ON INTEGRABILITY OF THE VECTOR SHORT PULSE EQUATION

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ABSTRACT. Using the Painlevé analysis preceded by appropriate transformations of nonlinear systems under investigation, we discover two new cases in which the Pietrzyk–Kanattšikov–Bandelow vector short pulse equation must be integrable due to the results of the Painlevé test. One of those two cases is technologically important because it describes the propagation of polarized ultra-short light pulses in usual isotropic silica optical fibres.

The short pulse equation (SPE), which has the form

$$u_{xt} = u + \tfrac{1}{6}\left(u^3\right)_{xx} \tag{1}$$

up to a scale transformation of its variables, was introduced recently by Schäfer and Wayne [1] as a model equation describing the propagation of ultra-short light pulses in silica optical fibres.[1] Unlike the celebrated nonlinear Schrödinger equation which models the evolution of slowly varying wave trains, the SPE is well applicable when the pulse spectrum is not narrowly localized around the carrier frequency, that is when the pulse is as short as a few cycles of its central frequency. Such ultra-short pulses are very important for future technologies of ultra-fast optical transmission of information.

The SPE (1) is an integrable equation possessing a Lax pair [3, 4] of the Wadati–Konno–Ichikawa type [5]. The transformation between the SPE and the sine-Gordon equation was discovered in [4], and later it was used in [6] for obtaining exact loop and pulse solutions of the SPE from the well-known kink and breather solutions of the sine-Gordon equation.[2] The recursion operator [4], Hamiltonian structures and conserved quantities [8, 9], multisoliton solutions [10] and periodic solutions [11] of the SPE were found and studied as well.

Very recently, Pietrzyk, Kanattšikov and Bandelow [12] introduced the vector short pulse equation (VSPE), a two-component nonlinear

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[1]However, for the first time equation (1) appeared in differential geometry, as one of Rabelo's equations describing pseudospherical surfaces [2].

[2]The derivation of that transformation was considerably simplified in [7], where analogous transformations to well-studied equations were also found for the other three equations of Rabelo.

wave equation that generalizes the scalar SPE (1) and describes the propagation of polarized ultra-short light pulses in cubically nonlinear anisotropic optical fibres, which can be written as

$$U_{n,xt} = c_{ni}U_i + c_{nijk}\left(U_iU_jU_k\right)_{xx}, \tag{2}$$

where $n, i, j, k = 1, 2$, and the summation over the repeated indices is assumed. Since the constant coefficients c_{ni} and c_{nijk} are determined by optical properties of the fibre's material, there is a wide variety of mathematically different cases of the VSPE (2), and it is interesting to find out which of them are integrable systems of coupled nonlinear wave equations.

The following three integrable cases of the VSPE (2) were obtained in [12] by direct construction of their Lax pairs:

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3 + 3uv^2\right)_{xx}, \\ v_{xt} &= v + \tfrac{1}{6}\left(3u^2v + v^3\right)_{xx}, \end{aligned} \tag{3}$$

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3 - 3uv^2\right)_{xx}, \\ v_{xt} &= v + \tfrac{1}{6}\left(3u^2v - v^3\right)_{xx}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3\right)_{xx}, \\ v_{xt} &= v + \tfrac{1}{2}\left(u^2v\right)_{xx}, \end{aligned} \tag{5}$$

where u and v denote U_1 and U_2. The following interesting and valuable remark was made in [12] on the nature of system (5): this case of the VSPE describes the propagation of a small perturbation v on the background of a solution u of the scalar SPE (1). However, contrary to what was proposed in [12], we cannot consider the VSPE (3) as a short pulse analogue of the Manakov system of coupled nonlinear Schrödinger equations. Indeed, in the new variables $p = u + v$ and $q = u - v$ the equations of system (3) become uncoupled and turn into two scalar SPEs (1) for p and q separately, whereas the polarization modes in the Manakov system do interact nonlinearly. In the variables $p = u+iv$ and $q = u - iv$ the equations of system (4) become uncoupled as well. It is easy to prove that the 4×4 zero-curvature representations, found in [12] for systems (3) and (4), can be brought by gauge transformations into the block-diagonal form with the 2×2 diagonal blocks corresponding to the zero-curvature representation of the scalar SPE (1) for p or q.

In the present paper, we show that there are at least two more cases of the VSPE (2) which can be strongly expected to be integrable due to the analytic properties of their general solutions, namely,

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3 + uv^2\right)_{xx}, \\ v_{xt} &= v + \tfrac{1}{6}\left(u^2v + v^3\right)_{xx}, \end{aligned} \tag{6}$$

and

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3\right)_{xx}, \\ v_{xt} &= v + \tfrac{1}{6}\left(u^2 v\right)_{xx}. \end{aligned} \tag{7}$$

The VSPE (6) represents the technologically important case, where the fibre is made of a cubically nonlinear isotropic optical material, such as the widely used silica glass, but the ultra-short light pulse is not linearly polarized. The VSPE (7) can be considered as the limiting case of system (6) for small values of v, i.e. the case of almost linearly polarized pulses. We discover these systems (6) and (7) by applying the Painlevé test for integrability of partial differential equations [13, 14] to the following two one-parameter classes of VSPEs:

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3 + cuv^2\right)_{xx}, \\ v_{xt} &= v + \tfrac{1}{6}\left(cu^2 v + v^3\right)_{xx}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} u_{xt} &= u + \tfrac{1}{6}\left(u^3\right)_{xx}, \\ v_{xt} &= v + \tfrac{c}{6}\left(u^2 v\right)_{xx}, \end{aligned} \tag{9}$$

where c is the parameter. Of course, this is far not the complete test for integrability of the whole variety of systems (2) but rather the first attempt to search for new integrable VSPEs methodically. In what follows, we make the computations with the Mathematica computer algebra system [15] and omit their inessential bulky details.

Let us consider the class of systems (8) first.

The Painlevé test cannot be applied to the VSPE (8) directly, because of an inappropriate dominant behaviour of solutions, and we must appropriately transform this nonlinear system in order to start the test. We follow the way of transformation similar to the way used in [7] for the scalar SPE (1) and the other three Rabelo equations. Making the change of the independent variable x,

$$u(x,t) = f(y,t), \quad v(x,t) = g(y,t), \quad y = y(x,t), \tag{10}$$

setting

$$y_t = \tfrac{1}{2}\left(f^2 + g^2\right) y_x \tag{11}$$

for the reason of symmetry[3] between f and g, and inverting $y = y(x,t)$ as $x = x(y,t)$, we obtain from system (8) and relation (11) the following

[3]One could use a general quadratic polynomial in f and g instead of $f^2 + g^2$ in relation (11), but this would have no effect on the dominant behaviour of solutions, positions of resonances and compatibility of recursion relations, found during the Painlevé analysis.

system of three coupled equations for $f(y,t)$, $g(y,t)$ and $x(y,t)$:

$$
\begin{aligned}
& 2x_t + f^2 + g^2 = 0, \\
& 6x_y^2 f_{yt} + (3-c) g^2 x_y f_{yy} \\
& -2cfg x_y g_{yy} + \left((c-3) g^2 f_y + 2cfg g_y\right) x_{yy} \\
& +(6-4c) g f_y g_y x_y - 2cfg_y^2 x_y - 6 f x_y^3 = 0, \\
& 6x_y^2 g_{yt} + (3-c) f^2 x_y g_{yy} \\
& -2cgf x_y f_{yy} + \left((c-3) f^2 g_y + 2cgf f_y\right) x_{yy} \\
& +(6-4c) f g_y f_y x_y - 2cgf_y^2 x_y - 6 g x_y^3 = 0.
\end{aligned}
\tag{12}
$$

Note that the fact of correspondence between the fifth-order system (12) and the fourth-order system (8) (we mean the total order of a system, or the number of arbitrary functions in its general solution) can be explained by the invariance of system (12) with respect to an arbitrary transformation $y \to Y(y)$, which just means that solutions of system (12) represent solutions of system (8) parametrically, with y being the parameter.

Substitution of the expansions

$$
\begin{aligned}
x &= x_0(t)\phi^{\alpha} + \cdots + x_r(t)\phi^{r+\alpha} + \cdots, \\
f &= f_0(t)\phi^{\beta} + \cdots + f_r(t)\phi^{r+\beta} + \cdots, \\
g &= g_0(t)\phi^{\gamma} + \cdots + g_r(t)\phi^{r+\gamma} + \cdots
\end{aligned}
\tag{13}
$$

with $\phi_y(y,t) = 1$ to system (12) determines the dominand behaviour of solutions near the singularity manifold $\phi(y,t) = 0$, i.e. admissible values of α, β, γ, x_0, f_0 and g_0, and the corresponding positions of resonances r, where arbitrary functions of t can enter the expansions. For all values of c except[4] $c = -1$ we obtain $\alpha = \beta = \gamma = -1$. When[5] $c \neq -1, 1, 3$, we find that

$$
\begin{aligned}
& x_0 = -(1+c)\phi_t, \quad f_0 = \pm i\sqrt{1+c}\,\phi_t, \quad g_0 = \pm i\sqrt{1+c}\,\phi_t, \\
& r = -1, 1, 4, \tfrac{1}{2}\left(5 - \sqrt{\tfrac{27+23c}{3-c}}\right), \tfrac{1}{2}\left(5 + \sqrt{\tfrac{27+23c}{3-c}}\right),
\end{aligned}
\tag{14}
$$

where the $\pm$ signs in the expressions for f_0 and g_0 are independent. When $c = 1$, we find that

$$
x_0 = -2\phi_t, \quad f_0^2 + g_0^2 = -4\phi_t^2, \quad r = -1, 0, 1, 4, 5, \tag{15}
$$

where either $f_0(t)$ or $g_0(t)$ is arbitrary due to the resonance $r = 0$.

In the case of relations (14), there are three posibilities to have four resonances in integer non-negative positions, namely, $c = -1, 0, 1$. However, the values $c = -1, 1$ have been excluded, whereas the value

[4] At present, we do not know how to transform the VSPE (8) with $c = -1$ in order to start the Painlevé test for it.

[5] The case of $c = 3$, i.e. the VSPE (3), is not of interest because the equations can be easily uncoupled.

$c = 0$ corresponds to the uninteresting case of uncoupled equations in the VSPE (8). On the other hand, in the case of relations (15) with $c = 1$, we have got the admissible positions of resonances, and then the usual way of constructing the recursion relations for the coefficients of expansions (13) and checking the compatibility of those relations at the resonances brings us to the conclusion that system (12) with $c = 1$ passes the Painlevé test well. Consequently, the VSPE (6) can be strongly expected to be integrable.

Let us proceed now to the class of systems (9).

Making the same transformation (10), setting

$$y_t = \tfrac{1}{2} f^2 y_x, \tag{16}$$

and inverting $y = y(x,t)$ as $x = x(y,t)$, we obtain from system (9) and relation (16) the following system[6] of three coupled equations for $f(y,t)$, $g(y,t)$ and $x(y,t)$:

$$\begin{gathered} 2x_t + f^2 = 0, \quad f_{yt} - f x_y = 0, \\ 6x_y^2 g_{yt} + (3-c) f^2 x_y g_{yy} \\ -2cgf x_y f_{yy} + \big((c-3) f^2 g_y + 2cgf f_y\big) x_{yy} \\ +(6-4c) f g_y f_y x_y - 2cg f_y^2 x_y - 6g x_y^3 = 0. \end{gathered} \tag{17}$$

Then, substituting expansions (13) with $\phi_y(y,t) = 1$ to system (17), we find that

$$\begin{gathered} \alpha = \beta = -1, \quad \gamma^2 - 3\gamma + 2 = 6/c, \\ x_0 = -2\phi_t, \quad f_0 = \pm 2i\phi_t, \\ r = -1, 0, 1, 4, 3 - 2\gamma. \end{gathered} \tag{18}$$

Note that the resonance $r = 0$ corresponds to the arbitrariness of $g_0(t)$, and that the resonance $r = 3 - 2\gamma$ must correspond to the arbitrary coefficient $g_{3-2\gamma}(t)$ due to the structure of recursion relations which follow from system (17) and expansions (13). Denoting the resonance position $3 - 2\gamma$ as m, we find from (18) that

$$\gamma = (3-m)/2, \quad c = 24/(m^2 - 1). \tag{19}$$

The admissible values of m are $m = 2, 3, 4, 5, \ldots$; $m = 0$ is excluded because it leads to the double resonance $r = 0, 0$ which indicates that the expansion for g must contain a logarithmic term, and $m = 1$ is excluded because it implies $c = \infty$. Though the even values of m correspond to non-integer values of γ, these cases must not be excluded, because by introducing the new variable $h = g^2$ one can improve the dominant behaviour of solutions.

Thus, we have found infinitely many cases of system (17), which are all characterized by some admissible dominant behaviour of solutions and admissible positions of resonances. Unfortunately, we cannot check

[6]It is interesting that the third equation of system (17) coincides with the third equation of system (12).

the compatibility of the recursion relations at the resonances for the whole infinite set of those cases at once. This situation is quite similar to the one observed in the Painlevé analysis of triangular systems of coupled Korteweg–de Vries equations [16]. On available computers, we were able to complete the Painlevé test for the cases $m = 2, 3, \ldots, 9, 10$ of system (17) with c given by relation (19). The recursion relations turned out to be compatible only in the cases $m = 3$ and $m = 5$, whereas some nontrivial compatibility conditions appeared in all other cases at the resonance $r = m$ as an indication of non-dominant logarithmic singularities of solutions. The case $m = 3$ with $\gamma = 0$ and $c = 3$ is the integrable VSPE (5) discovered in [12]. The case $m = 5$ with $\gamma = -1$ and $c = 1$ is our new VSPE (7), which can be strongly expected to be integrable due to the result of the Painlevé test.

Let us remind, however, that the Painlevé property does not prove the integrability of a nonlinear equation but only gives a strond indication that the equation must be integrable. Consequently, the new probably integrable nonlinear systems (6) and (7), discovered in this paper, deserve further investigation, especially taking into account their importance for physics and technology.

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