Max-Planck-Institut für Mathematik
Bonn

The spinning particle with curved target

by

Ezra Getzler

Max-Planck-Institut für Mathematik
Preprint Series 2016 (21)

The spinning particle with curved target

Ezra Getzler

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Northwestern University
Evanston, Illinois
USA

MPIM 16-21

THE SPINNING PARTICLE WITH CURVED TARGET

EZRA GETZLER

ABSTRACT. We extend our previous calculation of the BV cohomology of the spinning particle with a flat target to the general case, in which the target carries a non-trivial pseudo-Riemannian metric and a magnetic field.

1. INTRODUCTION

Unlike in other models which have been investigated, the BV cohomology of the spinning particle with a flat target is nontrivial in all negative degrees [4], raising the question of whether our understanding of the BV formalism is incomplete. In this paper, we show that these results extend to the spinning particle with general target, in which the target carries a non-trivial pseudo-Riemannian metric carrying a possibly non-zero magnetic field.

The quantum theory associated to this model is familiar to mathematicians as the Dirac operator on a manifold; the magnetic field corresponds to twisting by a complex line bundle.

The BV formalism associates to a solution of the classical master equation

$$\left\{ \int S \, dt, \int S \, dt \right\} = 0$$

a vector field s on the space of fields, given by the explicit formula

$$\mathsf{s} = \sum_i (-1)^{\mathrm{p}(\Phi_i)} \sum_{\ell=0}^{\infty} \left(\partial^\ell \left(\frac{\delta S}{\delta \Phi_i} \right) \frac{\partial}{\partial(\partial^\ell \Phi_i^+)} + \partial^\ell \left(\frac{\delta S}{\delta \Phi_i^+} \right) \frac{\partial}{\partial(\partial^\ell \Phi_i)} \right) . \tag{1}$$

In Section 2, we show in complete generality that the classical master equation implies that $\mathsf{s}^2 = 0$. Our proof of this statement employs a modified Batalin-Vilkovisky (anti)bracket which differs from the usual one by a total derivative, and satisfies the graded Jacobi formula on densities, without the need for any total derivative corrections. This bracket was introduced (in the ungraded setting) by Soloviev [6] and applied to BV geometry in [3].

In Section 3, we derive the master action of the spinning article. With these technical details out of the way, we calculate the BV cohomology of the spinning particle in Section 4: it turns out that the description is essentially identical to the special case discussed in [4].

P. Mnëv has remarked (private communication) that the model considered in this paper may also be constructed by the method of Alexandrov et al. [1]. We discuss this reformulation of the theory at the end of Section 3.

In Section 4, we discuss the quantum master equation for the spinning particle. One expects neither anomalies nor renormalization in a quantum mechanical system, and this is confirmed by our calculations: there is a potential contribution to the full action at one-loop (which in fact vanishes for typical regularization schemes), and no higher-loop contributions.

2. THE BATALIN-VILKOVISKY FORMALISM

In the Batalin-Vilkovisky formalism, there are fields Φ_i, of ghost number $\mathrm{gh}(\Phi_i) \in \mathbb{Z}$ and parity $\partial(\Phi_i) \in \mathbb{Z}/2$, along with the corresponding antifields Φ_i^+, of ghost number $\mathrm{gh}(\Phi_i^+) = 1-\mathrm{gh}(\Phi_i)$, and parity $\mathrm{p}(\Phi_i^+) = 1 - \mathrm{p}(\Phi_i)$.

We focus on the classical BV formalism for a single independent variable t (classical mechanics). Let ∂ denote the total derivative with respect to t. Denote by $\mathcal{A}^j$ the superspace of all differential expressions in the fields and antifields with $\mathrm{gh}(S) = j$. The sum $\mathcal{A}$ of the superspaces $\mathcal{A}^j$ for $j \in \mathbb{Z}$ is a graded superalgebra. A vector field is a graded derivation of the graded superalgebra $\mathcal{A}$. An example is the total derivative ∂.

We denote by $\partial_{k,\Phi} : \mathcal{A}^j \to \mathcal{A}^j$ the partial derivative

$$\partial_{k,\Phi} = \frac{\partial}{\partial(\partial^k \Phi)},$$

and by $\delta_{k,\Phi} : \mathcal{A}^j \to \mathcal{A}^j$ the higher Euler operators of Kruskal et al. [5]

$$\delta_{k,\Phi} = \sum_{\ell=0}^{\infty} \tbinom{k+\ell}{k} (-\partial)^\ell \partial_{k+\ell,\Phi}.$$

When $k = 0$, $\delta_{0,\Phi} = \delta_\Phi$ is the classical variational derivative.

A vector field ξ is called **evolutionary** if it commutes with ∂. Such a vector field is determined by its value on the fields Φ and the antifields Φ^+:

$$\begin{aligned} \xi &= \sum_i \sum_{k=0}^{\infty} \Big(\partial^k\big(\xi(\Phi_i)\big)\, \partial_{k,\Phi_i} + \partial^k\big(\xi(\Phi_i^+)\big)\, \partial_{k,\Phi_i^+} \Big) \\ &= \sum_i \mathrm{pr} \left(\xi(\Phi_i) \frac{\partial}{\partial \Phi_i} + \xi(\Phi_i^+) \frac{\partial}{\partial \Phi_i^+} \right). \end{aligned}$$

The operation pr is called **prolongation**.

The **Soloviev bracket** is defined by the formula

$$\{\!\{f,g\}\!\} = \sum_i (-1)^{(\mathrm{p}(f)+1)\,\mathrm{p}(\Phi_i)} \sum_{k,\ell=0}^{\infty} \Big(\partial^\ell\big(\partial_{k,\Phi_i} f\big) \partial^k\big(\partial_{\ell,\Phi_i^+} g\big) + (-1)^{\mathrm{p}(f)} \partial^\ell\big(\partial_{k,\Phi_i^+} f\big) \partial^k\big(\partial_{\ell,\Phi_i} g\big) \Big).$$

It is proved in [3] that the bracket $\{\!\{f,g\}\!\}$ satisfies the following equations:

skew symmetry: $\{\!\{f,g\}\!\} = -(-1)^{(\mathrm{p}(f)+1)(\mathrm{p}(g)+1)} \{\!\{g,f\}\!\}$

Jacobi: $\{\!\{f,\{\!\{g,h\}\!\}\}\!\} = \{\!\{\{\!\{f,g\}\!\},h\}\!\} + (-1)^{(\mathrm{p}(f)+1)(\mathrm{p}(g)+1)} \{\!\{g,\{\!\{f,h\}\!\}\}\!\}$

linearity over ∂: $\{\!\{\partial f,g\}\!\} = \{\!\{f,\partial g\}\!\} = \partial\{\!\{f,g\}\!\}$

The superspace $\mathcal{F} = \mathcal{A}/\partial\mathcal{A}$ of functionals is the graded quotient of $\mathcal{A}$ by the subspace $\partial\mathcal{A}$ of total derivatives. The image of $f \in \mathcal{A}$ in $\mathcal{F}$ is denoted by $\int f\,dt$, and the bracket induced on $\mathcal{F}$ by the Soloviev bracket is denoted

$$\{\textstyle\int f\,dt, \int g\,dt\}.$$

This bracket may also be written directly in terms of the variational derivatives:

$$\{\textstyle\int f\,dt, \int g\,dt\} = \sum_i (-1)^{(\mathrm{p}(f)+1)\,\mathrm{p}(\Phi_i)} \displaystyle\int \Big(\big(\partial_{\Phi_i} f\big)\big(\partial_{\Phi_i^+} g\big) + (-1)^{\mathrm{p}(f)} \big(\partial_{\Phi_i^+} f\big)\big(\partial_{\Phi_i} g\big) \Big)\, dt.$$

The Batalin-Vilkovisky formalism for classical field theory involves the selection of a solution of the classical master equation

$$\{\textstyle\int S\,dt, \int S\,dt\} = 0,$$

where $S \in \mathcal{A}^0$ is an element with $\mathrm{p}(S) = 0$. When the antifields are set to zero, the expression $S(\Phi, 0)$ is the classical action.

Stated in terms of the Soloviev bracket, the classical master equation becomes the equation

$$\tfrac{1}{2}\{\!\{S, S\}\!\} = \partial\tilde{S}, \tag{2}$$

where $\tilde{S} \in \mathcal{A}^1$ is an element with $\mathrm{p}(\tilde{S}) = 1$.

Proposition 2.1. *The differential operator* $\mathrm{ad}(f) = \{\!\{f, -\}\!\}$ *is given by the formula*

$$\mathrm{ad}(f) = \sum_{k=0}^{\infty} \partial^k \mathsf{f}_k,$$

where f_k *is the sequence of evolutionary vector fields*

$$\mathsf{f}_k = \sum_i (-1)^{(\mathrm{p}(f)+1)\,\mathrm{p}(\Phi_i)}\,\mathrm{pr}\left(\left(\delta_{k,\Phi_i} f\right)\frac{\partial}{\partial\Phi_i^+} + (-1)^{\mathrm{p}(f)}\left(\delta_{k,\Phi_i^+} f\right)\frac{\partial}{\partial\Phi_i}\right).$$

Proof. We see that

$$\begin{aligned}
&\sum_{j,k,\ell=0}^{\infty} (-1)^j \tbinom{k+j}{k}\,\partial^k\left(\partial^{\ell+j}\left(\partial_{k+j,\Phi} f\right)\partial_{\ell,\Phi^+} g\right) \\
&\qquad = \sum_{i,j,k,\ell=0}^{\infty} (-1)^j \tbinom{k+j}{k}\tbinom{k}{i}\,\partial^{\ell+i+j}\left(\partial_{k+j,\Phi} f\right)\,\partial^{k-i}\left(\partial_{\ell,\Phi^+} g\right) \\
&\qquad = \sum_{i,j,\ell,m=0}^{\infty} (-1)^j \tbinom{m}{m-i-j}\tbinom{i+j}{j}\,\partial^{\ell+i+j}\left(\partial_{m,\Phi} f\right)\,\partial^{m-i-j}\left(\partial_{\ell,\Phi^+} g\right) \\
&\qquad = \sum_{\ell,m=0}^{\infty} \partial^{\ell}\left(\partial_{m,\Phi} f\right)\,\partial^{m}\left(\partial_{\ell,\Phi^+} g\right),
\end{aligned}$$

and the analogous equation holds with the roles of Φ and Φ^+ exchanged. Summing over the fields Φ_i, the result follows. □

Given a solution of the classical master equation (4), the functions S and $\tilde{S}$ give rise to the evolutionary vector fields s_k and $\tilde{\mathsf{s}}_k$ respectively, where the vector field s_0 is the vector field s of (1). Define the vector fields

$$\sigma_k = \tilde{\mathsf{s}}_k - \tfrac{1}{2}\sum_{\ell=0}^{k+1}[\mathsf{s}_\ell, s_{k-\ell+1}].$$

Lemma 2.2.

$$\mathsf{s}^2 = \sum_{k=0}^{\infty} \partial^{k+1}\sigma_k \tag{3}$$

Proof. The equation $(d + \mathrm{ad}(\mathsf{S}))^2 = 0$ implies that $\mathrm{ad}(S)^2 + \partial\,\mathrm{ad}(\tilde{S}) = 0$. In other words,

$$\mathsf{s}^2 + \sum_{k=0}^{\infty}\sum_{\ell=0}^{k+1} \partial^{k+1}\mathsf{s}_\ell\mathsf{s}_{k-\ell+1} = \sum_{k=0}^{\infty}\partial^{k+1}\tilde{\mathsf{s}}_k,$$

which proves the result after a little rearrangement. □

We can now prove the main result of this section.

Theorem 2.3. *If S is a solution of the classical master equation* (2)*, then the associated vector field* s *satisfies the equation* $\mathsf{s}^2 = 0$.

Proof. The idea of the proof is that whereas the right-hand side is a vector field of (3) is a vector field, the left-hand side is a differential operator of degree > 1. Taking the symbols of both sides, we see that the symbol of this differential operator must vanish.

We now prove by downward induction in k that the vector fields σ_k vanish. Let K be the largest integer such that σ_K is nonzero. (For the solution of the classical master equation associated to a first-order field theory, $K = 1$.) Let Φ be one of the fields of the theory having $\mathrm{p}(\Phi) = 0$ (that is, a bosonic field), and take the $(K+2)$-fold commutator of both sides of (3) with Φ. The differential operator s^2 is a vector field, so the left-hand side vanishes, while the right-hand side equals

$$(K+2)!\,(\partial\Phi)^{K+1}\sigma_K(\Phi).$$

It follows that $\sigma_K(\Phi) = 0$.

Next, we take the commutator with the antifield Φ^+ followed by the $(K+1)$-fold commutator with Φ: again, the left-hand side vanishes, while the right-hand side equals

$$(K+1)!\,(\partial\Phi)^K\Big((\partial\Phi)\sigma_K(\Phi^+) + (K+1)(\partial\Phi^+)\sigma_K(\Phi)\Big).$$

We have already shown that the second of the two term vanishes, and we conclude that $\sigma_K(\Phi^+) = 0$.

The vanishing of $\sigma_K(\Phi)$ and $\sigma_K(\Phi^+)$ may be proved for fields Φ with $\mathrm{p}(\Phi) = 1$ (fermionic fields) by exchanging the rôles of Φ and its antifield Φ^+ in the above argument. In this way, we see that $\sigma_K = 0$. Arguing by downward induction, we conclude that $\sigma_k = 0$ for all $k \geq 0$, proving the theorem. □

The vector field s induces a differential on $\mathcal{F}$, whose cohomology $H^*(\mathcal{F}, \mathsf{s})$ is the Batalin-Vilkovisky cohomology of the model. By Proposition 2.1, s equals the differential $\mathrm{ad}(S)$ induced by taking Soloviev bracket with the solution S of the classical master equation.

We may calculate the BV cohomology groups $H^*(\mathcal{F}, \mathsf{s})$ using the complex

$$\mathcal{V}^j = \mathcal{A}^j \oplus \tilde{\mathcal{A}}^{j+1}\,\varepsilon,$$

where

$$\tilde{\mathcal{A}}^j = \begin{cases} \mathcal{A}^0/\mathbb{C}, & j = 0, \\ \mathcal{A}^j, & j \neq 0, \end{cases}$$

with differential

$$d\big(f + g\,\varepsilon\big) = (-1)^{\mathrm{p}(g)}\,\partial g.$$

The symbol ε is understood to have odd parity and ghost number -1, so that the parities of the superspace $\tilde{\mathcal{A}}^{j+1}$ are reversed in $\mathcal{V}^j$. This complex is a shifted differential graded Lie algebra, with respect to the extension of the Soloviev bracket to $\mathcal{V}$:

$$\{\!\{f_0 + g_0\,\varepsilon, f_1 + g_1\,\varepsilon\}\!\} = \{\!\{f_0, f_1\}\!\} + \{\!\{f_0, g_1\}\!\}\,\varepsilon + (-1)^{\mathrm{p}(f_1)+1}\,\{\!\{g_0, f_1\}\!\}\,\varepsilon.$$

The differential satisfies

$$d\{\!\{a, b\}\!\} = \{\!\{da, b\}\!\} + (-1)^{\mathrm{p}(a)+1}\{\!\{a, db\}\!\}.$$

Lemma 2.4. *If $\int S\,dt \in \mathcal{F}$ is a solution of the classical master equation* (2)*, then*

$$\mathsf{S} = S + \tilde{S}\,\varepsilon \in \mathcal{V}^0$$

is a solution of the master equation

$$d\mathsf{S} + \tfrac{1}{2}\{\!\{\mathsf{S},\mathsf{S}\}\!\} = 0. \tag{4}$$

Proof. Applying the operator $\mathrm{ad}(S)$ to both sides of (2), we see that

$$\tfrac{1}{2}\{\!\{S,\{\!\{S,S\}\!\}\}\!\} = \{\!\{S,\partial\tilde{S}\}\!\} = \partial\{\!\{S,\tilde{S}\}\!\},$$

and hence that $\{\!\{S,\tilde{S}\}\!\} = 0$. □

For example, the Poisson structure of the KdV hierarchy (Dickey [2]; cf. [3]) gives a solution of the classical master equation (4) with $\mathrm{gh}(S) = -2$ instead of 0, and $\mathrm{gh}(\varepsilon) = 1$ instead of -1:

$$\mathsf{S} = x^+\partial^3x^+ + xx^+\partial x^+ + x^+\partial x^+\partial^2x^+\,\varepsilon.$$

The differentials $d+\mathsf{s}$ and $d+\mathrm{ad}(\mathsf{S})$ on $\mathcal{V}^*$ are equivalent, by the following proposition.

Proposition 2.5. *Let P be the automorphism of $\mathcal{V}^*$ defined by the formula*

$$P(f+g\varepsilon) = f + g\,\varepsilon + (-1)^{\mathrm{p}(f)}\sum_{k=0}^{\infty}\partial^k\mathsf{s}_{k+1}f\,\varepsilon.$$

Then the differentials $d+\mathrm{ad}(S)$ and $d+\mathsf{s}$ on $\mathcal{V}$ are related by the equation

$$d+\mathrm{ad}(\mathsf{S}) = P(d+\mathsf{s})P^{-1}.$$

Proof. Written out in full, we have

$$\begin{aligned}(d+\mathrm{ad}(\mathsf{S}))\big(f+g\varepsilon\big) &= \{\!\{S,f\}\!\} + (-1)^{\mathrm{p}(g)}\,\partial g + \Big((-1)^{\mathrm{p}(f)}\,\{\!\{\tilde{S},f\}\!\} + \{\!\{S,g\}\!\}\Big)\,\varepsilon \\ &= \sum_{k=0}^{\infty}\partial^k\mathsf{s}_kf + (-1)^{\mathrm{p}(g)}\,\partial g + \sum_{k=0}^{\infty}\Big((-1)^{\mathrm{p}(f)}\,\partial^k\tilde{\mathsf{s}}_kf + \partial^k\mathsf{s}_kg\Big)\,\varepsilon.\end{aligned}$$

We see that

$$\begin{aligned}(d+\mathrm{ad}(S))P(f+g\varepsilon) &= \mathsf{s}f + (-1)^{\mathrm{p}(g)}\,\partial g + \sum_{k=0}^{\infty}\Big((-1)^{\mathrm{p}(f)}\partial^k\big(\sigma_kf - \mathsf{s}_{k+1}\big) + \partial^{\mathsf{k}}\mathsf{s}_{\mathsf{k}}\mathsf{g}\Big)\,\varepsilon \\ &= \mathsf{s}f + (-1)^{\mathrm{p}(g)}\,\partial g + \sum_{k=0}^{\infty}\Big((-1)^{\mathrm{p}(f)+1}\,\partial^k\mathsf{s}_{k+1} + \partial^{\mathsf{k}}\mathsf{s}_{\mathsf{k}}\mathsf{g}\Big)\,\varepsilon \\ &= P(d+\mathsf{s})(f+g\varepsilon),\end{aligned}$$

where on the second line, we have used the vanishing of the vector fields σ_k. □

3. The classical master equation for the spinning particle in curved target

In this section, we construct the solution of the classical master equation associated to the spinning particle in a curved target.

Let $\mathbb{R}^d$ be a vector space with constant pseudo-metric $\eta_{ab} = \eta(e_a,e_b)$. The target of the spinning particle is an open subset U of $\mathbb{R}^d$, carrying a Riemannian pseudo-metric $g_{\mu\nu} = g(\partial_\mu,\partial_\nu)$ with the same signature as η. Let $g^{\mu\nu} = g(dx^\mu,dx^\nu)$ be the metric induced by g on the tangent bundle. In other words,

$$g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu_\mu.$$

Similarly, let $\eta^{ab} e_a \otimes e_b$ be the pseudo-metric induced on $(\mathbb{R}^d)^*$ by η.

We will represent the pseudo-metric $g^{\mu\nu}$ by a moving frame $\omega^a = \omega^a_\mu \, dx^\mu$. Geometrically speaking, a moving frame is an isometry between the trivial bundle $U \times \mathbb{R}^d$ with constant pseudo-metric η and the tangent bundle of U. Equivalently, the one-forms $\{\omega^a\}$ satisfy the equation

$$g(\omega^a, \omega^b) = \eta^{ab},$$

or

$$g_{\mu\nu} = \eta_{ab} \omega^a_\mu \omega^b_\nu.$$

We denote by ω^μ_a the inverse of ω^a_μ, in the sense that

$$\omega^a_\mu \omega^\mu_b = \delta^a_b.$$

We may use the frame ω^a_μ and its inverse ω^μ_a to exchange contravariant and covariant indices μ with upper and lower internal indices a: for example, $A_a = \omega^\mu_a A_\mu$.

The physical fields of the spinning particle (fields of ghost number 0) are as follows:

a) the position x^μ, which is a field of even parity taking values in U;
b) fields p_a and θ^a, respectively of even and odd parity;
c) the graviton e and gravitino ψ, respectively even and odd.

In addition, the model has ghosts c and γ (fields of ghost number 1), corresponding respectively to diffeomorphism in the independent variable t and local supersymmetry, which are respectively odd and even.

The connection one-form $\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu \in \Omega^1(U, \mathrm{End}(\mathbb{R}^d))$ is a matrix of one-forms on U characterized in terms of the frame ω^a_μ by two conditions: it is **skew-symmetric**

$$\omega^b{}_a = -\eta_{a\tilde{a}} \eta^{b\tilde{b}} \omega^{\tilde{a}}{}_{\tilde{b}},$$

and **torsion-free**, that is, satisfies the first Cartan structure equation

$$d\omega^a + \omega^a{}_b \wedge \omega^b = 0.$$

Written in terms of components, this equation becomes

$$\partial_\mu \omega^a_\nu - \partial_\nu \omega^a_\mu + \omega_\mu{}^a{}_b \omega^b_\nu - \omega_\nu{}^a{}_b \omega^b_\mu = 0.$$

The curvature $R^a{}_b = \frac{1}{2} R_{\mu\nu}{}^a{}_b dx^\mu dx^\nu \in \Omega^2(U, \mathrm{End}(\mathbb{R}^d))$ is a skew-symmetric matrix of two-forms defined by the second Cartan structure equation

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b.$$

Written in terms of components, this equation reads

$$\partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b = R_{\mu\nu}{}^a{}_b.$$

We will need the Bianchi identities for the curvature $R_{\mu\nu ab}$: the antisymmetrizations of the expressions $\omega^a_\lambda R_{\mu\nu ab}$ and

$$\partial_\lambda R_{\mu\nu ab} + \omega_\lambda{}^c{}_a R_{\mu\nu cb} - \omega_\lambda{}^c{}_b R_{\mu\nu ca}$$

in the indices $\{\lambda, \mu, \nu\}$ vanish.

We also introduce a magnetic potential (connection one-form)

$$A = A_\mu dx^\mu \in \Omega^1(U)$$

on U, with associated field-strength (curvature) $F = dA$, or in terms of components,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We now turn to the construction of the solution S of the classical master equation associated to the moving frame ω_μ^a and magnetic field A_μ. In all of our calculations, the antifield x_μ^+ enters via the expression

$$X_a^+ = \omega_a^\mu \big(x_\mu^+ + \omega_\mu{}^b{}_c \, p_b p^{+c} - \omega_\mu{}^b{}_c \, \theta_b^+ \theta^c \big).$$

Lemma 3.1. *Let $\Sigma \in \mathcal{A}^0$ and $\mathsf{G} \in \mathcal{A}^{-2}$ be given by the formulas*

$$\begin{aligned} \Sigma &= \big(p_\mu + A_\mu\big)\partial x^\mu - \tfrac{1}{2}\big(\eta_{ab}\theta^a\partial\theta^b + \omega_{\mu ab}\,\partial x^\mu \theta^a\theta^b\big) + \partial e^+ c + \partial\psi^+\gamma \\ \mathsf{G} &= X_a^+ p^{+a} + \tfrac{1}{4} p^{+a}p^{+b}\theta^c\theta^d R_{abcd} + \tfrac{1}{2}p^{+a}p^{+b}F_{ab} - \tfrac{1}{2}\eta^{ab}\theta_a^+\theta_b^+ + c^+ e + \gamma^+\psi. \end{aligned}$$

Then $\{\!\{\Sigma,\Sigma\}\!\} = \{\!\{\mathsf{G},\mathsf{G}\}\!\} = 0$, and $\{\!\{\Sigma,\mathsf{G}\}\!\} = \mathsf{T}$, where

$$\begin{aligned} \mathsf{T} = &-x_\mu^+\partial x^\mu + \partial p^{+a} p_a + \tfrac{1}{2}\big(\partial\theta_a^+\theta^a - \theta_a^+\partial\theta^a\big) \\ &+ \partial e^+ e + \partial c^+ c + \partial\psi^+\psi + \partial\gamma^+\gamma + \partial\big(A_a p^{+a}\big). \end{aligned}$$

Proof. We may decompose both Σ and G into two parts, the first of which only involves the fields $\{x^\mu, p_\mu, x_\mu^+, p^{+\mu}\}$, and the second of which involves the remaining fields:

$$\begin{aligned} \Sigma_0 &= \big(p_\mu + A_\mu\big)\partial x^\mu - \tfrac{1}{2}\big(\eta_{ab}\theta^a\partial\theta^b + \omega_{\mu ab}\,\partial x^\mu\theta^a\theta^b\big), \\ \Sigma_1 &= \partial e^+ c + \partial\psi^+\gamma, \\ \mathsf{G}_0 &= X_a^+ p^{+a} + \tfrac{1}{4}p^{+a}p^{+b}\theta^c\theta^d R_{abcd} + \tfrac{1}{2}p^{+a}p^{+b}F_{ab} - \tfrac{1}{2}\eta^{ab}\theta_a^+\theta_b^+, \\ \mathsf{G}_1 &= c^+ e + \gamma^+\psi. \end{aligned}$$

The formulas $\{\!\{\Sigma,\Sigma\}\!\} = \{\!\{\mathsf{G}_1,\mathsf{G}_1\}\!\} = 0$ and

$$\{\!\{\Sigma_1,\mathsf{G}_1\}\!\} = \partial e^+ e + \partial c^+ c + \partial\psi^+\psi + \partial\gamma^+\gamma$$

are easily verified, and it is also clear that $\{\!\{\Sigma_i,\mathsf{G}_j\}\!\} = \{\!\{\mathsf{G}_i,\mathsf{G}_j\}\!\} = 0$ if $i \neq j$.

The formulas

$$\{\!\{\Sigma_0,\mathsf{G}_0\}\!\} = -x_\mu^+\partial x^\mu + \partial p^{+a}p_a + \tfrac{1}{2}\big(\partial\theta_a^+\theta^a - \theta_a^+\partial\theta^a\big) + \partial\big(A_a p^{+a}\big)$$

and $\{\mathsf{G}_0,\mathsf{G}_0\} = 0$ are a consequence of the structure equations and the Bianchi identities, together with the corresponding equations $F = dA$ and $dF = 0$ for the magnetic potential and its field strength. □

The interest of this result is that $\operatorname{ad}(\mathsf{T}) = \mathsf{t}_0 + \partial\mathsf{t}_1$ where $\mathsf{t}_0 = \partial$ and

$$\begin{aligned} \mathsf{t}_1 = &-\big(x_\mu^+ - p^{+a}\partial_\nu A_a\big)\frac{\partial}{\partial x_\nu^+} - \big(p_a + A_a\big)\frac{\partial}{\partial p_a} - \frac{1}{2}\left(\theta_a^+\frac{\partial}{\partial\theta_a^+} - \theta^a\frac{\partial}{\partial\theta^a}\right) \\ &- e\frac{\partial}{\partial e} - c\frac{\partial}{\partial c} - \psi\frac{\partial}{\partial\psi} - \gamma\frac{\partial}{\partial\gamma}. \end{aligned}$$

The following proposition gives a method of constructing solutions of the classical master equation.

Proposition 3.2. *Let $\mathsf{W} \in \mathcal{A}^1$ satisfy the equations $\{\!\{\Sigma,\mathsf{W}\}\!\} = 0$ and*

$$\{\!\{\{\!\{\mathsf{G},\mathsf{W}\}\!\},\mathsf{W}\}\!\} = 0.$$

Then $\mathsf{S} = \Sigma + \{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\} + (\mathsf{W} + \mathsf{t}_1\mathsf{W})\varepsilon$ *is a solution of the classical master equation*

$$d\mathsf{S} + \tfrac{1}{2}\{\!\!\{\mathsf{S},\mathsf{S}\}\!\!\} = 0.$$

Proof. The proposition is implied by Lemma 2.4, if we can prove the equation

$$\tfrac{1}{2}\{\Sigma + \{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\}, \Sigma + \{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\}\}\!\!\} = \partial(\mathsf{W} + \mathsf{t}_1\mathsf{W}).$$

By the graded Jacobi relation, we see that

$$\begin{aligned}\tfrac{1}{2}\{\Sigma + \{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\}, \Sigma + \{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\}\}\!\!\} &= \big(d\Sigma + \tfrac{1}{2}\{\!\!\{\Sigma,\Sigma\}\!\!\}\big) + \tfrac{1}{4}\{\!\!\{\{\!\!\{\{\!\!\{\mathsf{G},\mathsf{G}\}\!\!\},\mathsf{W}\}\!\!\},\mathsf{W}\}\!\!\} \\ &\quad - \{\!\!\{G,\{\!\!\{\Sigma,\mathsf{W}\}\!\!\}\}\!\!\} - \tfrac{1}{2}\{\!\!\{\mathsf{G},\{\!\!\{\{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\},\mathsf{W}\}\!\!\}\}\!\!\} + \{\!\!\{\mathsf{T},\mathsf{W}\}\!\!\}.\end{aligned}$$

Both terms on the first line vanish by Lemma 3.1, while the first two terms on the second line vanish by hypothesis. The result follows from the formula

$$\{\!\!\{\mathsf{T},\mathsf{W}\}\!\!\} = \partial\mathsf{W} + \partial\mathsf{t}_1\mathsf{W}. \qquad \square$$

We now consider the expression

$$\mathsf{W} = \tfrac{1}{2}\eta^{ab}p_a p_b c + \tfrac{1}{2}F_{ab}\theta^a\theta^b c + p_a\theta^a\gamma - e^+\gamma^2 \in \mathcal{A}^1. \tag{5}$$

It is clear that $\{\!\!\{\Sigma,\mathsf{W}\}\!\!\} = 0$, and a somewhat lengthier calculation shows that

$$\{\!\!\{\{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\},\mathsf{W}\}\!\!\} \in \mathcal{A}^2$$

vanishes as well. It follows that

$$\begin{aligned}\mathsf{S} &= \Sigma + \{\!\!\{\mathsf{G},\mathsf{W}\}\!\!\} + (\mathsf{W} + \mathsf{t}_1\mathsf{W})\varepsilon \\ &= \big(\omega^a_\mu p_a + A_\mu\big)\,\partial x^\mu - \tfrac{1}{2}\big(\eta_{ab}\theta^a\partial\theta^b + \omega_{\mu ab}\partial x^\mu\theta^a\theta^b\big) \\ &- \tfrac{1}{2}e\left(\eta^{ab}p_a p_b + F_{ab}\theta^a\theta^b\right) + \psi p_a\theta^a \\ &+ \big(\partial e^+ - \eta^{ab}X^+_a p_b + \tfrac{1}{2}p^{+a}p_b\theta^c\theta^d R_a{}^b{}_{cd} - p^{+a}p_b F_a{}^b + \theta^+_a\theta^b F^a{}_b + \tfrac{1}{2}\omega^\lambda_a p^+_a\theta^b\theta^c\nabla_\lambda F_{bc}\big)c \\ &+ \big(\partial\psi^+ - X^+_a\theta^a + \eta^{ab}\theta^+_a p_b + 2e^+\psi - p^{+a}\theta^b F_{ab}\big)\gamma - c^+\gamma^2 \\ &- \big(\eta^{ab}p_a p_b c + \tfrac{3}{2}p_a\theta^a\gamma - e^+\gamma^2 - \tfrac{1}{2}F_{ab}\theta^a\theta^b c + \eta^{ab}A_a p_b c + A_a\theta^a\gamma\big)\varepsilon\end{aligned}$$

satisfies the classical master equation $d\mathsf{S} + \tfrac{1}{2}\{\!\!\{\mathsf{S},\mathsf{S}\}\!\!\} = 0$. In this equation, we have denoted by ∇F the covariant derivative of the two-tensor F with respect to the Levi-Civita connection $\omega_\mu{}^a{}_b$.

Corollary 3.3. *If* $\int f\,dt \in \mathcal{F}^k$ *is a cocycle in the complex* $(\mathcal{F},\mathsf{s})$, *where* s *is the vector field associated to the solution* S *of the classical master equation, then* $\int \{\!\!\{\mathsf{G},f\}\!\!\}\,dt \in \mathcal{F}^{k-1}$ *is a cocycle in* $\mathcal{F}$, *called the* **transgression** *of* f. *In particular, the long exact sequence*

$$\cdots \longrightarrow H^{-1}(\mathcal{A},\mathsf{s}) \xrightarrow{\partial} H^{-1}(\mathcal{A},\mathsf{s}) \longrightarrow H^{-1}(\mathcal{F},\mathsf{s}) \longrightarrow$$

$$\longrightarrow H^0(\mathcal{A}/\mathbb{C},\mathsf{s}) \xrightarrow{\partial} H^0(\mathcal{A},\mathsf{s}) \longrightarrow H^0(\mathcal{F},\mathsf{s}) \longrightarrow$$

$$\longrightarrow H^1(\mathcal{A},\mathsf{s}) \xrightarrow{\partial} H^1(\mathcal{A},\mathsf{s}) \longrightarrow H^1(\mathcal{F},\mathsf{s}) \longrightarrow \cdots$$

splits, in the sense that the morphisms ∂ *vanish.*

Proof. Since $\{\!\{\mathsf{G},\mathsf{G}\}\!\}=0$, we have the equation

$$\{\!\{\Sigma+\{\!\{\mathsf{G},\mathsf{W}\}\!\},\{\!\{\mathsf{G},f\}\!\}\}\!\}=\{\!\{\mathsf{T},f\}\!\}-\{\!\{\mathsf{G},\{\!\{\Sigma+\{\!\{\mathsf{G},\mathsf{W}\}\!\},f\}\!\}\}\!\}.$$

By hypothesis, $\{\!\{\Sigma+\{\!\{\mathsf{G},\mathsf{W}\}\!\},f\}\!\}=\partial g$ is a total derivative. Thus

$$\{\!\{\Sigma+\{\!\{\mathsf{G},\mathsf{W}\}\!\},\{\!\{\mathsf{G},f\}\!\}\}\!\}=\partial(f+\mathsf{t}_1 f+g).$$

Hence $\{\!\{\mathsf{G},f\}\!\}$ descends to a cocycle in $\mathcal{F}^{k-1}$. This shows that the connecting morphisms ∂ in the long-exact sequence vanish. $\square$

We close this section by showing how to rewrite S as an AKSZ action. In AKSZ models, the fields may be assembled into differential forms of homogeneous total degree: in our case, the sum of a 0-form of ghost number k and a 1-form of ghost number $k-1$. These differential forms are as follows:

$$\begin{gathered}\mathbf{x}^\mu=x^\mu+dt\,\{\!\{\mathsf{G},x^\mu\}\!\}\quad \boldsymbol{\theta}^a=\theta^a+dt\,\{\!\{\mathsf{G},\theta^a\}\!\}\quad \mathbf{p}_a=p_a+dt\,\{\!\{\mathsf{G},p_a\}\!\}\\ \mathbf{c}=c+dt\,\{\!\{\mathsf{G},c\}\!\}\qquad\qquad \boldsymbol{\gamma}=\gamma+dt\,\{\!\{\mathsf{G},\gamma\}\!\}\\ \mathbf{e}^+=e^++dt\,\{\!\{\mathsf{G},e^+\}\!\}\quad \boldsymbol{\psi}^+=\psi^++dt\,\{\!\{\mathsf{G},\psi^+\}\!\}\end{gathered}$$

The action S is the one-form component of the differential form

$$\begin{gathered}\big(\omega^a_\mu(\mathbf{x})\mathbf{p}_a+A_\mu(\mathbf{x})-\tfrac12\omega_{\mu ab}(\mathbf{x})\boldsymbol{\theta}^a\boldsymbol{\theta}^b\big)d\mathbf{x}^\mu-\tfrac12\eta_{ab}\boldsymbol{\theta}^a d\boldsymbol{\theta}^b+\mathbf{c}\,d\mathbf{e}^++\boldsymbol{\gamma}\,d\boldsymbol{\psi}^+\\ +\tfrac12\eta^{ab}\mathbf{p}_a\mathbf{p}_b\mathbf{c}+\tfrac12F_{ab}(\mathbf{x})\boldsymbol{\theta}^a\boldsymbol{\theta}^b\mathbf{c}+\mathbf{p}_a\boldsymbol{\theta}^a\boldsymbol{\gamma}-\mathbf{e}^+\boldsymbol{\gamma}^2,\end{gathered}$$

where we recognize the expressions Σ and W of Lemma 3.1 and (5) respectively on the first and second lines. The resemblance between the action in an AKSZ model and the Chern-Simons action is clear after changing variables from the field $\mathbf{p}_a$ to the field

$$\mathbf{P}_\mu=\omega^a_\mu(\mathbf{x})\mathbf{p}_a+A_\mu(\mathbf{x})-\tfrac12\omega_{\mu ab}(\mathbf{x})\boldsymbol{\theta}^a\boldsymbol{\theta}^b.$$

4. Calculation of BV cohomology

The method of [4, Section 7] may be used to calculate the BV cohomology of the spinning particle in the general case. Let $\mathcal{O}$ be the ring of functions on the target $U\subset\mathbb{R}^d$ of the spinning particle: we may take any of the standard structure rings of geometry, namely algebraic, analytic or infinitely-differentiable functions, or even power series. Let $\mathcal{A}$ is the graded polynomial algebra over $\mathcal{O}$ generated by the remaining variables of the theory, namely

$$\begin{aligned}&\{\partial^\ell x^\mu\}_{\ell>0}\cup\{\partial^\ell\theta^a,\partial^\ell p_a,\partial^\ell x^+_\mu,\partial^\ell\theta^+_a,\partial^\ell p^{a+}\}_{\ell\ge0}\\ &\qquad\qquad\cup\{\partial^\ell e,\partial^\ell\psi,\partial^\ell e^+,\partial^\ell\psi^+\}_{\ell\ge0}\cup\{\partial^\ell c,\partial^\ell\gamma,\partial^\ell c^+,\partial^\ell\gamma^+\}_{\ell\ge0}.\end{aligned}$$

Let

$$\mathcal{A}^*_\gamma=\mathcal{A}^*\otimes_{\mathbb{C}[\gamma]}\mathbb{C}[\gamma,\gamma^{-1}]$$

be the localization of $\mathcal{A}^*$, obtained by inverting the ghost γ.

Given a vector v with components v_a, define

$$\iota(v)=\eta^{ab}v_a\frac{\partial}{\partial\theta^b}.$$

If $f\in\mathcal{O}$, denote by ∇f the vector with components

$$(\nabla f)_a=\omega^\mu_a\big(\partial_\mu+\omega_\mu{}^b{}_b f\big).$$

We may interpret the function f as representing a section of a line bundle over U with connection form $\omega_\mu{}^b{}_b dx^\mu$.

Let $\Omega = \theta^1 \dots \theta^d$. Given a function $f \in \mathcal{O}$ and $k \geq 0$, consider the following elements of $\mathcal{A}_\gamma^{-k-1}$:

$$
\begin{aligned}
A_k(f) &= (\psi^+)^{k+1} c f \Omega \gamma^{-1}, \\
Z_k(f) &= (k+1)(\psi^+)^k f \Omega \gamma^{-1} + (\psi^+)^{k+1} c \iota(\nabla f) \Omega \gamma^{-1}.
\end{aligned}
$$

After application of the BV differential s to these expressions, the poles in γ cancel, showing that the following expressions are cocycles in $\mathcal{A}^{-k}$ with respect to the differential s:

$$
\alpha_k(f) = \mathsf{s}(A_k(f)), \qquad \zeta_k(f) = \mathsf{s}(Z_k(f)).
$$

Consider also the transgressions of these cocycles:

$$
\tilde{\alpha}_k(f) = \{\!\{ \mathsf{G}, \alpha_{k-1}(f) \}\!\}, \qquad \tilde{\zeta}_k(f) = \{\!\{ \mathsf{G}, \zeta_{k-1}(f) \}\!\}.
$$

Let $\mathcal{R}$ be the quotient of the differential graded superalgebra $\mathcal{A}^*$ by the differential ideal generated by the fields

$$
\{e, \psi, c\} \cup \{x_\mu^+, \theta_a^+, p^{+a}, e^+, \psi^+, c^+, \gamma^+\}
$$

Denote by P_a, Θ^a, X^μ and Γ the zero-modes $\int p_a \, dt$, $\int \theta^a \, dt$, $\int x^\mu \, dt$ and $\int \gamma \, dt$ respectively. Then $\mathcal{R}$ is the graded superalgebra

$$
\mathcal{O}[\Theta^a, \mathsf{P}_a, \Gamma] / \big(\mathsf{P}_a \Theta^a, \eta^{ab} \mathsf{P}_a \mathsf{P}_b + F_{ab}(\mathsf{X}) \Theta^a \Theta^b, \Gamma^2 \big)
$$

with differential $\Gamma \mathsf{Q}$, where Q is the differential operator

(6)
$$
\mathsf{Q} = \omega_a^\mu(\mathsf{X}) \Theta^a \frac{\partial}{\partial \mathsf{X}^\mu} + \eta^{ab} \mathsf{P}_a \frac{\partial}{\partial \Theta^b} + \omega_c^\mu(\mathsf{X}) \omega_\mu{}^a{}_b(\mathsf{X}) \Theta^c \left(\mathsf{P}_a \frac{\partial}{\partial \mathsf{P}_b} - \Theta^b \frac{\partial}{\partial \Theta^a} \right) + F_{ab}(\mathsf{X}) \Theta^a \frac{\partial}{\partial \mathsf{P}_b}.
$$

We denote the element $\Theta^1 \dots \Theta^d$ of $\mathcal{R}^0$ by the same symbol Ω as in $\mathcal{A}^0$.

The map ξ^0 from $\mathcal{O}[\Theta^a, \mathsf{P}_a]$ to $\mathcal{A}^0$ which takes a function u to the corresponding function $\xi^0(u)$ in the variables $\{x^\mu, \theta^a, p_a\}$ induces a map from $H^0(\mathcal{R})$ to $H^0(\mathcal{A}, \mathsf{s})$. Observe that $\xi^0(\iota(P)\Omega) = -\zeta_0(1)$.

Similarly, the map from $\mathcal{O}[\Theta^a, \mathsf{P}_a]$ to $\mathcal{A}^1$ which takes a function v to the element

$$
\xi^1(v) = \gamma v + c \mathsf{Q} v
$$

induces a map from $H^1(\mathcal{R})$ to $H^1(\mathcal{A}, \mathsf{s})$. Define the transgressions of the classes $\xi^0(u)$ and $\xi^1(v)$:

$$
\tilde{\xi}^{-1}(u) = \{\!\{ \mathsf{G}, \xi^0(u) \}\!\} \qquad \tilde{\xi}^0(v) = \{\!\{ \mathsf{G}, \xi^1(v) \}\!\}.
$$

The following theorem has the same form as in the special case where $g^{\mu\nu}$ is constant and $A_\mu = 0$, discussed in [4].

Theorem 4.1.

$$H^{-k}(\mathcal{F},\mathsf{s}) = \begin{cases} \Big\{ \int \big(\alpha_k(f) + \zeta_k(g) + \tilde{\alpha}_k(\tilde{f}) + \tilde{\zeta}_k(\tilde{g})\big)\,dt \,\Big|\, f,g,\tilde{f},\tilde{g} \in \mathcal{O} \Big\} & k > 1, \\ \Big\{ \int \big(\tilde{\xi}^{-1}(u) + \alpha_1(f) + \zeta_1(g) + \tilde{\alpha}_1(\tilde{f}) + \tilde{\zeta}_1(\tilde{g})\big)\,dt \,\Big| \\ \qquad u \in H^0(\mathcal{R}/\mathbb{C}), f,g,\tilde{f} \in \mathcal{O}, \tilde{g} \in \mathcal{O}/\mathbb{C} \Big\} & k = 1, \\ \Big\{ \int \big(\xi^0(u) + \tilde{\xi}^0(v) + \alpha_0(f) + \zeta_0(g)\big)\,dt \,\Big| \\ \qquad u \in H^0(\mathcal{R}), v \in H^1(\mathcal{R}), f \in \mathcal{O}, g \in \mathcal{O}/\mathbb{C} \Big\} & k = 0, \\ \Big\{ \int \xi^1(v)\,dt \,\Big|\, v \in H^1(\mathcal{R}) \Big\} & k = -1, \\ 0 & k < -1. \end{cases}$$

The proof of the theorem follows along the same lines as in Section 7 of [4]. We use the filtration on the complex $(\mathcal{A}^*, \mathsf{s})$ associated to the parameter $\sigma = 0$, which assigns bidegrees to the fields and their derivatives according to the following table:

Φ	(p,q)	(p^+,q^+)
x	$(0,0)$	$(0,-1)$
θ	$(0,0)$	$(0,-1)$
p	$(0,0)$	$(0,-1)$
e	$(2,0)$	$(-1,0)$
ψ	$(2,0)$	$(-1,0)$
c	$(2,-1)$	$(-1,-1)$
γ	$(2,-1)$	$(-1,-1)$

Here, p and p^+ are the filtration degrees of a field Φ and its antifield Φ^+, and q and q^+ are the complentary degrees, such that $\mathrm{gh}(\Phi) = p + q$ and $\mathrm{gh}(\Phi^+) = p^+ + q^+$. We obtain a spectral sequence E_r^{pq} such that $E_r^{pq} = 0$ if $q > 0$, and $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$.

It is not *a priori* evident that this spectral sequence converges. We will see that, as in [4], d_r vanishes for $r \geq 3$. its convergence is proved by lifting the cohomology classes in E_3 to the explicit nontrivial cocycles in the original complex that were introduced above.

The differential $d_0 : E_0^{pq} \to E_0^{p,q+1}$ of the initial page E_0 is as follows:

$$\begin{aligned} d_0 = &-\big(\partial p_\mu + \tfrac{1}{2}\partial_{[\mu}\omega_{\nu]ab}\partial x^\nu \theta^a\theta^b - \omega_{\mu ab}\theta^a\partial\theta^b - F_{\mu\nu}\partial x^\nu\big)\frac{\partial}{\partial x^+_\mu} \\ &+ \big(\eta_{ab}\partial\theta^b + \omega_{\mu ab}\partial x^\mu\theta^b\big)\frac{\partial}{\partial\theta^+_a} + \omega^a_\mu\partial x^\mu\frac{\partial}{\partial p^{a+}} \\ &+ \partial e^+\frac{\partial}{\partial c^+} + \partial\psi^+\frac{\partial}{\partial\gamma^+} - \partial c\frac{\partial}{\partial e} + \partial\gamma\frac{\partial}{\partial\psi}. \end{aligned}$$

It follows that E_1 is the tensor product of the algebra $\mathcal{O}$, with generators $\mathsf{X}^\mu = \int x^\mu \, dt$, and the free graded commutative algebra with the following generators:

gh	generators
-1	$\mathsf{E}^+ = \int e^+ \, dt$, $\Psi^+ = \int \psi^+ \, dt$
0	$\Theta^a = \int \theta^a \, dt$, $\mathsf{P}_a = \int p_a \, dt$
1	$\mathsf{C} = \int c \, dt$, $\Gamma = \int \gamma \, dt$

The differential $d_1 : E_1^{pq} \to E_1^{p+1,q}$ is given by the formula

$$d_1 = -\tfrac{1}{2} \left(\eta^{\mu\nu} \mathsf{P}_\mu \mathsf{P}_\nu + F_{ab}(\mathsf{X}) \Theta^a \Theta^b \right) \frac{\partial}{\partial \mathsf{E}^+} - \mathsf{P}_\mu \Theta^\mu \frac{\partial}{\partial \Psi^+}.$$

Cohomology classes in $E_2 = H^*(E_1, d_1)$ take the general form

$$z = [b_0] + \sum_{j>0} \Big([\mathbb{A}_j(f_j)] + [\mathbb{B}_j(g_j)] \Big),$$

where $[b_0]$ is an element of the ring

$$\mathcal{O}[\Theta^a, \mathsf{P}_a, \mathsf{C}, \Gamma] / \big(\mathsf{P}_a \Theta^a, \eta^{ab} \mathsf{P}_a \mathsf{P}_b + F_{ab}(\mathsf{X}) \Theta^a \Theta^b \big)$$

and

$$\mathbb{A}_j(f) = 2j(\Psi^+)^{j-1} \mathsf{E}^+ f \, \Omega - (\Psi^+)^j f \iota(\mathsf{P}) \Omega, \qquad \mathbb{B}_j(g) = (\Psi^+)^j f \, \Omega,$$

for $f, g \in \mathcal{O}[\mathsf{C}, \Gamma]$.

The differential $d_2 : E_2^{pq} \to E_2^{p+2,q-1}$ is given by the formula

$$d_2 = -\mathsf{C}\mathsf{Q}^2 - \Gamma \mathsf{Q} + \Gamma^2 \frac{\partial}{\partial \mathsf{C}} + 2\mathsf{E}^+ \Gamma \frac{\partial}{\partial \Psi^+},$$

where Q is the differential operator introduced in (6). The remainder of the proof of the theorem is as in [4].

5. The quantum master equation

The Batalin-Vilkovisky formalism for quantization of a solution S of the classical master equation involves a series

$$\mathbb{S} = S + \sum_{n=1}^{\infty} \hbar^n S_n$$

satisfying the quantum master equation

$$\tfrac{1}{2} \{ \textstyle\int \mathbb{S} \, dt, \int \mathbb{S} \, dt \} + \hbar \int \Delta \mathbb{S} \, dt = 0.$$

Expanding in powers of $\hbar$, we see that this amounts to the sequence of equations

$$\mathsf{s} \textstyle\int S_{n+1} \, dt + \tfrac{1}{2} \displaystyle\sum_{k=1}^{n} \{ \textstyle\int S_k \, dt, \int S_{n-k+1} \, dt \} + \hbar \int \Delta S_n \, dt = 0, \quad n \geq 0.$$

Here, Δ is the differential operator

$$\Delta \textstyle\int f \, dt = \displaystyle\sum_{\Phi_i} (-1)^{\mathrm{p}(\Phi_i)} \int \sum_{k,\ell} \lim_{s \to t} \partial_s^k \partial_t^\ell \delta(s,t) \frac{\partial^2 f}{\partial(\partial^k \Phi_i(s)) \partial(\partial^\ell \Phi_i^+(t))} \, dt.$$

The operator Δ is ill-defined, owing to ultra-violet divergences. But in the case of the spinning particle, there is a great simplification, since the only contribution to ΔS comes from the terms $-\eta^{ab}X_a^+p_bc$ and $-X_a^+\theta^a\gamma$ of S, and we have

$$\begin{aligned}\Delta S &= \Delta\left(-\omega_a^\mu x_\mu^+\big(\eta^{ab}p_bc+\theta^a\gamma\big)+\omega_a^\mu\omega_\mu{}^c{}_d\big(-\eta^{ab}p_cp^{d+}p_bc+\theta_c^+\theta^d\theta^a\gamma\big)\right)\\ &= C_\Lambda\big(-\partial_\mu\omega_a^\mu+\omega_a^\mu\omega_\mu{}^a{}_b\big)\big(\eta^{bc}p_cc+\theta^b\gamma\big)\\ &= -C_\Lambda\,\mathsf{s}\log\det\big(\omega_\mu^a\big).\end{aligned}$$

where C_Λ is a function of the cut-off Λ. (In fact, C_Λ vanishes in the heat-kernel regularization, since the world-line $\mathbb{R}$ is odd-dimensional.) We see that $S_1 = C_\Lambda\log\det\big(\omega_\mu^a\big)$. Since $\{S_1,S_1\}$ and ΔS_1 both clearly vanish, we also see that $S_n=0$, $n>1$. This shows that the solution to the quantum master equation associated for the spinning particle with curved target is

$$\mathsf{S} = S + C_\Lambda\log\det\big(\omega_\mu^a\big).$$

Acknowledgements. I am grateful to Chris Hull for introducing me to the first-order formalism of the spinning particle. This research is partially supported by EPSRC Programme Grant EP/K034456/1 "New Geometric Structures from String Theory" and Collaboration Grant #243025 of the Simons Foundation. This paper was written while the author was a member of the program "Higher Structures in Geometry and Physics" at the Max-Planck-Institut in Bonn in 2016.

REFERENCES

[1] M. Alexandrov, A. Schwarz, O. Zaboronsky, and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, Internat. J. Modern Phys. A **12** (1997), no. 7, 1405–1429.

[2] L. A. Dickey, *Poisson brackets with divergence terms in field theories: three examples*, Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), Contemp. Math., vol. 227, Amer. Math. Soc., Providence, RI, 1999, pp. 67–78.

[3] Ezra Getzler, *A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, Duke Math. J. **111** (2002), no. 3, 535–560.

[4] ———, *The Batalin-Vilkovisky cohomology of the spinning particle*, to appear, JHEP, available at arXiv:1511.02135.

[5] M. D. Kruskal, R. M. Miura, C. S. Gardner, and N. J. Zablusky, *Korteweg-de Vries equation and generalizations. V. Uniqueness and nonexistence of polynomial conservation laws*, J. Math. Phys. **11** (1970), 952–960.

[6] Vladimir O. Soloviev, *Boundary values as Hamiltonian variables. I. New Poisson brackets*, J. Math. Phys. **34** (1993), no. 12, 5747–5769.

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS, USA
E-mail address: getzler@northwestern.edu