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ON A REMARKABLE CLASS OF LEFT-SYMMETRIC ALGEBRAS

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Abstract. *We discuss locally simply transitive affine actions of Lie groups G on finite-dimensional vector spaces such that the commutator subgroup $[G,G]$ is acting by translations. In other words, we consider left-symmetric algebras satisfying the identity $[x,y]\cdot z=0$. We derive some basic characterizations of such left-symmetric algebras and we highlight their relationships with the so-called Novikov algebras and derivation algebras.*

1 Introduction

Let M be a differentiable manifold together with a flat torsion-free connection ∇, and let $\mathfrak{X}(M)$ denote the space of all vector fields on M. Endowing $\mathfrak{X}(M)$ with the product defined by $X\cdot Y=\nabla_X Y$, we observe that the flatness of ∇ yields the identity

$$(X\cdot Y)\cdot Z-X\cdot(Y\cdot Z)=(Y\cdot X)\cdot Z-Y\cdot(X\cdot Z), \tag{1}$$

for all $X,Y,Z\in\mathfrak{X}(M)$.

A finite-dimensional vector space together with a product satisfying the above identity is called *a left-symmetric algebra.* The concept of a left-symmetric algebra has its roots in the paper [8], and the term was first introduced in [22] (see also [23]).

If $(A,\cdot)$ is a left-symmetric algebra over a field $\mathbb{F}$, then the binary operation defined by $[x,y]=x\cdot y-y\cdot x$ makes A into a Lie algebra called the associated Lie algebra. Conversely, if $\mathcal{G}$ is a Lie algebra together with a left-symmetric product such that $[x,y]=x\cdot y-y\cdot x$, then we say that this left-symmetric product is compatible with the Lie structure of $\mathcal{G}$. For instance, if ∇ is a flat torsion-free connection on a differentiable manifold M, then it is known that the space $\mathfrak{X}(M)$ is an infinite dimensional Lie algebra under the standard Lie bracket of vector fields. In this case, the left-symmetric product on $\mathfrak{X}(M)$ defined by $X\cdot Y=\nabla_X Y$ is compatible with the standard Lie structure of $\mathfrak{X}(M)$.

Let now $M=G$ be a finite dimensional Lie group with Lie algebra $\mathcal{G}$, and let ∇ be a left-invariant affine connection (i.e., a left-invariant flat torsion-free

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connection) on G. As mentioned above, by identifying the Lie algebra $\mathcal{G}$ of G with the set of all left-invariant vector fields on G, we see that $\mathcal{G}$ becomes a finite dimensional left-symmetric subalgebra of $\mathfrak{X}(G)$. Conversely, if $\mathcal{G}$ is endowed with a left-symmetric product which is compatible with the Lie structure of $\mathcal{G}$, then by means of the conditions

$$(fX) \cdot Y = f(X \cdot Y), \qquad X \cdot (fY) = X(f) \cdot Y + f(X \cdot Y),$$

for all $X, Y \in \mathfrak{X}(G)$ and $f \in C^{\infty}(G)$, we can extend this left-symmetric product to a left-invariant affine connection on G.

To summarize, we deduce that giving a left-invariant affine connection on a Lie group G is precisely equivalent to giving a compatible left-symmetric product on its Lie algebra $\mathcal{G}$. Moreover it turns out that, under this equivalence, bi-invariant affine connections on G correspond exactly to associative produtcs on $\mathcal{G}$ (cf. [16], p. 186; see also [15], Proposition 1.1).

Another geometric source of left-symmetric algebras comes from the geometry of locally simply transitive actions of Lie groups by affine transformations on vector spaces. It turns out that a simply connected Lie group G admits a left-invariant affine connection if and only if G acts locally simply transitively on a vector space E by affine transformations (cf. [14]). Accordingly, if G is a simply connected Lie group with Lie algebra $\mathcal{G}$, then giving a locally simply transitive affine action of G on a vector space E can be interpreted as giving a compatible left-symmetric product on $\mathcal{G}$. In this respect, it has been proved in [2] that a simply connected Lie group G which acts simply transitively by affine transformations of $\mathbb{R}^n$ is necessarily solvable. This result leads to the question raised in [16] of whether every solvable simply connected Lie group G can act simply transitively by affine transformations of $\mathbb{R}^n$. A negative answer to this question can be found in [3].

The object of this paper is to introduce a new class of left-symmetric algebras, namely, the class of finite-dimensional left-symmetric algebras A over a field $\mathbb{F}$ satisfying the identity

$$(x \cdot y) \cdot z = (y \cdot x) \cdot z, \quad \text{for all } x, y, z \in A, \tag{2}$$

or, equivalently,

$$[x, y] \cdot z = 0, \quad \text{for all } x, y, z \in A. \tag{3}$$

In terms of affine actions, this is equivalent to consider locally simply transitive affine actions of Lie groups G on a vector space E such that the commutator subgroup $[G, G]$ is acting by translations. Left-symmetric algebras over $\mathbb{R}$ satisfying the identity (2) are classified up to dimesnsion 4 in [1].

The paper is organized as follows. In Section 2, after recalling the definition of left-symmetric algebras, we give the definitions of the so-called Novikov algebras and derivation algebras. These are defined as left-symmetric algebras satisfying the identities $(x \cdot y) \cdot z = (x \cdot z) \cdot y$ and $(x \cdot y) \cdot z = (z \cdot y) \cdot x$, respectively. We then consider left-symmetric algebras satisfying the identity (2).

We give some examples showing that these three conditions are pairwise independent, and we prove that if any two of these three conditions are satisfied for a given left-symmetric algebra then the third condition is satisfied as well. We also give a criterion that connects associative algebras satisfying (2) and Novikov assocative algebras, and we deduce that the Lie algebra associated to a Novikov assocative algebra is necessarily 2-step solvable.

In Section 3, we introduce the notion of center for a left-symmetric algebra and we turn our attention to the question as to when this center coincides with the center of the associated Lie algebra. For instance, we show that these two centers coincide if the left-symmetric algebra is Novikov or derivation, and we give an example of a left-symmetric algebra satisfying (2) so that its center does not coincide with the center of its associated Lie algebra. We also show that the center of a left-symmetric algebra that is Novikov or derivation is a two-sided ideal, and we give an example of a left-symmetric algebra satisfying the (2) (and which is of course neither Novikov nor derivation) so that its center fails to be a two-sided ideal. In the last two subsections of this section, we introduce the translational center of a left-symmetric algebra. In terms of affine actions, this corresponds to the set of translations lying in the center of the group. We show that every left-symmetric algebra satisfying (2) whose associated Lie algebra is nilpotent can be obtained as a central extension of a smaller dimensional left-symmetric algebra satisfying (2). We also show that the translational center of a noncommutative derivation algebra is non-trivial, and we deduce from this that a derivation algebra whose associated Lie algebra is nonsingular nilpotent is necessarily complete, that is, all right multiplications are nilpotent transformations.

In Section 4, we introduce two notions of a radical for a left-symmetric algebra A, namely the Koszul radical $R(A)$ and the right-nilpotent radical $N(A)$. If A is a left-symmetric algebra satisfying (2), we show that $R(A)$ is a two-sided ideal of A containing the derived Lie algebra $[A, A]$. As a consequence, we conclude that every left-symmetric algebra satisfying (2) can be obtained as an extension of a commutative associative algebra by a complete left-symmetric algebra satisfying (2). This allows us to classify all low-dimensional left-symmetric algebras satisfying (2). If A is a Novikov algebra with associated Lie algebra $\mathcal{G}_A$, we show that these two radicals coincide and are equal to the kernel of the homomorphism $L : A \to \mathfrak{gl}(\mathcal{G}_A)$, where L_x is the left-multiplication defined by $L_x y = x \cdot y$.

Finally, in Section 5, we make some comments on simple left-symmetric algebras. In [24], Zelmanov proved that a finite-dimensional simple Novikov algebra A over a field $\mathbb{F}$ of characteristic zero is one-dimensional (i.e., A coincides with $\mathbb{F}$). This is true when $\mathbb{F}$ is algebraically closed. However, the two-dimensional commutative algebra $A_{2,\mathbb{R}}$ over $\mathbb{R}$ defined by the following multiplication table: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2 \cdot e_1 = e_2$, $e_2 \cdot e_2 = -e_1$ is a Novikov algebra which turns out to be simple. In fact, we shall show that this is the unique simple Novikov algebra over $\mathbb{R}$ of dimension ≥ 2. We shall also show that a simple left-symmetric algebra over $\mathbb{R}$ which is derivation or satisfying(2) is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$. We close this section by pointing out that every complete left-symmetric algebra over a field of characteristic zero that is Novikov, derivation, or satisfying (2) is not simple.

2 Left-symmetric algebras

Throughout this paper, $\mathbb{F}$ will denote a field of characteristic zero. As we mentioned above, recall that a finite-dimensional algebra $(A, \cdot)$ over $\mathbb{F}$ is called *left-symmetric* if it satisfies the identity

$$(x, y, z) = (y, x, z), \quad \text{for all } x, y, z \in A,$$

where (x, y, z) denotes the associator $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$. In this case, the commutator $[x, y] = x \cdot y - y \cdot x$ defines a bracket that makes A into a Lie algebra. We denote by $\mathcal{G}_A$ this Lie algebra and call it *the associated Lie algebra to* A. Conversely, if $\mathcal{G}$ is a Lie algebra endowed with a left-symmetric product satisfying the condition $[x, y] = x \cdot y - y \cdot x$, then we say that this left-symmetric product is *compatible* with the Lie structure of $\mathcal{G}$.

Let A be a left-symmetric algebra over a field $\mathbb{F}$, and let L_x and R_x denote the left and right multiplications by the element $x \in A$, respectively. The identity (1) is now equivalent to the formula

$$[L_x, L_y] = L_{[x,y]}, \quad \text{for all } x, y \in A, \tag{4}$$

or, in other words, the linear map $L : \mathcal{G}_A \to End\,(A)$ is a representation of Lie algebras.

We notice that (1) is also equivalent to the formula

$$[L_x, R_y] = R_{x \cdot y} - R_y \circ R_x, \quad \text{for all } x, y \in A. \tag{5}$$

We say that A is *complete* if R_x is a nilpotent operator, for all $x \in A$. In this context, if G is an n-dimensional simply connected Lie group with Lie algebra $\mathcal{G}$, then giving a compatible complete left-symmetric product on $\mathcal{G}$ can be interpreted as giving a complete left-invariant affine connection on G or, equivalently, as giving a simply transitive affine action of G on an n-dimensional vector space E.

We say that A is *a Novikov algebra* if it satisfies the identity

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y, \quad \text{for all } x, y, z \in A. \tag{6}$$

In terms of left and right multiplications, (6) is equivalent to any one of the following formulas

$$\begin{aligned} [R_x, R_y] &= 0, \quad \text{for all } x, y \in A, & (7)\\ L_{x \cdot y} &= R_y \circ L_x, \quad \text{for all } x, y \in A. & (8) \end{aligned}$$

The left-symmetric algebra A is called *a derivation algebra* if it satisfies the identity

$$(x \cdot y) \cdot z = (z \cdot y) \cdot x, \quad \text{for all } x, y, z \in A. \tag{9}$$

or, equivalently, all left and right multiplications L_x and R_x are derivations of the ssociated Lie algebra $\mathcal{G}_A$. This is also equivalent to the formula

$$L_{x\cdot y} = R_x \circ R_y, \quad \text{for all } x, y \in A. \tag{10}$$

2.1 Left-symmetric algebras satisfying $[x, y] \cdot z = 0$

As we mentioned before, in this paper we are mainly concerned with locally simply transitive affine actions of Lie groups G on vector spaces such that the commutator subgroup $[G, G]$ is acting by translations. In terms of left-symmetric structures, we are concerned with the class of finite-dimensional left-symmetric algebras satisfying the identity (2). If A is such an algebra, then recall that (2) is equivalent to the identity $[x, y] \cdot z = 0$ for all $x, y, z \in A$, which in turn is equivalent to say that the restriction of any right multiplication R_x to the derived ideal $[A, A]$ is identically zero. In light of (4), this is also equivalent to the identity

$$[L_x, L_y] = 0, \quad \text{for all } x, y \in A. \tag{11}$$

The simplest examples of left-symmetric algebras satisfying (2) are the commutative algebras. This is stated in the following lemma, whose proof is straightforward.

Lemma 1 *A left-symmetric algebra A over a field $\mathbb{F}$ is commutative if and only if the associated Lie algebra $\mathcal{G}_A$ is abelian. In that case, A is associative, Novikov, derivation, and satisfying (2).*

However, an arbitrary left-symmetric algebra which satisfies (2) need not be Novikov nor derivation at all. In fact, the conditions (6), (9), and (2) are pairwise independent, as the following examples show.

Example 2 *Over any field $\mathbb{F}$, there is up to isomorphism just one two-dimensional non-abelian Lie algebra, namely the Lie algebra $\mathcal{G}_2$ of the affine group of the line. It has a basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_2$. Let A_{21} be the left-symmetric algebra whose associated Lie algebra is $\mathcal{G}_2$ and which has multiplication table: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2$. It is easy to check that A_{21} satisfies (2) but A_{21} is neither Novikov nor derivation. Of course, A_{21} is not complete. We shall give later an example of a complete left-symmetric algebra satisfying (2) which is neither Novikov nor derivation (see Example 12 below).*

Example 3 *Let $\mathcal{G}_2$ be as in the example above, and let A_{22} be the left-symmetric algebra whose associated Lie algebra is $\mathcal{G}_2$ and which has multiplication table: $e_1 \cdot e_1 = -e_1$, $e_2 \cdot e_1 = -e_2$. It is easy to check that A_{22} is a Novikov algebra which is not derivation and does not satisfy (2).*

Example 4 *Let $\mathcal{G}_3$ be the real Lie algebra with basis $\{e_1, e_2, e_3\}$ and non-trivial Lie brackets $[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$. Let $A_{3,\gamma}$ be the left-symmetric algebra whose associated Lie algebra is $\mathcal{G}_3$ and which has multiplication table: $e_1 \cdot e_1 =$*

$\alpha e_2 + \beta e_3$, $e_1 \cdot e_2 = e_2 + \gamma e_3$, $e_2 \cdot e_1 = \gamma e_3$, where α, β, γ are arbitrary real numbers. It is easy to check that $A_{3,\gamma}$ is a derivation algebra for any γ and that $A_{3,\gamma}$ satisfies (2) if and only if $\gamma = 0$. It therefore follows from Proposition 5 below that $A_{3,\gamma}$ is Novikov if and only if $\gamma = 0$. Hence, assuming $\gamma \neq 0$, we obtain a derivation left-symmetric algebra $A_{3,\gamma}$ which is not Novikov and does not satisfy (2).

Although identities (6), (9), and (2) are pairwise independent, we have the following remarkable fact.

Proposition 5 *Let A be a left-symmetric algebra over a field $\mathbb{F}$. If any two of the conditions (6), (9), (2) are satisfied, then the third condition is satisfied as well.*

Proof. Assume first that A satisfies (6) and (9), that is A is Novikov and derivation. Using (6), (9), and once again (6), it follows that

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y = (y \cdot z) \cdot x = (y \cdot x) \cdot z,$$

that is (2) is satisfied. The other cases can be proved in a similar way. ■

In [10], it was shown that the Lie algebra associated to a derivation algebra is necessarily solvable. In the case of Novikov algebras, a similar result can be deduced from [24] (see [5], Proposition 2.7). In the case of left-symmetric algebras satisfying (2), we can be more specific.

Proposition 6 *Let A be a left-symmetric algebra satisfying (2). Then, the associated Lie algebra $\mathcal{G}_A$ is 2-step solvable.*

Proof. By (3), $L_x = 0$ for all $x \in \mathcal{DG}_A = [\mathcal{G}_A, \mathcal{G}_A]$. It follows that $[x, y] = x \cdot y - y \cdot x = L_x y - L_y x = 0$ for all $x, y \in \mathcal{DG}_A$. Therefore, $\mathcal{DG}_A$ is abelian, that is $\mathcal{G}_A$ is 2-step solvable. ■

The following crucial observation is an immediate consequence of (3).

Proposition 7 *Let A be a left-symmetric algebra over a field $\mathbb{F}$ satisfying (2). Then $[A, A]$ is a two-sided ideal of A.*

2.2 Associative algebras satisfying $[x, y] \cdot z = 0$

Let $(A, \cdot)$ be an arbitrary algebra over a field $\mathbb{F}$. Define a new product $\circ$ on A as follows:

$$x \circ y = -y \cdot x, \quad \text{for all } x, y \in A.$$

It is obvious that $(A, \circ)$ is an algebra over $\mathbb{F}$, and that $(A, \cdot)$ is left (resp. right)-symmetric if and only if $(A, \circ)$ right (resp. left)-symmetric, where right symmetry means here that $(x, y, z) = (x, z, y)$ for all $x, y, z \in A$. Particularly, we have that $(A, \cdot)$ is associative if and only if $(A, \circ)$ is so. In that case, we have the following:

Proposition 8 *Let $(A,\cdot)$ be an associative algebra over a field $\mathbb{F}$, and let $(A,\circ)$ be the associative algebra over $\mathbb{F}$ defined as above. Then, $(A,\cdot)$ satisfies (2) if and only if $(A,\circ)$ is Novikov.*

Proof. Let $(A,\cdot)$ be an associative algebra, and define the associative product $\circ$ on A as above. Using the associativity of the product $\circ$, we have

$$\begin{aligned}[x,y]\cdot z &= (x\cdot y)\cdot z-(y\cdot x)\cdot z\\ &= z\circ(y\circ x)-z\circ(x\circ y)\\ &= (z\circ y)\circ x-(z\circ x)\circ y,\quad \text{for all } x,y,z\in A.\end{aligned}$$

It follows from this equality that $(A,\cdot)$ satisfies (2) if and only if $(z\circ y)\circ x-(z\circ x)\circ y=0$ for all $x,y,z\in A$, that is $(A,\circ)$ is Novikov, as desired. ■

As a consequence of Proposition 6 and Proposition 8, we have the following:

Corollary 9 *Let $\mathcal{G}$ be a Lie algebra which can admit an associative Novikov structure. Then, $\mathcal{G}$ is 2-step solvable.*

3 The center of a left-symmetric algebra

Recall that the center $Z(A)$ of an arbitrary algebra A (see for instance [20]) is defined to be the set of all z in A which commute and associate with all elements in A; that is, the set of all z in A with the property $x\cdot z=z\cdot x$ for all $x\in A$, and which satisfy the additional identity

$$(z,x,y)=(x,z,y)=(x,y,z)=0,\ \text{for all } x,y\in A. \tag{12}$$

We first begin by showing that if A is a left-symmetric algebra and $\mathcal{Z}_A$ is the center of the Lie algebra $\mathcal{G}_A$ associated to A, then (12) simplifies.

Lemma 10 *Let A be a left-symmetric algebra. Then*

$$Z(A)=\{z\in\mathcal{Z}_A:(z,x,y)=0,\quad \textit{for all } x,y\in A.\}.$$

Proof. By definition, $Z(A)\subseteq\mathcal{Z}_A$. Conversely, for all $z\in\mathcal{Z}_A$, we have

$$\begin{aligned}(x,y,z) &= (x\cdot y)\cdot z-x\cdot(y\cdot z)\\ &= z\cdot(x\cdot y)-x\cdot(z\cdot y)\\ &= (z\cdot x)\cdot y-(x\cdot z)\cdot y\\ &= (x\cdot z)\cdot y-(x\cdot z)\cdot y\\ &= 0.\end{aligned}$$

The lemma follows now by observing that $(z,x,y)=(x,z,y)$ for all $x,y,z\in A$. ■

In case A is a derivation algebra, it turns out that $Z(A)$ coincides with $\mathcal{Z}_A$ (see [15], 2.2.7). We show here that this is also true for a Novikov algebra.

Lemma 11 *If A is a Novikov or derivation algebra, then $Z(A) = \mathcal{Z}_A$.*

Proof. Assume that A is a Novikov algebra. Then, according to Lemma 10, we only have to show that $(z, x, y) = 0$ for all $z \in \mathcal{Z}_A$ and $x, y \in A$. To do this, we use identity (6) to obtain

$$\begin{aligned}(z, x, y) &= (z \cdot x) \cdot y - z \cdot (x \cdot y) \\ &= (z \cdot x) \cdot y - (x \cdot y) \cdot z \\ &= (z \cdot x) \cdot y - (x \cdot z) \cdot y \\ &= (z \cdot x) \cdot y - (z \cdot x) \cdot y \\ &= 0.\end{aligned}$$

■

If A is a left-symmetric algebra satisfying (2), then $Z(A)$ can be different from $\mathcal{Z}_A$. Here is an example.

Example 12 *On the Lie algebra $\mathcal{H}_3 \times \mathbb{R}$ with basis $\{e_1, e_2, e_3, e_4\}$ such that $[e_2, e_3] = e_1$, define a left-symmetric product as follows: $e_2 \cdot e_3 = e_3 \cdot e_4 = e_4 \cdot e_3 = e_1$, $e_4 \cdot e_4 = e_2$. The resulting left-symmetric algebra that we denote by A satisfies (2), since $[A, A] = \mathbb{R}e_1$ and $L_{e_1} = 0$. However, it is easy to verify that A is neither Novikov nor derivation. Now, we notice that $\mathcal{Z}_A = span\{e_1, e_4\}$; and to show that $Z(A) \neq \mathcal{Z}_A$ we need to check that $e_4 \notin Z(A)$. To this end, it suffices to show that $(e_4, x, y) \neq 0$ for some $x, y \in A$. Indeed, as desired, we have*

$$\begin{aligned}(e_4, e_4, e_3) &= (e_4 \cdot e_4) \cdot e_3 - e_4 \cdot (e_4 \cdot e_3) \\ &= e_2 \cdot e_3 - e_4 \cdot e_1 \\ &= e_1.\end{aligned}$$

Remark 13 *It is remarkable that (3) (or equivalently (2)) is partially true for Novikov algebras and derivation algebras in the sense that, in both these cases, R_z is identically zero on the derived ideal $[A, A]$, for all z in the center $Z(A)$ of A. This has been established in [15], 2.2.8, for derivation algebras, and in [6], Lemma 2.6, for Novikov algebras but with z in $\mathcal{Z}_A$ instead of $Z(A)$.*

It is clear that $Z(A)$ is always a commutative associative subalgebra of A. However, even if A is a left-symmetric algebra, $Z(A)$ need not be a left (resp. right or two-sided) ideal of A. An example is the following:

Example 14 *Let A_4 be the four-dimensional real left-symmetric algebra given by the following multiplication table:*

$$e_1 \cdot e_4 = e_4 \cdot e_1 = e_2, \quad e_2 \cdot e_3 = e_1, \quad e_2 \cdot e_4 = -e_3, \quad e_3 \cdot e_3 = e_2.$$

It is clear that $\mathcal{Z}_{A_4} = \mathbb{R}e_1$ is neither a left ideal nor a right ideal of A_4, and that $Z(A_4) = \mathcal{Z}_{A_4}$.

Remark 15 *It is easy to check that the left-symmetric algebra A_4 of the above example does not satisfy (2) and is neither Novikov nor derivation.*

Now, we specialize to the case when A is a Novikov algebra or a derivation algebra. In that case, we have the following result first proved in [6] for the center $\mathcal{Z}_A$ of the Lie algebra associated to a Novikov algebra A.

Proposition 16 *If A is a Novikov algebra or a derivation algebra, then $Z(A) = \mathcal{Z}_A$ is a two-sided ideal of A.*

Proof. For an arbitrary left-symmetric algebra A, identity (1) is clearly equivalent to the identity

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \quad x, y, z \in A.$$

Now, by assuming that A is a Novikov algebra or a derivation algebra and using Lemma 11 and Lemma 13, we deduce that $Z(A) = \mathcal{Z}_A$ and $[x, y] \cdot z = 0$ for all $x, y \in A$ and $z \in Z(A)$. Thus, the above identity reduces to the identity

$$x \cdot (y \cdot z) = y \cdot (x \cdot z), \quad x, y \in A \text{ and } z \in Z(A). \tag{13}$$

In case A is Novikov, using (13) we get

$$\begin{aligned} (x \cdot z) \cdot y - x \cdot (z \cdot y) &= (z \cdot x) \cdot y - z \cdot (x \cdot y) \\ &= (z \cdot x) \cdot y - (x \cdot y) \cdot z \\ &= (z \cdot x) \cdot y - (x \cdot z) \cdot y \\ &= 0, \end{aligned}$$

for all $x, y \in A$ and $z \in Z(A)$.

In case A is derivation, using (13) again we get

$$\begin{aligned} (x \cdot z) \cdot y - x \cdot (z \cdot y) &= (z \cdot x) \cdot y - z \cdot (x \cdot y) \\ &= (z \cdot x) \cdot y - (x \cdot y) \cdot z \\ &= (y \cdot x) \cdot z - (x \cdot y) \cdot z \\ &= [y, x] \cdot z \\ &= 0, \end{aligned}$$

for all $x, y \in A$ and $z \in Z(A)$.

In summary, we have shown that if A is a Novikov algebra or a derivation algebra, then

$$(x \cdot z) \cdot y = x \cdot (z \cdot y), \quad x, y \in A \text{ and } z \in Z(A). \tag{14}$$

Assume now that A is a Novikov algebra or a derivation algebra, and let $x, y \in A$ and $z \in Z(A)$. Then, by combining (13) with (14), we obtain

$$\begin{aligned} (x \cdot z) \cdot y &= x \cdot (z \cdot y) \\ &= x \cdot (y \cdot z) \\ &= y \cdot (x \cdot z), \end{aligned}$$

which is equivalent to the identity $[x \cdot z, y] = 0$, from which we conclude that $x \cdot z \in \mathcal{Z}_A = Z(A)$. That is, $Z(A) = \mathcal{Z}_A$ is a left ideal of A.

Since $z \cdot x = x \cdot z$, we conclude that $Z(A) = \mathcal{Z}_A$ is also a right ideal of A. Hence $Z(A) = \mathcal{Z}_A$ is a two-sided ideal of A, as desired. ■

If A is a left-symmetric algebra satisfying (2) which is neither Novikov nor derivation, then $Z(A)$ need not be a two-sided ideal of A. Here is an example.

Example 17 *On the Lie algebra $\mathcal{H}_3 \times \mathbb{R}$ with basis $\{e_1, e_2, e_3, e_4\}$ such that $[e_2, e_3] = e_1$, define a left-symmetric product as follows: $e_2 \cdot e_3 = e_1$, $e_3 \cdot e_2 = 2e_1$, $e_3 \cdot e_3 = e_4$, $e_3 \cdot e_4 = e_4 \cdot e_3 = e_2$, $e_4 \cdot e_4 = 2e_1$. The resulting left-symmetric algebra that we denote by A satisfies (2), since $[A, A] = \mathbb{R}e_1$ and $L_{e_1} = 0$. However, it is easy to check that A is neither Novikov nor derivation. It is also easy to verify that $Z(A) = \mathcal{Z}_A = span\{e_1, e_4\}$, and that $Z(A)$ is neither a left ideal nor a right ideal of A*

3.1 The translational center of a left-symmetric algebra

Given a left-symmetric algebra A over a field $\mathbb{F}$, we consider the set

$$T(A) = \{x \in A : x \cdot y = 0, \quad \text{for all } y \in A\}.$$

Since $T(A)$ is both a right ideal of A and a Lie ideal of the associated Lie algebra $\mathcal{G}_A$, it follows that $T(A)$ is a two-sided ideal of A.

Definition 18 *The subset $C(A)$ of A given by*

$$C(A) = T(A) \cap Z(A)$$

will be called the translational center *of A.*

We note that, in [14], the center of A is defined to be the subset $T(A) \cap \mathcal{Z}_A$, but it follows from Lemma 10 that this coincides with $C(A)$. On the other hand, the translational center has the following geometric interpretation:

We know that a transitive (resp. simply transitive) affine action of a Lie group G on $\mathbb{R}^n$ corresponds to the existence of a left-symmetric (resp. complete left-symmetric) product on the Lie algebra $\mathcal{G}$ of G (see [14]), and it turns out that the exponential of $C(A)$ corresponds exactly to central translations in G. For instance, we know by [12] that if G is a 3-dimensional nilpotent Lie group acting simply transitively on $\mathbb{R}^3$ by affine transformations, then G contains a non-trivial central translation. This answers affirmatively an old conjecture of Auslander in dimension 3. However, an example of a simply transitive affine action of a unipotent Lie group on $\mathbb{R}^3$ which contains no central translation was constructed in [11].

Definition 19 *Let $\mathcal{G}$ be a Lie algebra with center $\mathcal{Z}(\mathcal{G})$, and define the lower central series of $\mathcal{G}$ to be the series of ideals $\{\mathcal{C}^i\mathcal{G}\}$ given by $\mathcal{C}^0\mathcal{G} = \mathcal{G}$, $\mathcal{C}^1\mathcal{G} = [\mathcal{G}, \mathcal{G}]$, and $\mathcal{C}^i\mathcal{G} = [\mathcal{G}, \mathcal{C}^{i-1}\mathcal{G}]$ for all $i \geq 1$. If there exists some integer $k \geq 1$ such that $\mathcal{C}^k\mathcal{G} \supsetneq \mathcal{C}^{k+1}\mathcal{G} = \{0\}$, then we say that $\mathcal{G}$ is k-step nilpotent. In that case, if in addition $\mathcal{Z}(\mathcal{G}) = \mathcal{C}^k\mathcal{G}$, then we say that $\mathcal{G}$ is* nonsingular.

Proposition 20 *Let A be a left-symmetric algebra whose associated Lie algebra $\mathcal{G}_A$ is k-step nilpotent. If A satisfies (2), then $\mathcal{C}^{k-1}\mathcal{G}_A \subseteq C(A)$. In particular $C(A) \neq \{0\}$.*

If in addition $\mathcal{G}_A$ is nonsingular, then $C(A) = Z(A) = \mathcal{Z}_A$.

Proof. By (3), we have $[\mathcal{G}_A, \mathcal{G}_A] = [A, A] \subseteq T(A)$; and consequently

$$\{0\} \neq \mathcal{C}^{k-1}\mathcal{G}_A = [\mathcal{G}_A, \mathcal{G}_A] \cap \mathcal{Z}_A \subseteq T(A) \cap \mathcal{Z}_A = C(A).$$

Assume now that $\mathcal{G}_A$ is nonsingular. On the one hand, since $\mathcal{G}_A$ is nonsingular and A satisfies (2) we have $\mathcal{Z}_A = \mathcal{C}^{k-1}\mathcal{G}_A \subseteq [\mathcal{G}_A, \mathcal{G}_A] \subseteq T(A)$, from which we deduce that $C(A) = T(A) \cap \mathcal{Z}_A = \mathcal{Z}_A$. On the other hand, we have $C(A) \subseteq Z(A) \subseteq \mathcal{Z}_A$. ■

As an immediate but useful consequence of Proposition 20, we have the following corollary which gives a method for classifying left-symmetric products satisfying (2) on low dimensional nilpotent Lie algebras.

Corollary 21 *Let A be a left-symmetric algebra satisfying (2) whose associated Lie algebra is nilpotent. Then, A can be obtained as a central extension of a left-symmetric algebra satisfying (2) of smaller dimension.*

3.2 Some comments on derivation algebras

In this subsection we make some comments on other existing results concerning derivation algebras. The first comment concerns a special family of derivation algebras, the so-called inner derivation algebras. In [15], a left-symmetric algebra A over a field $\mathbb{F}$ is called an *inner derivation algebra* if all left (resp. right) multiplications are inner derivations of the associated Lie algebra $\mathcal{G}_A$. If A is an inner derivation algebra, then it is easy to verify that the left-symmetric product is given by

$$x \cdot y = ad_{f(x)}y = [f(x), y], \quad \text{for all } x, y \in A,$$

where f is an endomorphism of the vector space A satisfying the conditions:

1. $[x, y] = [f(x), y] + [x, f(y)]$,
2. $f([x, y]) - [f(x), f(y)] \in \mathcal{Z}_A$, for all $x, y \in A$.

Proposition 22 *Let A be an inner derivation algebra whose associated Lie algebra $\mathcal{G}_A$ is 2-step nilpotent. Then, A satisfies (2).*

Proof. Let f be an endomorphism of the vector space A such that $L_x = ad_{f(x)}$, for all $x \in A$; and let $x, y, z \in A$. From the above second condition on f, we have that $f([x, y]) = [f(x), f(y)] + z'$, with $z' \in \mathcal{Z}_A$. It follows that

$$\begin{aligned} [x, y] \cdot z &= [f([x, y]), y] \\ &= [[f(x), f(y)] + z', z] \\ &= 0, \end{aligned}$$

given that $\mathcal{G}_A$ is 2-step nilpotent (i.e., $[A, A] \subseteq \mathcal{Z}_A$). Thus A satisfies (2), as desired. ■

The second and third comments concern derivation algebras that are not necessarily inner. These can be derived from the following result proved in [15].

Theorem 23 *A derivation algebra A over a field $\mathbb{F}$ splits uniquely as a direct sum of two-sided ideals A_0 and A_* such that A_0 is complete and contains the derived ideal $[A, A]$ and A_* is commutative with identity and contained in the center $Z(A)$. Moreover, we have that $T(A) \subseteq A_0$ with $T(A) = \{0\}$ if and only if $A_0 = \{0\}$.*

Corollary 24 *Let A be a noncommutative derivation algebra over a field $\mathbb{F}$. Then $T(A) \neq \{0\}$.*

Proof. By Theorem 23, we have $[A, A] \subseteq A_0$. Since A is noncommutative, we deduce that $A_0 \neq \{0\}$. Again, by Theorem 23, this implies that $T(A) \neq \{0\}$. ■

Now, we consider a derivation algebra A such that the center $\mathcal{Z}_A$ of its associated algebra $\mathcal{G}_A$ satisfies the inclusion $\mathcal{Z}_A \subseteq [A, A]$. In fact, there are a lot of Lie algebras (even nilpotent of any step of nilpotency) which satisfy the above inclusion. In that case, by Theorem 23 we have that $A_* \subseteq \mathcal{Z}(A) \subseteq [A, A] \subseteq A_0$, from which we deduce that $A_* = \{0\}$, that is, A is complete. As a special case of this, let us consider a derivation algebra A with nonsingular k-step nilpotent associated Lie algebra $\mathcal{G}_A$. In this case, we have

$$\mathcal{Z}_A = \mathcal{C}^{k-1}\mathcal{G}_A = \mathcal{Z}_A \cap [A, A] \subseteq [A, A].$$

Thus, as an immediate consequence of Theorem 23, we can also state the following corollary.

Corollary 25 *Let A be a derivation algebra over a field $\mathbb{F}$ whose associated Lie algebra $\mathcal{G}_A$ is nonsingular nilpotent. Then, A is complete.*

4 Radicals of a left-symmetric algebra

The radical of an associative algebra A is the unique nilpotent ideal of A which is maximal, that is it contains all nilpotent ideals of A. No such ideal exists in an arbitrary nonassociative algebra, and so the radical of such an algebra has never been defined. For the case of left-symmetric algebras, different types of radicals have been defined. We shall consider here three of them.

4.1 The (Koszul) radical of a left-symmetric algebra

The radical of a left-symmetric algebra was firstly defined by J. L. Koszul (see [13]). Given a left-symmetric algebra A over a field $\mathbb{F}$, one defines *the radical* $R(A)$ of A to be the largest left ideal contained in the subset

$$I(A) = \{a \in A : tr(R_a) = 0\}.$$

It turns out that $R(A)$ is nothing but the largest complete left ideal of A. This has been proved in [13] for the case $\mathbb{F} = \mathbb{C}$, and in [9] for $\mathbb{F} = \mathbb{R}$.

In general, the radical of an arbitrary left-symmetric algebra is not a two-sided ideal (cf. [13]). In [18], it has been announced that if A is a left-symmetric algebra over $\mathbb{C}$ whose associated Lie algebra is solvable, then $R(A)$ of A is a two-sided ideal. However, this ressult turns out to be false as the following example, excerpted from [4], shows.

Example 26 *Consider the 4-dimensional left-symmetric algebra A over $\mathbb{C}$ given by the following multiplication table: $e_1 \cdot e_3 = e_3$, $e_1 \cdot e_4 = -e_4$, $e_2 \cdot e_2 = 2e_2$, $e_2 \cdot e_3 = e_3$, $e_2 \cdot e_4 = e_4$, $e_3 \cdot e_4 = e_4 \cdot e_3 = e_2$. It is easy to verify that the associated Lie algebra $\mathcal{G}_A$ is 2-step solvable, and that $R(A) = \mathbb{C}e_1$. Clearly, $R(A)$ is not a right ideal of A.*

In the same context, we also note that it has been announced in [17] and [19] that if A is a left-symmetric algebra over $\mathbb{C}$ or $\mathbb{R}$ whose associated Lie algebra is nilpotent, then $R(A)$ is a two-sided ideal of A containing the derived Lie algebra $[A, A]$. Although we do not know whether this result is true or false, there are some special cases of it where the radical is a two-sided ideal. For instance, as we will see later, we can deduce from [24] that the radical of a Novikov algebra over a field of characteristic zero is a two-sided ideal. This is also the case for derivation algebras (see [15]). In the case of a left-symmetric algebra satisfying (2), we have the following result.

Theorem 27 *Let A be a left-symmetric algebra satisfying (2). Then, $R(A)$ is a two-sided ideal of A containing the derived Lie algebra $[A, A]$.*

Proof. By (3), the right multiplication R_x is identically zero on $[A, A]$ for all $x \in A$. Therefore, $[A, A]$ is a complete subalgebra of A. On the other hand, , by Proposition 7, $[A, A]$ is a two-sided ideal of A. It follows that $[A, A] \subseteq R(A)$. This in turn implies that $R(A)$ is a Lie ideal, and since it is a left ideal, we deduce that $R(A)$ is a two-sided ideal containing $[A, A]$. ■

Corollary 28 *Every left-symmetric algebra satisfying (2) can be obtained as an extension of a commutative associative algebra by a complete left-symmetric algebra satisfying (2).*

4.2 The left (resp. right)-nilpotent radical

Let A be a left-symmetric algebra over a field $\mathbb{F}$ of characteristic zero, and I a two-sided ideal of A. We say that I is *left-nilpotent* if there exists some fixed integer $n \geq 1$ such that $L_{a_1} \cdots L_{a_n} = 0$ for all $a_i \in I$. It is not difficult to show that every finite-dimensional left-symmetric algebra A has a unique maximal left-nilpotent ideal $L(A)$, called the *left radical* of A (see [9]).

It is also clear that if I is a left-nilpotent ideal of a left-symmetric algebra A, then the left multiplications L_a are nilpotent for all $a \in I$. On the other hand, it is well known that given a left-symmetric algebra A, then we have: all left multiplications L_a are nilpotent if and only if all the right multiplications R_a are nilpotent and the associated Lie algebra $\mathcal{G}_A$ is nilpotent (see [21]; see also [14], Theorem 2.1 and Theorem 2.2). We deduce from these two facts that the left radical $L(A)$ of an arbitrary left-symmetric algebra A is a complete ideal; and consequently we have that $L(A) \subseteq R(A)$.

Similarly we define an ideal I to be *right-nilpotent* if there exists some fixed integer $n \geq 1$ such that $R_{a_1} \cdots R_{a_n} = 0$ for all $a_i \in I$. It follows immediately that any right-nilpotent algebra is complete. However, unlike the left-nilpotent case, the largest right-nilpotent ideal need not exist for an arbitrary left-symmetric algebra, because the sum of any two right-nilpotent ideals need not be right-nilpotent. However, it was shown in [24] that a Novikov algebra A has always a unique maximal right-nilpotent two-sided ideal $N(A)$, called the *right radical* of A. In the same paper, it was also shown that $I(A) = \{a \in A : tr(R_a) = 0\}$ is a two-sided ideal that is right-nilpotent. From these two facts, we can deduce the following:

Proposition 29 *Let A be a Novikov algebra over a field $\mathbb{F}$ of characteristic zero. Then, we have $N(A) = R(A) = I(A)$.*

Proof. As mentioned above, since A is a Novikov algebra then the right radical $N(A)$ exists and the two-sided ideal $I(A)$ is right-nilpotent. It follows from maximality of $N(A)$ that $I(A) \subseteq N(A)$. On the other hand, since $N(A)$ is right-nilpotent, then right multiplications R_a are nilpotent, and consequently $N(A)$ is complete. Thus, $N(A) \subseteq R(A)$. But the definition of $R(A)$ says that $R(A) \subseteq I(A)$. Hence, we have $I(A) \subseteq N(A) \subseteq R(A) \subseteq I(A)$, as desired. ■

Remark 30 *Let A be as in Example 26. Recall that we have seen that $R(A) = \mathbb{C}e_1$, and that $R(A)$ is not a right ideal of A. It is now not difficult to see that $N(A)$ exists and that $N(A) = \{0\}$. This example shows that, in some cases, $N(A)$ can exist without being equal to $R(A)$.*

As we mentioned above, a derivation algebra A can always be uniquely decomposed into a direct sum of two-sided ideals A_0 and A_* such that A_0 is complete and contains the derived ideal $[A, A]$ and A_* is commutative with identity and contained in the center $\mathcal{Z}(A)$. In particular, for a derivation algebra A we have $A_0 \subseteq R(A)$. We give here an example of a derivation algebra A such that $A_0 \subsetneq R(A)$.

Example 31 *Over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, consider the two-dimensional left-symmetric algebra A defined by following multiplication table: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2 \cdot e_1 = e_2$. Since the Lie algebra associated to A is commutative, it follows that A is a derivation algebra (it is also Novikov and satisfies 2). It is now easy to see that $N(A) = R(A) = \mathbb{F}e_2$. When we look at A as a derivation algebra, we see that $A_0 = \{0\}$. This shows that $A_0 \subsetneq R(A)$. On the other hand, we should also notice that, being a Novikov algebra, A can be obtained as an extension of the field $\mathbb{F}e_1$ by $N(A) = \mathbb{F}e_2$.*

5 Simple left-symmetric algebras

An algebra A over a filed $\mathbb{F}$ is called simple if it has no proper two-sided ideal and A is not the zero algebra of dimension 1. Therefore, since $A^2 = A \cdot A$ is a two-sided ideal of A, we have $A^2 = A$ in case A is simple.

In [24], the following result was proved.

Theorem 32 *A simple Novikov algebra A over a field $\mathbb{F}$ of characteristic zero is isomorphic to $\mathbb{F}$.*

It is worth mentioning that when the field $\mathbb{F}$ is not algebraically closed, then simple Novikov algebras over $\mathbb{F}$ of dimension ≥ 2 can exist. Here is an example of a two-dimensional simple Novikov algebra over $\mathbb{R}$.

Example 33 (A two-dimensional simple Novikov algebra over $\mathbb{R}$) *Over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, let us consider the two-dimensional commutative associative algebra $A_{2,\mathbb{F}}$ defined by the following multiplication table: $e_1 \cdot e_1 = e_1$, $e_1 \cdot e_2 = e_2 \cdot e_1 = e_2$, $e_2 \cdot e_2 = -e_1$. Being commutative, $A_{2,\mathbb{F}}$ is a Novikov algebra; and by setting $e_1' = \frac{1}{2}(e_1 + ie_2)$, $e_2' = \frac{1}{2}(e_1 - ie_2)$, we can easily see that $A_{2,\mathbb{C}}$ is a direct sum of fields, that is $A_{2,\mathbb{C}} \cong \mathbb{C} \oplus \mathbb{C}$. However, it is not difficult to show that $A_{2,\mathbb{R}}$ is simple.*

Remark 34 *It is worth pointing out that, in the example above, $A_{2,\mathbb{C}}$ is nothing but the complexification of $A_{2,\mathbb{R}}$. It follows that the complexification of a simple left-symmetric algebra (even Novikov) need not be simple.*

Anyway, the following theorem will show that $A_{2,\mathbb{R}}$ is the only simple Novikov algebra over $\mathbb{R}$ of dimension ≥ 2. To prove the theorem, we need to recall the notion of complexification of a left-symmetric algebra.

Let A be a real left-symmetric algebra of dimension n, and let $A^{\mathbb{C}}$ denote the real vector space $A \oplus A$. Let $J : A \oplus A \to A \oplus A$ be the linear map on $A \oplus A$ defined by $J(x, y) = (-y, x)$.

For $\alpha + i\beta \in \mathbb{C}$ and $x, x', y, y' \in A$, we define

$$(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \alpha y + \beta x) \tag{15}$$

and

$$(x, y) \cdot (x', y') = (xx' - yy', xy' + yx') \tag{16}$$

We endow the set $A^{\mathbb{C}}$ with the componentwise addition, multiplication by complex numbers defined by (15), and the product defined by (16). It is then straightforward to verify that, if endowed with the product defined by (16), $A^{\mathbb{C}}$ becomes a complex left-symmetric algebra of dimension n that we call *the complexification* of A. In that case, the left-symmetric algebra A can be identified with the set of elements in $A^{\mathbb{C}}$ of the form $(x, 0)$, where $x \in A$. Furthermore, if $e_1, \ldots, e_n$ is a basis in A, then the elements $(e_1, 0), \ldots, (e_n, 0)$ form a basis in the complex vector space $A^{\mathbb{C}}$.

Theorem 35 *Let A be simple Novikov algebra over $\mathbb{R}$. Then, A is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$.*

Proof. Let $A^{\mathbb{C}}$ be the complexification of A. Since A is Novikov, then it is easy to check that $A^{\mathbb{C}}$ is Novikov as well. Since, by Proposition 29 we have $N(A) = I(A)$, it follows that $N(A)^{\mathbb{C}} = N(A^{\mathbb{C}})$, where $N(A)^{\mathbb{C}}$ is the complexification of $N(A)$ and $N(A^{\mathbb{C}})$ is the right radical of $A^{\mathbb{C}}$. But since A is simple, we have either $N(A) = \{0\}$ or $N(A) = A$. From this we deduce that either $N(A)^{\mathbb{C}} = \{0\}$ or $N(A)^{\mathbb{C}} = A^{\mathbb{C}}$.

If $N(A)^{\mathbb{C}} = A^{\mathbb{C}}$, then $A^{\mathbb{C}}$ is right-nilpotent. By Proposition 1 of [24], $(A^{\mathbb{C}})^2$ is nilpotent. If $(A^{\mathbb{C}})^2 = \{0\}$, then $A^2 = \{0\}$ which implies that A is not simple, a contradiction. Theus, we necessarily have $(A^{\mathbb{C}})^2 \neq \{0\}$. It follows that there exists $k \geq 1$ such that $(A^{\mathbb{C}})^{2k} \supsetneq (A^{\mathbb{C}})^{2k+2} = \{0\}$. This implies that $(A^{\mathbb{C}})^{2k}$ is a non-trivial two-sided ideal of $A^{\mathbb{C}}$. It follows that A^{2k} is a non-trivial two-sided ideal of A, which leads to a contradiction since A is assumed to be simple. Thus, $N(A)^{\mathbb{C}} = \{0\}$. In this case, Proposition 2 of [24] tells us that $A^{\mathbb{C}}$ is a direct sum of fields. Hence, $A^{\mathbb{C}}$ is commutative associative. It follows that A is commutative associative, and therefore by applying Lemma 3.1 of [7] we deduce that A is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$, as desired. ■

For derivation algebras and left-symmetric algebras satisfying (2), we have the following immediate consequence of Corollary 24 and Proposition 7.

Lemma 36 *A simple left-symmetric algebra over a field $\mathbb{F}$ which is derivation or satisfying (2) is necessarily commutative.*

If the field is not algebraically closed, we have the following immediate consequence of Theorem 35 and Lemma 36.

Proposition 37 *A simple left-symmetric algebra over $\mathbb{R}$ which is derivation or satisfying (2) is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$.*

If the field is algebraically closed, we have the following immediate consequence of Theorem 32 and Lemma 36.

Proposition 38 *Let A be a simple left-symmetric algebra over an algebraically closed field $\mathbb{F}$ which is derivation or satisfying (2). Then, A is isomorphic to $\mathbb{F}$.*

We close this section with the following propositions concerning completeness of simple left-symmetric algebras.

Proposition 39 *A complete Novikov algebra over a field of characteristic zero is not simple.*

Proof. Let A be a complete Novikov algebra over a field $\mathbb{F}$ of characteristic zero. Since A is complete, then A is right-nilpotent. By Proposition 1 of [24], A^2 is nilpotent. If $A^2 = \{0\}$, then A is obviously not simple. If $A^2 \neq \{0\}$, then there exists $k \geq 1$ such that $A^{2k} \supsetneq A^{2k+2} = \{0\}$. Thus A contains the non-trivial two-sided ideal A^{2k}, and hence A is not simple, as desired. ■

Proposition 40 *Let A be a complete left-symmetric algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If A is derivation or satisfies (2), then A is not simple.*

Proof. Suppose to the contrary that A is simple. In the case $\mathbb{F} = \mathbb{R}$, we derive from Proposition 37 that A is isomorphic to either $A_{2,\mathbb{R}}$ or the field $\mathbb{R}$. In the case $\mathbb{F} = \mathbb{C}$, we derive from Proposition 38 that A is isomorphic to the field $\mathbb{F} = \mathbb{C}$. In both cases, this leads to a contradiction since $A_{2,\mathbb{R}}$ and the field $\mathbb{F}$ are not complete. It follows that A is not simple, as desired. ■

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