Hom_R(F,R) \otimes_R M \cong Hom_R(F,M)$$

for any quadratic R–module M.

<u>Proof</u>: We define the natural isomorphism χ as follows. Let $a,b \in Hom_R(F,R)$, $m \in M(e)$, $n \in M(e,e)$. Then $\chi(a\otimes m) = \alpha$ is given by $\alpha(x) = M_e(a(x))(m)$ and $\alpha(x,y) = M_{ee}(a(x),a(y))H(m)$ for $x,y \in F$. Moreover $\chi([a,b]\otimes n) = \beta$ is given by $\beta(x) = PM_{ee}(a(x),b(x))(n)$ and $\beta(x,y) = M_{ee}(a(x),b(y))(n) + M_{ee}(a(y),b(x))(n)$.

(5.6) <u>Example:</u> Let V be a finitely generated free abelian group and let $V^{\#} = Hom(V,\mathbb{Z})$. Then a quadratic form $V^{\#} \longrightarrow \Gamma(V)$ can be identified via (5.5) with an element in $\Gamma(V)$ since $Hom(V^{\#},\mathbb{Z}) = V$. Now consider a closed 1–connected 4–dimensional manifold X and let $b : H_4X = \mathbb{Z} \longrightarrow \Gamma(H_2X)$ be Whitehead's secondary boundary operator. Here $H_2X = V$ is finitely generated free abelian and $b(1) \in \Gamma(H_2X)$ corresponds to a quadratic form $(H_2X)^{\#} = H^2(X) \longrightarrow \mathbb{Z}^{\Gamma}$ which actually is the intersection form of X.

Let $\underline{\underline{A}}$ be an additive category and let $F : \underline{\underline{A}}^{op} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor. For a small subringoid $\underline{\underline{R}} \subset \underline{\underline{A}}$ the quadratic $\underline{\underline{R}}^{op}$–module $F\{\underline{\underline{R}}^{op}\}$ is defined as in (3.5) by $\underline{\underline{R}}^{op} \subset \underline{\underline{A}}^{op}$. On the other hand each object U in $\underline{\underline{A}}$ gives as the $\underline{\underline{R}}^{op}$–module $[\underline{\underline{R}},\underline{\underline{U}}]$ as in (4.3). We now define the map

(5.7) $$\lambda : F(U) \longrightarrow Hom_{\underline{\underline{R}}^{op}}([\underline{\underline{R}},U],F\{\underline{\underline{R}}^{op}\})$$

as follows. For $\xi \in F(U)$ let $\lambda(\xi)$ be given by the functions α_X, $\alpha_{X,Y}$ ($X,Y \in \underline{\underline{R}}^{op}$) with $\alpha_X(a) = a^*(\xi) = F(a)(\xi)$, $a \in [X,U]$ and $\alpha_{X,Y}(a,b) = F(a|b)H(\xi)$, $b \in [Y,U]$. The next proposition corresponds to (4.4).

(5.8) Proposition: The homomorphism λ in (5.7) is well defined and natural. Moreover λ is an isomorphism if $U = X_1 \vee ... \vee X_r$ is a finite biproduct with $X_i \in \underline{\underline{R}}$ for $i = 1,...,r$ and if $\underline{\underline{R}}$ is a full subringoid of $\underline{\underline{A}}$.

This result is a crucial property of the Hom–group (5.1) which shows that definition (5.1) is again naturally derived from the notion of a quadratic functor. We leave it to the reader to formulate a corollary of (5.8) corresponding to (4.6). Moreover we get as in (4.7) the following example. Let R be a commutative ring and let $F : \underline{\underline{M}}(R)^{op} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor. Then the quadratic R–module $F\{R\}$ is defined and we derive from (5.7) the natural transformation

$$(5.9) \qquad \lambda : F(A) \longrightarrow \mathrm{Hom}_R(A, F\{R\})$$

where $A \in \underline{\underline{M}}(\underline{\underline{R}})$, compare (4.7). By (5.8) this map is an isomorphism if A is a finitely generated free R–module. We call (5.9) the Hom–approximation of the quadratic functor F.

§6 *The quadratic chain functors*

In this section we associate with each quadratic $\underline{\underline{R}}$–module M quadratic chain functors M_* and M^*. The definition of M_* and M^* is motivated by applications in homotopy theory [4]. The quadratic chain functors as well form a first step for the construction of the derived functors in §7 and §8.

Let $\underline{\underline{R}}$ be a ringoid with a zero object denoted by 0. A chain complex $X_* = (X_*, d)$ in $\underline{\underline{R}}$ is a sequence of maps in $\underline{\underline{R}}$

$$(6.1) \qquad \cdots \longrightarrow X_n \xrightarrow{d} X_{n-1} \xrightarrow{d} \cdots \quad (n \in \mathbb{Z})$$

with $dd = 0$. A chain map $F : X_* \longrightarrow Y_*$ is given by maps $F = F_n : X_n \longrightarrow Y_n$ with $dF = Fd$ and a chain homotoy $\alpha : F \simeq G$ is given by maps $\alpha = \alpha_n : X_{n-1} \longrightarrow Y_n$ with $-F_n + G_n = \alpha_n d + d\alpha_{n+1}$. The chain complex X_* is positive (negative) if $X_i = 0$ for $i < 0$ ($X_i = 0$ for $i > 0$). A negative chain complex is also called a cochain complex X^* where we write $X^n = X_{-n}$, $d : X^n \longrightarrow X^{n+1}$. Let $\underline{\underline{R}}_*$ ($\underline{\underline{R}}^*$) be the category of positive (negative) chain complexes and let $\underline{\underline{R}}_*/\simeq$ ($\underline{\underline{R}}^*/\simeq$) be its homotopy category.

We also need the category $\underline{\text{Pair}(\underline{\underline{R}})}$ of pairs in $\underline{\underline{R}}$; objects are morphisms d in $\underline{\underline{R}}$ and maps $F : d \longrightarrow d'$, $F = (F_A, F_B)$, are commutative diagrams

$$
(6.2) \qquad
\begin{array}{ccc}
A & \xrightarrow{\ F_A\ } & A' \\
d \big\downarrow & F_B & \big\downarrow d' \\
B & \xrightarrow{\qquad} & B'
\end{array}
$$

A homotopy $\alpha : F \simeq G$ is a map $\alpha : B \longrightarrow A$ with $-F_A + G_A = \alpha d$, $-F_B + G_B = d'\alpha$. We have full inclusions of $\underline{\text{Pair}(\underline{\underline{R}})}/\simeq$ into $R_*/\simeq$ and $R^*/\simeq$ which carry d to the chain complex $d : A = X_1 \longrightarrow B = X_0$ and to the cochain complex $d : A = X^0 \longrightarrow B = X^1$ respectively.

(6.3) Definition: Let M be a quadratic $\underline{\underline{R}}$–module. The quadratic clain functors associated to M are the functors

(1) $$M_* : \underline{\text{Pair}(\underline{\underline{R}})} \longrightarrow \underline{\text{Ab}}_*\, , \quad M^* : \underline{\text{Pair}(\underline{\underline{R}})} \longrightarrow \underline{\text{Ab}}^*$$

which are defined as follows. For an object $d : X_1 \longrightarrow X_0$ in $\underline{\text{Pair}(\underline{\underline{R}})}$ we define the chain complex $M_*(d)$ by $M_i(d) = 0$ for $i > 2$ and by

$$
(2) \qquad
\begin{array}{ccccc}
M(X_1, X_1) & \xrightarrow{(P,-(1,d)_*)} & M(X_1) \oplus M(X_1, X_0) & \xrightarrow{(d_*, P(d,1)_*)} & M(X_0) \\
\| & & \| & & \| \\
M_2(d) & & M_1(d) & & M_0(d)
\end{array}
$$

otherwise. On the other hand we define for an object $d : X^0 \longrightarrow X^1$ in $\underline{\text{Pair}(\underline{\underline{R}})}$ the cochain complex $M^*(d)$ by $M^i(d) = 0$ for $i > 2$ and by

$$(3)\qquad \begin{array}{ccccc} M(X^0) & \xrightarrow{(d_*,(d,1)_*H)} & M(X^1)\oplus M(X^1,X^0) & \xrightarrow{(H,(1,d)_*)} & M(X^1,X^1) \\ \| & & \| & & \| \\ M^0(d) & & M^1(d) & & M^2(d) \end{array}$$

otherwise. For a map $F : d \longrightarrow d'$ in $\underline{\underline{Pair}}(\underline{\underline{R}})$ the induced chain maps $M_*(F)$ and $M^*(F)$ are defined in the obvious way. One readily checks that the composition of maps in (2) and (3) respectively is the trivial map 0. The definition of M_*, M^* is motivated by the examples in (5.5), (6.5) and (6.10) in [4].

We point out that the definition of M^* above is dual to the definitiion of M_*; here duality is obtained by reversing arrows and by the interchange of H and P.

(6.4) Theorem. The quadratic chains functors (6.3) induce functors

$$M_* : \underline{\underline{Pair}}(\underline{\underline{R}})/\simeq \longrightarrow \underline{\underline{Ab}}_*/\simeq\ ,\ \ M^* : \underline{\underline{Pair}}(\underline{\underline{R}})/\simeq \longrightarrow \underline{\underline{Ab}}^*/\simeq$$

between homotopy categories.

Proof: Let $f = (f_1,f_0)$ and $g = (g_1,g_0)$ be maps $d \longrightarrow d'$ in $\underline{\underline{Pair}}(\underline{\underline{R}})$ and let $\alpha : f \simeq g$ be a homotopy. We can define a homotopy

$$(1)\qquad \beta : M_*(f) \simeq M_*(g)$$

by the matrices (2) and (3).

$$(2)\qquad \beta_1 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{with} \quad \begin{cases} B_1 = \alpha_* \\ B_2 = (\alpha,f_0)_*H \end{cases}$$

$$(3)\qquad \beta_2 = (A_1,A_2) \quad \text{with} \quad \begin{cases} A_1 = (\alpha d,f_1)_*H \\ A_2 = -(g_1,\alpha)_* + T(f_1,\alpha)_* \end{cases}.$$

For the proof of (1) we have to check the following equations (4)...(9).

$$(4)\qquad -f_{0*} + g_{0*} = d_*B_1 + P(d,1)_*B_2$$

$$(5)\qquad -(f_1,f_1)_* + (g_1,g_1)_* = A_1P - A_2(1,d)_*$$

$$(6)\qquad -f_{1*} + g_{1*} = PA_1 + B_1d_*$$

$$(7)\qquad -(f_1,f_0)_* + (g_1,g_0)_* = -(1,d)_*A_2 + B_2P(d,1)_*$$

$$(8)\qquad 0 = PA_2 + B_1P(d,1)_*$$

$$(9)\qquad 0 = -(1,d)_*A_1 + B_2d_*\ .$$

Originally we found the formulas in (2) ... (3) as a solutions of the system of equations (4)...(9). We now check (4).

(10) $d_*\alpha_* + P(d,1)_*(d,f_0)_*H = (d\alpha)_* + P(d\alpha,f_0)_*H$
$= (d\alpha)_* + (d\alpha + f_0)_* - (d\alpha)_* - f_{0*} = g_{0*} - f_{0*}.$

Here we use (3.1)(5) and $d\alpha = -f_0 + g_0$. Next we obtain (5) by $\alpha d = -f_1 + g_1$ and by (3.1)(3):

(11) $(\alpha d,f_1)_*HP + (g_1,\alpha)_*(1,d)_* - T(f_1,\alpha)_*(1,d)_* =$
$(\alpha d,f_1)_*T + (\alpha d,f_1)_* + (g_1,\alpha d)_* - T(f_1,\alpha d)_* =$
$(\alpha d,f_1)_* + (g_1,\alpha d)_* = (-f_1 + g_1,f_1)_* + (g_1,-f_1+g_1)_*$
$= -(f_1,f_1)_* + (g_1,g_1)_* .$

In the last equation we use the biadditivity of the functor M_{ee} in (3.1). For equation (6) we consider

(12) $P(\alpha d,f_1)_*H + \alpha_*d_* = (\alpha d+f_1)_* - (\alpha d)_* - f_{1*} + (\alpha d)_* = -f_{1*} + g_{1*} .$

Next equation (7) follows from

(13) $-(1,d)_*(g_1,\alpha)_* - (1,d)_*T(f_1,\alpha)_* + (\alpha,f_0)_*HP(d,1)_* =$
$(g_1,d\alpha)_* - (\alpha,df_1)_*T + (\alpha d,f_0)_* + (\alpha,f_0d)_*T =$
$(g_1,-f_0+g_0)_* + (-f_1+g_1,f_0)_* = (g_1,g_0)_* - (f_1,f_0)_*.$

Moreover we obtain (8) by

(14) $-P(g_1,\alpha)_* + PT(f_1,\alpha)_* + \alpha_*P(d,1)_* =$
$-P(g_1,\alpha)_* + P(f_1,\alpha)_* + P(\alpha d,\alpha)_* = 0$

In the last equation we use $\alpha d = -f_1 + g_1$. Finally we obtain (9) by

(15) $-(1,d)_*(\alpha d,f_1)_*H + (\alpha,f_0)_*Hd_* = -(\alpha d,df_1)_*H + (\alpha d,f_0d)_*H = 0$

Here we use $df_1 = f_0d$. This completes the proof of theorem (6.4) for M_*. The proof for M^* uses the 'dual' arguments. Let $f = (f^0,f^1)$, $g = (g^0,g^1)$ be maps $d' \longrightarrow d$ in $\underline{\text{Pair}}(\underline{R})$ and let $\alpha : f \simeq g$ be a homotopy. Then we define a homotopy

(16) $$\beta : M^*(f) \simeq M^*(g)$$

by the matrices (17) and (18).

(17) $$\beta^0 = (B_1,B_2) \text{ with } \begin{cases} B_1 = \alpha_* \\ B_2 = P(\alpha,f^0)_* \end{cases}$$

(18) $$\beta^1 = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \text{ with } \begin{cases} A_1 = P(\alpha d',f^1)_* \\ A_2 = -(g^1,\alpha)_* + T(f^1,\alpha)_* . \end{cases}$$

One can check as above that (16) is satisfied.

§7 *Twelve quadratic functors induced by a quadratic $\mathbb{Z}$–module*

For a $\mathbb{Z}$–module M one has the functors which carry an abelian group A to the group

$$A \otimes M, \ A*M, \ \mathrm{Hom}(A,M) \ \text{and} \ \mathrm{Ext}(A,M)$$

respectively. We now introduce for a quadratic $\mathbb{Z}$–module M twelve quadratic functors which generalize these classical functors. Using short free resolutions we obtain functors

(7.1) $\quad i \ : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Pair}}(\underline{\underline{Ab}})/\simeq$ and $i^{op} : \underline{\underline{Ab}}^{op} \longrightarrow \underline{\underline{Pair}}(\underline{\underline{Ab}}^{op})/\simeq$

as follows. For each abelian group A we choose a short exact sequence

$$G \overset{d_A}{\rightarrowtail} F \overset{q}{\twoheadrightarrow} A$$

where G and F are free abelian groups and we set $i(A) = d_A$. For a homomorphism $\varphi : A \longrightarrow B$ we can choose a map $f : d_A \longrightarrow d_B$ in $\underline{\underline{Pair}}(\underline{\underline{Ab}})$ which induces φ. The homotopy class $\{f\}$ of f is well defined by φ and we set $i(\varphi) = \{f\}$. The functor i is actually full and faithful. The functor i^{op} is induced by i.

A quadratic $\mathbb{Z}$–module M yields the quadratic functors

(7.2) $\quad (\_)\otimes_{\mathbb{Z}} M : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ and $\mathrm{Hom}(\_,M) : \underline{\underline{Ab}}^{op} \longrightarrow \underline{\underline{Ab}}$

which as well yield a quadratic $\underline{\underline{Ab}}$–module $\{\_\}\otimes_{\mathbb{Z}} M$ and a quadratic $\underline{\underline{Ab}}^{op}$–module $\mathrm{Hom}\,\{\_,M\}$ respectively, compare (4.2)(5), (6) and (5.2)(5), (6). We now use (6.4) and (7.1) for the definition of the <u>quadratic chain functors</u>

(7.3)
$$\begin{aligned}
&(\{_\}\otimes_{\mathbb{Z}} M)_* \, i : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}_*/\simeq \,, \\
&(\{_\}\otimes_{\mathbb{Z}} M)^* \, i : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}^*/\simeq, \\
&(\mathrm{Hom}\{_,M\})_* i^{op} : \underline{\underline{Ab}}^{op} \longrightarrow \underline{\underline{Ab}}_*/\simeq \,, \\
&(\mathrm{Hom}\{_,M\})^* i^{op} : \underline{\underline{Ab}}^{op} \longrightarrow \underline{\underline{Ab}}^*/\simeq .
\end{aligned}$$

The (co)homology groups of these four quadratic chain functors yield six functors $\underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ and six functors $\underline{\underline{Ab}}^{op} \longrightarrow \underline{\underline{Ab}}$ which we denote as follows where $d_A = i(A)$ as in (7.1) and where $j = 0,1$, resp. 2.

(7.4) $H_j(\{\_\}\otimes_{\mathbb{Z}}M)_* d_A$ = A⊗M, A∗'M, resp. A∗"M,
$H^j(\{\_\}\otimes_{\mathbb{Z}}M)^* d_A$ = A∗M, A⊗'M, resp. A⊗"M,
$H_j(\mathrm{Hom}\{\_,M\})_* d_A^{op}$ = Ext(A,M), Hom'(A,M), resp. Hom"(A,M),
$H^j(\mathrm{Hom}\{\_,M\})^* d_A^{op}$ = Hom(A,M), Ext'(A,M), resp. Ext"(A,M).

The first nine of these functors appear in the exact sequences of [4], see (2.5), (3.5) and (4.5) in [4]. For the convenience of the reader we now describe explicitly the chain complexes used in (7.4). For this we choose $d = d_A : G \longrightarrow F$ as in (7.1).

(1) The chain complex $(\{\_\}\otimes_{\mathbb{Z}}M)_* d_A$ is given by

$$G\otimes G\otimes M_{ee} \xrightarrow{(P,-d_*)} G\otimes_{\mathbb{Z}}M \oplus G\otimes F\otimes M_{ee} \xrightarrow{(d_*,Pd_*)} F\otimes_{\mathbb{Z}} M.$$

(2) The cochain complex $(\{\_\}\otimes_{\mathbb{Z}}M)^* d_A$ is given by

$$F\otimes F\otimes M_{ee} \xleftarrow{(H,-d_*)} F\otimes_{\mathbb{Z}}M \oplus F\otimes G\otimes M_{ee} \xleftarrow{(d_*,d_*H)} G\otimes_{\mathbb{Z}} M.$$

(3) The chain complex $(\mathrm{Hom}\{\_,M\})_* d_A^{op}$ is given by

$$\mathrm{Hom}(F\otimes F,M_{ee}) \xrightarrow{(P,-d^*)} \mathrm{Hom}_{\mathbb{Z}}(F,M)\oplus\mathrm{Hom}(F\otimes G,M_{ee}) \xrightarrow{(d^*,Pd^*)} \mathrm{Hom}_{\mathbb{Z}}(G,M).$$

(4) The cochain complex $(\mathrm{Hom}\{\_,M\})^* d_A^{op}$ is given by

$$\mathrm{Hom}(G\otimes G,M_{ee}) \xleftarrow{(H,-d^*)} \mathrm{Hom}_{\mathbb{Z}}(G,M) \oplus \mathrm{Hom}(G\otimes F,M_{ee}) \xleftarrow{(d^*,d^*H)} \mathrm{Hom}_{\mathbb{Z}}(F,M)\,.$$

Here d_*, d^* denote the maps induced by d and the formulas for H and P are described in (4.2)(7) and (5.2)(7) respectively. The degree of the group at the right hand side in each sequence above is 0.

(7.5) Remark: The notation in (7.4) is chosen since there is the following compatibility with classical functors. Assume M is a $\mathbb{Z}$–module, that is $M_{ee} = 0$, then one readily verifies that the groups
A⊗M = A⊗'M, A∗M = A∗'M, Hom(A,M) = Hom'(A,M), Ext(A,M) = Ext'(A,M)
are given by the corresponding classical functors for abelian groups. Moreover all groups A∗"M, A⊗"M, Hom"(A,M) and Ext"(A,M) with $j = 2$ in (7.4) are trivial for $M_{ee} = 0$.

(7.6) Proposition: One has natural isomorphisms $A \otimes M = A \otimes_{\mathbb{Z}} M$ and $\mathrm{Hom}(A,M) = \mathrm{Hom}_{\mathbb{Z}}(A,M)$ where the right hand side is defined by (4.1) and (5.1) respectively. Compare also (8.10).

(7.7) Proposition: All functors in (7.4) are additive in M and quadratic in A. The quadratic cross effects are naturally given by

$$\begin{aligned}
&(A\,|\,B)\otimes M = A\otimes B\otimes M_{ee} = (A\,|\,B)\otimes'' M\\
&(A\,|\,B)*M = A*B*M_{ee} = (A\,|\,B)*'' M\\
&\mathrm{Ext}(A\,|\,B,M) = \mathrm{Ext}(A*B,M_{ee}) = \mathrm{Ext}''(A\,|\,B,M)\\
&\mathrm{Hom}(A\,|\,B,M) = \mathrm{Hom}(A\otimes B,M_{ee}) = \mathrm{Hom}''(A\,|\,B,M)\\
&(A\,|\,B)*'M = H_1(d_A\otimes d_B,M_{ee}) = (A\,|\,B)\otimes' M\\
&\mathrm{Hom}'(A\,|\,B,M) = H^1(d_A\otimes d_B,M_{ee}) = \mathrm{Ext}'(A\,|\,B,M).
\end{aligned}$$

Here d_A denotes as well the chain complex (X_*,d) with $d = d_A : X_1 = G \longrightarrow X_0 = F$, $X_i = 0$ for $i \geq 2$. The Künneth formula yields natural exact sequences

(1) $$(A*B)\otimes M_{ee} \rightarrowtail H_1(d_A\otimes d_B,M_{ee}) \twoheadrightarrow (A\otimes B)*M_{ee},$$

(2) $$\mathrm{Ext}(A\otimes B,M_{ee}) \rightarrowtail H^1(d_A\otimes d_B,M_{ee}) \twoheadrightarrow \mathrm{Hom}(A*B,M_{ee})$$

These sequences are split, the splitting however is not natural. There is a natural isomorphism

(3) $$H_1(d_A\otimes d_B,M_{ee}) = \mathrm{Trip}(A,B,M_{ee})$$

where the right hand side is the triple torsion product of Mac Lane [12].

Proof of (7.7): We consider for $N = \{\_\}\otimes_{\mathbb{Z}} M$ the functor $N_*: \underline{\mathrm{Pair}(\underline{\mathrm{Ab}})}/\simeq \longrightarrow \underline{\mathrm{Ab}}_*/\simeq$, see (7.3). This functor is quadratic and its quadratic cross effect admits a weak equivalence

$$\Phi : N_*(d_A\,|\,d_B) \xrightarrow{\sim} d_A\otimes d_B\otimes M_{ee}$$

of chain complexes. For $d_A : X_1 \longrightarrow X_0$ and $d_B : Y_1 \longrightarrow Y_0$ and $C_* = N_*(d_A\,|\,d_B)$ we have

$$\begin{aligned}
C_0 &= X_0\otimes Y_0\otimes M_{ee}\\
C_1 &= X_1\otimes Y_1\otimes M_{ee}\oplus X_1\otimes Y_0\otimes M_{ee}\oplus Y_1\otimes X_0\otimes M_{ee}\\
C_2 &= X_1\otimes Y_1\otimes M_{ee}\oplus Y_1\otimes X_1\otimes M_{ee}
\end{aligned}$$

The differential $d_i : C_i \longrightarrow C_{i-1}$ is given by

$$d_2(x_1 \otimes y_1 \otimes n) = x_1 \otimes y_1 \otimes n - x_1 \otimes d_B y_1 \otimes n$$
$$d_2(y_1 \otimes x_1 \otimes n) = x_1 \otimes y_1 \otimes Tn - y_1 \otimes d_A x_1 \otimes n$$
$$d_1(x_1 \otimes y_1 \otimes n) = d_A x_1 \otimes d_B y_1 \otimes n$$
$$d_1(x_1 \otimes y_0 \otimes n) = d_A x_1 \otimes y_0 \otimes n$$
$$d_1(y_1 \otimes x_0 \otimes n) = x_0 \otimes d_B y_1 \otimes Tn$$

where $y_i \in Y_i$, $x_i \in X_i$, $n \in M_{ee}$. The map Ψ is given by the identity in degree 0 and by

$$\Psi_2(x_1 \otimes y_1 \otimes n) = 0$$
$$\Psi_2(y_1 \otimes x_1 \otimes n) = x_1 \otimes y_1 \otimes Tn$$
$$\Psi_1(x_1 \otimes y_1 \otimes n) = x_1 \otimes d_B y_1 \otimes n$$
$$\Psi_1(x_1 \otimes y_0 \otimes n) = x_1 \otimes y_0 \otimes n$$
$$\Psi_1(y_1 \otimes x_0 \otimes n) = x_0 \otimes y_1 \otimes Tn$$

Since $H_j N_*(d_A | d_B)$ is the cross effect in $H_j N_*(d_A \oplus d_B)$ we obtain $(A|B) \otimes M$, $(A|B)*'M$ and $(A|B)*''M$ by the weak equivalence Ψ and by the Künneth formula. In a similar way one obtains the other cross effects in (7.7).

(7.8) Proposition: There are natural inclusions and projections of abelian groups

$$A*''M \rightarrowtail A * A * M_{ee},$$
$$A\otimes''M \twoheadleftarrow A \otimes A \otimes M_{ee},$$
$$\mathrm{Hom}''(A,M) \rightarrowtail \mathrm{Hom}(A\otimes A, M_{ee}),$$
$$\mathrm{Ext}''(A,M) \twoheadleftarrow \mathrm{Ext}(A * A, M_{ee}).$$

Proof: We only consider the first inclusion. For this we see by (7.4)(1), that $A*''M$ is the intersection $(d_* = 1 \otimes d \otimes 1)$.

$$\ker(P) \cap \ker(-d_*) \subset G \otimes (A*M_{ee}) \subset G \otimes G \otimes M_{ee}$$

where $\ker(-d_*) = G \otimes (A*M_{ee})$. We have to show $(d \otimes 1 \otimes 1)(A*''M) = 0$. Then the first inclusion in (7.8) is given. Let $T : G \otimes G \otimes M_{ee} \longrightarrow G \otimes G \otimes M_{ee}$ be the interchange map with $T(x \otimes y \otimes n) = y \otimes x \otimes Tn$. Since $HP = 1 + T$ we see that T restricted to $\ker(P)$ is -1. Whence we get for $x \in A*''M$

$(d \otimes 1 \otimes 1)(x) = -(d \otimes 1 \otimes 1)T(x) = -T(1 \otimes d \otimes 1)(x) = 0.$

(7.9) Remark: Using (7.7) it is easy to compute the functors (7.4) for finitely generated abelian groups A. For this we need only to consider cyclic groups $\mathbb{Z}/n = A$ with the presentation $d_A = n : \mathbb{Z} = G \longrightarrow \mathbb{Z} = F$. In this case we have $\mathbb{Z} \otimes_{\mathbb{Z}} M = M_e$ and

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},M) = M_e$; therefore the chain complexes (7.4)(1)...(4) can be expressed in terms of H,P in the quadratic $\mathbb{Z}$–module M. In particular (7.4)(1), resp. (2), is given for $d_A = n$ by

(1) $M_{ee} \xrightarrow{(P,-n)} M_e \oplus M_{ee} \xrightarrow{(n_*,nP)} M_e$, resp.

(2) $M_{ee} \xleftarrow{(H,-n)} M_e \oplus M_{ee} \xleftarrow{(n_*,nH)} M_e$

where n_* is defined in (2.1). In addition we can use the following formulas for the computation.

(7.10) <u>Proposition</u>: Let A be a finite abelian group and let $A^E = \mathrm{Ext}(A,\mathbb{Z})$. Then one has the natural isomorphisms

$$\mathrm{Ext}\,(A,M) = A^E \otimes M, \qquad \mathrm{Hom}(A,M) = A^E * M,$$
$$\mathrm{Hom}'(A,M) = A^E *' M, \qquad \mathrm{Ext}'(A,M) = A^E \otimes' M,$$
$$\mathrm{Hom}''(A,M) = A^E *'' M, \qquad \mathrm{Ext}''(A,M) = A^E \otimes'' M.$$

There is a non natural isomorphism $A^E \cong A$.

<u>Proof</u>: Since A is finite we obtain a presentation of $\mathrm{Ext}(A,\mathbb{Z})$ by $d_A^* : F^\# = \mathrm{Hom}(F,\mathbb{Z}) \longrightarrow G^\# = \mathrm{Hom}(G,\mathbb{Z})$. Using (5.5) we can replace $\mathrm{Hom}_{\mathbb{Z}}(F,M)$ by $F^\# \otimes_{\mathbb{Z}} M$. This way the chain complex (7.4)(3) for d_A is the same as the chain complex (7.4)(1) for d_A^*. This proves the left hand side of equations in (7.10).

(7.11) <u>Remark</u>: The 12 functors in (7.4) evaluated on $A = \mathbb{Z}$ are given by the table

$\mathbb{Z}\otimes M = M_e$	$\mathbb{Z}*'M = 0$	$\mathbb{Z}*''M = 0$
$\mathbb{Z}*M = 0$	$\mathbb{Z}\otimes'M = \ker H$	$\mathbb{Z}\otimes''M = \mathrm{cok}\, H$
$\mathrm{Ext}(\mathbb{Z},M) = 0$	$\mathrm{Hom}'(\mathbb{Z},M) = \mathrm{cok}\, P$	$\mathrm{Hom}''(\mathbb{Z},N) = \ker P$
$\mathrm{Hom}(\mathbb{Z},M) = M_e$	$\mathrm{Ext}'(\mathbb{Z},M) = 0$	$\mathrm{Ext}''(\mathbb{Z},M) = 0.$

Here H,P are the maps of the quadratic $\mathbb{Z}$–module M.

(7.12) <u>Examples</u>: Eilenberg–Mac Lane introduced quadratic functors $R : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$ and $\Omega : \underline{\underline{Ab}} \longrightarrow \underline{\underline{Ab}}$, compare § 22 and § 13 respectively in [9]. For these functors one has a new interpretation since there are natural isomorphisms (see (3.15) and (3.7) in [4])

$$R(A) \cong A*'\mathbb{Z}^{\Gamma} \text{ and } \Omega(A) \cong A\otimes'\mathbb{Z}^{\Gamma}.$$

Here $\mathbb{Z}^{\Gamma}$ is the quadratic $\mathbb{Z}$–module in (2.11); we do not know, whether for example also $A*'\mathbb{Z}^S$ and $A \otimes'\mathbb{Z}^S$ play a role in the literature. Generalizations of the functors R and Ω are described by J. Decker, see III (4.2)[7] and also [5].

§8 *Quadratic derived functors*

In this section we associate with a quadratic $\underline{\underline{R}}$–module M a chain functor and a cochain functor. If we apply these functors to a projective (resp. injective) resolution we get the quadratic derived functors which coincide with the classical derived functors in case $M_{ee} = 0$. We understand that Dold–Puppe [8] obtained derived functors of non additive functors which as well generalized the classical derived functors of an additive functor; the construction of the quadratic derived functors below is different and relies on the structure of a quadratic module.

Let $\underline{\underline{R}}$ be a ringoid with a zero object. An $\underline{\underline{R}}$–module M yields the following chain functors which are as well denoted by M

(8.1) $$M : \underline{\underline{R}}_*/\simeq \;\longrightarrow\; \underline{\underline{Ab}}_*/\simeq \text{ and } M : \underline{\underline{R}}^*/\simeq \;\longrightarrow\; \underline{\underline{Ab}}^*/\simeq,$$

compare the notation in (6.1). For a chain complex X_* in $\underline{\underline{R}}_*$ we define $M(X_*)$ simply by setting $M(X_*)_n = M(X_n)$. The differential d_* in $M(X_*)$ is induced by the differential d in X_*, $d_* = M(d)$. Similarly we get induced chain maps $M(F)$ with $M(F)_n = M(F_n)$ and induced chain homotopies $M(\alpha)$ with $M(\alpha)_n = M(\alpha_n)$. Since M is an additive functor one readily observes that this chain functor is well defined. In the same way one gets the cochain functor M which carries $X^* \in \underline{\underline{R}}^*$ to the cochain complex $M(X^*)$.

Now let M be a quadratic $\underline{\underline{R}}$–module. We associate with M the quadratic chain functors M as in (8.1) which again are simply denoted by M, see (8.2) and (8.3). In fact, if $M_{ee} = 0$ these chain functors coincide with the additive functors above.

(8.2) Definition: For X_* in $\underline{\underline{R}}_*$ the chain complex $C_* = M(X_*)$ is given by the abelian groups $(n \geq 2)$

$$(1)\begin{cases} C_0 = M(X_0) \\ C_1 = \mathrm{cok}\{(P,-(1,d)_*) \ : M(X_1, X_1) \to M(X_1) \oplus M(X_1,X_0)\} \\ C_n = \mathrm{cok}\{P\oplus(1,d)_*: M(X_n,X_n)\oplus M(X_n,X_1) \longrightarrow M(X_n)\oplus M(X_n,X_0)\} \end{cases}$$

The differential $d = d_n : C_n \longrightarrow C_{n-1}$ is induced by the maps

$$(2) \qquad \begin{cases} d_1 = (d_*, P(d,1)_*), \\ d_n = d_*\oplus (d,1)_*, \ n \geq 2. \end{cases}$$

For a chain map $F : X_* \longrightarrow Y_*$ we get the induced chain map $M(F) : MX_* \longrightarrow MY_*$ by

$$(3) \qquad \begin{cases} (MF)_0 = (F_0)_* , \\ (MF)_n = (F_n)_* \oplus (F_n,F_0)_*, \ n \geq 1 . \end{cases}$$

Finally a chain homotopy $\alpha : F \simeq G$, $\alpha_n : X_{n-1} \longrightarrow Y_n$ in $\underline{\underline{R}}_*$ yields a chain homotopy $M\alpha : MF \simeq MG$ by

$$(4) \qquad \begin{cases} (M\alpha)_1 = ((\alpha_1)_*, (\alpha_1, F_0)_* H), \\ (M\alpha)_n = (\alpha_n)_* \oplus (\alpha_n, F_0)_*, \ n \geq 2. \end{cases}$$

The next definition is dual to (8.2).

(8.3) Definition: For X^* in $\underline{\underline{R}}^*$ the cochain complex $C^* = MX^*$ is given by the abelian groups $(n \geq 2)$

$$(1)\begin{cases} C^0 = M(X^0) \\ C^1 = \mathrm{ker}\{(H,-(1,d)_*) \ : \ M(X^1) \oplus M(X^1,X^0) \longrightarrow M(X^1,X^1)\} \\ C^n = \mathrm{ker}\{H\oplus(1,d)_* \ : \ M(X^n) \oplus M(X^n,X^0) \longrightarrow M(X^n,X^n)\oplus M(X^n,X^0)\} \end{cases}$$

The differential $d = d^n : C^n \longrightarrow C^{n+1}$ is induced by the maps

$$(2) \qquad d^1 = (d_*,(d,1)_* H), \ d^n = d_*\oplus(d,1)_*, \ n\geq 2 .$$

For a chain map $F : X^* \longrightarrow Y^*$ we get the induced chain map $M(F) : MX^* \longrightarrow MY^*$ by

$$(3) \qquad (MF)^0 = (F^0)_*, \ (MF)^n = (F^n)_*\oplus(F^n,F^0)_*, \ n\geq 1 .$$

Finally a chain homotopy $\alpha : F \simeq G$ $(\alpha^n : X^{n+1} \longrightarrow Y^n)$ in $\underline{\underline{R}}^*$ yields a chain homotopy $M\alpha : MF \simeq MG$ by

$$(4) \qquad (M\alpha)^0 = ((\alpha^0)_*, P(\alpha^0,F^0)_*), \ (M\alpha)^n = (\alpha^n)_*\oplus(\alpha^n,F^0)_*, \ n\geq 1.$$

(8.4) Proposition: The definitions (8.2) and (8.3) yield well defined functors $M : \underline{\underline{R}}_*/\simeq \longrightarrow \underline{\underline{Ab}}_*/\simeq$ and $M : \underline{\underline{R}}^*/\simeq \longrightarrow \underline{\underline{Ab}}^*/\simeq$ respectively.

The functors M in (8.4) are quadratic, the cross effect of these functors is described in §9 below. The proof of (8.4) is similar to the proof of (6.4), in fact (6.4) can be used for the 1–dimensional part of the proposition, compare (8.5) below.

We point out that the definition of the quadratic chain functors relies on the structure maps H and P of the quadratic $\underline{\underline{R}}$–module M; a functor $\underline{\underline{R}} \longrightarrow \underline{\underline{Ab}}$ which is merely quadratic is not appropriate for the definition of the functors in (8.4).

(8.5) Remark: The quadratic chain functors M_* and M^* in (6.3) are related to the quadratic chain functors M in (8.4) as follows. Let $d_1 : X_1 \longrightarrow X_0$ and $d^0 : X^0 \longrightarrow X^1$ be given by X_* and X^* respectively. Then the 1–dimensional part of MX_*, resp. of MX^*, coincides with the map

$$M_1(d_1)/\text{boundaries} \longrightarrow M_0(d_1), \text{ resp. } M^0(d^0) \longrightarrow \text{cycles} \subset M^1(d^0),$$

compare the definition in (6.3) and (8.2), (8.3). This shows that for $X_i = 0$, $X^i = 0$, $i \geq 2$, one has isomorphic homology groups $H_iMX_* = H_iM_*(d_1)$, $H^iMX^* = H^iM^*(d^0)$ for $i = 0,1$. The homology $H_2M_*(d_1)$ and $H^2M^*(d^0)$, however, cannot be obtained by MX_* and MX^* respectively.

We now assume that the additive category $\underline{\underline{A}}$ is an abelian category with enough projectives and injectives respectively, for example $\underline{\underline{A}} = \underline{\underline{M}}(\underline{\underline{R}})$. The homology of chain complexes in $\underline{\underline{A}}$ is defined. We say that X_* is a projective resolution of $X \in \mathrm{Ob}(\underline{\underline{A}})$ if a chain map $\epsilon : X_* \longrightarrow X$ in $\underline{\underline{A}}_*$ is given (which induces an isomorphism in homology) where all X_i of X_* are projective in $\underline{\underline{A}}$ and where X is the chain complex concentrated in degree 0. On the other hand X^* is an injective resolution of X if a chain map $\epsilon : X \longrightarrow X^*$ in $\underline{\underline{A}}^*$ is given (which induces an isomorphism in cohomology) where all X^i of X^* are injective in $\underline{\underline{A}}$. It is well known that the choice of resolutions X_*, X^* yields functors $i : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}^*/\simeq$ and $j : A \longrightarrow A_*/\simeq$ which are well defined up to canonical isomorphisms.

(8.6) Definition: Let $\underline{\underline{A}}$ be an abelian category as above and let $M : \underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}$ be a quadratic functor. Then (3.5) shows that M yields a quadratic $\underline{\underline{A}}$–module

$M = M\{A\}$ as well denoted by M. Using the resolution functors i,j above and using (8.4) one gets functors

(1) $$Mi : \underline{A} \longrightarrow \underline{Ab}_*/\simeq \quad \text{and} \quad Mj : \underline{A} \longrightarrow \underline{Ab}^*/\simeq .$$

The n–th (co)homology of these functors yields the <u>quadratic</u> <u>derived</u> <u>functors</u> $L_nM : \underline{A} \longrightarrow \underline{Ab}$, $R^nM : \underline{A} \longrightarrow \underline{Ab}$ respectively, $n \geq 0$. For $X \in Ob(\underline{A})$ one has

(2) $$(L_nM)X = H_nMX_* \quad \text{and} \quad (R^nM)X = H^nMX^*$$

where X_*, X^* are resolutions as above. The chain complexes MX_*, MX^* are defined as in (8.2), (8.3).

(8.7) <u>Remark</u>: In case M in (8.6) is an additive functor, that is $M_{ee} = 0$, the derived functors coincide with the classical derived functors of M, see for example [6], [10]. For a quadratic functor M Dold–Puppe [8] as well defined derived functors; their construction, however, is different to the one in (8.6) and is available for any non additive functor $\underline{A} \longrightarrow \underline{Ab}$. Our definition in (8.6) is adapted especially to quadratic functors.

(8.8) <u>Definition</u>: Let $\underline{A}$ be an abelian category and let $M : \underline{A} \longrightarrow \underline{Ab}$ be a quadratic functor. We say that M is <u>quadratic</u> <u>right</u> <u>exact</u> if each exact sequence $X_1 \xrightarrow{d} X_0 \xrightarrow{q} X \longrightarrow 0$ in $\underline{A}$ induces an exact sequence

$$M(X_1) \oplus M(X_1 | X_0) \xrightarrow{(d_*, P(d,1)_*)} M(X_0) \xrightarrow{q_*} M(X) \longrightarrow 0 .$$

We say that M is <u>quadratic</u> <u>left</u> <u>exact</u> if each exact sequence $0 \longrightarrow X \xrightarrow{i} X^0 \xrightarrow{d} X^1$ in $\underline{A}$ induces an exact sequence

$$0 \longrightarrow M(X) \xrightarrow{i_*} M(X^0) \xrightarrow{(d_*, (d,1)_* H)} M(X^1) \oplus M(X^1 | X^0) .$$

The definitions immediately imply as in the classical case:

(8.8) <u>Lemma</u>: Let $M : \underline{A} \longrightarrow \underline{Ab}$ be quadratic right exact then one has the natural isomorphism $M \cong L_0M$. Dually if M is quadratic left exact one has the natural isomorphism $M \cong R^0M$.

As examples of quadratic derived functors we obtain the following <u>quadratic</u> <u>Tor</u> <u>and</u> <u>Ext</u> <u>functors</u> for a small ringoid $\underline{R}$, $n \geq 0$.

(8.9) $$\mathrm{Tor}_n^{\underline{\underline{R}}} : \underline{\underline{M}}(\underline{\underline{R}})^{op} \times \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}},$$
$$\mathrm{Ext}^n_{\underline{\underline{R}}} : \underline{\underline{M}}(\underline{\underline{R}})^{op} \times \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}.$$

For M in $\underline{\underline{QM}}(\underline{\underline{R}})$ these functors are derived from the quadratic functors

(1) $$\_ \otimes_{\underline{\underline{R}}} M : \underline{\underline{M}}(\underline{\underline{R}}^{op}) \longrightarrow \underline{\underline{Ab}},$$

(2) $$\mathrm{Hom}_{\underline{\underline{R}}}(\_,M) : \underline{\underline{M}}(\underline{\underline{R}})^{op} \longrightarrow \underline{\underline{Ab}},$$

that is, for a projective resolution X_* of X in $\underline{\underline{M}}(\underline{\underline{R}}^{op})$ and for a projective resolution Y_* of Y in $\underline{\underline{M}}(\underline{\underline{R}})$ we set

(3) $$\mathrm{Tor}_n^{\underline{\underline{R}}}(X,M) = L_n(\_\otimes_{\underline{\underline{R}}} M)(X) = H_n((\_\otimes_{\underline{\underline{R}}} M)(X_*)),$$

(4) $$\mathrm{Ext}^n_{\underline{\underline{R}}}(Y,M) = R^n\mathrm{Hom}_{\underline{\underline{R}}}(\_,M)(Y) = H^n(\mathrm{Hom}_{\underline{\underline{R}}}(\_,M)(Y_*)).$$

In (4) we consider Y_* as an injective resolution in $\underline{\underline{M}}(\underline{\underline{R}})^{op}$ and we use (8.3). Clearly the groups (3), (4) are trivial for $n \geq 1$ in case X and Y are projective objects in $\underline{\underline{A}}$. The functors (8.9) are quadratic in the first variable and additive in the second variable.

(8.10) Proposition: The functor $\_\otimes_{\underline{\underline{R}}} M$ is quadratic right exact and the functor $\mathrm{Hom}_{\underline{\underline{R}}}(\_,M)$ is quadratic left exact so that we have natural isomorphisms (see (8.8))

$$\mathrm{Tor}_0^{\underline{\underline{R}}}(X,M) = X \otimes_{\underline{\underline{R}}} M \quad \text{and} \quad \mathrm{Ext}^0_{\underline{\underline{R}}}(Y,M) = \mathrm{Hom}_{\underline{\underline{R}}}(Y,M)\,.$$

In case M is an $\underline{\underline{R}}$–module, that is $M_{ee} = 0$, the Tor and Ext groups above coincide with the classical groups, see [10].

(8.11) Example: Let $\underline{\underline{R}} = \mathbb{Z}$ be the ring of integers and let M be a quadratic $\mathbb{Z}$–module. For an abelian group A one gets (see (7.4)) $\mathrm{Tor}_1^{\mathbb{Z}}(A,M) = A*'M$ and $\mathrm{Ext}^1_{\mathbb{Z}}(A,M) = \mathrm{Ext}'(A,M)$. This follows since d_A in (7.4) is a projective resolution of A, see (8.5). Clearly $\mathrm{Tor}_n^{\mathbb{Z}} = 0 = \mathrm{Ext}^n_{\mathbb{Z}}$ for $n \geq 2$ since the chain complex d_A is 1–dimensional.

We also introduce functors

$$(8.12)\quad \begin{cases} \mathrm{Tor}_n^{\underline{\underline{R}}} : \underline{\underline{M}}(\underline{\underline{R}})^{op} \times \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}, \\ \mathrm{Ext}_{\underline{\underline{R}}}^n : \underline{\underline{M}}(\underline{\underline{R}})^{op} \times \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}. \end{cases}$$

For $A \in \underline{\underline{M}}(\underline{\underline{R}}^{op})$, resp. $A \in \underline{\underline{M}}(\underline{\underline{R}})^{op}$, these functors are derived from the additive functors

(1) $\qquad A \otimes_{\underline{\underline{R}}} \_ : \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$, resp.

(2) $\qquad \mathrm{Hom}_{\underline{\underline{R}}}(A,\_) : \underline{\underline{QM}}(\underline{\underline{R}}) \longrightarrow \underline{\underline{Ab}}$.

That is, for a projective resolution M_*, resp. injective resolution M^* of $M \in \underline{\underline{QM}}(\underline{\underline{R}})$ we set

(3) $\qquad \mathrm{TOR}_n^{\underline{\underline{R}}}(A,M) = L_n(A\otimes_{\underline{\underline{R}}}\_)M = H_n(A\otimes_{\underline{\underline{R}}}M_*)$,

(4) $\qquad \mathrm{EXT}_{\underline{\underline{R}}}^n(A,M) = R^n(\mathrm{Hom}_{\underline{\underline{R}}}(A,\_))M = H^n(\mathrm{Hom}_{\underline{\underline{R}}}(A,M^*))$.

Clearly these groups are trivial for $n \geq 1$ if M is a projective, resp. injective, object in $\underline{\underline{QM}}(\underline{\underline{R}})$. Moreover, since (1), (2) are additive functors, all the usual results of homological algebra are available, for example in (8.16) we apply the long exact sequences induced by short exact sequences, see IV. (6.1) [10].

(8.13) Proposition. The functor $A\otimes_{\underline{\underline{R}}}\_$ is right exact and the functor $\mathrm{Hom}_{\underline{\underline{R}}}(A,\_)$ is left exact, so that $\mathrm{TOR}_0^{\underline{\underline{R}}}(A,M) = A\otimes_{\underline{\underline{R}}}M = \mathrm{Tor}_0^{\underline{\underline{R}}}(A,M)$, and

$\mathrm{EXT}_{\underline{\underline{R}}}^0(A,M) = \mathrm{Hom}_{\underline{\underline{R}}}(A,M) = \mathrm{Ext}_{\underline{\underline{R}}}^0(A,M)$.

It is a classical result (see for example [10]) that for $M_{ee} = 0$ there are also natural isomorphisms $\mathrm{TOR}_n^{\underline{\underline{R}}} = \mathrm{Tor}_n^{\underline{\underline{R}}}$ and $\mathrm{Ext}_{\underline{\underline{R}}}^n = \mathrm{EXT}_{\underline{\underline{R}}}^n$, $n \geq 1$. The next example shows that this is not tree for $M_{ee} \neq 0$.

(8.14) Example: Let $\underline{\underline{R}} = \mathbb{Z}$ and $M = Q$, see (2.2). Then Q is a free (whence projective) Q–module and therefore $\mathrm{TOR}_n^{\mathbb{Z}}(A,Q) = 0$ for $n \geq 1$. On the other hand

$Tor_1^{\mathbb{Z}}(A,Q) = A*'Q$ has the cross effect $(A|B)*'Q = (A*B)\otimes\mathbb{Z}^3$, see (7.7). Therefore $TOR_1^{\mathbb{Z}} \neq Tor_1^{\mathbb{Z}}$.

(8.15) Example: One has a short exact sequence $0 \to \mathbb{Z}^S \to \mathbb{Z}^{\tilde{\Gamma}} \to \mathbb{Z} \to 0$ in $\underline{QM}(\mathbb{Z})$, see (2.11). This sequence induces the exact sequence

$$TOR_1^{\mathbb{Z}}(A,\mathbb{Z}) \to A\otimes\mathbb{Z}^S \to A\otimes\mathbb{Z}^{\tilde{\Gamma}} \to A\otimes\mathbb{Z} \to 0$$

where $TOR_1^{\mathbb{Z}}(A,\mathbb{Z}) = A*\mathbb{Z} = 0$. Whence one gets this way the top row of (2.10), see (4.9). On the other hand the exact sequence $0 \to \mathbb{Z}^S \to \mathbb{Z}^\Gamma \to \mathbb{Z}/2 \to 0$ in $\underline{QM}(\mathbb{Z})$, induces the exact sequence

$$TOR_1^{\mathbb{Z}}(A,\mathbb{Z}^\Gamma) \to TOR_1^{\mathbb{Z}}(A,\mathbb{Z}/2) \xrightarrow{\delta} A\otimes\mathbb{Z}^S \xrightarrow{i} A\otimes\mathbb{Z}^\Gamma \to A\otimes\mathbb{Z}/2 \to 0$$

where $TOR_1^{\mathbb{Z}}(A,\mathbb{Z}/2) = A*\mathbb{Z}/2$. We now show that i is injective so that $\delta = 0$. In fact i is injective if A is cyclic and whence also if A is finitely generated. Since $\_\otimes M$ commutes with direct limits we see that i is injective for any abelian group A.

§9 *The cross effect of quadratic derived functors*

We introduce biderived functors which describe the cross effects of the quadratic derived functors in §8. Moreover we discuss various exact sequences for these functors. We assume that $\underline{R}$ is a ringoid with a zero object.

(9.1) Definition: Let M be an $\underline{R}\otimes\underline{R}$–module, see (1.7). Then we define the additive functor

$$M : \underline{R}_*/\simeq \otimes \underline{R}_*/\simeq \to \underline{Ab}_*/\simeq$$

(as well denoted by M) as follows. For chain complexes X_*, Y_* in $\underline{R}_*$ we get $C_* = M(X_*,Y_*)$ by $(n \geq 2)$

$$(1)\quad \begin{cases} C_0 = M(X_0,Y_0) \\ C_1 = \mathrm{cok}\{((1,d)_*;-(d,1)_*): M(X_1,Y_1) \to M(X_1,Y_0)\oplus M(X_0Y_1)\} \\ C_n = \mathrm{cok}\{(1,d)_*\oplus(d,1)_*: M(X_n,Y_1)\oplus M(X_1,Y_n) \to M(X_n,Y_0)\oplus M(X_0,Y_n)\} \end{cases}$$

The differential $d = d_n : C_n \to C_{n-1}$ is induced by the maps

$$(2)\qquad \begin{cases} d_1 = ((d,1)_*,\ (1,d)_*)\,, \\ d_n = (d,1)_* \oplus (1,d)_*, \quad n \geq 2. \end{cases}$$

For chain maps $F : X_* \longrightarrow X'_*$, $G : Y_* \longrightarrow Y'_*$ we get the induced chain map $M(F\otimes G) : M(X_*,Y_*) \longrightarrow M(X'_*,Y'_*)$ by

$$(3)\qquad \begin{cases} M(F\otimes G)_0 = (F_0,G_0)_* \\ M(F,G)_n = (F_n,G_0)_* \oplus (F_0,G_n)_*, \quad n \geq 1. \end{cases}$$

Finally, chain homotopies $\alpha : F \simeq F'$, $\beta : G \simeq G'$ yield a chain homotopy $M(\alpha,\beta) : M(F \otimes G) \simeq M(F'\otimes G')$ by

$$(4)\qquad \begin{cases} M(\alpha,\beta)_1 = ((\alpha_1\,,G_0)_*,(F_0,\beta_1)_*), \\ M(\alpha,\beta)_n = (\alpha_n,G_0)_* \oplus (F_0,\beta_n)_*, \ n \geq 2. \end{cases}$$

The next definition is dual to (9.1).

(9.2) <u>Definition</u>: We now associate with an $\underline{\underline{R}}\otimes\underline{\underline{R}}$–module M the additive functor

$$M : \underline{\underline{R}}^*/\simeq \otimes\, \underline{\underline{R}}^*/\simeq \longrightarrow \underline{\underline{Ab}}^*/\simeq$$

(as well denoted by M) as follows. For cochain complexes X^*, Y^* in $\underline{\underline{R}}^*$ we get $C^* = M(X^*,Y^*)$ by $(n \geq 2)$

$$(1)\qquad \begin{cases} C^0 = M(X^0,Y^0) \\ C^1 = \ker\{((1,d)_*,-(d,1)_*) : M(X^1,Y^0) \longrightarrow M(X^0,Y^1) \longrightarrow M(X^1,Y^1)\} \\ C^n = \ker\{(1,d)_* \oplus (d,1)_* : M(X^n,Y^0) \oplus M(X^0,Y^n) \longrightarrow M(X^m,Y^1) \oplus M(X^1,Y^n)\} \end{cases}$$

The differential $d = d_n : C^n \longrightarrow C^{n+1}$ is induced by the maps

$$(2)\qquad \begin{cases} d^1 = ((d,1)_*,(1,d)_*)\,, \\ d^n = (d,1)_* \oplus (1,d)_*, \quad n \geq 2 \end{cases}$$

For chain maps $F : X^* \longrightarrow X'^*$, $G : Y^* \longrightarrow Y'^*$ we get the induced chain map $M(F\otimes G) : G(X^*,Y^*) \longrightarrow M(X'^*,Y'^*)$ by

$$(3)\qquad \begin{cases} M(F\otimes G)^0 = (F^0,G^0)_*\,, \\ M(F,G)^n = (F^n,G^0)_* \oplus (F^0,G^n)_*, \ n \geq 1. \end{cases}$$

Finally chain homotopies $\alpha : F \simeq F'$, $\beta : G \simeq G'$ yield a chain homotopy

$M(\alpha,\beta) : M(F\otimes G) \simeq M(F'\otimes G')$ by

(4)
$$\begin{cases} M(\alpha,\beta)^0 = ((\alpha^0,G^0)_*,(F^0,\beta^0)_*), \\ M(\alpha,\beta)^n = (\alpha^n,G^0)_*\oplus(F^0,\beta)_*,\ n \geq 1. \end{cases}$$

As in (8.4) one can readily check:

(9.3) <u>Proposition</u>: The functors in (9.1) and (9.2) are well defined and additive.

The crucial property of the functors (9.1) and (9.2) is described by the next result.

(9.4) <u>Theorem</u>: Let M be a quadratic $\underline{\underline{R}}$–module and let $M(X_*\,|\,Y_*)$ and $M(X^*\,|\,Y^*)$ be cross effects of the quadratic functors M in (8.2) and (8.3) respectively. Then there are natural isomorphism

$$\Psi : M(X_*\,|\,Y_*) \cong M_{ee}(X_*,Y_*) \ \text{ and } \ \chi : M_{ee}(X^*,Y^*) \cong M(X^*\,|\,Y^*)$$

of chain complexes. Here M_{ee} is the $\underline{\underline{R}}\otimes\underline{\underline{R}}$–module given by M, see (3.1) and (1.7), and $M_{ee}(X_*,Y_*)$ and $M_{ee}(X^*,Y^*)$ are defined by (9.1) and (9.2) respectively.

Similarly as in (8.6) we can use the functors in (9.1), (9.2) for the definitions of derived functors. Let $\underline{\underline{A}}$ be an abelian category with enough projective and injectives.

(9.5) <u>Definition</u>: Let M be an $\underline{\underline{A}}\otimes\underline{\underline{A}}$–module. Using the resolution functors $i : \underline{\underline{A}} \longrightarrow \underline{\underline{A}}_*/\simeq$ and $j : \underline{\underline{A}} \longrightarrow A^*/\simeq$ one gets the additive functors

(1)
$$M(i\otimes i){:}\underline{\underline{A}}\otimes\underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}_*/\simeq \ \text{ and } \ M(j\otimes j){:}\underline{\underline{A}}\otimes\underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}^*/\simeq .$$

The n–th (co)homology of these functors yields the <u>biderived functors</u>

$$L_nM : \underline{\underline{A}} \otimes \underline{\underline{A}} \longrightarrow \underline{\underline{Ab}} \ \text{ and } \ R^nM : \underline{\underline{A}}\otimes\underline{\underline{A}} \longrightarrow \underline{\underline{Ab}}$$

respectively, $n \geq 0$. For $X,Y \in Ob(\underline{\underline{A}})$ one has

(2)
$$\begin{cases} (L_nM)(X,Y) = H_nM(X_*,Y_*)\ , \\ (R^nM)(X,Y) = H^nM(X^*,Y^*) \end{cases}$$

where X_*,Y_* (resp. X^*,Y^*) are projective (resp. injective) resolutions of X,Y. The chain complexes $M(X_*,Y_*)$, $M(X^*,Y^*)$ are defined in (9.1), (9.2).

As a corollary of (9.4) one gets immediately.

(9.6) Corollary: Let M be a quadratic $\underline{\underline{A}}$–module. Then the quadratic derived functors (8.6) have the cross effects

$$(L_nM)(X\,|\,Y) = (L_nM_{ee})(X,Y)$$
$$(R^nM)(X\,|\,Y) = (R^nM_{ee})(X,Y)$$

where M_{ee} is the $\underline{\underline{A}}\otimes\underline{\underline{A}}$–module given by M.

In addition to (9.6) one gets the following natural exact sequences for quadratic derived functors, they correspond to the classical exact sequences for derived functors in case $M_{ee} = 0$. To this end we consider a short exact sequence

(9.7) $$S = (0 \longrightarrow X \xrightarrow{i^s} Y \xrightarrow{q^s} Z \longrightarrow 0)$$

in $\underline{\underline{A}}$ and maps $S \longrightarrow S'$ between such sequences.

(9.8) Theorem: Let M be a quadratic $\underline{\underline{A}}$–module. Then S in (9.7) yields the following natural commutative diagram in which the rows and columns are long exact sequences $(n \in \mathbb{Z})$.

$$\begin{array}{ccccccccccc}
 & & & \downarrow & & & & \downarrow & & & \\
 & & & L_{n+1}M_{ee}(X,Y) & & & & L_nM_{ee}(X,Z) & & & \\
 & & & \downarrow{\scriptstyle\partial} & & & & \downarrow & & & \\
\rightarrow & L_{n+1}Mq^s & \xrightarrow{\partial} & L_nMX & \xrightarrow{i^s_*} & L_nMY & \longrightarrow & L_nMq^s & \xrightarrow{\partial} & L_{n-1}MX & \rightarrow \\
 & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & \\
\rightarrow & L_{n+1}MZ & \xrightarrow{\partial} & L_nMi^s & \longrightarrow & L_nMY & \xrightarrow{q^s_*} & L_nMZ & \xrightarrow{\partial} & L_{n-1}Mi^s & \rightarrow \\
 & & & \downarrow & & & & \downarrow{\scriptstyle\partial} & & & \\
 & & & L_nM_{ee}(X,Z) & & & & L_{n-1}M_{ee}(X,Z) & & & \\
 & & & \downarrow & & & & \downarrow & & &
\end{array}$$

We leave it to the reader to write down the dual diagram for right derived functors R^n; for this we simply replace L_* by R^* in such a way that ∂ raises the degree by 1. If $M_{ee} = 0$ we see that the rows of the diagram are isomorphic, in this case the row coincides with the classical exact sequence for left derived functors, see IV §6 [10]. In case the sequence S is split all boundaries ∂ are trivial and the remaining short exact sequences are split, this yields (9.6).

Proof of (9.8): We can choose a short exact sequence of projective resolutions

(1) $$0 \longrightarrow X_* \xrightarrow{i} Y_* \xrightarrow{r} Z_* \longrightarrow 0$$

of S, compare the proof of (IV. 6.1) [10]. As a module we have $Y_n = X_n \oplus Z_n$. The differential of Y_* is given by

(2) $$(d \oplus d) + i_1 \xi r_2 : X_n \oplus Z_n = Y_n \longrightarrow X_{n-1} \oplus Z_{n-1} = Y_{n-1}$$

Here d denotes the differential of X_* and Y_* respectively. We now derive from (1) the following commutative diagram in which rows and columns are short exact sequences of chain complexes

$$\begin{array}{ccccc}
 & & & & M(X_*,Z_*) \\
 & & & & \downarrow j \\
MX_* & \overset{i_*}{\rightarrowtail} & MY_* & \twoheadrightarrow & cok(i_*) \\
\downarrow & & \| & & \downarrow \\
ker(q_*) & \rightarrowtail & MY_* & \underset{q_*}{\twoheadrightarrow} & MZ_* \\
\downarrow j & & & & \\
M(X_*,Z_*) & & & &
\end{array}$$

The maps j are well defined chain maps since we have (2) for the boundary in Y_*. We now set

(4) $$L_n Mi^s = H_n ker(q_*),\ L_n Mq^s = H_n cok(i_*)\ .$$

Now (9.8) is obtained by the long exact sequences associated to short exact sequences of chain complexes.

There are the following examples of biderived functors. We associate with M in $\underline{\underline{M}}(\underline{\underline{R}} \otimes \underline{\underline{R}})$ the additive functors

(9.9) $$\begin{cases} \_ \otimes_{\underline{\underline{R}} \otimes \underline{\underline{R}}} M : \underline{\underline{M}}(\underline{\underline{R}}^{op}) \otimes \underline{\underline{M}}(\underline{\underline{R}}^{op}) \longrightarrow \underline{\underline{Ab}} \\ Hom_{\underline{\underline{R}} \otimes \underline{\underline{R}}}(\_, M) : \underline{\underline{M}}(\underline{\underline{R}})^{op} \otimes \underline{\underline{M}}(\underline{\underline{R}})^{op} \longrightarrow \underline{\underline{Ab}} \end{cases}$$

which carry the object (X,Y) to $(X \otimes Y) \otimes_{\underline{\underline{R}} \otimes \underline{\underline{R}}} M$ and $Hom_{\underline{\underline{R}} \otimes \underline{\underline{R}}}(X \otimes Y, M)$ respectively, compare (4.2)(3) and (5.2)(3). The biderived functors of (9.9) are denoted by

(1) $$Tor_n^{\underline{\underline{R}} \otimes \underline{\underline{R}}}(X,Y,M) = L_n(\_ \otimes_{\underline{\underline{R}} \otimes \underline{\underline{R}}} M)(X,Y),$$

(2) $\quad \mathrm{Ext}^n_{\underline{\underline{R}}\otimes\underline{\underline{R}}}(X,Y,M) = R^n(\mathrm{Hom}_{\underline{\underline{R}}\otimes\underline{\underline{R}}}(\_,M)(X,Y)$.

Using (9.6) one obtains for a quadratic $\underline{\underline{R}}$–module M the cross effects ($n \geq 0$)

(3) $\quad \mathrm{Tor}^{\underline{\underline{R}}}_n(X\,|\,Y,M) = \mathrm{Tor}^{\underline{\underline{R}}\otimes\underline{\underline{R}}}_n(X,Y,M_{ee})$,

(4) $\quad \mathrm{Ext}^n_{\underline{\underline{R}}}(X\,|\,Y,M) = \mathrm{Ext}^n_{\underline{\underline{R}}\otimes\underline{\underline{R}}}(X,Y,M_{ee})$.

As an example of (1) we get for $\underline{\underline{R}} = \mathbb{Z}$ the triple torsion product of Mac Lane [12]

(5) $\quad \mathrm{Tor}^{\mathbb{Z}}_1(X,Y,M) = \mathrm{Trp}(X,Y,M) = H_1(d_X\otimes d_Y,M)$,

compare (7.7)(3). We also can apply theorem (9.8) for the functors in (3), (4); this leads for $\underline{\underline{R}} = \mathbb{Z}$ to the following results on the functors in (7.4), see (8.11).

(9.10) <u>Theorem:</u> Let M be a quadratic $\mathbb{Z}$–module and let

$S : 0 \longrightarrow X \xrightarrow{\ i\ } Y \xrightarrow{\ q\ } Z \longrightarrow 0$ be an exact sequence of abelian groups. Then one has the following commutative diagrams in which the rows and the rectangle sequences of broken arrows are exact sequences of abelian groups. Moreover these diagrams are natural in S.

$$\begin{array}{ccccccccccc}
H_1(d_X\otimes d_Z,M_{ee}) & & \overset{\partial}{- - - - - - - - \longrightarrow} & & & X\otimes M & & & & & \\
\uparrow & & & & & \downarrow & & & & & \\
0\to i*'M & \longrightarrow & Y*'M & \xrightarrow{q_*} & Z*'M \xrightarrow{\partial} & i\otimes M & \longrightarrow & Y\otimes M & \xrightarrow{q_*} & Z\otimes M \to 0 & \\
\uparrow & & & & & \downarrow & & & & & \\
X*'M & \longleftarrow - - - - & 0 & & \longleftarrow - - - - - - & X\otimes Z\otimes M_{ee} & & & & &
\end{array}$$

$$\begin{array}{ccccccccccc}
H^1(d_X\otimes d_Z,M_{ee}) & & \overset{\partial}{\longleftarrow - - - - - - - -} & & & \mathrm{Hom}(X,M) & & & & & \\
\downarrow & & & & & \uparrow & & & & & \\
0\leftarrow \mathrm{Ext}(i,M) & \leftarrow & \mathrm{Ext}(Y,M) & \xleftarrow{q^*} & \mathrm{Ext}(Z,M) \xleftarrow{\partial} & \mathrm{Hom}(i,M) & \leftarrow & \mathrm{Hom}(Y,M) & \xleftarrow{q_*} & \mathrm{Hom}(Z,M)\leftarrow 0 & \\
\downarrow & & & & & \uparrow & & & & & \\
\mathrm{Ext}(X,M) & - - - - \longrightarrow & 0 & & - - - - - - \longrightarrow & \mathrm{Hom}(X\otimes Z,M_{ee}) & & & & &
\end{array}$$

In case $M_{ee} = 0$ the diagram above correspond exactly to the classical six term exact sequences. We can apply these exact sequences for example if M is the quadratic $\mathbb{Z}$–module $M = \mathbb{Z}^{\Gamma}$. In this case the torsion product $Y*'\mathbb{Z}^{\Gamma} = R(Y)$ correspond to the functor R of Eilenberg–Mac Lane, see (7.13).

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Hans–Joachim Baues
Max–Planck–Institut für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3