A CONDITION ON SURFACES THE UNIT NORMAL PROJECTION OF WHICH IS A HARMONIC MORPHISM *

Xiaohuan Mo
School of Mathematical Sciences,
Peking University, Beijing 100871, China
E-mail: moxh@pku.edu.cn

Abstract

We show that surfaces with natural projections of unit normal bundles to be harmonic morphisms are composed of minimal points and totally umbilical points and answer an interesting problem proposed by S. Gudmundsson. We also establish sufficiant conditions for reducing the codimension of an isometric immersion with conformal second fundamental form.

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1 Introduction

A class of harmonic maps with a strong geometric flavour is the class of harmonic morphisms. Harmonic morphisms between Riemannian manifolds are solutions to an over-determined non-linear system of partial differential equations. For this reason they have no general existence theory, not even locally.

For any isotropic holomorphic map $\xi : M \to CP^n$ from a Riemannian surface M to an n-dimensional complex space CP^n, Gudmundsson constructed in [8] (cf.(4.1)) a sphere bundle M_ξ over M with projection map that is a harmonic morphism with totally geodesic fibres and was later generalized by Aprodu-Aprodu([1]). Eells and Sampson proved in [3] that any holomorphic map between Kaehler manifolds is harmonic. Eells and Wood manufactured non-holomorphic harmonic surfaces in CP^n([4]). A natural and interesting problem is that

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when ξ is an isotropic harmonic map, whether or not the natural projection of M_ξ is a harmonic morphism[7].

Let $\phi : M \to R^{n+1}$ be an isometric immersion with $n \geq 3$. Then its unit normal bundle $TM_1^\perp$ is exactly the same as M_ξ where ξ is the Gauss map of ϕ. As a natural extension of a result in [8], Gudmundsson and the author proved in [10] that the projection map of the unit normal bundle of a minimal or totally umbilical surface in a Riemannian manifold is a harmonic morphism with totally geodesic fibres (see Proposition 4.2).

In this paper we show that the converse holds, i.e., for any immersed surface in a Riemannian manifold, if the projection map of unit normal bundle is a harmonic morphism, then the surface can be split into two parts, one is minimal and the other is totally umbilical(see Theorem 5.1).

Then we apply Ruh-Vilms' theorem and construct an isotropic harmonic map ξ from the torus $S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}})$ to CP^3 such that the projection map $\pi : (S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}}))_\xi \to S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}})$ is not a harmonic morphism which gives a negative answer to the problem mentioned above.

Our approach is to discuss the geometry of submanifolds with conformal second fundamental forms (Definition 3.1). First we give structures of surfaces with conformal second fundamental forms (Proposition 3.3) which will be used in the proof of our main result (Theorem 5.1). Meanwhile Proposition 3.3 means that the dimensions of the first normal spaces for surfaces with conformal second fundamental forms are smaller than in the general case. After noting this interesting fact we naturally invastigate the dimensions of the first normal spaces for submanifolds with conformal second fundamental forms (Proposition 3.7). Finally, using dimensional estimation of the first normal spaces and J. Erbacher's reduction we establish sufficient conditions for reducing the codimension of an isometric immersion with conformal second fundamental form in a constant curvature space (Proposition 3.8).

2 Submanifolds in a Riemannian manifold

Let M be a submanifold immersed into a Riemannian manifold (N, h) by a mapping $\phi : M \to N$. We denote the second fundamental form (resp. the mean curvature vector) of ϕ by B (resp. H). ϕ is said to have *constant mean curvature* if H is parallel with respect to the normal connection. In particular, if H vanishes identically, then ϕ is said to be *minimal*.

Let ξ be a normal vector field of $\phi(M)$. Then we define the quadratic differential form by $B_\xi = h(B, \xi)$. If B_H is proportional to the induced Riemannian metric $\phi^*h := g$, that is, if $B_H = \rho g$ then the immersion ϕ is said to be *pseudo-umbilical*. In particular, if B_ξ is proportional to g for all normal vector ξ, ϕ is said to be *totally umbilical*. It is easy to see that the immersion ϕ is totally umbilical if and only if $B(X, Y) = g(X, Y)H$ for all tangent vector X, Y of $\phi(M)$, and any minimal submanifold is pseudo-umbilical.

Remark 2.1 Let $S^{n-1}(r) := \{x \in R^n, \|x\| = r\}$ be an Euclidean spheres in an Euclidean space R^n. The including map $i : S^{n-1}(r) \to R^n$ is totally umbilical. Minimal submanifold in Euclidean spheres are submanifolds with constant mean curvature in Euclidean spaces (see[11], page 167, Remark 2).

Let $\phi : (M, g) \to (N, h)$ be an isometric immersion and ξ a normal vector field of $\phi(M)$. We denote the shape operator in the direction ξ by A_ξ, i.e.

$$g(A_\xi X, Y) = B_\xi(X, Y) \qquad X, Y \in \Gamma(TM) \tag{2.1}$$

A simple calculation yields that

$$\operatorname{tr} A_\xi = \operatorname{tr} B_\xi = mh(H, \xi) \tag{2.2}$$

where $m = \dim M$. The following fact will be used in the next section.

Lemma 2.2 *Let $\phi : (M, g) \to (N, h)$ be an isometric immersion.*

(1). *Let ξ be a normal vector field which is orthogonal to H. Then $tr A_\xi = 0$;*

(2). *ϕ is a minimal immersion if and only if $tr A_H = 0$;*

(3). *ϕ is a pseudo-umbilical immersion if and only if A_H is proportional to the identity;*

(4). *ϕ is a totally umbilical immersion if and only if A_ξ is proportional to the identity for all the normal vector fields ξ;*

(5). *Suppose that ϕ is pseudo-umbilical. Then ϕ is totally umbilical if and only if $A_\xi = 0$ for all the normal vector field ξ which are orthogonal to H.*

We omit the proof as it is only direct calculations by using (2.1) and (2.2).

3 Isometric immersions with conformal second fundamental forms

As a natural generalization of totally umbilical submanifold (cf. (4) of Lemma 2.2) , we introduce the following (see [10]):

Definition 3.1 Let $(M,\, g)$ and $(N,\, h)$ be Riemannian manifolds, and $\phi : M \to N$ be an isometric immersion. The second fundamental form of ϕ is said to be *conformal* if the square of all the corresponding shape operators along ϕ are proportional to the identity.

Example 3.2 Any minimal surface in a Riemannian manifold has conformal second fundamental form(cf. [10]).

Conversely, we have following

Proposition 3.3 *Let M be an immersed surface in a Riemannain manifold $(N,\, h)$. Then M has conformal second fundamental form if and only if $M = M_1 \cup M_2$ where M_1 is minimal and M_2 is totally umbilical. In particular M is a pseudo-umbilical surface.*

We require the following result, the proof which is omitted.

Lemma 3.4 *Let A be a real and symmetric $2 \times 2-$matrix. Then A^2 is proportional to the identity matrix if and only if either the trace of A vanishes or A is proportional to the identity matrix.*

Proof of Proposition 3.3 Sufficiency is an immediate consequence of Theorem 4.5 and Example 4.6 in [10]. Let $x \in M$ and $\{e_1,\, e_2\}$ the orthogonal basis of T_xM. Let ξ be a normal vector field of $\phi(M)$. Denote the matrix of $(A_\xi)_x : T_xM \to T_xM$ with respect to $\{e_1,\, e_2\}$ by $\tilde{A}_\xi$. Then the conformality of second fundamental form implies that $\tilde{A}_\xi^2$ is proportional to the identity matrix. We work at a non-minimal point. Then (2) in Lemma 2.2 implies that $\operatorname{tr} \tilde{A}_H \neq 0$. Together with Lemma 3.4 we have

$$\tilde{A}_H \propto \text{identity matrix} \tag{3.1}$$

It follows that, from (3) of Lemma 2.2, ϕ is pseudo-umbilical. Consider a normal vector field ξ which is orthogonal to H. (1) of Lemma 2.2 tells us that

$$\operatorname{tr} \tilde{A}_\xi = 0 \tag{3.2}$$

At x, we set $\eta = \frac{H}{\|H\|}\cos\theta + \xi \sin\theta$ where $\theta =$ constant $\in (0,\, \frac{\pi}{2})$. Then (2.1) yields that

$$\tilde{A}_\eta = \frac{\cos\theta}{\|H\|}\tilde{A}_H + \sin\theta \tilde{A}_\xi \tag{3.3}$$

Together with (2.2) and (3.2) we get

$$\begin{aligned}\mathrm{tr}\tilde{A}_\eta &= \tfrac{\cos\theta}{\|H\|}\mathrm{tr}\tilde{A}_H + \sin\theta\,\mathrm{tr}\tilde{A}_\xi \\ &= 2\cos\theta\|H\| \neq 0\end{aligned}$$

By the conformality of the second fundamental form of ϕ and Lemma 3.4 ones get

$$\tilde{A}_\eta \propto \text{identity matrix} \tag{3.4}$$

On the other hand, (3.3) deduces that

$$\tilde{A}_\xi = \tilde{A}_\eta \csc\theta - \tilde{A}_H \frac{ctg\theta}{\|H\|}$$

¿From which together with (3.1) and (3.4) it follows that $\tilde{A}_\xi$ is proportional to the identity matrix. Together with (3.2) we have $\tilde{A}_\xi = 0$. ¿From (5) of Lemma 2.2, x is a totally umbilical point of ϕ.

By using Obata's result ([13]), we have following

Corollary 3.5 *Suppose that a surface M is isometrically immersed into a space form with conformal second fundamental form. Then the Gauss map is conformal.*

By an elementary observation ones get ([15])

Corollary 3.6 *Let M be a surface in a Riemannain manifold (N, h) with conformal second fundamental form. Then at each point the dimension of the first normal space, i.e. the image of the second fundamental form of M, is at most two.*

In the general case, we have

Proposition 3.7 *Let M be a submanifold in a Riemannain manifold (N, h) with conformal second fundamental form. If the dimension of M is odd or if N has constant sectional curvature and the unit mean curvature vector field is parallel in the normal bundle, then at each point the dimension of the first normal space is at most one.*

Proof. Let ξ be a normal vector field which is orthogonal to H. Then $\mathrm{tr}\,A_\xi = 0$ (cf. Lemma 2.2). Choose an orthonormal frame on M such that the shape operator in the direction ξ is diagonalized, that is,

$$\tilde{A}_\xi = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_m)$$

where $m = \dim M$. By the conformality of the second fundamental form of M, we have

$$\lambda_1^2 = \cdots = \lambda_n^2 = \mu^2$$

where μ is a locally non-negative function. If $\tilde{A}_\xi = \mu I$, then $\tilde{A}_\xi = 0$ using $\operatorname{tr} A_\xi = 0$ where I denotes identity matrix. Otherwise we can assume

$$\lambda_1 = \cdots = \lambda_p = -\lambda_{p+1} = \cdots = -\lambda_m = \mu$$

Then

$$0 = \operatorname{tr} \tilde{A}_\xi = (2p - m)\mu$$

If $m :=$ odd, then we have $\mu = 0$. It implies that $A_\xi = 0$. Together with (2.1) we have

$$\|H\|^2 B = B_H H$$

Hence the first normal space is generated by H. In the general case, using the conformality of the second fundamental form of M we have

$$\tilde{A}_\xi \tilde{A}_H + \tilde{A}_H \tilde{A}_\xi = \tilde{A}^2_{\xi+H} - \tilde{A}^2_\xi - \tilde{A}^2_H = \tau_\xi I \tag{3.4}$$

$$\tilde{A}^2_H = \sigma I \tag{3.5}$$

If $(N,\, h)$ has constant sectional curvature and the unit mean curvature vector field is parallel in the normal bundle, then[16, 18]

$$\tilde{A}_\xi \tilde{A}_H = \tilde{A}_H \tilde{A}_\xi \tag{3.6}$$

Substitute (3.6) into (3.4) we have

$$\tilde{A}_\xi \tilde{A}_H = \frac{1}{2}\tau_\xi I \tag{3.7}$$

The existence of the unit mean curvature vector implies that $\|H\| \neq 0$. Using (2.2) we get $\operatorname{tr}\tilde{A}_H \neq 0$. Together with (3.5) ones have $\det\tilde{A}_H \neq 0$. Thus from (3.5) and (3.7) we see

$$\tilde{A}_\xi = \frac{1}{2}\tau_\xi \tilde{A}_H^{-1} = \frac{\tau_\xi}{2\sigma}\tilde{A}_H$$

Combine with (2.1) we have

$$B = \Sigma_\alpha B_{e_\alpha} e_\alpha = \Sigma_\alpha \frac{\tau_{e_\alpha}}{2\sigma} B_H e_\alpha = \frac{1}{2\sigma} B_H V$$

where

$$V := \Sigma_\alpha \tau_{e_\alpha} e_\alpha$$

and $\{e_\alpha\}$ an orthonormal frame of the normal bundle of M.

We now use Proposition 3.7 to show the following reduction of the codimension of an isometric immersion with conformal second fundamental form. We assume that M is a connected manifold.

Proposition 3.8 *Let N be a constant curvature space and M an m-dimensional submanifold of N with conformal second fundamental form. If m is odd and M has constant mean curvature or if M has flat normal bundle and the first normal spaces of M have the constant dimension. Then either*

(1). M is totally geodesic, i.e. $B \equiv 0$, or

(2). M lies in an $(m+1)$-dimensional totally geodesic submanifold with conformal second fundamental form.

Proof. Suppose that $\dim M =$ odd. It is easy to see that M has constant mean curvature implies that either

(i). M is minimal or

(ii). $\|H\|$ is non-zero constant and M has parallel unit mean curvature vector field.

If M is minimal, then using a similar way in the proof in Proposition 3.7 we have $A_\xi = 0$ for all normal vector field ξ. Hence M is a totally geodesic submanifold from (2.1). Suppose that $\|H\|$ is non-zero constant and M has parallel unit mean curvature vector field. Then from the proof of Proposition 3.7 the image of the second fundamental form of M is generated by the mean curvature vecter field. Using J. Erbacher's reduction theorem [5] we have M is a hypersurface of a totally geodesic submanifold, denoted by P, of N. Moreover the shape operator of M in P along normal vector field ξ is exactly the same as the shape operator M in N along ξ. Hence M is a hypersurface in P with conformal second fundamental form. Assume that M has flat normal bundle and the dimensions of the first normal spaces are constant. If M is not totally geodesic, then $A_\xi \neq 0$ for some normal vector field ξ. Since N has constant sectional curvature, the normal bundle of M is flat if and only if all the shape operators can be diagonalized simultaneously[16]. Using a similar way in the proof of Proposition 3.7 we have all first normal spaces of M consist of a rank one vector bundle. The flatness of the normal bundle implies that this bundle is parallel with respect to the normal connection. Again using J. Erbacher's reduction theorem ones get (2).

4 Sphere bundles over surfaces and harmonic maps

Let CP^n be the n-dimensional complex projective space with the Fubini-Study metric and $Q_{n-1}(\subset CP^n)$ be the $(n-1)$-dimensional complex quadratic. Let M be a surface. A map $\xi : M \to CP^n$ is said to be *isotropic* if $\xi(M) \subset Q_{n-1}$. Notice that Q_{n-1} with induced Kaehler metric is isometric to the real Grassmannian $G_2(R^{n+1})$([17]). Using the standard inner product in R^{n+1}, we associate the isotropic map $\xi : M \to CP^n$ to the S^{n-2}-bundle (cf. [8])

$$M_\xi = \{(z, x) \in M \times S^n | \xi(z) \perp x\} \tag{4.1}$$

Let $n \geq 3$ and $\phi : M \to R^{n+1}$ be an isometric immersion of a surface and ξ be its Gauss map. Then ξ is isotropic and the unit normal bundles of M

$$TM_1^\perp := \{(z, x) \in TM^\perp; \|x\| = 1\}$$

is the S^{n-2}-bundle associated with ξ([8,10]).

Let P and Q be Riemannian manifolds. A map $f : P \to Q$ is *harmonic* if it is an extremal of the energy. As an example, for an isometric immersions $\phi : M \to R^{n+1}$ with constant mean curvature the Gauss map of ϕ is harmonic[14]. A map $f : P \to Q$ between Riemannian manifold is called a *harmonic morphism* if for any harmonic function $\psi : U \to$R defined on an open subset U of Q with $f^{-1}(U)$ non-empty, $\psi \circ f : f^{-1}(U) \to$R is a harmonic function. Recall (cf.[6, 12]) that a map $f : P \to Q$ between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and *horizontally* (*weakly*) *conformal*, in the sense that at any point $p \in P$ not contained in the critical set $C_f = \{p \in P | df_p = 0\}$ of f the restriction of df_p to the orthogonal complement

$$\{X \in T_pP | < X, Y >_P = 0 \text{ for all } Y \in \operatorname{Ker} df_p\}$$

of $\operatorname{Ker} df_p$ is surjective and conformal onto the tangent space $T_{f(p)}Q$. It is easy to see that horizontally (weakly) conformal maps embrace Riemannian submersions.

Proposition 4.1 ([8]) *Let $\xi : M \to CP^n$ be an isotropic holomorphic curve and M_ξ its associated sphere bundle. Then the bundle map from M_ξ to M is a harmonic morphism.*

Proposition 4.2 ([10]) *Let (M, g) be a minimal or totally umbilical surface in a Riemannian manifold (N, h) and $TM_1^\perp$ be the unit normal bundle of M in (N, h). Then the projection map*

$$\pi : (TM_1^\perp, ds^2) \to (M, g)$$

is a harmonic morphism with totally geodesic fibres where ds^2 is the restriction of the Sasaki metric on TN to $TM_1^\perp$ as a submanifold.

For more interesting examples of harmonic morphisms we refer the reader to [9] on the World Wide Web.

5 Main result and its application

One of the problems in the theory of harmonic morphisms was suggested to me by S. Gudmundsson: Given an isotropic harmonic map $\xi : M \to CP^n$ from a Riemann surface, is the bundle map $\pi : M_\xi \to M$ (cf. (4.1)) a harmonic morphism[7]?

In this section, we firstly prove the main result of this note (see Theorem 5.1) which completes Proposition 4.2. As an application, we give a negative answer of the problem mentioned above.

Theorem 5.1 *Let $\phi : (M, g) \to (N, h)$ be an isometric immersion from a surface M and $(TM_1^\perp, ds^2)$ the unit normal bundle where ds^2 is the restriction of the Sasaki metric on TN to $TM_1^\perp$. Then the projection map $\pi : (TM_1^\perp, ds^2) \to (M, g)$ is a harmonic morphism if and only if $M = M_1 \cup M_2$ where M_1 is minimal and M_2 is totally umbilic, i.e. M is composed of minimal points and totally umbilical points.*

Proof. By Proposition 3.3 it is sufficient to show that π is a harmonic morphism if and only if ϕ has a conformal second fundamental form. Suppose that π is a harmonic morphism. By using Proposition 4.2 in [10] we have ϕ has a conformal second fundamental form. Conversely, it is an immediate consequence of Proposition 4.4 in [10] since dimension of M is two.

An important application of Theorem 5.1 is to give a negative answer of the interesting above-mentioned problem.

Let $S^q(r)$ denote a q-dimensional sphere in R^{q+1} with radius r. Let m and n be positive integers such that $m < n$ and let $M_{m,n-m} = S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$. The image of the imbedding $M_{m,n-m} \to R^{n+2}$ lies in the unit sphere $S^{n+1}(1)$. We call $M_{m,n-m}(\subset S^{n+1}(1))$ a $Clifford\ hypersurface$. We have following

Proposition 5.2 *Let M be a Clifford hypersurface in $S^{n+1}(1) \subset R^{n+2}$. As a submanifold in R^{n+2} with codimension two, then*

(i). *M is a pseudo-umbilical submanifold;*

(ii). *M has constant mean curvature;*

(iii). *There are neither totally umbilical points nor minimal points on M.*

Proof. Let e_0, e_1, $\cdots$, e_m be an orthonormal frame field for R^{m+1} such that e_0 is normal to $S^m(\sqrt{\frac{m}{n}})$. Similarly, for $S^{n-m}(\sqrt{\frac{n-m}{n}})$ in R^{n-m+1}, we choose an orthonormal frame field e_{m+1}, $\cdots$, e_{n+1} such that e_{n+1} is normal to $S^{n-m}(\sqrt{\frac{n-m}{n}})$. Put

$$\xi = \sqrt{\frac{m}{n}}e_0 + \sqrt{\frac{n-m}{n}}e_{n+1}, \quad \eta = \sqrt{\frac{n-m}{n}}e_0 - \sqrt{\frac{m}{n}}e_{n+1}$$

Then ξ is normal to $S^{n+1}(1)$ and η is normal to $M_{m,n-m}$. On $M_{m,n-m}$, $\{\xi, \eta\}$ is an orthonormal normal frame. By the total umbilicality of $S^m(\sqrt{\frac{m}{n}})$, $S^{n-m}(\sqrt{\frac{n-m}{n}})$ and $S^{n+1}(1)$, we have, with respect to $\{e_1, \cdots, e_n\}$(cf.[2]),

$$\tilde{A}_\eta = \text{diag}\,(\underbrace{\lambda, \cdots, \lambda}_{m}, \underbrace{\mu, \cdots, \mu}_{n-m}), \qquad \tilde{A}_\xi = \text{identity matrix}$$

where $\lambda = \sqrt{(n-m)/m}$ and $\mu = -\sqrt{m/(n-m)}$. Combine with (2.2) we obtain that ξ is the mean curvature vector field of submanifold $M_{m,n-m} \subset R^{n+2}$. Now (ii) is an immediate consequence of $\operatorname{tr}\tilde{A}_\eta = 0$ and Remark 2.1. (i) follows from (3) of Lemma 2.2. Using (2) and (5) of Lemma 2.2 and $\operatorname{tr}\tilde{A}_\xi \neq 0, \tilde{A}_\eta \neq 0$ we now obtain (iii).

Consider $n = 2$ and $m = 1$, i.e. $M_{1,1} = S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}}) \to R^4$. We have following

Corollary 5.3 *There is an isotropic harmonic map $\xi : M_{1,1} = S^1(\sqrt{\frac{1}{2}}) \times S^1(\sqrt{\frac{1}{2}}) \to CP^3$ such that the bundle map $\pi : (M_{1,1})_\xi \to M_{1,1}$ is not a harmonic morphism.*

Proof Let $\phi : M_{1,1} \to R^4$ be include map and $\xi : M_{1,1} \to CP^3$ its Gauss map. Then ϕ has constant mean curvature from Proposition 5.2. Together with Ruh-Vilms' theorem [14] we get ξ is isotropic and harmonic. Recall that Q_2 with induced Kaehler metric is isometric to the real Grassmannian $G_2(R^4)$([17]). Again using Proposition 5.2 ϕ has neither totally geodesic points nor minimal points. Theorem 5.1 tells us that the unit normal projection $\pi : (TM_1^\perp, ds^2) \to (M, g)$ is not a harmonic morphism. As we mention in section 4, the unit normal bundle of $M_{1,1}$ is exactly the same as the S^1-bundle associated with ξ. It follows that the bundle map $\pi : (M_{1,1})_\xi \to M_{1,1}$ is not a harmonic morphism.

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References

[1] M. A. Aprodu and M. Aprodu, *Implicity defined harmonic PHH submersions*, Manu. Math. 100(1999), 103-121.

[2] S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds on a sphere with second fundamental form of constant length*, Shiing-shen Chern Selected Papers, 393-409.

[3] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. 86(1964), 109-160.

[4] J. Eells and J. C. Wood, *Harmonic maps from from surfaces to complex projective spaces*, Adv. in Math. 49(1983), 217-263.

[5] J. Erbacher, *Reduction of the codimension of an isometric immersion*, J. Diff. Geom. 5(1971), 333-340.

[6] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier, 28(1978)107-144.

[7] S. Gudmundsson, private communication.

[8] S. Gudmundsson, *Harmonic morphisms as sphere bundles over compact Riemannsurfaces*, Internat. J. Math. 8(1997), 421-433.

[9] S. Gudmundsson, E. Loubeau, S. Montaldo, M. T. Mustafa, *The Atlas of Harmonic Morphisms*, http://www.maths.lth.se/matematiklu/personal/sigma/harmonic /atlas.html.

[10] S. Gudmundsson and X. Mo, *Harmonic morphisms as unit normal bundles of minimal surfaces*, Manuscripta Math. 100(1999), 323–333.

[11] D. A. Hoffman, *Surfaces of constant mean curvature of constant curvature*, J. Diff. Geom. 8(1973), 161-176.

[12] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto. Univ. 19(1979) , 215-229.

[13] M. Obata, *The Gauss map of immersions of Riemannian manifolds in spaces of constant curvature.* II, J. Diff. Geom. 2(1968), 217-223.

[14] E. A. Ruh and J. Vilms, *The tension field of the Gauss map*, Trans. Amer. Math. Soc. 149 (1970), 569-573.

[15] M. Sakaki, *Remarks on the rigidity and stablity of minimal submanifolds*, Proc. Amer. Math. Soc. 106(1989), 793-795.

[16] C. L. Terng, *Submanifolds with flat normal bundle*, Math. Ann. 277(1987), 95-111.

[17] J. G. Wolfson, *Harmonic maps of the two-sphere into the complex hyperquadric* , J. Diff. Geom. 24(1986), 141-152.

[18] S. T. Yau, *Submanifolds with constant mean curvature* I, Amer. J. Math. 96(1974), 346-366.