To the memory of Andrey Tyurin

Finite rank vector bundles on inductive limits of grassmannians

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Abstract

If $\mathbb{P}^\infty$ is the projective ind-space, i.e. $\mathbb{P}^\infty$ is the inductive limit of linear embeddings of complex projective spaces, the Barth-Van de Ven-Tyurin (BVT) Theorem claims that every finite rank vector bundle on $\mathbb{P}^\infty$ is isomorphic to a direct sum of line bundles. We extend this theorem to general sequences of morphisms between projective spaces by proving that, if there are infinitely many morphisms of degree higher than one, every vector bundle of finite rank on the inductive limit is trivial. We then establish a relative version of these results, and apply it to the study of vector bundles on inductive limits of grassmannians. In particular we show that the BVT Theorem extends to the ind-grassmannian of subspaces commensurable with a fixed infinite dimensional and infinite codimensional subspace in $\mathbb{C}^\infty$. We also show that for a class of twisted ind-grassmannians, every finite rank vector bundle is trivial.

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1 Introduction

About 30 years ago the study of "infinitely extendable" vector bundles of finite rank on projective spaces and grassmannians was initiated. In particular the following remarkable theorem was proved: any finite rank vector bundle on the infinite complex projective space $\mathbb{P}^\infty$ (or equivalently, any finite rank vector bundle on $\mathbb{P}^n$ which admits an extension to $\mathbb{P}^m$ for large enough $m > n$) is isomorphic to a direct sum of line bundles. For rank two bundles this was established by W. Barth and A. Van de Ven in [BV], and for arbitrary finite rank bundles the theorem was proved by A. Tyurin in [T]. In what follows we refer to this result as to the Barth-Van de Ven-Tyurin Theorem, or as to the BVT Theorem. Some first steps were made also towards understanding finite rank vector bundles on infinite grassmannians: R. Hartshorne conjectured that every finite rank bundle on an infinite grassmannian $G(k,\infty)$ is homogeneous, see [BV]. For rank two bundles this conjecture is proved in [BV], and in the general case the conjecture is proved by E. Sato, [S2]. Sato

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established also a partial analog of Hartshorne's conjecture for the infinite grassmanians of the classical simple groups, and reproved the BVT Theorem, see [S2] and [S1].

The purpose of the present note is to revive this discussion and relate it with the more recent duscussion of homogeneous ind-spaces of locally linear ind-groups, see [DPW], [DP] and the references therein. Our starting point is an infinite sequence

$$G_1 \subset G_2 \subset \dots$$

of complex linear algebraic groups and a subsequence

$$P_1 \subset P_2 \subset \dots$$

of parabolic subgroups. This yields a sequence of morphisms

$$G_1/P_1 \longrightarrow G_2/P_2 \longrightarrow \dots \, . \tag{1}$$

Let G/P denote the inductive limit of (1). We restrict ourselves here to the case when $G_N \simeq GL(n_N)$ and P_N are maximal parabolic subgroups, i.e. G_N/P_N are grassmannians.

The study of line bundles on G/P gives some first hints on what general finite rank bundles on G/P might look like. An essential difference with the cases studied in [BV], [T], [S1], and [S2] is that the restriction maps

$$\mathrm{Pic}(G_N/P_N) \longrightarrow \mathrm{Pic}(G_{N-1}/P_{N-1})$$

on the Picard groups induced by inclusions in (1) are only injective and in general not surjective. Therefore $\mathrm{Pic}(G/P)$, the Picard group of the inductive limit, is isomorphic to $\mathbb{Z}$ or equals zero. In the first case the call the sequence (1) *linear* and in the second case we call it *twisted*.

Here is a brief description of our results. The simplest case for a sequence (1) is when all G_N/P_N are projective spaces. Consider more generally an arbitrary sequence of morphisms of projective spaces

$$\mathbb{P}^{i_1} \xrightarrow{\varphi_1} \mathbb{P}^{i_2} \xrightarrow{\varphi_2} \dots \, . \tag{2}$$

If (2) is linear, i.e. all but finitely many Picard groups map isomorphically, the inductive limit is isomorphic to the projective ind-space $\mathbb{P}^\infty$. Here the BVT Theorem claims that any finite rank bundle is isomorphic to a direct sum of line bundles. If (2) is twisted, i.e. the Picard group of the inductive limit $\mathbb{P}^\infty_{tw}$ of (2) equals zero, the problem of describing all finite rank vector bundles on $\mathbb{P}^\infty_{tw}$ was not posed in the 70's.

Our first main result (Theorem 3.1) claims that every finite rank vector bundle on $\mathbb{P}^\infty_{tw}$ is trivial. This theorem, together with the BVT Theorem, can be extended to the relative case and yields a complete description of finite rank vector bundles on any inductive limit of relative projective spaces.

In Section 4 we apply the above result to the study of finite rank vector bundles on inductive limits of grassmannians $G(k_N, n_N)$. We consider two types of morphisms of grassmannians in (1): standard inclusions and a certain class of twisted homogeneous morphisms which we call twisted extensions. For standard inclusions we show that, if

$\lim_{N\to\infty} k_N = \lim_{N\to\infty}(n_N - k_N) = \infty$, any finite rank vector bundle on the inductive limit is isomorphic to a direct sum of line bundles. An interesting ind-variety which arises as the inductive limit of a sequence of standard inclusions satisfying the above condition is the ind-grassmanian $G(V, \infty)$ of subspaces $V' \subset \mathbb{C}^\infty$ commensurable with a fixed infinite dimensional and infinite codimensional subspace $V \subset \mathbb{C}^\infty$. Therefore, any finite rank vector bundle on $G(V, \infty)$ is isomorphic to a direct sum of line bundles. Finally, we prove that in the case of twisted extensions of grassmannians every finite rank vector bundle on the inductive limit is trivial.

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2 Preliminary results

2.1 Notation

The ground field is $\mathbb{C}$ and we work in the category of complex analytic spaces. A *vector bundle* always means a vector bundle of finite rank. If E is a vector bundle, we denote by E^* the dual vector bundle. For the tensor power of a line bundle E we write simply E^k. More generally, the convention $E^{-k} = (E^*)^k$ enables us to assume that $k \in \mathbb{Z}$. By $H^i(E)$ we denote the i-th cohomology group of the sheaf of local sections of E, and put $h^i(E) = \dim H^i(E)$, and $\chi(E) := \sum_i (-1)^i h^i(E)$.

By $G(k, n)$ we denote the grassmannian of k-dimensional subspaces in $\mathbb{C}^n$. For $k = 1$ we have the projective space $\mathbb{P}^{n-1} := G(1, n)$. The grassmannian $G(k, n)$ is isomorphic to $G(n-k, n)$ via the isomorphism

$$\{V \subset \mathbb{C}^n\} \mapsto \{V^\perp \subset \mathbb{C}^n\},$$

where $\mathbb{C}^n$ is identified with its dual space. Under a *projective subspace* of $G(k, n)$ we understand the set of k-dimentional subspaces $V \subset \mathbb{C}^n$ such that $U \subset V \subset W$, where $U \subset W$ are fixed subspaces of $\mathbb{C}^n$ with $\dim U = k-1$, $\dim W > k$, or $\dim W = k+1$, $\dim U < k$. The projective subspace is a *line* if $\dim W - \dim V = 2$.

Let S_k be the vector bundle on $G(k, n)$ with fiber V at the point $V \in G(k, n)$. There is a canonical inclusion $S_k \subset \tilde{\mathbb{C}}^n$, where $\tilde{\mathbb{C}}^n$ the trivial bundle on $G(k, n)$ with fiber $\mathbb{C}_n$. We set $S_{n-k} := (\tilde{\mathbb{C}}^n/S_k)^*$. By definition, S_k and S_{n-k} are the *tautological bundles* on $G(k, n)$. The Picard group $\mathrm{Pic}(G(k, n))$ is isomorphic to $\mathbb{Z}$, and both maximal exterior powers $\wedge^k S_k$ and $\wedge^{n-k} S_{n-k}$ are isomorphic generators of $\mathrm{Pic}(G(k, n))$. We set $\mathcal{O}_{G(k,n)}(-1) := \wedge^k S_k \simeq \wedge^{n-k} S_{n-k}$ and $\mathcal{O}_{G(k,n)}(m) := \mathcal{O}_{G(k,n)}(-1)^{-m}$ for $m \in \mathbb{Z}$.

An *ind-space* is the union $\cup_N X_N$ of analytic spaces X_N related by closed immersions

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots . \tag{3}$$

For instance, the projective ind-space $\mathbb{P}^\infty$ is the union $\cup_N \mathbb{P}_N$ where $\mathbb{P}^N \subset \mathbb{P}^{N+1}$ is the standard closed immersion. A *morphism of ind-spaces* $\varphi : X \to Y$ is a map whose restriction to each X_N is a morphism of X_N into $Y_{j(N)}$ for some $j(N)$. In this paper we will more generally consider sequences (3) of arbitrary morphisms. A *vector bundle* on the system (3) is a collection $\{E_N\}$ of vector bundles E_N on X_N such that $\varphi_N^* E_{N+1} = E_N$. If (3) determines an ind-space X, we speak of a vector bundle on X.

2.2 The Barth-Van de Ven-Tyurin Theorem

Theorem 2.1. *If $X \simeq \mathbb{P}^1$ or $X \simeq \mathbb{P}^\infty$, every vector bundle E on X is isomorphic to a unique direct sum of line bundles, i.e. $E \simeq \oplus_j \mathcal{O}_X(d_j)$ for some unique integers $d_1 \geq ... \geq d_{\mathrm{rk}\,E}$.*

For $X \simeq \mathbb{P}^1$ this is a classical result due to A. Grothendieck. For $X \simeq \mathbb{P}^\infty$ the theorem has been proved by A. Tyurin in [T] (and earlier by W. Barth and A. Van de Ven, [BV], for vector bundles of rank two). E. Sato also presents a proof in [S1].

We will make extensive use of Theorem 2.1. In particular, if $l \subset Y$ is a rational curve in a complex manifold or ind-space Y and E is a vector bundle on Y, we call the numbers $d_1, ..., d_{\mathrm{rk}\,E}$, together with the multiplicities with which they occur in the isomorphism $E|_l \simeq \oplus_j \mathcal{O}_Y(d_j)$, the *splitting data* of $E|_l$. We always denote by d_1 the maximal among all d_j, and by D – the difference $d_1 - d_{\mathrm{rk}\,E}$. When we need to emphasize the dependence of d_j and D on l, we write $d_j(E|_l)$ and $D(E|_l)$.

We will use the same convention when Y is an ind-space and l is replaced by an ind-subspace $P \subset Y$ isomorphic to $\mathbb{P}^\infty$. Note that, if E' is a vector bundle on $X \simeq \mathbb{P}^1$ or $X \simeq \mathbb{P}^\infty$ with splitting data $d_1 \geq ... \geq d_{\mathrm{rk}\,E}$, and ν_1 is the multiplicity of d_1, there is a unique subbundle E'_1 of E' isomorphic to the direct sum of ν_1 copies of $\mathcal{O}_X(d_1)$. Indeed, consider the twisted bundle $E' \otimes \mathcal{O}_X(-d_1)$ and its unique maximal subbundle E'' generated by global sections. Then $E'_1 = E'' \otimes \mathcal{O}_X(d_1)$.

We say that a vector bundle E on a grassmannian $G(k,n)$ is *uniform*, if the splitting data $d_1(E|_l), ..., d_{\mathrm{rk}\,E}(E|_l)$ for a line $l \subset G(k,n)$ does not depend on the choice of l. The bundle E is *linearly trivial* if its restriction $E|_l$ is trivial for any line $l \subset G(k,n)$. A linearly trivial bundle is necessarily trivial: for $k = 1$ this is a well-known result, and an induction argument on k yields the result also for $G(k,n)$, see for instance [P].

2.3 Local rigidity of direct sums of line bundles on $\mathbb{P}^N$

Let $\mathcal{S}$ be an analytic space and π_1 and π_2 denote respectively the projections of $\mathbb{P}^1 \times S$ onto the first and the second factor. If E is a vector bundle on $\mathbb{P}^1 \times \mathcal{S}$, $E(x)$ denotes the restriction of E to $\pi_2^{-1}(x) \simeq \mathbb{P}^1$. In [D1] the following statement is proved.

Proposition 2.2. *Let E be a vector bundle on $\mathbb{P}^1 \times \mathcal{S}$, where $\mathcal{S}$ is a connected analytic space. Fix $x_0 \in \mathcal{S}$. Then*

a) there exists an open neighborhood $U \ni x_0$ such that $d_1(E(x)) \leq d_1(E(x_0))$ for $x \in U$, i.e. $d_1(E(x))$ locally decreases;

b) $\sum_i^{\mathrm{rk}\,E(x)} \nu_i(E(x)) d_i(E(x)) = c_1(E(x))$, where $c_1(E(x))$ is the first Chern class of $E(x)$ and $\nu_i(E(x))$ is the multiplicity of $d_i(E(x))$ (in particular $\sum_i^{\mathrm{rk}\,E(x)} \nu_i(E(x)) d_i(E(x))$

does not depend on $x \in S$).

Proposition 2.2, applied to E^* implies that $d_{\operatorname{rk} E}(E(x)) = -d_1(E^*(x))$ locally increases, hence $D(E(x)) = d_1 - d_{\operatorname{rk} E}$ also locally decreases.

Proposition 2.3. *Let E be a vector bundle on $\mathbb{P}^N$, $N > 1$, which is isomorphic to a direct sum of line bundles. Then E is locally rigid.*

Proof. Note that $H^1(\operatorname{End}(E)) = 0$. This follows from the fact that $\operatorname{End}(E)$ is isomorphic to a direct sum of line bundles, as $H^1(\mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on $\mathbb{P}^N$ for $N > 1$. Therefore the parameter space of the versal deformation of E consists of a single reduced point, see [D2]. □

2.4 Estimates related to vector bundles on $\mathbb{P}^1 \times \mathbb{P}^1$

In the case of $\mathbb{P}^1 \times \mathbb{P}^1$ we call the fibers of π_1 (respectively, π_2) vertical (resp., horizontal) sections of $\mathbb{P}^1 \times \mathbb{P}^1$, and for any vector bundle E on $\mathbb{P}^1 \times \mathbb{P}^1$ we denote the twisted bundle $E \otimes \pi_1^*(\mathcal{O}(k)) \otimes \pi_2^*(\mathcal{O}(l))$ by $E(k, l)$.

Lemma 2.4. *Let E be a vector bundle on $\mathbb{P}^1 \times \mathbb{P}^1$. Let P be a vertical section and $\mathcal{S}$ be a horizontal section of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $d_1(E|_P), d_1(E|_S) \geq -1$ implies*

$$h^0(E) \leq (\operatorname{rk} E)(d_1(E|_P) + 1)(d_1(E|_S) + 1). \tag{4}$$

If $d_1(E|_P) < 0$ or $d_1(E|_S) < 0$, we have $h^0(E) = 0$.

Proof. The second statement in the Lemma follows immediately from Proposition 2.2, since $h^0(F) = 0$ for any vector bundle F on $\mathbb{P}^1$ when $d_1(F) < 0$. If $d_1(E|_P)$ or $d_1(E|_S)$ equals -1, then (4) is obvious, therefore we can assume that $d_1(E|_P), d_1(E|_S) \geq 0$.

For any k there is the exact sequence

$$0 \longrightarrow E(0, -k-1) \longrightarrow E(0, -k) \longrightarrow E|_S \longrightarrow 0.$$

The corresponding cohomology sequence gives

$$h^0(E(0, -k)) - h^0(E(0, -k-1)) \leq h^0(E|_S) \leq (\operatorname{rk} E)(d_1(E|_S) + 1).$$

Summation from k to zero yields

$$h^0(E) - h^0(E(0, -k-1)) \leq (\operatorname{rk} E)(k+1)(d_1(E|_S) + 1). \tag{5}$$

When $k = d_1(E|_P)$ we have, $h^0(E(0, -k-1)) = 0$, therefore (5) implies (4). □

Lemma 2.5. *Let E, P, and S be as in Lemma 2.4. Suppose $d_1(E|_P), d_1(E|_S) \geq 0$ and $d_{\operatorname{rk} E}(E|_P), d_{\operatorname{rk} E}(E|_S) \leq 0$. If P' is another vertical section of $\mathbb{P}^1 \times \mathbb{P}^1$, then*

$$D(E|_{P'}) \leq 4(\operatorname{rk} E)(D(E|_P) + 2)(D(E|_S) + 1) - \chi(E) - \chi(E^*). \tag{6}$$

Proof. From the cohomology sequence of the exact sequence

$$0 \longrightarrow E \longrightarrow E(1,0) \longrightarrow E|_{P'} \longrightarrow 0$$

we have

$$h^0(E) + h^0(E|_{P'}) \leq h^0(E(1,0)) + h^1(E).$$

This, together with the equality $h^2(E) = h^0(E^*(-2,-2))$ (Serre duality), gives

$$d_1(E|_{P'}) \leq h^0(E|_{P'}) \leq h^0(E(1,0)) - \chi(E) + h^0(E^*(-2,-2)). \tag{7}$$

Note that $d_1((E^*(-2,-2)|_P) = -d_{\mathrm{rk}\,E}(E|_P) - 1$ and $d_1((E^*(-2,-2)|_S) = -d_{\mathrm{rk}\,E}(E|_S) - 1$, so Lemma 2.4 yields

$$h^0(E^*(-2,-2)) \leq (\mathrm{rk}\,E)(-d_{\mathrm{rk}\,E}(E|_P)(-d_{\mathrm{rk}\,E}(E|_S). \tag{8}$$

Combining (7) and (8) we have

$$d_1(E|_{P'}) \leq (\mathrm{rk}\,E)(d_1(E|_P)+2)(d_1(E|_S)+1) - \chi(E) + (\mathrm{rk}\,E)(-d_{\mathrm{rk}\,E}(E|_P))(-d_{\mathrm{rk}\,E}(E|_S)).$$

Since $d_1(E|_P), -d_{\mathrm{rk}\,E}(E|_P) \leq D(E|_P)$ and $d_1(E|_S), -d_{\mathrm{rk}\,E}(E|_S) \leq D(E|_S)$, we obtain

$$d_1(E|_{P'}) \leq 2(\mathrm{rk}\,E)(D(E|_P)+2)(D(E|_S)+1) - \chi(E). \tag{9}$$

By adding to (9) the analogous inequality for E^* we obtain (6). $\square$

3 The case of a twisted projective ind-space

Our first main result is the following theorem.

Theorem 3.1. *Let*

$$\mathbb{P}^{i_1} \xrightarrow{\varphi_1} \mathbb{P}^{i_2} \xrightarrow{\varphi_2} \ldots$$

be a twisted sequence of projective spaces, i.e. $\deg \varphi_N > 1$ *for infinitely many* N, *and let* $E = \{E_N\}$ *be a vector bundle on this inductive system. Then* E *is trivial, i.e. all* E_N *are trivial.*

Lemma 3.2. *Let* E *be as in Theorem 3.1. Then all Chern classes* $c_q(E_N)$ *equal zero.*

Proof. By definition, $c_q(E_N) \in H^{2q}(\mathbb{P}^{i_N}, \mathbb{Z})$. For each $j > N$, the homomorphism

$$H^{2q}(\mathbb{P}^{i_j}, \mathbb{Z}) \longrightarrow H^{2q}(\mathbb{P}^{i_N}, \mathbb{Z}) \tag{10}$$

induced by the composition $\varphi_{j,N} := \varphi_N \circ \ldots \circ \varphi_j$ maps $c_q(E_j)$ to $c_q(E_N)$. On the other hand, we can identify $H^{2q}(\mathbb{P}^{i_N}, \mathbb{Z})$ and $H^{2q}(\mathbb{P}^{i_j}, \mathbb{Z})$ in a standard way with $\mathbb{Z}$, and (10) considered as an endomorphism of $\mathbb{Z}$ is nothing but multiplication by $(\deg \varphi_j) \cdot \ldots \cdot (\deg \varphi_N)$. Therefore $c_q(E_N) \in \mathbb{Z}$ is divisible by $(\deg \varphi_j) \cdot \ldots \cdot (\deg \varphi_N)$ for all $j > N$. According to our assumption, $\lim_{j\to\infty}((\deg \varphi_j) \cdot \ldots \cdot (\deg \varphi_N)) = \infty$, consequently, $c_q(E_N) = 0$. $\square$

Proof of Theorem 3.1. Without restriction of generality we assume that $i_1 = 1$. Fix N with $i_N > 2$ and set

$$D_N := \max_{l'} D(E_N|_{l'}),$$

where l' runs over all lines in $\mathbb{P}^{i_N}$. Let l be a line in $\mathbb{P}^{i_N}$ such that $D(E|_l) = D_N$, and Q be a projective subspace in $\mathbb{P}^{i_N}$ of dimension $i_N - 2$ not intersecting l. We can choose homogeneous coordinates $z_0, ..., z_{i_N}$ in $\mathbb{P}^{i_N}$ such that l is defined by $z_i = 0$, $i \geq 2$, and Q is defined by $z_0 = z_1 = 0$. Furthermore, we fix a morphism $f : \mathbb{P}^1 \to \mathbb{P}^{i_N}$ of degree $\deg \varphi_{1,n}$ such that

$$f(\mathbb{P}^1) \cap l = \emptyset, \quad f(\mathbb{P}^1) \cap Q = \emptyset, \tag{11}$$

and

$$D(f^* E_N) \leq D(E_1). \tag{12}$$

To see that f exists, note that by Proposition 2.2 there is a neighborhood of unity U in the group of linear transformation of $\mathbb{P}^{i_N}$ such that for any $g \in U$ the morphism $g \circ \varphi_{1,N} : \mathbb{P}^1 \to \mathbb{P}^{i_N}$ (where $\varphi_{1,N} = \varphi_N \circ \varphi_{N-1} \circ ... \circ \varphi_1$) satisfies (12). Moreover, it is obvious that one can choose g in such a way that (11) holds.

We now extend the morphism f to a morphism $\tilde{f} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{i_N}$ in the following way. Let f be given in coordinates as $z_i = f_i(x_0, x_1)$, where x_0, x_1 are homogeneous coordinates on the space $\mathbb{P}^1$ which we identify with the second factor of $\mathbb{P}^1 \times \mathbb{P}^1$. Let t_0, t_1 be homogeneous coordinates on the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$. We define $\tilde{f}$ by putting $z_i = t_0 f_i(x_0, x_1)$ for $i = 0, 1$ and $z_i = t_1 f_i(x_0, x_1)$ for $i \geq 2$. Condition (11) ensures that $\tilde{f}$ is well-defined. Set $\widetilde{E}_N := \tilde{f}_N^* E_N$.

The morphism $\tilde{f}$ has the following properties.

a) The restriction of $\tilde{f}$ to the vertical fiber P over the point $(t_0, t_1) = (1, 1)$ coincides with f. Therefore (12) implies

$$D(\widetilde{E}_N|_P) \quad \leq \quad D(E_1). \tag{13}$$

b) The restriction of $\tilde{f}$ to the vertical fiber P' over the point $(t_0, t_1) = (1, 0)$ is a morphism of P' onto l of degree $\deg \varphi_{1,N}$. Thus

$$D(\widetilde{E}_N|_{P'}) \quad = \quad (\deg \varphi_{1,N}) D_N. \tag{14}$$

c) The restriction of $\tilde{f}$ to any horizontal fiber S is a linear embedding. Therefore

$$D(\widetilde{E}_N|_S) \quad \leq \quad D_N. \tag{15}$$

The hypotheses of Lemma 2.5 hold for $\widetilde{E}_N$, since $c_1(\widetilde{E}_N|_P) = c_1(\widetilde{E}_N|_S) = 0$ by Lemma 3.2. Therefore Lemma 2.5, together with (13), (14), and (15), yields

$$(\deg \varphi_{1,N}) D_N \leq 4(\operatorname{rk} E_N)(D(E_1) + 2)(D_N + 1) - \chi(E_N) - \chi(E_N^*). \tag{16}$$

By the Riemann-Roch Theorem and the vanishing of all Chern classes of E_N (Lemma 3.2), we have

$$\chi(E_N) = \chi(E_N^*) = n\chi(\mathcal{O}_{P^N}) = \operatorname{rk} E_N.$$

Therefore (16) turns into

$$(\deg \varphi_{1,N})D_N \leq 4(\operatorname{rk} E_N)(D(E_1) + 2)(D_N + 1) - 2\operatorname{rk} E_N. \tag{17}$$

As $\operatorname{rk} E_N = \operatorname{rk} E$ and $D(E_1)$ do not depend on N, we see that the right hand side of (17) is a linear function of D_N, while the left hand side grows with N faster than a linear function of D_N, as $\lim_{N\to\infty} \deg \varphi_{1,N} = \infty$ by our hypothesis. Therefore, inequality (17) can hold only if D_N equals zero for large enough N and hence for all N. This means that all vector bundles E_N are linearly trivial, and thus trivial. □

Theorem 3.1 together with Theorem 2.1 leads to the following general description of a vector bundle on an inductive limit of relative projective spaces. Let $p_N : M_N \to S_N$ be a relative projective space, i.e. a locally trivial fibration with base a connected complex analytic space S_N and with fiber the projective space $\mathbb{P}^{i_N}$. Let furthermore

$$\begin{array}{ccccc} M_1 & \xrightarrow{\varphi_1} & M_2 & \xrightarrow{\varphi_2} & \dots \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_2}\big\downarrow & & \\ S_1 & \longrightarrow & S_2 & \longrightarrow & \dots \end{array} \tag{18}$$

be a commutative diagram. Note that if the restriction of φ_N to a fiber P_N of p_N has degree d, then the restriction of φ_N to any fiber of p_N has also degree d. Therefore it makes sense to call (18) *linear* if the degree of the restriction of φ_N on the fibers of p_N equals one for almost all N, and *twisted* otherwise.

Theorem 3.3. *Let $E = \{E_N\}$ be a vector bundle on the upper row of (18). Then there exist integers $d_1 \geq \dots \geq d_{\operatorname{rk} E}$ such that, for large enough N and any fiber P_N of p_N, $E_N|_{P_N} \simeq \oplus_r \mathcal{O}_{P_N}(d_r)$. Furthermore, if (18) is twisted, $d_1 = \dots = d_{\operatorname{rk} E} = 0$.*

Proof. The fact that $E_N|_{P_N}$ is isomorphic to a direct sum of line bundles follows from Theorems 2.1 and 3.1. The fact that the splitting data of $E_N|_{P_N}$ does not depend on P_N follows from Proposition 2.3. If (18) is linear, the degree of P_N equals one for large enough N, thus the splitting data $E_N|_{P_N}$ does also not depend on N for large enough N. Finally, if (18) is twisted, each $d_r(E_N|_{P_N})$ is divisible by $(\deg \varphi_N) \cdot \dots \cdot (\deg \varphi_j)$ for all $j > N$, and therefore $d_r(E_N|_{P_N}) = 0$. □

In general it is not true that E is isomorphic to a direct sum of line bundles, or that E admits a filtration whose associated quotient are line bundles. The reader will easily prove that the latter holds only under the additional assumption that all d_r are distinct. Finally, if (18) is twisted, Theorem 3.3 implies that each E_N is the pull-back $p_N^* E'_N$ for some vector bundle $E' = \{E'_N\}$ on the lower row of (18).

4 The case of ind-grassmannians

In this section we consider two different types of closed immersions

$$G(k_1,n_1) \xrightarrow{f_1} G(k_2,n_2) \xrightarrow{f_2} ... \tag{19}$$

and characterize vector bundles on the corresponding ind-spaces.

4.1 Standard extensions of grassmannians

We define a *standard extension* of grassmannians as a closed immersion of the form

$$\lambda_{r,m} : G(k,n) \to G(k+r,n+m), \qquad \{V \subset \mathbb{C}^n\} \mapsto \{V \oplus W \subset \mathbb{C}^n \oplus \mathbb{C}^m\},$$

where $W \subset \mathbb{C}^m$ is a fixed subspace of dimension $r \geq 0$.

Proposition 4.1. *Assume that a sequence (19) is given, where f_N are standard extensions. Let $E = \{E_N\}$ be a vector bundle on (19). Then for each N, E_N is a uniform bundle on $G(k_N, n_N)$.*

Proof. For each N consider the natural diagram

$$\begin{array}{ccc} F(k_N-1,k_N,n_N) & \xrightarrow{\pi_N} & G(k_N,n_N) \\ {\scriptstyle p_N}\big\downarrow & & \\ G(k_N-1,n_N), & & \end{array} \tag{20}$$

where $F(k_N-1,k_N,n_N)$ stands for the space of all flags of type (k_N-1,k_N) in $\mathbb{C}^{n_N}$. Note that f_N induce morphisms between the diagrams (20) for N and $N+1$. Furthermore, the embeddings $F(k_N-1,k_N,n_N) \to F(k_{N+1}-1,k_{N+1},n_{N+1})$ and the morphisms p_N define a relative projective ind-space. The vector bundles $\pi_N^* E_N$ define a vector bundle on this relative projective ind-space. By Theorem 3.3 the splitting data of the restriction $\pi_N^* E_N|_{P_N}$ on each fiber P_N of p_N does not depend on N and P_N. This implies the result, as any projective line $l \subset G(k_N,n_N)$ is a line in some fiber P_N, and the splitting data of $E|_l$ and $\pi_N^* E_N|_{P_N}$ are equal. $\square$

Theorem 4.2. *Assume that a sequence (19) is given, where f_N are standard extensions and $\lim_{N\to\infty} k_N = \lim_{N\to\infty}(n_N - k_N) = \infty$. Then any vector bundle $E = \{E_N\}$ on the inductive limit of (19) is isomorphic to a direct sum of line bundles.*

Proof. By Proposition 4.1, each E_N is a uniform bundle. We will prove the Theorem by induction on $\operatorname{rk} E = \operatorname{rk} E_N$. Fix N and let $d_1, ..., d_{\operatorname{rk} E_N}$ be the splitting data for the restriction $E_N|_l$, where $l \subset G(k_N,n_N)$ is any line. Denote by ν_1 the multiplicity of d_1. As each f_j induces an isomorphisms of Picard groups, the splitting data $d_1, ..., d_{\operatorname{rk} E_N}$ does not depend on N. Fix $x \in G(k_N,n_N)$. Any line $l \subset G(k_N,n_N)$ passing through x determines a subspace $E'_{l,x}$ in the fiber $(E_N)_x$: this is the fiber of the unique subbundle of $E_N|_l$

isomorphic to the direct sum of ν_1 copies of $\mathcal{O}_l(d_1)$, see Subsection 2.2. The variety of all lines l passing through x is isomorphic to the direct product $\mathbb{P}^{k_N-1} \times \mathbb{P}^{n_N-k_N-1}$, hence we have a morphism

$$\psi_x : \mathbb{P}^{k_N-1} \times \mathbb{P}^{n_N-k_N-1} \longrightarrow G(\nu_1, \mathrm{rk}\, E_N), \qquad l \mapsto E'_{l,x}.$$

By our assumption, $\lim_{N\to\infty} k_N = \lim_{N\to\infty}(n_N - k_N) = \infty$. Therefore, for large N, the morphism ψ_x is trivial, i.e. it determines a fixed subspace $(E'_N)_x$ in $(E_N)_x$. In this way we obtain a subbundle $E'_N \subset E_N$. Clearly $f_N^*(E'_{N+1}) = E'_N$, therefore $E' := \{E'_N\}$ is a well-defined subbundle of E.

If $\mathrm{rk}\, E' = \mathrm{rk}\, E$, twisting by the line bundle $\{\mathcal{O}_{G(k_N,n_N)}(-d_1)\}$ yields a bundle whose restriction to any line in $G(k_N, n_N)$ is trivial for all N. Therefore the twisted bundle is trivial on each $G(k_N, n_N)$, and hence trivial. Thus E is isomorphic to a direct sum of ν_1 copies of $\{\mathcal{O}_{G(k_N,n_N)}(d_1)\}$. If $\mathrm{rk}\, E' < \mathrm{rk}\, E$, the induction assumption implies that both E' and E/E' are isomorphic to direct sums of line bundles. Finally, the observation that there are no non-trivial extensions of line bundles on $G(k_N, n_N)$ for $k_N > 1$ (by the Bott-Borel-Weil Theorem, $H^1(\mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on $G(k_N, n_N)$ unless $k_N = n_N - k_N = 1$) implies that E is isomorphic to a direct sum of line bundles. $\square$

Here is an important special case of Theorem 4.2. If $V \subset \mathbb{C}^\infty$ is an arbitrary fixed subspace, a subspace $V' \subset \mathbb{C}^\infty$ is *commensurable* with V, if there exists a finite dimensional subspace $U \subset \mathbb{C}^\infty$ such that $V \subset V' + U$, $V' \subset V + U$, and $\dim V \cap U = \dim V' \cap U$. The set of all subspaces V' comensurable with V is, by definition, the ind-grassmannian $G(V, \infty)$. If V is finite dimensional and $\dim V = k$, then $G(V, \infty) = G(k, \infty)$. In [DP] an explicit construction of $G(V, \infty)$ as an ind-space is given. Moreover, $G(V, \infty)$ is the inductive limit of standard extensions, and the conditions of Theorem 4.2 are satisfied for $G(V, \infty)$ if and only if $\dim V = \mathrm{codim}_{\mathbb{C}^\infty} V = \infty$. Therefore, by Theorem 4.2, we conclude that in the latter case every vector bundle on $G(k, \infty)$ is isomorphic to a direct sum of line bundles.

More generally, if $V_1 \subset ... \subset V_r$ is any fixed flag in $\mathbb{C}^\infty$, the ind-variety $F(V_1, ..., V_r, \infty)$ of flags $V'_1 \subset ... \subset V'_r$ comensurable with $V_1 \subset ... \subset V_r$, is constructed in [DP]. We leave it to the reader to prove the following corollary of Theorem 4.2 by double induction on $\mathrm{rk}\, E$ and r.

Corollary 4.3. *Let* $\dim V_1 = \dim(V_2/V_1) = ... = \dim(\mathbb{C}^\infty/V_r) = \infty$. *Then every vector bundle* E *on* $F(V_1, ..., V_r, \infty)$ *is isomorphic to a direct sum of line bundles.*

4.2 Twisted extensions of grassmannians

An alternative definition of a grassmannian is as a homogeneous space $GL(n)/P$ for a maximal parabolic subgroup $P \subset GL(n)$. We call a morphism $f : G(k_1, n_1) \to G(k_2, n_2)$ *homogeneous* if it is induced by a group homomorphism $\tilde{f} : GL(n_1) \to GL(n_2)$. A homogeneous morphism is a closed immersion or its image is a point.

Consider a homogeneous morphism $f : G(k_1, n_1) \to G(k_2, n_2)$, where $G(k_1, n_1) = GL(n_1)/P_1$, $G(k_2, n_2) = GL(n_2)/P_2$, and P_1, P_2 are maximal parabolic subgroups such what $\tilde{f}(P_1) \subset P_2$. The reductive part of P_1 is isomorphic to $GL(k_1) \times GL(n_1 - k_1)$ and the reductive part of P_2 is isomorphic to $GL(k_2) \times GL(n_2 - k_2)$. Note that $\tilde{f}(GL(k_1) \times GL(n_1 - k_1))$ is contained in one component of the direct product $GL(k_2) \times GL(n_2 - k_2)$ if and only if the induced morphism $f : G(k_1, n_1) \to G(k_2, n_2)$ is a morphism into a point. In the sequel we adopt a convention: whenever we write a homogeneous morphism as $f : G(k_1, n_1) \to G(k_2, n_2)$ and f is a closed immersion, we automatically assume that $\tilde{f}(GL(k_1)) \subset GL(k_2)$ and $\tilde{f}(GL(n_1 - k_1)) \subset GL(n_2 - k_2)$. Furthermore, we call a morphism $\varphi : \mathbb{P}^N \to G(k, n)$ *k-split* if $\varphi^* S_k$ is isomorphic to a direct sum of line bundles on $\mathbb{P}^N$. For example, Theorem 2.1 implies that any morphism $\mathbb{P}^1 \to G(k, n)$ is k-split.

Proposition 4.4. *a) Any homogeneous morphism $\varphi : \mathbb{P}^N = G(1, N) \to G(k, n)$ is k-split.*

b) Let $f : G(k_1, n_1) \to G(k_2, n_2)$ be a homogeneous embedding of grassmannians and $\varphi : \mathbb{P}^N \to G(k_1, n_1)$ be a k_1-split morphism. Then the composition $f \circ \varphi : \mathbb{P}^N \to G(k_2, n_2)$ is k_2-split.

Proof. a) Let $\mathbb{P}^N = GL(N+1)/P_1$, and $G(k, N) = GL(n)/P_2$. The reductive parts of P_1 and P_2 are isomorphic respectively to $GL(1) \times GL(N)$ and $GL(k) \times GL(n-k)$. According to our convention, the group homomorphism $\tilde{\varphi} : GL(N+1) \to GL(n)$ which induces φ maps $GL(1)$ into $GL(k)$, and $GL(N)$ into $GL(n-k)$. Therefore the structure group of $\varphi^* S_k$ is reduced to $GL(1)$, which implies that $\varphi^* S_k$ is a direct sum of line bundles.

b) Let $\tilde{f} : GL(n_1) \to GL(n_2)$ be a group homomorphism which induces the morphism f. Since $\tilde{f}(GL(k_1)) \subset GL(k_2)$, the $GL(n_1)$-homogeneous bundle $f^* S_{k_2}$ is a direct summand in the tensor product of several copies of S_{k_1} and $S_{k_1}^*$. This, together with the fact, that the morphism φ is k_1-split, implies that $(f \circ \varphi)^* S_{k_2}$ is a direct summand of a bundle isomorphic to a direct sum of line bundles on $\mathbb{P}^N$. Hence $(f \circ \varphi)^* S_{k_2}$ is isomorphic itself to a direct sum of line bundles, i.e. the morphism $f \circ \varphi$ is k_2-split. □

Here is a coordinate form of a k-split map. Let $t = (t_0, ..., t_N)$ be homogeneous coordinates on $\mathbb{P}^N$, and $v_1, ..., v_n$ be a basis in $\mathbb{C}^n$. Then the reader will check that a morphism $\varphi : \mathbb{P}^N \to G(k, n)$ is k-split if and only if φ can be presented in the form

$$\varphi(t) = \mathrm{span}\Big\{ \sum_{j=1}^{n} \varphi_{i,j}(t) v_j, \ i = 1, ..., k \Big\} \tag{21}$$

for some homogeneous polynomials $\varphi_{i,j}(t)$ in t. It is clear that if $\varphi : \mathbb{P}^N \to G(k, n)$ is represented in the form (21), then $\deg \varphi_{i,j}$ does not depend on j and $\deg \varphi = \sum_{i=1}^{k} \deg \varphi_{i,j}$.

Lemma 4.5. *Let $\varphi : \mathbb{P}^N \to G(k, n)$ be a k-split morphism and $\lambda_{0,k} : G(k, n) \to G(k, n+k)$ be a standard extension (see Subsection 4.1). Then the composition $\lambda_{0,k} \circ \varphi : \mathbb{P}^N \to G(k, n+k)$ factors through a standard extension $\mathbb{P}^N \to \mathbb{P}^{N+1}$, via a k-split morphism $\varphi : \mathbb{P}^{N+1} \to G(k, n+k)$.*

Proof. Let φ be presented in the form (21), and let the vectors $v_{n+1}, ...v_{n+k}$ extend the set $v_1, ..., v_n$ to a basis in $\mathbb{C}^{n+k}$. Fix homogeneous coordinates $t' = (t_0, ..., t_{N+1})$ in $\mathbb{P}^{N+1}$

and identify $\mathbb{P}^N$ with the hyperplane $t_{N+1} = 0$. Then $\tilde{\varphi} : \mathbb{P}^{N+1} \to G(k, n+k)$ is given by the formula

$$\tilde{\varphi}(t') = \operatorname{span}\Big\{ \sum_{j=1}^{n} \varphi_{i,j}(t) v_j + t_{N+1}^{\delta_i} v_{n+i},\ i = 1, ..., k \Big\},$$

where $\delta_i := \deg \varphi_{i,j}$. □

We need one last definition. We call a homogeneous morphism $f : G(k_1, n_1) \to G(k_2, n_2)$ a *twisted extension* if f decomposes as $\lambda_{0,n_2-n} \circ h$, where $h : G(k_1, n_1) \to G(k_2, n)$ is a homogeneous morphism for $n_2 - n \geq k_2$, and $\lambda_{0,n_2-n} : G(k_2, n) \to G(k_2, n_2)$ is a standard extension.

Proposition 4.6. *Assume that a sequence (19) is given, where f_N are twisted extensions, infinitely many of which have degree greater than one. Suppose in addition that $\lim_{N\to\infty} k_N = \lim_{N\to\infty}(n_N - k_N) = \infty$. Then any vector bundle $E = \{E_N\}$ on the inductive limit of (19) is trivial.*

Proof. Let $E = \{E_N\}$ be a vector bundle on the inductive limit of (19). Fix N and a $(k_N - 1)$-dimensional subspace of $\mathbb{C}^{n_N}$. This determines a projective subspace $\mathbb{P}^{n_N - k_N}$ of $G(k_N, n_N)$, and the closed immersion $\varphi_N : \mathbb{P}^{n_N-k_N} \to G(k_N, n_N)$ is k_N-split as $\varphi^* S_{k_N} \simeq (k_N - 1)\mathcal{O}_{\mathbb{P}^{n_N-k_N}} \oplus \mathcal{O}_{\mathbb{P}^{n_N-k_N}}(-1)$. Furthermore, let $f_N = \lambda_{0,m_N} \circ h_N$, where $h_N : G(k_N, n_N) \to G(k_{N+1}, n_{N+1} - m_N)$ is a homogeneous closed immersion and $m_N \geq k_{N+1}$. Proposition 4.4 b) implies that the composition $h_N \circ \varphi_N$ is k_{N+1}-split. Then, by Lemma 4.5, the morphism $f_N \circ \varphi_N = \lambda_{0,m_N} \circ (h_N \circ \varphi_N)$ factors through a standard extension $\mathbb{P}^{n_N-k_N} \to \mathbb{P}^{n_N-k_N+1}$ via a homogeneous immersion $\varphi_{N+1} : \mathbb{P}^{n_N-k_N+1} \to G(k_{n+1}, n_{N+1})$. Proceeding by induction on j, we build homogeneous immersions $\varphi_{N+j} : \mathbb{P}^{n_N-k_N+j} \to G(k_{N+j}, n_{N+j})$ and standard extensions $\mathbb{P}^{n_N-k_N+j} \to \mathbb{P}^{n_N-k_N+j+1}$ which form the commutative diagram

$$\begin{array}{ccccc}
\mathbb{P}^{n_N-k_N} & \longrightarrow & \mathbb{P}^{n_N-k_N+1} & \longrightarrow & \dots \\
{\scriptstyle\varphi_N}\big\downarrow & & {\scriptstyle\varphi_{N+1}}\big\downarrow & & \\
G(k_N, n_N) & \xrightarrow{f_N} & G(k_{N+1}, n_{N+1}) & \xrightarrow{f_{N+1}} & \dots .
\end{array} \tag{22}$$

Applying the BVT theorem to the inductive limit of the upper row of (22), we conclude that $E_N|_{\mathbb{P}^{n_N-k_N}}$ is isomorphic to a direct sum of line bundles. Furthermore, one sees exactly as in Proposition 4.1 that E_N is a uniform bundle. Now the argument in the proof of Theorem 4.2 shows that, since $\lim_{N\to\infty} k_N = \lim_{N\to\infty}(n_N - k_N) = \infty$, E_N is isomorphic to a direct sum of line bundles. Therefore E is also isomorphic to a direct sum of line bundles. Finally, as $\deg f_N > 1$ for infinitely many N, a line bundle on this inductive limit is necessarily trivial, hence E is trivial. □

Here are two examples to Proposition 4.6.

Let $\mathbb{C}^w$ be a proper subspace of $\mathbb{C}^m$. Consider the closed immersion

$$G(k, n) \longrightarrow G(kw, nm), \qquad \{V \subset \mathbb{C}^n\} \mapsto \{V \otimes \mathbb{C}^w \subset \mathbb{C}^n \otimes \mathbb{C}^m\}. \tag{23}$$

It factors, via a standard extension, through the closed immersion

$$G(k,n) \longrightarrow G(kw,nw), \qquad \{V \subset \mathbb{C}^n\} \mapsto \{V \otimes \mathbb{C}^w \subset \mathbb{C}^n \otimes \mathbb{C}^w\}.$$

Fixing a sequence of pairs $\mathbb{C}^{w_N} \subset \mathbb{C}^{m_N}$, determines a sequence of immersions (23). If the set $\{w_N\}$ is bounded, for sufficiently large N all embeddings in this sequence are twisted extensions as

$$nm_1...m_{N+1} - nw_1...w_{N+1} \geq kw_1...w_{N+1}$$

for large enough N. Therefore Proposition 4.6 implies that any vector bundle on the inductive limit is trivial.

Finally consider a closed immersion of the form

$$G(k,n) \longrightarrow G(k(k+1)/2, n^2), \qquad \{V \subset \mathbb{C}^n\} \mapsto \{S^2(V) \subset \mathbb{C}^n \otimes \mathbb{C}^n\}, \tag{24}$$

where S^2 stands for symmetric square. By iteration of (24) we obtain another example of a sequence of twisted extensions. Indeed, (24) factors, via a standard extension, through the closed immersion

$$G(k,n) \longrightarrow G(k(k+1)/2, n(n+1/2), \qquad \{V \subset \mathbb{C}^n\} \mapsto \{S^2(V) \subset S^2(\mathbb{C}^n)\},$$

and the inequality

$$k(k+1)/2 \leq n^2 - n(n+1)/2$$

needed in the definition of a twisted extension, is trivially satisfied. Proposition 4.6 implies that any vector bundle on the corresponding inductive limit is also trivial.

5 Conclusion

The results of this paper lead naturally to the following question. Consider an arbitrary sequence of morphisms

$$G(k_1,n_1) \xrightarrow{f_1} G(k_2,n_2) \xrightarrow{f_2} ... \,. \tag{25}$$

Is it true that, if $E = \{E_N\}$ is an arbitrary vector bundle on (25) then each E_N is a homogeneous bundle on $G(k_N, n_N)$? In fact, the BVT Theorem, both main theorems of Sato (see [S2]), Theorems 3.1 and 4.2, as well as Proposition 4.6, are all equivalent to the affirmative answer to the above question for the cases they apply to. More precisely, the description of vector bundles on sequences (25) in each specific case considered in the above statements, is equivalent to the descriptions of systems of homogeneous bundles $\{E_N\}$ with $f_{N+1}^* E_{N+1} = E_N$. Note that Theorem 3.1 applies to not necessarily homogeneous morphisms, while in all other statements homogeneous morphisms are considered. It would be very interesting to give an answer to the above general question even under the assumption that all morphisms f_N are homogeneous.

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