HECKE ALGEBRAS FOR p-ADIC LOOP GROUPS

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Let G be a split reductive algebraic group over $\mathbf{Z}$. For every field F we have the group $G(F)$ of F-points of G, and the study of representations of such groups for various fields F is a classical subject of representation theory. For example, when F is locally compact (i.e., is a finite extension of $\mathbf{R}$, $\mathbf{F}_p((t))$ or $\mathbf{Q}_p$), then so is $G(F)$; thus $G(F)$ possesses a Haar measure, which serves as a crucial ingredient for the deep and well-developed harmonic analysis on groups of this type. An early but important result here is the determination of the algebra of functions on G bi-invariant with respect to a maximal compact subgroup K. For the case when F is non-Archimedean, this algebra is known as the unramified Hecke algebra. It is commutative and in fact naturally identified with the Grothendieck ring of finite-dimensional algebraic representations of the Langlands dual group ${}^LG$ (Satake's isomorphism).

The next in difficulty is the case when F is a complete discrete valued field whose residue field k is locally compact, for example $F = \mathbf{C}((t))$ or $\mathbf{Q}_p((t))$ is the field of formal Laurent series with complex or p-adic coefficients. In this case the groups $G(F)$ are called (complex or p-adic) loop groups. They are not locally compact and hence do not possess any invariant measure, although there exists a very interesting representation theory of complex loop groups [PS]. In particular, the standard definition of the Hecke algebra cannot be applied here since it involves convolution with respect to the Haar measure.

The aim of the present paper is to show how to associate to a p-adic loop group $G(F)$ a natural Hecke algebra $\mathcal{H}$ and to describe this algebra completely, generalizing the Satake isomorphism. As in the classical case, $\mathcal{H}$ consists of certain $G(F)$-invariant integral operators on the homogeneous space $G(F)/K$ where K is an appropriate analog of the maximal compact subgroup (see below). For this choice of K the set $G(F)/K$ is the set of vertices of a so-called double Bruhat-Tits building in the sense of A.N. Parshin [Pa1-2], and the geometry of this building is our first main tool. Another ingredient, needed to avoid appealing to the (non-existent) Haar measure on $G(F)$, is the systematic use of so-called Poisson measures on the boundaries of Bruhat-Tits complexes for ordinary p-adic groups. The Poisson measure associated to a vertex can be interpreted in terms of the Brownian motion on the complex, as the probability that the Brownian particle eventually exits into the given region of the boundary (so it is a particular case of the so-called exit measure on the Martin boundary known in the theory of Markov processes [Do] [Dy] [Fu]). Even though our constructions are purely algebraic, the appearance of concepts with such probabilistic interpretation is quite natural since we are dealing here with (a certain algebraic version of) the integration over the loop space.

Let us describe our results more precisely. The residue field of F is denoted by k. It is a locally compact non-archimedean field, so it is, in its turn, a complete discrete valued field whose residue field is a finite field $\mathbf{F}_q$. Let $\mathcal{O} = \mathcal{O}_F$ and $\mathcal{O}_k$ be the rings of integers of F and k respectively, so that we have a surjection

$$\pi : \mathcal{O} \to k.$$

Denote by $\mathcal{O}' \subset \mathcal{O}$ the preimage $\pi^{-1}(\mathcal{O}_k)$, so we have surjections

$$\rho : \mathcal{O}' \to \mathbf{F}_q, \quad \rho_G : G(\mathcal{O}') \to G(\mathbf{F}_q).$$

The subgroup $G(\mathcal{O}')$ will be denoted by K' and will serve as the analog of a maximal compact subgroup in a p-adic Lie group. It was proved by Parshin for the case $G = PGL_n$ (see below for general case) that the double coset space $K' \backslash \mathcal{G} / K'$ is discrete and does not depend on the number of elements in the last residue field $\mathbf{F}_q$. Our Hecke algebra $\mathcal{H}$ will consist of functions on $K' \backslash \mathcal{G} / K'$ satisfying certain finiteness conditions.

In contrast with the classical case the algebra $\mathcal{H}$ is not commutative, but is related to the Heisenberg algebra. More precisely, let $T \subset G$ be a split maximal torus, X its lattice of characters and $X^\vee$ the lattice of 1-parameter subgroups in T. Denote by $T^\vee$ the dual torus to T, i.e., the spectrum of the group algebra of $X^\vee$. Denote, as usual, by W the Weyl group of G and by $\Delta \subset X$ (resp. Δ^+) the system of roots (resp. positive roots) of G. Then there is a $\mathbf{Z}$-valued bilinear form Ψ on $X^\vee$ defined entirely in terms of the root system Denote by $\tilde{\mathcal{A}}$ the semidirect product of two copies of the group algebra of $X^\vee$, generated by monomials z^a, w^b, $a, b \in X^\vee$ with the relations

$$z^a z^b = z^{a+b}, \quad w^a w^b = w^{a+b}, \quad w^a z^b = q^{\Psi(a,b)} z^b w^a.$$

We call $\tilde{\mathcal{A}}$ the Heisenberg algebra. The Hecke algebra $\mathcal{H}$ turns out to be very closely related to $\tilde{\mathcal{A}}$. More precisely, let $X_+^\vee \subset X^\vee$ be the set of positive coweights. Let $\mathcal{A} \subset \tilde{\mathcal{A}}$ be the subalgebra consisting of polynomials of the form $\sum_{a \in X_+^\vee} f_a(z) w^a$ with each $f_a(z)$ being invariant under the subgroup $W_a \subset W$ preserving a. In particular, f_0 is W-invariant. Now our results are as follows.

Theorem 1. *The associated graded algebra of an appropriate filtration of $\mathcal{H}$ is isomorphic to the Heisenberg algebra $\mathcal{A}$.*

Theorem 2. *The algebra $\mathcal{H}$ can be embedded into the completion of $\mathcal{A}$ as the algebra of series $\sum_{a \in X_+^\vee} f_a(z) w^a$ where the summation is over a finite subset of dominant coweights, and each $f_a(z)$ is a Laurent series in z invariant under W_a.*

These results open the way to the study of a new class of infinite-dimensional representations of loop groups which are analogs of principal series representations for real or p-adic Lie groups. Recall that for a p-adic group a representation from the unramified principal series can be realized as the space of functions on the vertices of the Bruhat-Tits complex which are common eigenfunctions of the unramified Hecke algebra (which is in this case commutative). In the situation of a p-adic loop group the Hecke algebra $\mathcal{H}$ is non-commutative so the analogs of the principal series representations should be parametrized by irreducible representations of $\mathcal{H}$ and be the multiplicity spaces of such representations in the space of functions on the Parshin building. Note that the only representations of loop groups which have been systematically studied are so-called integrable representations of complex loop groups and they are analogs of algebraic finite-dimensional representations of algebraic groups over $\mathbf{C}$. In particular, their construction cannot be modified to give complex representations of p-adic loop groups. On the contrary, the "principal series representations" arising from harmonic analysis on the Parshin building are much larger in size and can be defined for p-adic loop groups.

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§1. PGL_2 over an ordinary local field: a reminder.

In this and the following section we work out in detail the case $G = PGL_2$, in order to give the reader a good feeling of what is going on. The aim of this section is to recall some well known material on ordinary local fields. We refer to the books [FN] [Se] and to Cartier's Bourbaki talk [Ca] for more details.

(1.1) Bruhat-Tits trees. Let F be a complete discrete valued field with ring of integers $\mathcal{O}_F$ and residue field k. We denote by x a uniformising element of F, so $x \in \mathcal{O}_F$ and $\mathrm{ord}(x) = 1$.

It is well known that the left cosets of $PGL_2(F)$ by $PGL_2(\mathcal{O}_F)$ form the set of vertices of a certain tree, called the Bruhat-Tits tree and denoted by $\mathcal{T}$. It has the following properties:

(1.1.1) The group $PGL_2(F)$ acts on $\mathcal{T}$ with the stabilizer of one vertex being $PGL_2(\mathcal{O}_F)$.

(1.1.2) For every vertex $v \in \mathcal{T}$ the set of edges incident with v is naturally a projective line over the field k.

(1.1.3) The set $\partial\mathcal{T}$ of "ends" of $\mathcal{T}$ is naturally identified with $P^1(F)$.

Recall that an end of $\mathcal{T}$ is an equivalence class of half-infinite edge paths without returns, where two such paths are called equivalent if they eventually coincide.

In a more invariant fashion, one may start with an arbitrary projective line P over F, instead of the standard P^1 (for instance, P can be the projectivization of a 2-dimensional F-vector space V for which an identification with F^2 is not chosen). Then we have the group $PGL(P)$ of projective automorphisms of P. The Bruhat-Tits tree $\mathcal{T}(P)$ has as vertices all maximal compact subgroups in $PGL(P)$, and its boundary is P itself.

(1.2) Distance, apartments and horocycles. For any two vertices v, v' of $\mathcal{T}$ there is a unique edge path $A(v, v')$ without returns which joins v and v'. Its length (i.e., number of edges) is denoted by $d(v, v')$. This makes the set of vertices of T into a metric space which is a non-Archimedean analog of the Lobachevski plane, the projective line ∂T playing the role of the absolute [Se]. For any $v \in T$ and $r \in \mathbf{Z}_+$ we denote by $S_r(v)$ the sphere with center v and radius r. For instance, $S_1(v)$ is identified with the set of edges issuing from v, which is a projective line over k.

For any two ends $p, p' \in \partial\mathcal{T}$ there is a unique edge path $A(p, p')$ (infinite in both directions) which "joins" p and p'. Such paths are called apartments. They are in bijection with split tori in $PGL_2(F)$: given such a torus T, the points p, p' are the directions of the two common eigenvectors of T. Let $D(p, p')$ be the set of vertices of $A(p, p')$. It has a natural structure of a $\mathbf{Z}$-torsor (depending on the ordering of p, p'). More precisely, let $T_\mathcal{O}$ be the (unique) maximal compact subgroup of T. The action of T preserves $D(p, p')$ with $T_\mathcal{O}$ acting trivially in such a way that the action of $T/T_\mathcal{O} = \mathbf{Z}$ is simply transitive.

Let p be a point of $\partial\mathcal{T}$. Then for any vertex v there is a unique half-infinite edge path $A(p,v)$ joining v and p. The "distance" between p and v, i.e., the length of $A(p,v)$ is of course infinite. Still, one can speak about the *difference* between such distances for two vertices v, v'. Indeed, the paths $A(p,v)$ and $A(p,v')$ eventually coincide. Suppose they coincide after a vertex w. Define the "difference" to be

$$\Delta_p(v,v') = d(v,w) - d(v',w) \in \mathbf{Z}.$$

It is clear that

$$\Delta_p(v,v') + \Delta_p(v',v'') = \Delta_p(v,v'').$$

Thus we can define a natural $\mathbf{Z}$-torsor $D(p)$ which is generated by the symbols $d(p,v)$, $v \in \mathcal{T}$ subject to the relations

$$d(p,v) - d(p,v') = \Delta_p(v,v').$$

For each $v \in \mathcal{T}$ the distance $d(p,v)$ is now a well defined element of the torsor $D(p)$. Note that $D(p)$, being a $\mathbf{Z}$-torsor, is equipped with a natural order.

Given $p \in \partial\mathcal{T} = P^1(F)$ and $r \in D(p)$, the *horocycle* $S_r(p)$ with center p and radius r is the set $S_r(p) = \{v \in T : d(p,v) = r\}$. Let $N_p \subset PGL_2(K)$ be the unipotent subgroup fixing p. Then horocycles with center p are just orbits of N_p. This implies the following fact.

(1.2.1) Proposition. *Take the line bundle $\mathcal{O}(-1)$ on $\partial\mathcal{T} = P^1(F)$ and its fiber at p. This is a 1-dimensional F-vector space, denote it L_p. So $L_p - \{0\}$ is a F^*-torsor. The $\mathbf{Z}$-torsor $D(p)$ is natrally identified with the torsor obtained from $L_p - \{0\}$ by the base change with respect to the valuation homomorphism* $\mathrm{ord} : F^* \to \mathbf{Z}$.

Proof: Let $G = PGL_2(F)$, $K = PGL_2(\mathcal{O}_F)$, B the Borel subgroup and $N \subset B$ the unipotent. Let T be the diagonal subgroup. We have a natural fibration $\pi : G/N \to G/B = P^1(F)$ with fibers principal homogeneous spaces over $T \simeq F^*$. From the Iwasawa decomposition $G = KTN$ it follows that $D(p)$ is obtained from $\pi^{-1}(p)$ by factorizing by $K \cap T$. But G/N is nothing but $A^2 - \{0\}$, the punctured affine plane which is the same as the total space of the bundle $\mathcal{O}(-1)$ with zero section deleted.

(1.3) Double cosets. Hecke algebra. The set of double cosets

$$PGL_2(\mathcal{O}_F)\backslash PGL_2(F)/PGL_2(\mathcal{O}_F)$$

is the set of all possible relative positions of two vertices v, v' on $\mathcal{T}$ (i.e., equivalence classes of pairs of vertices modulo the action of $PGL_2(F)$). Such relative position is uniquely determined by the distance $d(v,v') \in \mathbf{Z}_+$. So we have the identification

$$\mathbf{Z}_+ \to PGL_2(\mathcal{O}_F)\backslash PGL_2(F)/PGL_2(\mathcal{O}_F), \quad n \mapsto PGL_2(\mathcal{O}_F)\begin{pmatrix} x^n & 0 \\ 0 & 1 \end{pmatrix} PGL_2(\mathcal{O}_F).$$

Here $x \in F$ is our uniformiser.

Assume that k is a finite field $\mathbf{F}_q$. Denote by $\mathcal{S}$ the space of all functions on $\mathrm{Vert}(\mathcal{T})$. The *unramified Hecke algebra* $\mathcal{H}_0 = H(PGL_2(F), PGL_2(\mathcal{O}))$ is, by definition, the algebra of compactly supported doubly $PGL_2(\mathcal{O})$-invariant functions on $PGL_2(F)$ with the multiplication given by the convolution. It can also be defined as the algebra of operators in $\mathcal{S}$ generated by the operators $T_n, n \geq 1$, where for $\phi \in \mathcal{S}$ we set

$$(T_n\phi)(v) = \sum_{v' : d(v,v')=n} \phi(v').$$

It is straightforward that if $m \neq n$, then

$$(1.3.1) \qquad T_mT_n = T_nT_m = T_{m+n} + (q-1)T_{m+n-2} +$$

$$+q(q-1)T_{m+n-4} + \ldots + q^{\mu-2}(q-1)T_{m+n-2\mu+2} + +q^{\mu-1}T_{m+n-2\mu},$$

where $\mu = \min(m,n)$. When $m = n$, we have

$$(1.3.2) \quad T_m^2 = T_{2m} + (q-1)T_{2m-2} + q(q-1)T_{2m-4} + \ldots + q^{m-2}(q-1)T_2 + q^{m-1}(q+1)\cdot 1.$$

(1.4) Stabilization of the Hecke algebra. Satake isomorphism. The relations (1.3.1) imply that $\mathcal{H}_0$ is the polynomial algebra in one generator T_1. The regular action of T_1 in the basis of the T_n is particularly simple:

$$(1.4.1) \qquad T_1T_n = \begin{cases} T_{n+1} + qT_{n-1}, & n \geq 2 \\ T_2 + (q+1)\cdot 1, & n = 1. \end{cases}$$

For later purposes it is convenient to extend the regular representation of $\mathcal{H}_0$ to a module M with basis t_n for all $n \in \mathbf{Z}$ and the action of T_1 given by the first case of (1.4.1), i.e.:

$$(1.4.2) \qquad T_1t_n = t_{n+1} + qt_{n-1}, \quad \forall n \in \mathbf{Z},$$

or, equivalently,

$$(1.4.3) \qquad T_mt_n = t_{m+n} + (q-1)t_{m+n-2} + \ldots + q^{m-2}(q-1)t_{n-m+2} + q^m t_{n-m}.$$

In other words, M is obtained by observing that the rule for multiplying T_m by T_n for $m < n$ is translation invariant with respect to n and then extending this rule by translational invariance to all $n \in \mathbf{Z}$. For this reason we call M the *stabilization* of $\mathcal{H}_0$. The translation invariance of relations in M means that M is an $(\mathcal{H}_0, \mathbf{C}[z,z^{-1}])$-bimodule, where $z \in \mathbf{C}[z,z^{-1}]$ acts by $t_nz = t_{n+1}$. Clearly, M is free of rank 1 over $\mathbf{C}[z,z^{-1}]$ and hence the bimodule structure gives a homomorphism of algebras

$$(1.4.4) \qquad S : \mathcal{H}_0 \to \mathrm{End}_{\mathbf{C}[z,z^{-1}]}(M) = \mathbf{C}[z,z^{-1}].$$

Its explicit form is immediately found from (1.4.3) to be:

$$(1.4.5) \qquad T_m \to z^m + (q-1)z^{m-2} + \dots + q^{m-2}(q-1)z^{-m+2} + q^m z^{-m}.$$

(1.4.6) Proposition. *The map S identifies $\mathcal{H}_0$ with the subalgebra of $\mathbf{C}[z, z^{-1}]$ consisting of $f(z)$ such that $f(qz^{-1}) = f(z)$.*

This identification is the simplest instance of the Satake isomorphism.

(1.5) Poisson measures on $\partial\mathcal{T}$. Assume that the residue field k is a finite field $\mathbf{F}_q$. Then every vertex $v \in T$ defines a natural measure μ_v on the absolute $\partial\mathcal{T} \simeq P^1(F)$. Namely, for every vertex $w \neq v$ consider the set

$$M_v(w) = \{p \in \partial T: \ A(p, w) \subset A(p, v)\}.$$

Such sets obviously form a basis of the topology on ∂T. The measure μ_v is uniquely specified by setting

$$\mu_v(M_v(w)) = \frac{1}{(q+1)q^{d(v,w)-1}}.$$

Thus we have $\mu_v(\partial\mathcal{T}) = 1$, so μ_v is a probability measure. In plain words, a choice of v represents $\partial\mathcal{T}$ as an inverse limit of the spheres

$$S_1(v) \leftarrow S_2(v) \leftarrow S_3(v) \leftarrow \dots$$

with all the fibers of all maps having cardinality q. Our measure is just associated with this inverse system.

The measure μ_v has the following probabilistic meaning [Ca]. Consider the (isotropic) Brownian motion on $\mathcal{T}$, i.e., the Markov chain whose states are vertices of $\mathcal{T}$ and such that the probability of the transition from a vertex v to any adjacent vertex w has the same value $1/(q+1)$. Then $\mu_v(U)$ for $U \subset \partial\mathcal{T}$ is the probability that the Brownian particle, having started from the point v, will, as the time goes to infinity, converge ("exit") to a boundary point from U. This concept of exit probability can be defined for any Markov chain or even continuous Markov process, the role of $\partial\mathcal{T}$ being played in general case by the so-called Martin boundary [Do] [Dy].

(1.5.1) Proposition. *The measure μ_v is the unique, up to scalar, measure on $\partial\mathcal{T}$ invariant under the compact subgroup in $PGL_2(F)$ preserving v.*

So μ_v is analogous to the Fubini-Study metric (volume form) on the projective line $\mathbf{R}P^1$ associated to a scalar product on $\mathbf{R}^2$. This volume form can also be interpreted as exit probability, this time for the Brownian motion in the Lobachevsky plane whose (Martin) boundary is $\mathbf{R}P^1$.

By construction, μ_v depends on v. For two vertices v, w the measures μ_v, μ_v are absolutely continuous with respect to each other, so that their ratio (Radon-Nikodim derivative) is just a nonvanishing function $b \mapsto (\mu_v/\mu_w)(b) = \Pi(v, w, b)$ on $\partial\mathcal{T}$ known as the Poisson kernel. Explicitly, one has

$$\Pi(v, w, b) = q^{d(v,b)-d(w,b)}. \tag{1.5.2}$$

Here the distances $d(v, b)$ and $d(w, b)$ in the exponent are elements of the $\mathbf{Z}$-torsor $D(b)$, so their difference is a well defined integer.

Given $b \in P^1(F)$, the functions $w \to \Pi(v, w, b)$ on $\mathrm{Vert}(\mathcal{T})$ for various v are scalar multiples of each other, so we have a well-defined 1-dimensional $\mathbf{C}$-vector space $\Pi_b \subset \mathcal{S}$ of such functions. Taken together, these 1-dimensional spaces form a $PGL_2(F)$-equivariant complex line bundle Π on $P^1(F)$.

(1.5.3) Proposition. *(a) As an equivariant bundle, Π is isomorphic to $|\mathcal{O}(-1)|$, the complex line bundle on $P^1(F)$ obtained from the F-line bundle $\mathcal{O}(-1)$ by the base change with respect to the norm homomorphism $F^* \to \mathbf{C}^*$, $a \mapsto |a|$.*
(b) For any $b \in P^1(F)$, ant $f \in \Pi_b$ is an eigenfunction of the Hecke algebra $\mathcal{H}_0$ with $T_1 f = (q+1)f$.
(c) More generally, for any $f \in \Pi_b$ and $s \in \mathbf{C}$ the complex power f^s is an eigenfunction of $\mathcal{H}_0$ with $T_1 f^s = (q^s + q^{1-s})f^s$.

Part (a) follows from (1.2.1), while (b) and (c) are checked explicitly.

Thus the Poisson kernel establishes an isomorphism between two realizations of the unramified principal series representations of $PGL_2(F)$. The first realization is as the space of sections $V_s = \Gamma(P^1(F), |\mathcal{O}(-1)|^s \otimes \mathrm{meas}^{-1})$, where $|\mathcal{O}(-1)|^s$ is the $\mathbf{C}$-line bundle obtained from $\mathcal{O}(-1)$ by the base change $F^* \to \mathbf{C}^*$, $a \mapsto |a|^s = q^{s \cdot \mathrm{ord}(a)}$ and meas is the sheaf of measures (isomorphic to $|\mathcal{O}(-1)|^2$). The second realization of the same representation can be given as the space of functions f on $\mathrm{Vert}(\mathcal{T})$ such that $T(1)f = (q^s + q^{1-s})f$.

§2. PGL_2 over a 2-dimensional local field.

(2.1) The Parshin tree. We return to the setup of the introduction, so F is a 2-dimensional local field, $\mathcal{O} = \mathcal{O}_F$ its ring of integers, $\mathbf{m}_F \subset \mathcal{O}$ the maximal ideal and k its residue field. The field k is a locally compact non-archimedean field) with ring of integers $\mathcal{O}_k$, maximal ideal $\mathbf{m}_k$ and finite residue field $\mathbf{F}_q$. We also denote by $\mathcal{O}' \subset \mathcal{O}$ the preimage of $\mathcal{O}_k$ under the natural projection.

Considering F as just a local field with residue field k, we associate to it the Bruhat-Tits tree $\mathcal{T}_F$, whose vertices correspond to left coset of $PGL_2(F)$ by $PGL_2(\mathcal{O})$. For any vertex $v \in \mathcal{T}_F$ the set of edges incident to v is a projective line P^1_v over k, so it is of "continuous" nature.

Let τ_v be the Bruhat-Tits tree with boundary P^1_v (and the valence of each vertex $q+1$). Let us think of this tree as "microscopic" and insert this tree, together with its boundary, instead of the neighborhood of the vertex v in $\mathcal{T}_F$. Do this for all the vertices. This way, each edge of $\mathcal{T}_F$ will become a "bridge" joining two boundary points of two neighboring microscopic trees. The infinite tree thus obtained is called the *Parshin tree* and denoted $\mathcal{P}$. The reader can consult [Pa1-2] for a more formal construction and a picture. Clearly, the group $PGL_2(K)$ acts on $\mathcal{P}$ by automorphisms.

Note that we have a continuous map

$$\pi : \mathcal{P} \to \mathcal{T}_F \tag{2.1.1}$$

which contracts each microscopic tree τ_v with its boundary into one vertex $v \in \mathcal{T}_F$.

In the sequel we will refer to vertices or edges of $\mathcal{P}$ which are among vertices or edges of some microscopic tree (not of its boundary) as *thin* and call the points on the boundary of microscopic trees as well as edges ("bridges") joining them *thick* vertices or edges of $\mathcal{P}$. Thus thin vertices are isolated and thick vertices are limits of thin ones. A thick edge is an edge joining two thick vertices. Thick edges of $\mathcal{P}$ are in bijection (induced by the map π above) with edges of $\mathcal{T}_F$

(2.1.2) Proposition. *The set of left cosets $PGL_2(F)/PGL_2(\mathcal{O}')$ is naturally identified with the set of thin vertices of $\mathcal{P}$.*

As in the case of an ordinary local field, for any two vertices w, w' (thick or thin) of $\mathcal{P}$ there is a unique edge path $A(w, w')$ joining them. This path can, however, have several infinite fragments which are separated by thick edges

More precisely, we have the following proposition.

(2.1.3) Proposition. *Let v, v' be two vertices of $\mathcal{T}_F$ and $\tau_v, \tau_{v'}$ the corresponding microscopic trees inside $\mathcal{P}$. Then all paths $A(w, w')$ for $w \in \tau_v$, $w' \in \tau_{v'}$ have the same set of thick vertices. In particular, the segments of these paths between the first and the last thick vertices are the same for all w, w'.*

(2.2) Distances, spheres and horocycles. We choose uniformising elements $x, y \in \mathcal{O}'$ such that $\mathrm{ord}_F(y) = 1, \mathrm{ord}_F(x) = 0$ and $\mathrm{ord}_k(x \bmod \mathbf{m}_F) = 1$. Then we have the decomposition

$$F^* = \{x^m y^n\} \cdot \mathcal{O}'^*.$$

We have the analog of the Cartan decomposition for the group $PGL_2(F)$ found by Parshin [Pa1-2]:

$$PGL_2(F) = PGL_2(\mathcal{O}') \left\{ \begin{pmatrix} x^m y^n & 0 \\ 0 & 1 \end{pmatrix} \right\} PGL_2(\mathcal{O}'). \tag{2.2.1}$$

Note that (2.2.1) is different from the kind of Cartan decompositions for p-adic loop groups found by Garland [Ga].

It follows that double cosets are labelled by equivalence classes of the group $\mathbf{Z} \oplus \mathbf{Z} = \{x^m y^n\}$ by the involution $(m, n) \mapsto (-m, -n)$. A set of respesentatives of monomials modulo this involution is provided by the semigroup

$$\Lambda = \left\{ (m, n) \in \mathbf{Z} \oplus \mathbf{Z} : \ m \geq 0 \text{ and if } m = 0, \text{ then } n \geq 0 \right\}. \tag{2.2.2}$$

We order Λ lexicographically, i.e., $(m, n) \leq (m', n')$ if $m < m'$ or $m = m'$ and $n \leq n'$. We will also write an element $(m, n) \in \Lambda$ as $m\infty + n$, so that the ordering becomes more intuitive. In terms of the tree $\mathcal{P}$ the above considerations can be summarized as follows.

(2.2.3) Proposition. *(a) For every two thin vertices $w, w' \in \mathcal{P}$ there is a well defined distance $d(w, w') \in \Lambda$, which satisfies the triangle inequality: $d(w, w'') \leq d(w, w') + d(w', w'')$ and is preserved by the action of the group $PGL_2(K)$.*
(b) Given two pairs of points (w_1, w_1') and (w_2, w_2') a necessary and sufficient condition for existence of $g \in PGL_2(F)$ taking $w_1 \mapsto w_2$ and $w_1' \mapsto w_2'$ is the equality $d(w_1, w_1') = d(w_2, w_2')$.
(c) If $d(w, w') = m\infty + n$, then in the path $A(w, w')$ there are exactly m thick edges.

For any $\lambda \in \Lambda$ and a thin vertex w of $\mathcal{P}$ we denote by $S_\lambda(w)$ the sphere of radius λ, i.e., the set of thin vertices w' such that $d(w, w') = \lambda$. Let v be the vertex of the "continuous" Bruhat-Tits tree $\mathcal{T}_F$ such that w lies in the microscopic tree τ_v. Then clearly

$$S_{m\infty+n}(w) \quad \subset \bigcup_{v' \in \mathrm{Vert}(\mathcal{T}_F):\, d_{\mathcal{T}_F}(v, v') = m} \tau_{v'}$$

where $d_{\mathcal{T}_F}$ is the $\mathbf{Z}_+$-valued distance in the tree $\mathcal{T}_F$. If $v' \in \mathcal{T}_F$ is any vertex entering into the above formula, denote by $\mathbf{e}(v, v')$ the last thick vertex on any path $A(w, w')$, $w \in \tau_v$, $w' \in \tau_{v'}$. This vertex lies on the boundary of $\tau_{v'}$.

(2.2.4) Proposition. *If $w \in \tau_v$ and $v' \in \mathcal{T}_F$ is such that $d_{\mathcal{T}_F}(v, v') = m$, then for any $n \in \mathbf{Z}$ the intersection $S_{m\infty+n}(w) \cap \tau_{v'}$ is a horocycle in $\tau_{v'}$ with center $\mathbf{e}(v, v')$.*

As we saw in §1, for any Bruhat-Tits tree $\mathcal{T}$ and any infinite point $p \in \partial T$ the "distances" parametrizing horocycles with center p are in fact elements of a certain $\mathbf{Z}$-torsor $D(p) = D_{\mathcal{T}}(p)$ (the second notation in introduced to emphasize the dependence of this torsor on $\mathcal{T}$). So the very possibility of attaching the Λ-valued distances from v to horocycles in $\tau_{v'}$ in (2.2.4) amounts to a non-trivial extra structure on the Parshin tree $\mathcal{P}$ which is not at all clear from the direct iterative construction of $\mathcal{P}$ described in (2.1). Let us describe this structure explicitly.

Recall that for any Abelian group A (written additively) the category of all A-torsors has a natural symmetric monoidal structure ("tensor product"), denoted $\odot$. Namely, if S_1, S_2 are two A-torsors then $S_1 \odot S_2$ is generated by symbols $s_1 \odot s_2$, $s_i \in S_i$ modulo the relations $(a + s_1) \odot s_2 = s_1 \odot (a + s_2)$. We will use this structure for $A = \mathbf{Z}$.

(2.2.5) Proposition. *For any thick edge e of $\mathcal{P}$ with ends p and p' (which thus lie on the absolutes of two adjacent microscopic trees τ_v and $\tau_{v'}$) there is an identification of $\mathbf{Z}$-torsors $\gamma_e : D_{\tau_v}(p) \odot D_{\tau_{v'}}(p') \to \mathbf{Z}$. This system of identifications is equivariant with respect to the action of the group $PGL_2(F)$ on $\mathcal{P}$.*

This identification comes about as follows. Given elements $a \in D_{\tau_v}(p)$ and $a' \in D_{\tau_{v'}}(p')$, we represent them as distances: $a = d(w, p)$, $a' = d(w', p')$ for some thin vertices $w \in \tau_v$, $w' \in \tau_{v'}$. Then the distance in $\mathcal{P}$ between w and w' has the form $d(w, w') = \infty + m$ for some $m \in \mathbf{Z}$, and we set $\gamma_e(a \odot a') = m$. This is well defined by 2.2.4.

Notice that unlike the case of ordinary Bruhat-Tits trees, neither the Λ-valued distance d nor the system of identifications $\{\gamma_e\}$ are preserved under the full group of automorphisms of $\mathcal{P}$. Clearly, the $\{\gamma_e\}$ determine d and vice versa. Thus it is of some interest to give a more geometric construction of the γ_e not appealing to matrix calculations. Such a construction can be obtained, by applying Proposition 1.2.1, from a statement concerning *ordinary* Bruhat-Tits trees.

Namely, let F be any local field with residue field k (so k is not assumed to have any extra structure) and $\mathcal{T}$ be the corresponding Bruhat-Tits tree, as in §1. For any vertices $v \in T$ the set of edges incident to v is, as we saw, a projective line over k or, more precisely, the set of k-points of an algebraic curve $\mathbf{P}_v$ over k isomorphic (not naturally) to P^1. If two vertices v, v' are adjacent and joined by an edge e, then e represents a point (v/e) on $\mathbf{P}_v$ as well as a point (v'/e) on $\mathbf{P}_{v'}$. Now the statement we mean is as follows.

(2.2.6) Proposition. *For every v, v', e as before there is a natural nondegenerate pairing of tangent spaces*

$$\beta_e : T_{(v/e)}\mathbf{P}_v \otimes T_{(v'/e)}\mathbf{P}_{v'} \to k,$$

and the system of these pairings is equivariant under the action of $PGL_2(F)$ on $\mathcal{T}$.

As we will see from the proof, the choice of these pairings will depend on the choice of a uniformising parameter $x \in F$.

To construct the γ_e (and thus the distance function) out of the β_e it is enough to notice that the tangent bundle on P^1 is isomorphic to $\mathcal{O}(2)$, so by Proposition 1.2.1 we get a pairing between $D_{\tau_v}(p)^{\odot 2}$ and $D_{\tau_{v'}}(p')^{\odot 2}$. But since the group $\mathbf{Z}$ has no torsion, such a pairing gives rise to a unique pairing between $D_{\tau_v}(p)$ and $D_{\tau_{v'}}(p')$.

Proof of (2.2.6): We can represent a vertex $v \in \mathcal{T}$ by a rank 2 vector bundle V on the formal curve $C = \mathrm{Spec}(\mathcal{O}_F)$. Let $0 \in S$ be the unique closed point. We denote by V_0 the fiber of V at 0. It is a 2-dimensional k-vector space. The set P_v is $P(V_0)$, the projectivization of (the set of lines in) V_0. Let $l \subset V_0$ be the line corresponding to the edge e. As for any projective space, the tangent space $T_{(v/e)}\mathbf{P}_v = T_l P(V_0)$ is naturally identified with $l^* \otimes (V_0/l)$. Further, the bundle V' corresponding to v' (the other end of e) is described (see [Se]) as the coherent subsheaf of V consisting of those sections s for which $s(0) \in l$. Therefore we have an exact sequence for the fiber of V' at 0:

$$0 \to (V_0/l) \otimes T_0^* C \to V_0' \to l \to 0,$$

where T_0^*C is the cotangent space to C at 0. Moreover, the line in V_0' corresponding to the same edge e, is the subspace $(V_0/l) \otimes T_0^*C$. Therefore $T_{(v'/e)}\mathbf{P}_{v'}$ is naturally identified with $((V_0)/l)^* \otimes l \otimes T_0 C$. But our choice of a uniformiser $x \in F$ identifies $T_0 C \to k$. Q.E.D.

(2.3) Measures on spheres. Let w be a thin vertex of $\mathcal{P}$ and $\lambda = m\infty + n \in \Lambda$. Our aim is to introduce a natural measure μ_w on the sphere $S_\lambda(w)$.

To do this, denote again by $\mathcal{T}_F$ the continuous Bruhat-Tits tree (corresponding to F considered as an ordinary local field) and $v = \pi(w) \in \mathrm{Vert}(\mathcal{T}_F)$ the vertex such that $w \in \tau_v$. For $m \in \mathbf{Z}_+$ let $S_{m,F}(v)$ be the sphere of radius m in $\mathcal{T}_F$. Then we have the following diagram of projections:

$$(2.3.1) \quad S_{m\infty+n}(w) \xrightarrow{\pi} S_{m,F}(v) \xrightarrow{\rho_m} S_{m-1,F}(v) \to \ldots S_{2,F}(v) \xrightarrow{\rho_2} S_{1,F}(v) \xrightarrow{\rho_1} S_{0,F}(v) = \{v\}.$$

As we saw, the fibers of π are horocycles, in particular, they are countable. The fiber of ρ_1 is $S_{1,F}(v)$, i.e., a projective line over the 1-dimensional local field k which is the same as $\partial\tau_v$. The fibers of ϕ_i, ≥ 2, are affine lines over k. We have seen in §1 that a choice of $w \in \mathrm{Vert}(\tau_v)$ defines a particular probability measure on $\partial\tau_v$, the Poisson measure.

Further, any fiber of ρ_2 over a point $b \in \partial\tau_v$ is described as follows. Let $\tau_{v'}$ be the unique microscopic tree adjacent to τ_v which has b on its absolute. Then $\rho_2^{-1}(b) = \partial\tau_{v'} - \{b\}$ We can identify this affine line with the set of all infinite edge paths $A(b, b')$ joining $\mathbf{e}$ with other points $b' \in \partial\tau_{v'}$. The basis of topology on this affine line is given by the following subsets:

$$(2.3.2) \qquad M_{w'} = \left\{ A(b,b') : \ w' \in A(b,b') \right\}$$

where $w' \in \tau_{v'}$ is a vertex. Now, the distance $d(w, w')$ from our fixed w to w' (this distance has form $\infty + n$, $n \in \mathbf{Z}$) gives us the canonical measure $\mu_{w,b}$ on $\partial\tau_{v'} - \{b\}$. Namely, we set:

$$(2.3.3) \qquad \mu_{w,b}(M_{w'}) = q^{-(d(w,w')-\infty)}.$$

Here $d(w, w') - \infty$ is an integer, so the formula makes sense. In other words, $\mu_{v,b}$ is a Lebesgue measure on the affine line $\partial\tau_{v'} - \{b\}$, and (2.3.3) serves to normalize it.

In a similar way, one defines the measure on each fiber of ρ_i, by subtracting $(i-1)\infty$ from the distances.

Finally, on the horocycles which are fibers of π we introduce the Dirac measure which assigns to each element the value 1 (so the integration by this measure is just the summation).

In this way we have constructed a canonical measure $\mu_{w,\lambda}$ on the whole sphere $S_\lambda(w)$, $\lambda = m\infty + n$ as the (Fubini) product of measures on the fibers of projections in (2.3.1).

(2.3.4) Proposition. *The measure μ_w is the unique, up to scalar, measure invariant under the subgroup K'_w in $PGL_2(F)$ preserving the vertex w.*

Such a subgroup is of course conjugate to $PGL_2(\mathcal{O}')$.

(2.4) The Hecke algebra. We are now going to define a kind of Hecke algebra for the subgroup $K' = PGL_2(\mathcal{O}') \subset PGL_2(F)$. To explain the construction, recall that for a locally compact group G and its compact subgroup H the Hecke algebra $\mathcal{H}(G, H)$ can be defined in one of three equivalent ways:

(1) As the algebra of compactly supported doubly H-invariant continuous functions on G with the operation given by convolution with respect to the chosen Haar measure.
(2) As the algebra of G-invariant integral operators in the space of all continuous functions on G/H.
(3) As the abstract algebra formed by kernels of such operators.

Let us comment on (3). If X is any topological space, we denote by $\mathcal{O}_X$ and $\mathcal{M}_X$ the sheaves of continuous functions and measures on X. Consider the sheaf $\mathcal{M}^{01} = p_1^*\mathcal{O}_X \otimes p_2^*\mathcal{M}_X$ on $X \times X$. Its sections will be called (0,1)-measures, i.e., functions in the first variable and measures in the second variable. We write them as $R(x, y)dy$. Such a measure is called properly supported in the projection of its support to the first factor is a proper map. Then kernels of G-invariant operators in (3) are just properly supported G-invariant (0,1)-measures on $(G/H) \times (G/H)$. The product of such measures is the convolution

$$(R(x, y)dy)(S(x, y)dy) = U(x, y)dy, \quad U(x, y) = \int_{z \in X} R(x, z)S(z, y)dz.$$

We now return to the case of $G = PGL_2(F), H = PGL_2(\mathcal{O}')$. As G has no Haar measure, the description (1) does not make sense; we are going to see that one can give sense to (2) and (3) although they will no longer be equivalent.

We start with (3). Let $\mathcal{V} = PGL_2(F)/PGL_2(\mathcal{O}')$ be the set of thin vertices of $\mathcal{P}$. Fix $\lambda \in \Lambda$. The $PGL_2(F)_w$-invariant measures $\mu_{w,\lambda}$ on the spheres $S_\lambda(w)$, $w \in \mathcal{V}$ fit together to form a $PGL_2(F)$-invariant (0,1)-measure T_λ on $\mathcal{V} \times \mathcal{V}$. This measure is not properly supported, so the possibility of taking products of the T_λ needs a special analysis. Call a subset $S \subset \Lambda$ well-ordered if any subset $S' \subset S$ has a maximal element.

(2.4.1) Proposition. *Let $\hat{\mathcal{H}}$ be the* **C**-*vector space of formal series $\sum_{\lambda\in\Lambda} c_\lambda T_\lambda$ whose support (i.e., the set of λ with $c_\lambda \neq 0$) is well-ordered. Then:*
(a) Convolution of (0,1)-measures on $\mathcal{V}\times\mathcal{V}$ makes $\hat{\mathcal{H}}$ into an algebra.
(b) $\hat{\mathcal{H}}$ has an algebra filtration F parametrized by Λ with $F_\lambda\hat{\mathcal{H}}$ consisting of series $\sum_{\lambda'\leq\lambda} c_{\lambda'}T_{\lambda'}$.

(c) $\hat{\mathcal{H}}$ has an algebra grading $\hat{\mathcal{H}} = \bigoplus_{m\in\mathbf{Z}_+}\hat{\mathcal{H}}_m$ where $\hat{\mathcal{H}}_m$ consists of sums $\sum_{n=\infty}^{N} c_n T_{m\infty+n}$.

Part (a) basically follows from the triangle inequality for the distance. Part (b) is obvious Property (c) means that $\hat{\mathcal{H}}_m\hat{\mathcal{H}}_{m'}\subset\hat{\mathcal{H}}_{m+m'}$. Note that a priori possible summands in the expansion of $T_\lambda T_\mu$ include all $T_{\lambda+\mu-\nu}$ with $\nu\leq\min(\lambda,\mu)$. Geometrically, T_μ is the averaging over all points on distance μ from the given one, so $T_\lambda T_\mu$ is obtained by first going to distance μ and then to a distance λ from there, whereby we can happen to first retrace back any number ν of steps of our path. The statement in (c) is that that unless ν is finite, the probability of such retracing is zero and thus the corresponding $T(\lambda+\mu-\nu)$ will not enter into the expansion. This is indeed true, since the coefficient at the corresponding $T(\lambda+\mu-\nu)$ will be obtained as an integral, with respect to a product of Lebesgue and Poisson measures,-of-a function supported on a veriety of positive codimension.

Note that the degree 0 part of $\hat{\mathcal{H}}$ is just the Hecke algebra $\mathcal{H}_0$ for $PGL_2(k)$ from §1.

(2.5) The subalgebra $\mathcal{H}\subset\hat{\mathcal{H}}$ and its action on functions. The algebra $\hat{\mathcal{H}}$ is of "complete" nature. In particular, it is not finitely generated. Let $\mathcal{H}\subset\hat{\mathcal{H}}$ be the subalgebra generated by T_1 and T_∞. We will now show how to make $\mathcal{H}$ act on an appropriate class of functions on the set $\mathcal{V} = PGL_2(F)/PGL_2(\mathcal{O}')$. This action does not seem to extend to the whole $\hat{\mathcal{H}}$.

(2.5.1) Definition. *A function $f:\mathcal{V}\to\mathbf{C}$ is called a Schwartz-Bruhat function, if the following condition holds: For any $w\in\mathcal{V}$ and any $n\in\mathbf{Z}$ the restriction of f to the sphere $S_{\infty+n}(w)$ (which is a* **Z**-*torsor over a p-adic projective line $\partial\tau_{\pi(w)}$) is a locally constant function with compact support.*

The space of Schwartz-Bruhat functions will be denoted $\mathcal{S}$.

It may be not obvious that Schwartz-Bruhat functions exist at all. So let us give a construction of a large class of them. Namely, we will construct many Schwartz-Bruhat functions f with the following property: in each microtree τ_v there is exactly one vertex $w(v)$ on which $f\neq 0$, and whenever $\tau_v,\tau_{v'}$ are adjacent, we have $d(w(v),w(v'))=\infty$. For this, start with some $w_0\in\tau_{v_0}$. The sphere $S_\infty(w_0)$ is a **Z**-torsor over $\partial\tau_{v_0}\simeq P^1(k)$. Choose a continuous section $\sigma_{v_0}:\partial\tau_{v_0}\to S_\infty(w_0)$ of this torsor. In this way, we mark, for any adjacent microtree τ_{v_1}, one point $w_1 = w(v_1)$ on distance ∞ from w_0. Forther, look at any such τ_{v_1} and at the **Z**-torsor $S_\infty(w_1)\to\partial\tau_{v_1}$. The point w_0 is an element of $S_\infty(w_1)$ lying over the point $p_{v_0,v_1}\in\partial\tau_{v_1}$ nearest to τ_{v_0}. Choose a continuous section $\sigma_{v_1}:\partial\tau_{v_1}\to S_\infty(w_1)$ such that $\sigma_{w_1}(p_{v_0,v_1}) = w_0$. In this way we mark, for any microtree τ_{v_2} with $d_{\mathcal{T}_F}(v_0,v_2)=2$, a point $w_2 = w(v_2)\in\tau_{v_2}$ on distance 2∞ from w and on distance ∞ from $w(v_1)$ where v_1 lies between v_0 and v_2. Continuing like this, we mark, for each

$v \in \mathcal{T}_F$, a point $w(v) \in \tau_v$.

Now we construct a function f. We take $f(w_0)$ to be arbitrary. To define $f(w(v_1))$ for v_1 adjacent to v_0, choose any continuous (i.e., locally constant) function $\phi_{v_0} : \partial\tau_{v_0} \to \mathbf{C}$ and put $f(\sigma)_{v_0}(p)) = \phi_{v_0}(p)$. Further, for any marked $w_1 \in \tau_{v_1}$ with $d(w_0, w_1) = \infty$, choose a locally constant function $\phi_{v_1} : \partial\tau_{v_1} \to \mathbf{C}$ with the property $\phi_{v_1}(p_{v_0,v_1}) = f(w_0)$ and put $f(\sigma_{v_1}(q)) = \phi_{v_1}(q)$, $q \in \partial\tau_{v_1}$ and so on. In this way we construct a Schwartz-Bruhat function f.

(2.5.2) Proposition. *The (0,1)-measures T_1, T_∞ and, more generally, $T_{m\infty+n}, m \leq 1$, give rise to well-defined integral operators preserving $\mathcal{S}$. Explicitly,*

$$(T_\lambda f)(w) = \int_{w' \in S_\lambda(w)} f(w') d\mu_{w,\lambda}, \quad \lambda = m\infty + n,\ m \leq 1, \infty.$$

These operators commute with the action of $PGL_2(F)$.

(2.6) Calculations in $\hat{\mathcal{H}}$ and $\mathcal{H}$. The Hecke algebra $\hat{\mathcal{H}}$ is not commutative, as one can see from the next proposition.

(2.6.1) Proposition. *We have the following equalities in $\mathcal{H}$:*

$$T_1 T_\infty = T_{\infty+1} + qT_{\infty-1}, \quad T_\infty T_1 = qT_{\infty+1} + T_{\infty-1} \tag{2.6.2}$$

More generally,

$$T_m T_{\infty+n} = T_{\infty+m+n} + (q-1)T_{\infty+m+n-2} + q(q-1)T_{\infty+m+n-4} + \ldots$$

$$\ldots + q^{m-1}(q-1)T_{\infty+m+n-2(m-1)} + q^m T_{\infty+n-m}. \tag{2.6.3}$$

$$T_{\infty+n}T_m = q^m T_{\infty+n+m} + (q^{m-1} - q^{m-2})T_{\infty+n+m-2} + (q^{m-2} - q^{m-3})T_{\infty+n+m-4} + \ldots$$

$$\ldots + (q-1)T_{\infty+n+m-2(m-1)} + 1 \cdot T_{\infty+n-m}. \tag{2.6.4}$$

The equality (2.6.3) is obtained in much the same way as (1.4.1): the combinatorial counting is the same.

Let us prove (2.6.4). For this, we have to take into account the change of the Poisson measure on the boundary of a microscopic tree when the vertex defining the measure is moved away to distance m. This change of given by formula (1.5.2). More precisely, start with some thin vertex $w \in \tau_v$. For every edge e incident to w let $M_e \subset \partial\tau_v$ be the set of p such that the shortest part $A(w,p)$ contains e. Then for the Poisson measure corresponding to w we have $\mu_w(M_e) = 1/(q+1)$. If w' is another vertex, then $\mu_{w'}$-measures of the same sets are, in virtue of (1.5.2), as follows:

$$\mu_{w'}(M_e) = \begin{cases} \frac{1}{(q+1)q^{d(w,w')}}, & \text{if} \quad e \notin A(w,w'), \\ 1 - \frac{1}{(q+1)q^{d(w,w')-1}}, & \text{if} \quad e \in A(w,w') \end{cases}$$

Now let us calculate $T_{\infty+n}T_m$. Since this is an integral operator, its value on the δ-function at w is a measure on the set of vertices of the neighboring microtrees $\tau_{v'}$. In order to find thie measure and compare it with those given by the $T_{\infty+i}$, choose in a continuous way, for each such $\tau_{v'}$, one vertex on the horocycle of points of distance $\infty + i$ from w. Let $M(i)$ be the set of such vertices and, for any edge e adjacent to w as before, let $M_e(i)$ be the subset of $M(i)$ consisting of points w' such that $A(w, w')$ contains e. Then

$$\int_{M_e(i)} T_{\infty+i}(\delta_w) = 1/(q+1),$$

where the integral just means the value of a measure on a set. Therefore the coefficient at $T_{\infty+n+m-i}$ in the expansion of $T_{\infty+n}T_m$ is the integral

$$\int_{M_e(\infty+n+m-i)} T_{\infty+n}T_m(\delta_w).$$

Let us first find the coefficient at $T_{\infty+n+m}$. In order to get from w to a point at distance $\infty + n + m$ we should go first to distance m and then from there to distance $\infty + n$ by a path without repetitions. Let w' be the intermediate vertex of such path, $d(w, w') = m$. Consider the set of those points p of $\partial\tau_v$ which are farther away from w than w', i.e., $w' \in A(w, p)$. The $\mu_{w'}$-measure of this set is $q/(q+1)$. Further, for a chosen e as above, the set of such possible w' leading to points from $M_e(\infty + m + n)$ is q^{m-1}. Thus the first term of expansion is:

$$T_{\infty+n}T_m = q^m T_{\infty+n+m} + \ldots$$

Let us find the next term. It corresponds to paths with one repetition: we first go to from w a vertex w' on distance m and then from there to a vertex $w'' \in M_e(\infty + m + n - 2)$ by retracing exactly one step of the path $A(w, w')$, The number of all paths of length m starting from w and passing through e is q^{m-1}. If we want to retrace exactly one step of such path $A(w, w')$, then after this retracing we have exactly $q - 1$ possibilities for the next ramification. For each such ramification the $\mu_{w'}$-measure of the set of all points from $M_e(\infty + m + n - 2)$ reachable by going further without repetitions is $1/q(q+1)$. This implies that the coefficient at $T_{\infty+m+n-2}$ is $q^{m-1}(1 - 1/q) = q^{m-1} - q^{m-2}$.

The next coefficient corresponds to paths with exactly two repetitions. As before, the set of w' reachable by going to distance m through e, has cardinality q^{m-1} and the set of possible ramifications after retracing two last steps of the path $A(w, w')$ has cardinality $q-1$. The $\mu_{w'}$-measure of the set of points of $M_e(\infty+m+n-4)$ reachable by going further after a choice of that ramification is $1/q^2(q+1)$, so we get the coefficient at $T_{\infty+n+m-4}$ to be $q^{m-2} - q^{m-3}$ and so on.

At the end the pattern will be slightly different: in this case we have to retrace the entire initial path $A(w, w')$, and in order that the composite path reaches a point from $M_e(\infty + n - m)$, the initial path should <u>not</u> pass through e. the number of paths with this property is q^m, while for the end w' of any such path we have $\mu_{w'}(M_e) = 1/q^m$, so the last coefficient will be 1. Proposition 2.6.1 is proved.

Formulas (2.6.2) imply the following statement.

(2.6.5) Theorem. *The associated graded algebra* $\mathrm{gr}^F(\hat{\mathcal{H}})$ *is isomorphic to the Heisenberg algebra generated by symbols* a_1, a_∞ *which are subject to the relations*

$$a_\infty a_1 = q a_1 a_\infty.$$

(2.7) The full multiplication table in $\hat{\mathcal{H}}$**.** The multiplication in $\hat{\mathcal{H}}$ is completely described by the following proposition.

(2.7.1) Proposition. *If* $a, b > 0$*, then*
(2.7.2)

$$T_{a\infty+m}T_{b\infty+n} = q^{an}\left(T_{(a+b)\infty+(m+n)} + (q-1)\sum_{i=1}^{\infty} q^{(1-2a)i-1}T_{(a+b)\infty+m+n-2i}\right).$$

If $b > 0$*, then*
(2.7.3)

$$T_mT_{b\infty+n} = T_{b\infty+m+n} + (q-1)T_{b\infty+m+n-2} + (q^2-q)T_{b\infty+m+n-4} + \ldots + q^m T_{b\infty+n-m}.$$

If $a \neq 0$*, then*
(2.7.4)

$$T_{a\infty+m}T_n = q^{an}\left(T_{a\infty+(m+n)} + (q-1)\sum_{i=1}^{n-1} q^{(1-2a)i-1}T_{a\infty+m+n-2i}\right) + q^{(1-a)n}T_{a\infty+m-n}$$

The proof is similar to that of (2.6.1) and left to the reader: one has to take to account the change in the Poisson measures on the boundary of the microtree as well as in the normalized Lebesgue measures on the punctured boundaries of the adjacent microtrees. It may be not immediately obvious that the multiplication given by the proposition is associative, and it is a good exercise to check it, for instance, to verify that $(T_\infty T_1)T_\infty = T_\infty(T_1T_\infty)$. A little later we will give a more conceptual explanation of this associativity.

Let $\mathcal{H}^{rat} \subset \hat{\mathcal{H}}$ be the subspace consisting of series $\sum_{m=1}^{N}\sum_{i=-\infty}^{M} a_{mi}T_{m\infty+i}$ with the property that for each m the series $\sum a_{mi}z^i$ represents a rational function in z. Proposition 2.7.1 implies that:

Proposition 2.7.5. *The subspace* $\mathcal{H}^{rat}$ *is a subalgebra.*

Let us note another consequence of 2.7.1:

Proposition 2.7.6. *Any* $T_{m\infty+i}$ *can be expressed as a non-commutative rational function in* T_1, T_∞.

Proof: By Prop. 2.7.1, we have

$$(2.7.7)\quad T_1T_{m\infty+i} = T_{m\infty+i+1} + qT_{m\infty+i-1}, \quad T_{m\infty+i}T_1 = q^m T_{m\infty+i+1} + q^{1-m}T_{m\infty+i-1}.$$

Therefore

$$T_{m\infty+i+1} = \frac{T_{m\infty+i}T_1 - q^{-m}T_1T_{m\infty+i}}{q^m - q^{-m}}, \quad T_{m\infty-i} = \frac{-T_{m\infty+i}T_1 + q^mT_1T_{m\infty+i}}{q^{1+m} - q^{1-m}}.$$

So it is enough to express $T_{m\infty}$ as a rational function in T_1, T_∞. Let us show how to do this for $m = 2$, the argument in the general case being similar. By repeated application of (2.7.1), we have:

$$(2.7.8) \qquad T_\infty^2T_1 = q^2T_{2\infty+1} + (q - 1 + q^{-1})T_{2\infty-1} + (1 - q^{-1} + q^{-2} - q^{-3})T_{2\infty-3}+$$
$$+(q^{-1} - q^{-2} + q^{-3} - q^{-4})T_{2\infty-5} + \dots$$

$$(2.7.9) \qquad T_\infty T_1 T_\infty = qT_{2\infty+1} + (2 - q^{-1}T_{2\infty-1} + (2 - 2q^{-1})T_{2\infty-3}+$$
$$+(2q^{-1} - 2q^{-2})T_{2\infty-5} + \dots$$

$$(2.7.10) \qquad T_1T_\infty^2 = T_{2\infty+1} + (q + q^{-1} - q^{-2})T_{2\infty-1} + (1 - q^{-1} + q^{-2} - q^{-3})T_{2\infty-3}+$$
$$+(q^{-1} - q^2 + q^{-3} - q^{-4})T_{2\infty-5} + \dots$$

This implies:

$$(2.7.11) \qquad T_\infty^2T_1 - T_1T_\infty^2 = (q^2 + 1)T_{2\infty+1} - (1 + q^{-2})T_{2\infty-1}$$

(2.7.12)
$$T_1^2T_\infty - (1+q^{-2})T_\infty T_1T_\infty + T_1T_\infty^2 = (q^2 - q + 1 + q^{-1})T_{2\infty+1} + (2q - 3 + 3q^{-1} + q^{-3})T_{2\infty-1}.$$

Therefore there are numbers c_1, c_2, c_3 such that

$$c_1T_\infty^2T_1 + c_2T_\infty T_1T_\infty + c_3T_1T_\infty^2 = T_{2\infty+1}$$

as well as numbers c_1', c_2', c_3' such that

$$c_1'T_\infty^2T_1 + c_2'T_\infty T_1T_\infty + c_3'T_1T_\infty^2 = T_{2\infty-1}$$

Now, since $T_1T_{2\infty+i} = T_{2\infty+i+1} + qT_{2\infty+i-1}$, we can find $T_{2\infty}$.

$$T_1T_{2\infty} = qT_{2\infty+1} + T_{2\infty-1} = a_1T_\infty^2T_1 + a_2T_\infty T_1T_\infty + a_3T_1T_\infty^2$$

for some a_1, a_2, a_3, so

$$T_{2\infty} = T_1^{-1}(a_1T_\infty^2T_1 + a_2T_\infty T_1T_\infty + a_3T_1T_\infty^2).$$

Proposition is proved.

(2.8) Bimodules and correspondences (preliminaries). Before going further into study of $\hat{\mathcal{H}}$, some preliminary discussion is in order. Given any (possibly non-commutative) graded algebra $A = \bigoplus_{n\geq 0} A_n$, each A_m is an A_0-bimodule. So in order to describe the structure of A, it is enough to describe each A_m as an A_0-bimodule together with multiplication maps $A_m \otimes_{A_0} A_{m'} \to A_{m+m'}$.

Suppose that A_0 is commutative. Then an A_0-bimodule is just an $A_0 \otimes_{\mathbf{C}} A_0$-module and it can be visualized as a coherent sheaf on $\mathrm{Spec}(A_0) \times \mathrm{Spec}(A_0)$. Further, tensor product of bimodules translates geometrically into "convolution" of sheaves on the product similar to the composition of kernels of integral operators, or of correspondences. More precisely, let S be any affine scheme over $\mathbf{C}$ and $\mathcal{F}, \mathcal{F}'$ be quasicoherent sheaves on $S \times S$ and M, M' the corresponding $\mathbf{C}[S]$-bimodules. Denote by $p_{12}, p_{13}, p_{23} : S \times S \times S \to S \times S$ obvious projections. Then the convolution sheaf

$$\mathcal{F} * \mathcal{F}' = p_{13*}(p_{12}^*\mathcal{F} \otimes p_{23}^*\mathcal{F}')$$

corresponds to the bimodule $M \otimes_{\mathbf{C}[S]} M'$.

(2.8.1) Example: Let A be the standard Heisenberg algebra of polynomials in z, w with $wz = qzw$, graded by $\deg(z) = 0$, $\deg(w) = 1$. Then $A_0 = \mathbf{C}[z]$, so $A_0 \otimes A_0 = \mathbf{C}[z', z'']$ where z' stands for the left action of z and z'' for the right action. So $\mathrm{Spec}(A_0 \otimes A_0)$ is the affine plane with coordinates z', z'' and A_m, as coherent sheaf on this plane, is just the structure sheaf of the line $z'' = q^m z'$. This line is the graph of the multiplication by q^m as a map from $\mathbf{C}$ to itself, and the tensor product of $A_m \otimes_{A_0} A_{m'}$ is the graph of the composition of such maps, so it is naturally identified with $A_{m+m'}$.

(2.9) Structure of the $\hat{\mathcal{H}}$ as bimodule over $\mathcal{H}_0$. We now apply the considerations of (2.8) to $\hat{\mathcal{H}}$. Let $\mathcal{H}'_m \subset \hat{\mathcal{H}}_m$ be the direct sum of $\mathbf{C} \cdot T_{m\infty+n}$ for all $n \in \mathbf{Z}$. Equivalently, $\mathcal{H}'_m$ is the sub-$\mathcal{H}_0$-bimodule in $\hat{\mathcal{H}}$ generated by $T_{m\infty}$. The graded component $\hat{\mathcal{H}}_m$ is just the completion of $\mathcal{H}'_m$, so we describe $\mathcal{H}'_m$ as a bimodule. By (2.7.3) we have that as a left $\mathcal{H}_0$-module, each $\mathcal{H}'_m$, $m > 0$, is isomorphic to M, the "stabilization of the Hecke algebra" from §1. To describe both right and left module structures, write $t' = T_1 \otimes 1$, $t'' = 1 \otimes T_1$ for the generators of $\mathcal{H}_0 \otimes \mathcal{H}_0$. Denote for short $T_{\infty+i}$ by e_i. Then the action of t', t'' on the basis vectors is found from (2.7.7) to be

$$\begin{cases} t'T_{m\infty+i} = T_{m\infty+i+1} + qT_{m\infty+i-1} \\ t''T_{m\infty+i} = q^m T_{m\infty+i+1} + q^{1-m} T_{m\infty+i-1} \end{cases} \tag{2.9.1}$$

Let z be the shift operator in $\mathcal{H}'_m$, namely $zT_{m\infty+i} = T_{m\infty+i+1}$. Then $\mathcal{H}'_m$ is a free $\mathbf{C}[z, z^{-1}]$-module of rank 1, and the action of t', t'' can be written as

$$\begin{cases} t' = z + qz^{-1} \\ t'' = q^m z + q^{1-m} z^{-1} \end{cases} \tag{2.9.2}$$

Now this equation can be seen as defining a parametrized curve $C(m)$ in the affine plane with coordinates t', t'' (so z is a parameter on the curve). This curve is the support of our bimodule. It is a hyperbola with asymptotas $t'' = q^{\pm m}t'$. This fact for $m = 1$ means that there are two ways of filtering the algebra generated by T_1, T_∞ so as to get the Heisenberg algebra $ba = qab$. For instance, $C(1)$ has the equation

$$(t' - qt'')(qt' - t'') = -(q^2 - 1)^2 \tag{2.9.3}$$

or, in the developed form,

$$qt'^2 - (q^2 + 1)t't'' + qt''^2 = -(q^2 - 1)^2. \tag{2.9.4}$$

Summarizing, we have the the following.

(2.9.5) Proposition. *The coherent oherent sheaf on* $\operatorname{Spec} \mathbf{C}[t', t'']$ *corresponding to the* $\mathcal{H}_0$*-bimodule* $\mathcal{H}'_m$ *is the structure sheaf of the curve* $C(m)$*. For instance,* $\mathcal{H}_1$ *is an* $\mathcal{H}_0$*-bimodule with one generator* T_∞ *and one relation*

$$qT_1^2T_\infty - (q^2 + 1)T_1T_\infty T_1 + qT_\infty T_1^2 = -(q^2 - 1)^2 T_\infty. \tag{2.9.6}$$

This relation is quite remarkable: it looks like the Serre relation in the quantum enveloping algebra $U_q(sl_3)$, but is inhomogeneous (has right hand side).

The group-theoretical meaning of $C(m)$ is as follows. Let $T^\vee \simeq \mathbf{C}^*$ be the maximal torus in $SL_2(\mathbf{C})$, the Langlands dual group of PGL_2, and let z be the coordinate in $T^\vee$. Denoting by $W = \{1, \sigma\}$ the Weyl group of SL_2, we have the Satake isomorphism $\mathcal{H}_0 = \mathbf{C}[T^\vee/W]$ where $\sigma \in W$ acts by $\sigma(z) = qz^{-1}$. This isomorphism is just read off the left $\mathcal{H}_0$-action on $\mathcal{H}_1$. Let $p : T^\vee \to T^\vee/W$ be the natural projection. For any $m \in \mathbf{Z}$ let $\tilde{C}(m) \subset T^\vee \times T^\vee$ be the shifted diagonal consisting of $(z, q^{-m}z)$. Then

$$C(m) = (p \times p)(\tilde{C}(m)) \subset (T^\vee/W) \times (T^\vee/W). \tag{2.9.7}$$

(2.10) The Heisenberg algebra $\mathcal{A}$. We now compare $\tilde{\mathcal{H}}$ with a simpler algebra $\mathcal{A}$. Note that there are natural isomorphisms

$$\tilde{\kappa}_{m,m'} : \mathcal{O}_{\tilde{C}(m)} * \mathcal{O}_{\tilde{C}(m')} \xrightarrow{\simeq} \mathcal{O}_{\tilde{C}(m+m')}, \quad m, m' \in \mathbf{Z}, \tag{2.10.1}$$

similar to Example 2.8.1. The corresponding algebra

$$\tilde{\mathcal{A}} = \bigoplus_{m \in \mathbf{Z}} \Gamma((T^\vee/W) \times (T^\vee)/W, \mathcal{O}_{\tilde{C}(m)})$$

is nothing but the Heisenberg algebra generated by $z^{\pm 1}, w^{\pm 1}$ with $wz = qzw$, and $\deg(z) = 0$, $\deg(w) = 1$. Let

$$\mathcal{A} = \bigoplus_{m \geq 0} \Gamma(T^\vee \times T^\vee, \mathcal{O}_{C(m)}).$$

Then $\mathcal{A}$ is also an algebra, the composition maps

$$\kappa_{m,m'} : \mathcal{O}_{C(m)} * \mathcal{O}_{C(m')} \to \mathcal{O}_{C(m+m')}$$

being induced by the $\tilde{\kappa}_{m,m'}$. More precisely, for $m > 0$ the projection $(p \times p) : \tilde{C}(m) \to C(m)$ is an isomorphism, so $\mathcal{A}_m = \tilde{\mathcal{A}}_m$, while $\mathcal{A}_0 = \tilde{\mathcal{A}}_0^W = \mathbf{C}[z, z^{-1}]^W$. Thus $\mathcal{A}$ is a subalgebra in $\mathcal{A}$.

Let $\tilde{\mathcal{A}}^{rat}$ denote the extension of $\tilde{\mathcal{A}}$ consisting of finite sums $\sum_{i \in \mathbf{Z}} \phi_i(z) w^i$ where each ϕ_i is a rational function in z, with the commutation rule given by $w\phi(z) = \phi(qz)w$. In $\tilde{\mathcal{A}}$ each ϕ_i is a Laurent polynomial. Let $\mathcal{A}^{rat}$ be the subalgebra of $\tilde{\mathcal{A}}^{rat}$ consisting of sums $\sum_{i \geq 0} \phi_i(z) w^i$ in non-negative powers of w in which ϕ_0 is actually a W-symmetric Laurent polynomial.

(2.11) Theorem. *The algebra $\mathcal{H}^{rat}$ is isomorphic to $\mathcal{A}^{rat}$.*

Proof: We identify $\mathcal{H}_0 \simeq \mathcal{A}_0^{rat} = \mathbf{C}[z, z^{-1}]^W$, with $T_1 \mapsto z + qz^{-1}$. Further, we have an isomorphisms of $\mathcal{H}_0$-bimodules $\psi_a : \mathcal{H}'_a \simeq \mathcal{A}_a$ which just takes $T_{a\infty+m} \mapsto z^m w^a$. However, these isomorphisms do not agree with the products $\mathcal{H}'_a \otimes_{\mathcal{H}_0} \mathcal{H}'_b \to \mathcal{H}^{rat}_{a+b}$. Let us denote by $\circ$ the new product in $\mathcal{A}^{rat}$ induced by the product in $\mathcal{H}^{rat}$, via the identifications ψ_a. From Proposition 2.7.1 we find that for $a, b > 0$ we have

$$(z^m w^a) \circ (z^n w^b) = R(q^a z) z^m w^a z^n w^b, \quad R(z) = 1 + (q-1) \sum_{i=1}^{\infty} q^{i-1} z^{-2i} = \frac{z^2 - 1}{z^2 - q},$$

while for $a = 0$ or $b = 0$ we have

$$(z^m w^a) \circ (z^n w^b) = z^m w^a z^n w^b.$$

Define now an isomorphism $\chi : \mathcal{A}^{rat} \to \mathcal{A}^{rat}$ of graded vector spaces as follows. On $\mathcal{A}_0^{rat}$ it is the identity. If $m > 0$, and $\alpha \in \mathcal{A}_m^{rat}$, then we define $\chi(\alpha) = R(z)^{-1}\alpha$. It follows that $\chi(\alpha\beta) = \chi(\alpha) \circ \chi(\beta)$. In other words, χ gives an algebra isomorphism $\mathcal{A}^{rat} \to \mathcal{H}^{rat}$. Theorem is proved.

§3. Bruhat-Tits buidings and Hecke algebras for arbitrary G.

(3.1) Notations. As in (1.1), we let F be a complete discrete valued field with ring of integers $\mathcal{O} = \mathcal{O}_F$ and residue field k. We denote by x a uniformising element of F. For an affine algebraic variety Z over K, a subset $Y \subset Z(F)$ is called bounded if the set $\{\mathrm{ord}(f(z)), z \in z(F)\}$ is bounded from below. When the residue field k is finite, being bounded is the same as being compact.

Let G be a split semisimple algebraic group over F. We introduce the standard paraphernalia related to G, see, e.g., [Sp]. Thus:

$T \subset G$ is a split maximal torus, $B \supset T$ a Borel subgroup, $N = [B, B]$ the maximal unipotent subgroup of B.

$X = X(T), X^\vee = X^\vee(T)$ are the lattices of characters of T (weights) and and 1-parameter subgroups of T (coweights) respectively. We denote $\mathbf{h}_\mathbf{R} = X^\vee \otimes \mathbf{R}$. For a lattice L we denote $L^\vee = \mathrm{Hom}(L, \mathbf{Z})$.

$\Delta \subset X$ is the root system of G. By $\Delta_{sim} \subset \Delta_+ \subset \Delta$ we denote the systems of simple roots, positive roots. Positive roots are the roots of the Lie algebra of B. We choose a numeration of simple roots:$\Delta_{sim} = \{\alpha_1, ..., \alpha_l\}$.

$\Delta^\vee \subset X^\vee$ is the system of coroots. For $\alpha \in \Delta$ we denote by $\alpha^\vee$ the corresponding coroot, and denote $\Delta^\vee_+, \Delta^\vee_{sim}$ the set of $\alpha^\vee$ for positive (or simple) roots α.

$Y \subset X$ is the lattice generated by Δ, and $Z \subset X^\vee$ is the lattice generated by $\Delta^\vee$. We can regard $Y^\vee$ as a lattice in $\mathbf{h}_\mathbf{R}$. Thus X/Y is the character group of the center of G while $X^\vee/Z$ is the fundamental group of G.

$\tilde{G}$ is the universal cover of G and G^{ad} is the adjoint group of G, so we have maps $\tilde{G} \to G \to G^{ad}$. By T^{ad} we denote the maximal torus in G^{ad} which is the image of T under the last map. Note that the lattice of characterts of T^{ad} is Y.

$X_+ \subset X, X^\vee_+ \subset X^\vee$ are the cones of dominant weights and coweights, i.e., those weights or coweights whose scalar product with each positive coroot or root is non-negative. Similarly for $Y_+, Y^\vee_+$ etc. We denote by ϵ_i the fundamental coweights, characterized by the condition that $(\alpha_i, \epsilon_j) = \delta_{ij}$. They form a semigroup basis of $Y^\vee_+$.

For a subset $I \subset \{1, ..., l\}$ we denote by P^I the standard parabolic subgroup corresponding to I. Its Lie algebra is generated by the Chevalley generators corresponding to all the positive roots and the negative roots $(-\alpha_i), i \in I$. For $i \in \{1, ..., l\}$ we denote the maximal parabolic subgroup $P^{\{1,...,l\}-\{i\}}$ simply by P_i. We denote P_i^{ad} the image of P_i in G^{ad}.

Not all maximal bounded subgroups of G are conjugate. More precisely, the group G^{ad} has only one conjugacy class of maximal bounded subgroups, while for $\tilde{G}$ this number equals $\prod(l_i + 1)$ where l_i are the ranks of quasi-simple factors of $\tilde{G}$. We will denote by K the bounded subgroup $G(\mathcal{O})$. Let $I \subset K$ be the standard Iwahori subgroup, i.e., the preimage, under the natural surjection $K = G(\mathcal{O}) \to G(k)$ of the Borel subgroup in $G(k)$. By an Iwahori subgroup we mean any subgroup conjugate to I; a parahoric subgroup is, by definition, a bounded subgroup containing an Iwahori subgroup. It is known that maximal parahoric subgroups are the same as maximal bounded subgroups.

(3.2) The (affine) building. To every G as before there is associated a natural cell complex $\mathcal{B}(G) = \mathcal{B}(G, F)$ with G-action known as the (affine) Bruhat-Tits building of G. It depends on G, F and the local field structure on F. Let us recall briefly its main properties which may be used to characterize it uniquely, see [BT] [Br] [Ro] for more details.

(3.2.1) $\mathcal{B}(G)$ is a contractible l-dimensional cell complex with $G(F)$-action whose ertices are in bijection with maximal bounded subgroups in $G(F)$ while cells of arbitrary dimension are in bijection with parahoric subgroups. In particular, maximal (l-dimensional) cells correspond to Iwahori subgroups. We denote by $\sigma(P)$ the simplex corresponding to a subgroup P.

(3.2.2) If G is quasi-simple (has no normal subgroups of positive dimension), then $\mathcal{B}(G)$ is a simplicial complex.

(3.2.3) For the product of two groups we have $\mathcal{B}(G_1 \times G_2) = \mathcal{B}(G_1) \times \mathcal{B}(G_2)$ (the product of cell complexes).

(3.2.4) As cell complex, $\mathcal{B}(G) = \mathcal{B}(G^{ad})$ depends only on G^{ad} (this is because there is a bijection between parahoric subgroups in $G(F)$ and $G^{ad}(F)$).

When G and F are fixed, we will denote the bulding $\mathcal{B}(G, F)$ just by $\mathcal{B}$. Note that the action of $G(F)$ on the maximal cells of $\mathcal{B}$ is always transitive while the action on cells of smaller dimension, e.g., on vertices, may be not. However, the action of $G^{ad}(F)$ is transitive on vertices.

(3.3) Apartments. To any choice of a split F-torus H in G there corresponds an *apartment* $A(H) \subset \mathcal{B}$ This is a subcomplex homeomorphic to the Euclidean space $\mathbf{R}^l$. More precisely, $A(H)$ has a natural structure of an affine space over $\mathbf{h}_\mathbf{R}$ with the subdivision given by the alcoves of the affine Weyl group of G. We will use the Killing form on $\mathbf{h}_\mathbf{R}$ to make $A(H)$ into a Euclidean affine space. It is known that for any two cells σ, σ' of $T(G)$ there always is an apartment containing them, any any two apartments with this property can be taken into each other by an element of G preserving σ, σ'.

(3.4) Spherical buildings. Links and the boundary of the affine building. For any field L (without any local field structure) we denote by $\Sigma(G, L)$ the spherical Bruhat-Tits building associated to G and L. It is defined in almost the same way as the affine building, only instead of parahoric subgroups one considers parabolic (in the usual sense) subgroups in $G(L)$. With this modification, the analogs of the properties (3.2.1-4) are all true with the exception that $\Sigma(G, L)$ is not contractible but rather is homotopy equivalent to a wedge of spheres.

For any cell complex C and any cell $\sigma \in C$ of dimension d the link $\mathrm{Lk}(\sigma/C)$ is the cell complex whose i-dimensional cells are in bijection with $(i+d+1)$-dimensional cells of C containing σ, with the same closure relation. The links in the affine building are described as follows.

(3.4.1) Proposition. *If P is a parahoric subgroup in $G(F)$ and $\sigma = \sigma(P)$ is the correspoding cell in the affine building $\mathcal{B}$, then $\mathrm{Lk}(\sigma/\mathcal{B})$ is identified with the spherical building $\Sigma(\bar{P}, k)$ where $\bar{P}$ is the semisimplification of the reduction of P modulo the maximal ideal of $\mathcal{O}$. In particular, for the vertex associated to the standard maximal bounded subgroup K the link is isomorphic to $\Sigma(G, k)$.*

This generalises property (1.1.2) that the set of edges of a Bruhat-Tits tree incident to a given vertex is a projective line over the residue field k. Note, in particular, that for each vertex v the edges coming out of v are subdivided into l types, one for each conjugacy class of a maximal parabolic subgroup in $G^{ad}(k)$. These conjugacy classes are labelled by simple roots, namely to a root α_i there corresponds the maximal parabolic subgroup P_i, see (3.1). We will refer to edges (coming out of x) corresponding to P_i as edges of type i. Note that this concept depends not just on the edge itself but also on the choice of the "beginning" x. More precisely, there is a well-defined involution $i \mapsto \bar{i}$ on $\{1, ..., l\}$ such that if (x, y) is of type i, then (y, x) is of type $\bar{i}$. This involution is an automorphism of the Dynkin diagram. For example, for the group PGL_n the situation is as follows. If we number the simple roots in the linear order with respect to the Dynkin diagram A_{n-1}, then $\bar{i} = n - 1 - i$.

We now briefly recall how the boundary $\partial\mathcal{B}$ is defined, see [Br] for more details. One calls a *ray* in $\mathcal{B}$ a subset $\mathbf{r} \subset \mathcal{B}$ which lies in some apartment A and represents a linearly embedded half-line $[0, \infty]$ with respect to the Euclidean structure of A. (In this case same statement will hold for any apartment containing $\mathbf{r}$. Two rays $\mathbf{r}, \mathbf{r}'$ are called parallel if they lie in a common apartment A and are parallel there (represent the same point on the sphere at the infinity of A). Being parallel is an equivalence relation, and equivalence classes are called *ideal points* of $\mathcal{B}$. Their set is denoted $\partial\mathcal{B}$. An ideal simplex in $\partial\mathcal{B}$ is a set obtained as follows. Take any apartment $A \subset \mathcal{B}$, any vertex $v \in A$, any cell σ containing v and form the cone with apex v by drawing all straight (with respect to the Euclidean structure on A) half-lines starting from v and passing through points of σ. Cones of this type are called conical cells in $\mathcal{B}$. The set of points on the sphere at the infinity of A represented by a conical cell is called an ideal cell of $\partial\mathcal{B}$. Now the generalization of the property (1.1.3) is as follows.

(3.4.2) Proposition. *The boundary $\partial\mathcal{B}(G, F)$ with the decomposition into ideal cells described above, is naturally identified with the spherical building $\Sigma(G, F)$. The apartments in $\mathcal{B}(G, F)$ and $\partial\mathcal{B}(G, F) = \Sigma(G, F)$ are in natural bijection.*

Note the particular case when the residue field k is finite. Then the affine building $\mathcal{B}$ is a locally compact CW-complex, with each link being a finite CW-complex. Stabilizers of vertices of $\mathcal{B}$ are all the maximal compact subgroups in $G(F)$. The set of cells (of all dimensions) of the spherical building $\Sigma(G,F)$ is just the disjoint union of $G(F)/P(F)$ for all conjugacy classes of parabolic subgroups $P \subset G$. In particular, this set has a natural topology induced by the valuation topology of F, with respect to which it is compact. Further, the spherical building $\Sigma(G,K)$ has a natural topology which mixes the standard topology on cells and the compact completely disconnected topology on the union of the G/P, see [Br]. With this topology $\Sigma(G,K)$ is compact. There is a natural way to topologize the union $\overline{\mathcal{B}} = \mathcal{B} \coprod \partial\mathcal{B}$ which is also compact, see Borel [Bo].

(3.5) Distances. Similarly to the case of Bruhat-Tits trees, for any two vertices $v, v' \in \mathcal{B}$ there is a well-defined "distance" $d(v,v')$ which is, however, not an integer but an element of $Y^\vee/W$. (Recall that $Y^\vee$ is the lattice of 1-parameter subgroups in T^{ad}, the maximal torus of G^{ad}.) Namely, the set of vertices of $\mathcal{B}$ is $G^{ad}(F)/K$, and we have the Iwasawa decomposition $G^{ad} = KT^{ad}N$, where N is the commutant of the Borel subgroup. Hence $K\backslash G^{ad}/K = X^\vee(T')/W$. The distance satisfies the invariance property $d(gv,gv') = d(v,v')$ for any $g \in Ad(G)$ and the following analog of the triangle inequality:

$$d(v,v'') \subset \mathrm{Conv}\Big(W \cdot (d(v,v') + d(v',v''))\Big).$$

We can, if we want, identify $Y^\vee/W$ with $Y_+^\vee$, the cone of dominant integer coweights of G^{ad}. For every $r \in Y^\vee$ we denote $r_+ \in Y_+^\vee$ the unique representative of the W-orbit Wr lying in $Y_+^\vee$.

Note the case when v' is a vertex joined to v by an edge. In this case this edge has one of l types, see above; if the type is i, then $d(v,v')$ is the fundamental coweight ϵ_i associated to the simple root α_i.

For every $r \in Y_+^\vee/W$ and any vertex $v \in \mathcal{B}$ we denote by $S_r(v)$ the "sphere" of radius r and center v, i.e.,

$$S_r(x) = \{v' : d(v,v') = r\}.$$

Given two vertices $v, v' \in \mathcal{B}$, we denote by $A(v,v')$ the intersection of all the apartments containing v, v'. This is the analog of the shortest path between two vertices on a tree. In our case $A(v,v')$ is a finite cell subcomplex in $\mathcal{B}$ which may have any dimension between 0 (when $v = v'$) and $l = \mathrm{rk}(G)$.

(3.5.1) Proposition. *Let $d(v,v') = \sum_{i=1}^l m_i\epsilon_i, m_i \geq 0$. Let A be any apartment containing v and v'. Then $A(v,v') \subset A$ is, with respect to the affine structure on A, a parallelotope, namely a translation of the following parallelotope in $\mathbf{h}_\mathbf{R}$:*

$$\left\{\sum_{i=1}^{l} \lambda_i\epsilon_i,\ \ 0 \leq \lambda_i \leq m_i\right\}.$$

In particular,
(a) The dimension of $A(v,v')$ is equal to l minus the number of simple roots vanishing on

$d(v, v')$.

(b) Consider all edge paths in $\mathcal{B}$ joining x and y and having minimal possible length. The set of vertices of $A(v, v')$ is the union of the sets of vertices of such minimal paths.

(c) If $v_0 = x, v_1, ..., v_N = v'$ is any minimal edge path joining v and v', then

$$d(x, y) = \sum_{i=0}^{N-1} \epsilon_{m(v_i, v_{i+1})},$$

where $m(v_i, v_{i+1}) \in \{1, ..., l\}$ is the type of the edge $[v_i, v_{i+1}]$.

(d) If $\dim(A(v, v')) = d$, then there exist exactly one d-dimensional cell of $A(v, v')$ containing v (resp. v').

The reader may consult [Ko] for a more thorough discussion of the example $G = PGL_3$.

We will denote the d-dimensional cell of $A(v, v')$ containing v, by $\mathrm{dir}(v, v')$ and call it the *direction* to v' from v. The d-dimensional cell of $A(v, v')$ containing v', will be denoted by $\mathrm{codir}(v, v') = \mathrm{dir}(v', v)$ and called the codirection from v to v'.

(3.5.2) Proposition. *Let v, v', v'' be three vertices of $\mathcal{B}$. Then the following conditions are equivalent:*

(i) $d(v, v') + d(v', v'') = d(v, v'')$.

(ii) The cells of the spherical building $\mathrm{Lk}(v'/\mathcal{B})$ corresponding to $\mathrm{codir}(v, v')$ and $\mathrm{dir}(v', v'')$, are in generic position.

(iii) v, v', v'' lie in a common apartment and $A(v, v') \cap A(v', v'') = \{v'\}$.

Let us explain the meaning of (ii). The set of cells of $\mathrm{Lk}(v'/\mathcal{B})$ of any given type is a flag variety $G(k)/P^I(k)$ over the residue field k. One says that two points $a \in G(k)/P^I(k)$ and $b \in G(k)/P^J(k)$ are in generic position, if they lie in the unique open orbit of G on $(G/P^I) \times (G/P^J)$.

The parallelotope $A(v, v')$ is the analog of the finite edge path joning two vertices of a Bruhat-Tits tree. We will need the analogs of (semi)infinite paths as well. Namely, if σ, τ are cells of the spherical building $\partial\mathcal{B}$, we denote by $A(\sigma, \tau) \subset \mathcal{B}$ the intersection of all the apartments containing σ, τ at the infinity. It is always an affine subspace in an apartment. For example, if σ, τ are maximal cells in generic position, then $A(\sigma, \tau)$ is an apartment (two complete flags in generic position determine a unique maximal torus). If v is a vertex of $\mathcal{B}$ and σ is a cell in $\partial\mathcal{B}$, then we define $A(v, \sigma)$ as the intersection of all the apartments containing x, σ. It is always a conical cell.

(3.5.3) Proposition. *Let $v, v' \in \mathcal{B}$ be two vertices and $K_{v,v'} \subset G(F)$ be the subgroup fixing both v and v'. Then the image of $K_{v,v'}$ in $\mathrm{Aut}(\mathrm{Lk}(v/\mathcal{B}))$ is the parabolic subgroup fixing the cell $\mathrm{dir}(v, v')$. Similarly, the image in $\mathrm{Aut}(\mathrm{Lk}(v'/\mathcal{B}))$ is the parabolic subgroup fixing $\mathrm{codir}(v, v')$.*

(3.6) Horocycles and mixed horocycles. A sphere in $\mathcal{B}$ can be defined as an orbit of a subgroup in $G^{ad}(F)$ conjugate to K^{ad}. Similarly, we call a horocycle an orbit of a subgroup conjugate to the unipotent subgroup N. Thus to specify a horocycle, one has first to specify a subgroup N' conjugate to N and second, an orbit of this subgroup. The set of all subgroups conjugate to N is nothing but the full flag variety $G(F)/B(F)$, and we think of the point of $G(F)/B(F)$ corresponding to N' as the *center* of the horocycle. In terms of $\mathcal{B}$ the center of a horocycle is just a maximal cell of the spherical building $\partial\mathcal{B}$.

Having fixed such a cell b (or, equivalently, asubgroup N'), we will distinguish various orbits of N' (i.e., horocycles with center b) by their "radii" which are elements of a certain $Y^\vee$-torsor $D(b)$. This torsor is obtained from the fiber of the projection $G/N \to G/B$ over b (this fiber is an T^{ad}-torsor) by quotienting by the maximal compact subgroup in T^{ad}.

In particular, for any $b \in G/B$ and any two vertices $v, v' \in \mathrm{Vert}(\mathcal{B})$ we have a well defined difference of the (infinite) distances from v and v' to b. This difference is an element of $Y^\vee$ and denoted by $d(v,b) - d(v,b)$, where $d(v,b), d(v',b)$ are elements of $D(b)$.

We will also need objects interpolating between spheres and horocycles (see [Kar] for the discussion of the archimedean case). More precisely, let $P \subset G^{ad}$ be any parabolic subgroup (not necessarily a standard one) and $N_P \subset P$ its unipotent radical. A *mixed horocycle* of type P is by definition, an orbit of a subgroup of the form $\tau^{-1}(K_p)$ where $K_P \subset P/N$ is any maximal bounded subgroup and $\tau : P \to P/N$ is the natural projection. Note that such K_P gives rise to a maximal bounded subgroup in any Levi complement to N_P. . Note also that we do not exclude here the case when $P = G^{ad}$ in which case a mixed horocycle is just a sphere.

Thus the "center" of a mixed horocycle is a pair (P, K_P). We prefer to encode this data geometrically in terms of the building. Namely, P corresponds just to a cell $\sigma \subset \partial\mathcal{B}$ of arbitrary dimension (the case of an honest sphere is obtained when $\sigma = \emptyset$). Further, a choice of $K_P \subset P/N_P$ is just a choice of a vertex of the Euclidean building $\mathcal{B}(P/N_P)$ associated to the semisimple group P/N_P over F. This Euclidean building has, as boundary, the link $\mathrm{Lk}(\sigma/\partial\mathcal{B})$. In terms of $\mathcal{B}$ itself, $\sigma \subset \partial\mathcal{B}$ is represented as an ideal cell, i.e., the part at infinity of a conical cell (sector) $\tilde{\sigma}$ in an apartment in $\mathcal{B}$. Two conical cells C, C' define the same ideal cell if they are parallel in the obvious sense. There is a trivial possibility to achieve this: take two conical cells in the same apartment which represent the same region at the sphere at the infinity but whose apexes are different. In this case $C \cap C'$ will be nonempty. We will say that two parallel conical cells C, C' are essentially different if $C \cap C' = \emptyset$. A choice of a vertex in $\mathcal{B}(P/N_P)$ is nothing but a choice of a conical cell C in the given class of parallel conical cells, whereby we distinguish only essentially different cells. Let us summarize this discussion as follows.

(3.6.1) Proposition. *Any conical cell $C \subset \mathcal{B}$ determines a center of a mixed horocycle. Two conical cells C, C' determine the same center if and only if they lie in a common apartment A and can be obtained from each other by an affine translation (with respect to the affine structure on A).*

In particular, vertices of $\mathcal{B}$ are conical cells, and the corresponding mixed horocycles are just spheres.

Having described what is a center C of a mixed horocycle, we go on to give a geometric interpretation of the radius as an element of some torsor. First, any parabolic subgroup has a type $I \subset \{1, ..., l\}$, i.e., it is conjugate to the standard parabolic subgroup P^I, see (3.1). Let

$$Y_I^\vee = \{\gamma \in Y^\vee : (\gamma, \alpha_i) = 0, i \notin I\} = \bigoplus_{i \in I} \mathbf{Z}\epsilon_i, \quad Y^\vee/I = \bigoplus_{i \notin I} \mathbf{Z}\epsilon_i \simeq Y^\vee/Y_I^\vee.$$

Let also $W_I \subset W$ be the Weyl group of G^I, the semisimplification of P^I, i.e., the subgroup generated by the simple reflections corresponding to $\alpha_i, i \in I$. It acts on $Y_I^\vee$. Let also ∂C be the cell in the spherical building $\partial \mathcal{B}$ represented by C. We can think of ∂C as an element of the generalized flag variety $G(F)/P^I(F)$. There is a principal fibration $G/P^I \to G/N^I$ with structure group G^I. Pass to the induced fibration with structure group being the torus $G_{ab}^I = G^I/[G^I, G^I]$. Note that the quotient of $G_{ab}^I(F)$ by the maximal bounded subgroup is a lattice naturally identified with $Y^\vee/I$. So by taking a further associated bundle, we get an $Y^\vee/I$-torsor over $G(F)/P^I(F)$ which we denote by D^∞. Its fiber over a point p will be denoted $D^\infty(p)$. Now define, for a conical cell C of type I,

$$D(C) = (Y_I^\vee/W_I) \times D^\infty(\partial C), \tag{3.6.2}$$

where we regard ∂C as a point of $G(F)/P^I(F)$.

(3.6.3) Proposition. *For a spherical cell C the set of possible mixed horocycles with center C is identified with $D(C)$.*

An element of $D(C)$ will be written as $r = (r', r'')$ according to the decomposition (3.6.2). For $r \in D(C)$ we write $S_r(C)$ for the horocycle with center C and radius r. Given C, any vertex $x \in \mathcal{B}$ lies on a unique horocycle $S_r(C)$. The corresponding value of r will be called the *distance* from x to C and denoted $d(x, C)$. We will write $d'(x, C) \in Y_I^\vee/W_I$ and $d''(x, C) \in D^\infty(\partial C)$ for the components of $d(x, C)$.

(3.6.4) Proposition. *Let $C \subset \mathcal{B}$ be a conical cell such that $\sigma = \partial C \subset \Sigma$ is invariant with respect to the standard torus $H_0 = T(F)$. Then every mixed horocycle with center C meets the standard apartment $A(H_0)$. Moreover, if σ is of type $I \subset \{1, ..., l\}$, the there is a unique intersection point of the form $x^a v_0$, where $a \in Y^\vee$ is I-dominant, i.e., $(\alpha_i, a) \geq 0$ for $i \in I$.*

This follows from the Iwasawa decomposition.

(3.7) Hecke operators. Assume now that the residue field of F is finite, of q elements. Then $G(F)$ is locally compact and has the Haar measure normalized by the requirement that K has measure 1. The Hecke algebra $\mathcal{H}_0(G)$ is defined to be the algebra of compactly supported doubly K-invariant functions on G with the operation given by the convolution. Since $K \backslash G/K \simeq X^\vee/W \simeq X_+^\vee$, a $\mathbf{C}$-basis of $\mathcal{H}_0(G)$ is formed by elements T_r, $r \in X_+^\vee$ which are just the characteristic functions of the corresponding double cosets.

Let $\mathcal{S}$ be the space of all functions on Vert($\mathcal{B}$) with finite support. The algebra $\mathcal{H}_0(G)$ can be realized as the algebra of operators in $\mathcal{S}$ with T_r being represented as the averaging operator

$$(T_r f)(v) = \sum_{v':d(v,v')=r} f(v'). \tag{3.7.1}$$

The operators T_r are known to commute with each other and form a polynomial algebra H with l generators. Note that all possible distances between vertices of $\mathcal{B}$ are given by elements of $Y_+^\vee \supset X_+^\vee$ so $\mathcal{H}_0(G)$ is a subalgebra in $\mathcal{H}_0(G^{ad})$ with the basis of that bigger algebra formed by $T_r, r \in Y_+^\vee$. In the treatment of questions related to Hecke algebras it is often convenient to treat the case of an adjoint group first and then specialize to the subalgebra $\mathcal{H}_0(G) \subset \mathcal{H}_0(G^{ad})$.

For instance, a system of polynomial generators of $\mathcal{H}_0(G^{ad})$ is given by the operators T_{ϵ_m}, $m = 1, ..., l$ corresponding to the fundamental coweights (3.1). Moreover, the sphere $S_{\epsilon_m}(x)$ with any center x is the set of $\mathbf{F}_q$-points of a generalized Grassmannian variety G/P_m. Thus the formula for the product $T_{\epsilon_m} T_r$ can be obtained from the decomposition of $(G/P_m)(\mathbf{F}_q)$ into Schubert cells. More precisely, let $W_m = W_{P_m} \subset W$ be the Weyl group of the Levi subgroup of P_m. It is nothing but the stabilizer of ϵ_m. The set of Schubert cells in G/P_m is identified with W/W_m. Let lt_m be the corresponding length function on W/W_m (giving the dimension of the corresponding Schubert cell). Then

$$T_{\epsilon_m} T_\alpha = T_\alpha T_{\epsilon_m} = \sum_{w \in W/W_m} q^{\mathrm{lt}_m(w)} T_{(\alpha + w(\epsilon_m))_+}. \tag{3.7.2}$$

Here the subscript "+" means the dominant representative in the W-orbit of a vector. Note that if α is dominant enough, then $r + w(\epsilon_m)$ is already dominant so nothing should be done with it. The equalities (3.7.2) completely describe $\mathcal{H}_0(G^{ad})$. The subalgebra $\mathcal{H}_0(G)$ for a non necessarily adjoint G does not possess such a simple multiplication table but can be analyzed using (3.7.2).

(3.8) Stabilizations of the Hecke algebra and the Satake isomorphism. Formula (3.7.2) implies the following.

(3.8.1) Proposition. *Given G and any basis vector $T_r \in \mathcal{H}_0(G)$, $r \in Y_+^\vee$, there exist numbers $c_{r,\beta}, \beta \in X^\vee$, almost all zero, such that whenever α is dominant enough (compared to r), then*

$$T_r T_\alpha = \sum_\beta c_{r\beta} T_{r+\beta}. \tag{3.8.2}$$

In other words, the multiplication by T_r eventually acts as a difference operator. We now define the $\mathcal{H}_0(G)$-module $\mathcal{M}(G)$ to be the $\mathbf{C}$-vector space with basis t_α for all $\alpha \in X^\vee$ and the action of T_r given by

$$T_r t_\alpha = \sum_\beta c_{r\beta} t_{r+\beta}. \tag{3.8.3}$$

The fact that $\mathcal{H}_0(G)$ is a module over itself together with (3.8.2) imply that (3.8.3) indeed defines a module. For example, if $G = G^{ad}$, then the action of T_{ϵ_m} on t_α is given by the same formula as in (3.8.2) but without ever passing to the dominant representative.

We call $\mathcal{M}(G)$ the *stabilization* of $\mathcal{H}_0(G)$. It is also known as the universal principal series representation, see [Kat1-2]. It is a free module of rank $|W|$.

Let $T^\vee$ be the torus dual to T, i.e., the spectrum of the group algebra $\mathbf{C}[X^\vee]$. We denote a typical point of $T^\vee$ by z so for $r \in X^\vee$ we denote z^r the monomial function on $T^\vee$ corresponding to r. Because the action of $\mathcal{H}_0(G)$ on $\mathcal{M}(G)$ is given by translation invariant difference operators, $\mathcal{M}(G)$ is in fact an $(\mathcal{H}_0(G), \mathbf{C}[X^\vee])$- bimodule where $X^\vee$ acts on the right by translations: $t_\alpha z^r = t_{\alpha+r}$. Clearly as a $\mathbf{C}[X^\vee]$-module it is free of rank 1 so the left module structure gives us an algebra homomorphism

$$S : \mathcal{H}_0(G) \to \mathbf{C}[X^\vee] = \mathbf{C}[T^\vee], \quad T_r \to \sum c_{r,\beta} z^\beta. \tag{3.8.3}$$

(3.8.4) Proposition (Satake isomorphism). *The map S identifies $\mathcal{H}_0(G)$ with the invariant subalgebra $\mathbf{C}[X^\vee]^W = \mathbf{C}[T^\vee/W]$ where the action of W on $T^\vee$ is defined by $w * z = q^{-\rho} w(q^\rho z)$.*

There is another way to stabilize the algebra $\mathcal{H}_0(G)$. Namely, take two basis vectors T_α, T_β where both α and β are dominant enough and look at their product. Obviously, it has the form

$$T_\alpha T_\beta = \sum_{\gamma \in X_+^\vee} c_{\alpha\beta}^\gamma T_{\alpha+\beta-\gamma}.$$

In other words, the maximal term in the product will be $T_{\alpha+\beta}$ while all the other summands will correspond to coweights less dominant than $\alpha + \beta$. The stabilization we have in mind is obtained by noticing that the the coefficients $c_{\alpha\beta}^\gamma$ actually depend only on γ, provided γ is well inside the region formed by all possible summands in $T_\alpha T_\beta$.

(3.8.5) Proposition. *(a) There exist numbers $c^{(\gamma)}, \gamma \in Y_+^\vee$ with the following property: for any finite subset $S \subset Y_+^\vee$ there exist an open cone $C \subset X_+^\vee \otimes \mathbf{R}$ such that whenever $\alpha, \beta \in C$ and $\gamma \in S$, we have $c_{\alpha\beta}^\gamma = c^{(\gamma)}$.*
(b) Explicitly, the $c^{(\gamma)}$ can be found from the expansion of the following rational function on $T^\vee$:

$$R(z) = \sum_{\gamma \in Y_+^\vee} c^{(\gamma)} z^\gamma = \prod_{\xi \in \Delta_+} \frac{1 - z^\xi}{1 - q z^\xi}.$$

Proof: Macdonald [Mac] has found the image of T_α under the Satake isomorphism S : $\mathcal{H}_0 \to \mathbf{C}[T_{ad}^\vee/W]$. Namely, for a strictly dominant α we have

$$S(T_\alpha)(z) = \frac{(q^\rho)^\alpha}{P(q^{-1})} \sum_{w \in W} z^{-w(\alpha)} \prod_{\xi \in \Delta_+} \frac{1 - q^{-1} z^{w(\xi)}}{1 - z^{w(\xi)}}$$

where $P(t) = \sum_{w \in W} t^{\mathrm{lt}(w)}$. Now note that for the "stabilization" we have in mind only one summand in Macdonald's formula is relevant, namely that corresponding to w being the unit element. In other words, the "stable" coefficients $c^{\gamma}_{\alpha\beta}$ will be the same as the coefficients obtained by multiplying these summands alone. Our statement follows from this.

Stabilizations of the kind described above will appear in the study of infinite Hecke operators in the next section. In fact we need a slight generalization of this construction in which we allow coweights not necessarily dominant but just lying far away in a cone given by partial dominance conditions. More precisely, for a set $I \subset \{1, ..., l\}$ we denote by $\Delta_+(I)$ the set of positive roots which are roots for the unipotent radical of the standard parabolic subgroup P^I, and let

$$R_I(z) = \prod_{\alpha \in \Delta_+(I)} \frac{1 - z^\alpha}{1 - qz^\alpha}. \tag{3.8.6}$$

(3.8.7) Proposition. *Let $I_1, I_2 \subset I$ be two subsets. Then the coefficients of the expansion*

$$\frac{R_{I_1}(z)R_{I_2}(z)}{R_{I_1 \cup I_2}(z)} = \sum_{\gamma \in Y_+^\vee} c^{(\gamma)}(I_1, I_2) z^\gamma$$

have the following interpretation. For any finite $S \subset Y_+^\vee$ there exist open cones $C_\nu \subset Y_+^\vee \cap \bigoplus_{i \in I_\nu} \mathbf{Z}\epsilon_i$ such that whenever $\alpha \in C_1$, $\beta \in C_2$ and $\nu \in S$, we have $c^{\gamma}_{\alpha\beta} = c^{(\gamma)}(I_1, I_2)$.

(3.9) Poisson measures. We continue to assume that the residue field k is finite, $k = \mathbf{F}_q$.

(3.9.1) Proposition-definition. *Let $x \in \mathcal{B}$ be any vertex. For any parabolic subgroup $P \subset G$ there is a unique probability measure μ_x^P on $G(F)/P(F)$ invariant with respect to $K_x \subset G$, the compact subgroup preserving the point x. This measure is called the Poisson measure.*

The mesure μ_x^P depends in the choice of x. For two vertices x, y the ratio μ_y^P/μ_x^P is a well defined function on G/P. This function, regarded as a function of x, y as well is known as the *Poisson kernel* and denoted

$$\Pi_P(x, y, b) = \frac{\mu_y^P}{\mu_x^P}(b), \quad x, y \in \mathrm{Vert}(\mathcal{B}),\ b \in G/P$$

(3.9.2) Proposition. *If P is a parabolic subgroup then for any $b \in G/P$ we have*

$$\Pi_P(x, y, b) = q^{\delta_P(d''(x,b) - d''(y,b))}$$

where $d''(x, b) \in D^\infty(b)$ was defined in (3.6) and δ_P is the sum of all roots entering the root decomposition of $\mathbf{n}_P$, the nilpotent subalgebra of P.

Proof: Note that $\mu_x(U) = \mu_{g(x)}(g(U))$ for any $x \in \mathrm{Vert}(\mathcal{B})$ and any $U \subset G/P$. Thus, if $g(b) = b$ then, taking for U an "infinitesimally small neighborhood" of b, we find:

$$\frac{\mu_x}{\mu_{g(x)}}(b) = |\det d_b(g)|,$$

where $d_b(g) : T_b(G/P) \to T_b(G/P)$ is the differential of g at b acting on the tangent space. Further, by homogeneity it is enough to verify the statement of the proposition at any one point $b \in G/P$. We take b to be a point invariant with respect to the standard torus $H = T(F)$, so it represents a cell at the boundary of the standard apartment $A = A(H)$. Also we take x to be a vertex of A and y to be of the form $h(x)$, $h \in H$. Then $h(b) = b$ and by the above $\Pi_P(x, y, b) = |\det(T_b h)|$. It remains to notice that the characters of the torus action on the tangent space $T_b(G/P)$ are precisely the roots of $\mathbf{n}_P$. Q.E.D.

The measure μ_{P_m} associated to the maximal parabolic subgroup P_m, $m = 1, ..., l$, has the following probabilistic interpretation. We consider the random walk $\mathcal{W}_m$ on $\mathrm{Vert}(\mathcal{B})$ in which a particle at a vertex x can move, with equal probability to any neighboring vertex y such that the corresponding edge is on type m. The Martin boundary for $\mathcal{W}_m$ is the Grassmannian $G(F)/P_m$ (lying inside the full boundary of $\mathcal{B}$ which is the spherical building). The measure μ_{P_m} is the exit measure corresponding to the Markov chain $\mathcal{W}_m$. Note that the Markov chains $\mathcal{W}_m$ are independent for different m, which is just a rephrasing of the fact that the Hecke operators T_{ϵ_m} commute with each other. The reader can consult [Fu] for a treatment of Archimedean symmetric spaces from this point of view.

(3.10) Measures on big cells. We consider the following sutuation:

$\mathcal{B} = \mathcal{B}(G, F)$ is the affine building of G over the local field F (with finite residue field $\mathbf{F}_q$).

$\Sigma = \partial\mathcal{B}$ is the corresponding spherical building.

$b \in \Sigma$ is a 0-dimensional cell, so it corresponds to a maximal parabolic subgroup $P_b \subset G(F)$. We assume that P_b is of type $i \in \{1, ..., l\}$, so P_b is conjugate to the standard subgroup P^I where $I = \{1, ..., l\} - \{i\}$. Let also $N_b \subset P_b$ be the unipotent radical.

The lattice $Y^\vee/I$ is naturally identified with $\mathbf{Z}$. So the torsor $D^\infty(b)$ parametrizing radii of horocycles with center b, is a $\mathbf{Z}$-torsor. Fix any type $j \in \{1, ..., l\}$ and let $\mathrm{Gr}_j \simeq G(F)/P_j$ be the set of 0-cells of Σ of type j. Let $U(b, j) \subset \mathrm{Gr}_j$ be the open subset consisting of $c \in \mathrm{Gr}_j$ which are in generic position with b (in the same sense as in Proposition 3.5.2 (ii)). Suppose that a maximal compact subgroup $K_b \subset P_b/N_b$ is chosen. Let $\tau : P_b \to P_b/N_b$ be the natural projection.

(3.10.1) Proposition. *The space of measures on $U(b, j)$ invariant under the subgroup $\tau^{-1}(K_b)$, is 1-dimensional.*

Proof: The set $U(b, j)$ is the set of F-points of a smooth quasiprojective variety over the p-adic field F, and the action of $\tau^{-1}(K_b)$ is by regular maps. Thus our statement follows from the next two facts: first, the action of $\tau^{-1}(K_b)$ on $U(b, j)$ is transitive, and second, for any fixed point c of any transformation $g \in \tau^{-1}(K_b)$ the Jacobian $\det(d_c g) \in F^*$ has norm 1. Both these facts are verified straightforwardly.

§4. Double Bruhat-Tits buildings and Hecke algebras for arbitrary G.

(4.1) The double building. We return to the situation of (2.1) and the introduction, so F is a 2-dimensional local field, k its locally compact residue field, $\mathbf{F}_q$ the finite residue field of k etc. We also keep all the notations of (3.1) related to our fixed reductive group G. Our first aim is to associate to G and F (as well as the 2-dimensional local field structure on F) a cell complex $\mathbf{B} = \mathbf{B}(G, F, k)$ called the *double Bruhat-Tits building* of G. For the case $G = PGL_n$ the complex $\mathbf{B}$ is closely related to (although not identical with) the higher building constructed by Parshin [Pa1-2].

We need some notation. Let P be any polyhedral ball (a CW-complex of dimension d which has one cell of dimension d and is homeomorphic to a d-ball). We associate to it a new polyhedral ball $\hat{P}$ which again has only one d-cell and whose boundary is a polyhedral sphere obtained as follows. We first take the barycentric subdivision of ∂P, getting a simplicial $(d-1)$-sphere, and then take the CW-decomposition of S^{d-1} dual to the one given by that barycentric subdivision. Thus vertices of $\hat{P}$ are in bijection with proper cells of P (of any dimension). If P is a convex polytope, then $\hat{P}$ can also be realized as a convex polytope. Namely, we first cut out, like with a knife, all vertices of P (so the each vertex will be replaced by a small new face), then make cuts parallel to the edges of P, then make cuts parallel to 2-faces etc. For example, if P is a simplex, then $\hat{P}$ is a permutohedron (the convex hull of a generic orbit of the symmetric group S_n in $\mathbf{R}^n$).

We now describe the construction of $\mathbf{B}$. Considering F as just a local field with residue field k, we associate to it (and G) the "continuous" Bruhat-Tits building $\mathcal{B}_F = \mathcal{B}(G, F)$. We will distinguish the objects related to this building by the subscript F, for instance, we will write $d_F(v, v')$ for the $Y_+^\vee$-valued distance in $\mathcal{B}_F$, as well as $A_F(v, v')$ for the intersection of all the apartments containing v, v' etc.

The link $\mathrm{Lk}(\sigma)$ of any cell $\sigma \in \mathcal{B}_F$ is thus a spherical building over the p-adic field k. As we know, there is a canonical locally finite Bruhat-Tits building β_σ whose boundary is $\mathrm{Lk}(\sigma)$. We will call it the *microscopic building* (or just the *microbuilding*) associated to σ. Let $\overline{\beta}_\sigma = \beta_\sigma \coprod \mathrm{Lk}(\sigma)$ be the compactification of β_σ obtained by attaching the boundary. We now take the disjoint union of $\hat{\sigma} \times \overline{\beta}_\sigma$ for all cells $\sigma \in \mathcal{B}_F$. Then, for any cells $\sigma \subset \sigma'$ of $\mathcal{B}_F$ we identify each face of $\hat{\sigma} \times \partial\beta_\sigma$ with the corresponding face of $\hat{\sigma}' \times \beta_\sigma$. The resulting topological space (with the topology induced, via the gluings, from the compact topologies on the $\hat{\sigma} \times \beta_\sigma$) is, by definition, the double bulding $\mathbf{B}$. Let us summarize its properties (obvious from the construction) in the following proposition.

(4.1.1) Proposition. *(a) The building* $\mathbf{B}$ *has a natural* $G(F)$*-action. It extends to a* $G^{ad}(F)$*-action.*
(b) There is a $G^{ad}(F)$*-equivariant projection*

$$\pi : \mathbf{B} \to \mathcal{B}_F, \tag{4.1.2}$$

whose fiber over any interior point of any cell $\sigma \in \mathcal{B}_F$ is naturally identified with $\overline{\beta_\sigma}$.

We will call a vertex of $\mathbf{B}$ thin if it is among the vertices of the microscopic buildings β_v where v is a vertex of $\mathcal{B}_F$. The set of thin vertices will be denoted $\mathbf{V}$. Clearly, $G(F)$ and $G^{ad}(F)$ act on $\mathbf{V}$. Recall that by construction, $\mathcal{B}_F$ has a distinguished vertex v_0 with stabilizer $G(\mathcal{O}_F)$. Thus β_{v_0} is canonically identified with the building $\mathcal{B}(G,k)$, and in particular it has a distinguished vertex w_0 whose stabilizer in $g(k)$ is $G(\mathcal{O}_k)$.

(4.1.3) Proposition. *(a) The stabilizer of w_0 in $G(F)$ is $G(\mathcal{O}')$.*
(b) The group $G^{ad}(F)$ acts on the set $\mathbf{V}$ of thin vertices of $\mathbf{B}$ in a transitive way.

For a vertex v of $\mathcal{B}_F$ we will denote G_v the stabilizer of v and by $\overline{G_v}$ the image of G_v in the group of automorphisms of the spherical building $\mathrm{Lk}(v/\mathcal{B}_F)$. Thus $\overline{G_v}$ is a group over k. In particular, $\overline{G_{v_0}} = G(k)$.

(4.2) Apartments in B. Let $H \subset G(F)$ be a split maximal torus. Denote by $A_F(H) \subset \mathcal{B}_F$ the corresponding apartment in $\mathcal{B}_F$. For every cell $\sigma \subset A_F(H)$ the link $\mathrm{Lk}(\sigma/A_F(H))$ is an apartment in the spherical building $\mathrm{Lk}(\sigma/\mathcal{B}_F) = \partial\beta_\sigma$ and thus gives an affine apartment $\mathbf{A}(H)_\sigma$ in β_σ. We will call it the *microscopic apartment* corresponding to σ. Let $\overline{\mathbf{A}(H)_\sigma}$ be the union of $\mathbf{A}(H)_\sigma$ and its boundary $\mathrm{Lk}(\sigma/A_F(H))$. So topologically it is a ball. It follows that the products $\hat{\sigma} \times \overline{\mathbf{A}(H)_\sigma}$ fit together to form a cell subcomplex $\mathbf{A}(H) \subset \pi^{-1}(A_F(H))$ which we call the double apartment corresponding to H. The fiber of the natural map (restriction of π)

$$\pi_H : \mathbf{A}(H) \to A_F(H) \tag{4.2.1}$$

over an interior point of a cell σ is, by construction, $\overline{\mathbf{A}(H)_\sigma}$.

(4.2.2) Proposition. *Any two cells of $\mathbf{B}$ are contained in a common double apartment.*

As in §3, we will use the notation $\mathbf{A}(\sigma,\tau)$ for the intersection of all the (double) apartments containing two given cells σ and τ.

The following construction will be important for describing spheres in $\mathbf{B}$.

(4.2.3) Proposition-definition. *Let $\mathbf{A} = \mathbf{A}(H)$ be a double apartment, $A_F = \pi(\mathbf{A}) \subset \mathcal{B}_F$ the corresponding ordinary apartment and v, v' be vertices of A_F. Let $\mathrm{dir}_F(v,v') \subset \mathrm{Lk}(v/\mathcal{B}_F)$ and $\mathrm{codir}_F(v,v') \subset \mathrm{Lk}(v'/\mathcal{B}_F)$ be the extreme cells of the parallelotope $A_F(v,v') \subset A$, see (3.5). Let $\overline{G_{v'}}$ be the the group defined in (4.1). It acts on $\mathrm{Lk}(v'/\mathcal{B}_F)$. Then each vertex $w \in \beta_v$ naturally gives rise to a maximal compact subgroup $\kappa_w(v')$ in the parabolic subgroup $\mathrm{Stab}(\mathrm{codir}_F(v,v')) \subset \overline{G_{v'}}$.*

Construction: As we saw in (3.6), for any cell $\sigma \subset \mathrm{Lk}(v'/\mathcal{B}_F) = \partial\beta_{v'}$ maximal compact subgroups in $\mathrm{Stab}(\sigma)$ correspond to conical cells in $\beta_{v'}$ defining σ. Now, let w be given and let $h = y^{d_F(v,v')} \in H(F)$. The translation by h gives a point $w' \in \beta_{v'}$. Consider the double parallelotope $\mathbf{A}(w,v')$. Its part lying in β_v is a conical cell defining the boundary cell $\mathrm{dir}_F(v,v')$. Translating it by h, we get a conical cell defining $h \cdot \mathrm{dir}_F(v,v')$ which is just the cell opposite to $\mathrm{codir}_F(v, v')$ in the spherical apartment $\mathrm{Lk}(v'/A_F)$. Now, for the stabilizers of two opposite cells in a spherical apartment one can naturally identify the semisimplifications and thus their maximal compact subgroups.

(4.3) Cartan decomposition. Distances in B. Let $\Gamma = F^*/(\mathcal{O}')^*$ be the valuation group of the 2-dimensional local field F. We denote by $\underline{\mathrm{ord}} : F^* \to \Gamma$ the natural projection. The group Γ is (non-canonically) isomorphic to $\mathbf{Z}^2$ and included into a canonical exact sequence

$$0 \to \mathbf{Z} \to \Gamma \to \mathbf{Z} \to 0.$$

Let us choose uniformizers $x, y \in \mathcal{O}'$, as in (2.2). Such a choice defines an identification

$$\mathbf{Z}^2 \to \Gamma, \quad (m, n) \mapsto \underline{\mathrm{ord}}(y^m x^n).$$

We will use this identification in the sequel, in particular, we will equip Γ with the lexicographical order and write (m, n) as $m\infty + n$ to highlight this order.

The quotient $T(F)/T(\mathcal{O}')$ is naturally identified with $X^\vee \otimes \Gamma$. We will write elements of $X^\vee \otimes \Gamma$ as $\lambda = \gamma\infty + \zeta$ with $\gamma, \zeta \in X^\vee$. The Weyl group W acts on $X^\vee \otimes \Gamma$ and we denote the quotient $(X^\vee \otimes \Gamma)/W$ by Λ_G. The quotient $\Lambda_{G^{ad}} = (Y^\vee \otimes \Gamma)/W$ will be denoted simply by Λ. We will identify Λ_G with the set of representatives of the form $\gamma\infty + \zeta$ where γ is dominant (i.e., $(\gamma, \alpha_i) \geq 0$ for any simple root α_i) and ζ is such that $(\zeta, \alpha_i) \geq 0$ whenever $(\gamma, \alpha_i) = 0$. We introduce a partial order on $Y^\vee$ by saying that $\gamma \leq \gamma'$ if $\gamma' - \gamma$ is dominant. By using this order, we order Λ (and thus Λ_G) lexicographically: $\gamma\infty + \zeta \leq \gamma'\infty + \zeta'$ iff $\gamma < \gamma'$ or $\gamma = \gamma'$ and $\zeta < \zeta'$.

(4.3.1) Proposition. *We have the following Cartan decomposition:*

$$G(F) = \coprod_{\gamma\infty + \zeta \in \Lambda_G} G(\mathcal{O}')(x^\zeta y^\gamma) G(\mathcal{O}').$$

Here x^ζ is the value of the 1-parameter subgroup $\zeta : \mathbf{G}_m \to T$ on x, and similarly for y^γ.

Proof: It is enough to consider the adjoint case. As with the Cartan decomposition for ordinary local fields, it is useful to restate the problem geometrically, in terms of the double building $\mathbf{B}$. let $w_0 \in \beta_{v_0}$ be the distinguished vertex of $\mathbf{B}$ and $\mathbf{A}_0$ be the standard doubla apartment through w_0. Geometrically, our statement says that any other thin vertex $w \in \beta_v$ can be brought by a transformation from $K' = \mathrm{Stab}(w_0) \subset G(F)$ to a unique point of the form $x^\zeta y^\gamma \cdot w_0 \in \mathbf{A}_0$ such that $\gamma\infty + \zeta \in \Lambda$.

From the Cartan decomposition for F cosidered as an ordinary local field, we conclude that there is $g \in K = \mathrm{Stab}(v_0)$ such that $gv = y^\gamma v_0$ for a unique $\gamma \in Y_+^\vee$. Therefore Of course, $g(w_0)$ may not equal w_0, we just know that it lies in β_{v_0}. So let $K_{v,v_0} \subset G(F)$ be the subgroup preserving v, v_0. By Proposition 3.5.3 the image of K_{v,v_0} in $\mathrm{Aut}(\beta_{v_0})$ is the parabolic subgroup in $G(k)$ preserving the generalized flag $\mathrm{dir}(v_0, v)$. By the Iwasawa decomposition for the local field k, the action of this parabolic subgroup on vertices of β_{v_0}

is transitive, so by composing g with an appropriate transformation from K_{v,v_0}, we get $h \in K' = \mathcal{G}(\mathcal{O}')$ such that $h(w_0) = w_0$, $h(w) \in \beta_{y^\gamma v_0}$. Denote $y^\gamma v_0$ simply by v' and the cell $\mathrm{codir}(v, v') \subset \mathrm{Lk}(v'/\mathcal{B}_F) = \partial\beta_{v'}$ by σ.

Further, let $K'_{v,v'} = K_{v,v'} \cap K'$; let $P_\sigma \subset \mathrm{Aut}(\beta_{v'})$ be the parabolic subgroup preserving σ; denote $N_\sigma \subset P_\sigma$ the unipotent radical and $\pi_\sigma : P_\sigma \to P_\sigma/N_\sigma$ the natural projection. Then the image of $K'_{v,v'}$ in $\mathrm{Aut}(\beta_{v'})$ lies in P_σ and coincides with the preimage $\pi_\sigma^{-1}(\kappa_{w_0,v'})$ where $\kappa_{w_0,v'} \subset P_\sigma/N_\sigma$ is the maximal compact subgroup described in (4.2.3). So its orbits are mixed horocycles is $\beta_{v'}$ whose center is a boundary cell fixed by the standard torus. So by Proposition 3.6.4 this horocycle meets the standard apartment in a unique γ-dominant point.

We can restate this as follows.

(4.3.2) Proposition. *There is a $G(F)$-invariant distance function $d : \mathbf{V} \times \mathbf{V} \to \Lambda$ satisfying the triangle inequality with respect to the lexicographic order on Λ. Moreover:*
(a) For $w_1, w_2, w'_1, w'_2 \in \mathbf{V}$ the existence of $g \in G^{ad}(F)$ such that $g(w_i) = w'_i$), is equivalent to the condition $d(w_1, w_2) = d(w'_1, w'_2)$.
(b) If $d(w_1, w_2) = \gamma\infty + \beta$, then in the continual building $\mathcal{B}_F$ we have $d_F(\pi(w_1), \pi(w_2)) = \gamma$.

Similarly to what we saw in (2.2), the existence of d means that we have some natural identifications of the distance torsors parametrising horocycles in neighboring microbuildings. As in (2.2.6), these identifications can be deduced from a statement about identifications of fibers of natural line bundles for two neighboring vertices in any ordinary Bruhat-Tits building. We leave this as an exercise for an interested reader.

(4.4) Spheres in B and horocycles in the microbuildings. Let $w \in \mathbf{V}$ be a thin vertex of $\mathbf{B}$ and $r \in \Lambda$. We denote by $S_r(w) = \{w' : d(w, w') = r\}$ the sphere of radius r with center w. let $r = \gamma\infty + \zeta$ and let $v = \pi(w)$, so w lies in the microbuilding β_v. Then, by Proposition 4.2.2 (b),

$$S_r(w) = \coprod_{d_F(v,v')=\gamma} S_r(w) \cap \beta_{v'}.$$

We are going to describe each of the parts of this decomposition.

(4.4.1) Proposition. *For any $v' \in \mathcal{B}_F$ such that $d_F(v, v') = \gamma$ the intersection $S_r(w) \cap \beta_{v'}$ is a mixed horocycle in $\beta_{v'}$. Its center is given by the cell*

$$\mathrm{codir}_F(v, v') \subset \mathrm{Lk}(v'/\mathcal{B}_F) = \partial\beta_{v'}$$

and by the maximal compact subgroup $\kappa_w(v')$ in the stabilizer of this cell.

This follows from the proof of the Cartan decomposition (4.3.1)

(4.5) Measures on spheres in B. Let $w \in \beta_v \subset \mathbf{B}$ be a thin vertex and let $r = \gamma\infty + \zeta \in \Lambda$. Denote by $K'_w \subset G(F)$ the subgroup preserving w. The sphere $S_r(w)$ has a natural structure of a locally compact completely disconnected topological space.

(4.5.1) Proposition. *The space of Borel measures on $S_r(w)$ invariant under the group K'_w, is 1-dimensional.*

Proof: Note first that we have a surjection

$$S_r(w) \to S_{\gamma,F}(v)$$

whose fibers are (mixed) horocycles. In particular, they are countable and discrete. Further, the weight γ being dominant, let us write it in the form $\gamma = \sum_{j=0}^{l} m_j \epsilon_j$ with $m_j \geq 0$. Let $\gamma_0 = 0, \gamma_1, ..., \gamma_{m_1+...+m_l} = \gamma$ be the sequence of vertices obtained by first going m_1 steps in the direction ϵ_1, then m_2 steps in the direction ϵ_2 and so on. We will say that γ_i is the jth pivot of this sequence, if $i = m_1 + ... + m_j$, i.e., our sequence changes direction at γ_i. If γ_i goes after the jth pivot but before the $(j+1)$st pivot, then the edge (γ_i, γ_{i+1}) has the type j (see (3.4) for the discussion of edge types). We have a sequence of fibrations

$$(4.5.2) \qquad S_{\gamma,F}(v) \overset{\rho_{m_1+...+m_l}}{\longrightarrow} ... \longrightarrow S_{\gamma_2,F}(v) \overset{\rho_2}{\to} S_{\gamma_1,F}(v) \overset{\rho_1}{\to} S_{\gamma_0,F}(v) = \{v\}.$$

Each fiber of each of the maps here is a big cell in some generalized Grassmannian. More precisely, let $v_i \in S_{\gamma_i,F}(v)$ and let $v_{i'}, i' < i$ be its images in the previous spheres. If γ_i is not a pivot and lies after the jth pivot, then the edge (v_i, v_{i-1}) has type $\bar{j}$ where the bar means the involution on $\{1, ..., l\}$ described in (3.4). This edge represents thus a 0-cell b_i of type j in the spherical building $\mathrm{Lk}(v_i/\mathcal{B}_F)$. The fiber $\rho_{i+1}^{-1}(v_i)$ is nothing but the big cell $U(b_i, j)$ in the generalized Grassmannian Gr_j, see (3.10). Similarly, if γ_i is a jth pivot, then (v_i, v_{i-1}) represents a 0-cell b_i in $\mathrm{Lk}(v_i/\mathcal{B}_F)$ but of type $\overline{j-1}$ and $\rho_{i+1}^{-1}(v_i)$ is $U(b_i, j)$ in Gr_j. Note that at every step we have a canonically defined maximal compact subgroup K_i in the stabilizer of b_i in the p-adic group acting on $\mathrm{Lk}(v_i/\mathcal{B}_F)$. Now our statement follows by repeated application of Proposition 3.10.1.

(4.6) The Hecke algebra $\mathcal{H}$. Let w_0 be the distinguished thin vertex of $\mathbf{B}$ (whose stabilizer is the subgroup K'). For $r \in \Lambda$ denote the 1-dimensional space of K'-invariant measures on $S_r(w_0)$ by $\mathcal{H}(r)$. Let $\gamma \in X_+^\vee$ be a dominant coweight. Denote $\mathcal{H}_\gamma$ to be the space of formal series $\sum_{\delta:\gamma\infty+\zeta\in\Lambda} h_{\gamma\infty+\zeta}$ where each $h_{\gamma\infty+\zeta}$ is an element of $H(\gamma\infty + \zeta)$ such that the set

$$\{\zeta : h_{\gamma\infty+\zeta} \neq 0\}$$

is contained in some translation of $(-X_+^\vee)$, the cone opposite to the cone of dominant coweights. Finally, let $\mathcal{H} = \bigoplus_{\gamma\in X_+^\vee} \mathcal{H}_\gamma$.

As in (2.4), we can view elements of each $\mathcal{H}_\lambda$ as $G(F)$-invariant (0,1)-measures on $(G(F)/K') \times (G(F)/K')$ so that formally the convolution of such measures is defined.

(4.6.1) Proposition. *The convolution of (0,1)-measures makes $\mathcal{H}$ into an algebra so that for $\gamma, \gamma' \in X_+^\vee$ we have $\mathcal{H}_\gamma\mathcal{H}_{\gamma'} \subset \mathcal{H}_{\gamma+\gamma'}$. In other words, the algebra $\mathcal{H}$ is $X_+^\vee$-graded. Further, it has a filtration F parametrized by the ordered semigroup Λ with $F_\lambda\mathcal{H}$ consisting of sums $\sum_{\lambda'<\lambda} h_{\lambda'}$.*

The proof is similar to (2.4.1).

We will call $\mathcal{H}$ the *double Hecke algebra* of the p-adic loop group $G(F)$. Note that $\mathcal{H}_0$ is the standard unramified Hecke algebra for $G(k)$, i.e., $\mathcal{H}_0 \simeq \mathbf{C}[T^\vee/W]$. As in §2, we denote $\mathcal{H}'_\gamma \subset \mathcal{H}_\gamma$ the subspace $\bigoplus_\zeta \mathcal{H}(\gamma\infty+\zeta)$ of finite sums. The following proposition is clear.

(4.6.2) Proposition. *Each $\mathcal{H}_\gamma$ is an $\mathcal{H}_0$-bimodule and $\mathcal{H}'_\gamma$ is a sub-bimodule.*

Let us introduce some distinguished elements of $\mathcal{H}$. First, we will retain the notations T_α, $\alpha \in X_+^\vee$ for the finite Hecke operators from $\mathcal{H}_0$. Second, if $r = \epsilon_m\infty + \alpha$ is such that its infinite part is a fundamental coweight, then for any thin vertex $w \in \beta_v$ the sphere $S_r(w)$ is a fibration over the p-adic generalized Grassmannian $S_{\epsilon_m,F}(v) \simeq (G/P_m)(k)$ with countable discrete fibers (which are mixed horocycles). So we define the measure $\mu_{w,r}$ on this sphere to be the (Fubini) product of the Poisson probability measure μ_w on the Grassmannian and the discrete Dirac measures on the horocycles. Therefore, for r of the described form we have specified certain elements $T_r \in \mathcal{H}(r)$.

For arbitrary $\gamma \in X_+^\vee$ it is difficult to normalize geometrically the measure on the $S_{\gamma\infty+\zeta}(w)$. However, let us note that, for a fixed γ, a choice of such a normalization for one particular value of ζ produces a normalization for all other values of ζ. This is because the spheres $S_{\gamma\infty+\zeta}$ and $S_{\gamma\infty+\zeta'}$ are fibered over the same continuous variety $S_{\gamma,F}(v)$, each with discrete fiber, and the measures we are talking about are products of the measures on this $S_{\gamma,F}(v)$ with the corresponding Dirac measures on the fibers. Therefore we choose once and for all some non-zero elements $T_{\gamma\infty} \in \mathcal{H}(\gamma\infty)$ and define $T_{\gamma\infty+\zeta}$ to be the image of $T_{\gamma\infty}$ under the identification just described. For $\gamma = \epsilon_m$ this is compatible with the above convention.

(4.7) Structure of $\mathcal{H}'_\gamma$ as an $\mathcal{H}_0$-bimodule. Let $\gamma \in X_+^\vee$, and let $X_\gamma^\vee \subset X^\vee$ be the cone of elements ξ such that $(\xi,\alpha) \geq 0$ for each positive root α such that $(\gamma,\alpha) = 0$. Thus, for a strictly dominant γ we have $X_\gamma^\vee = X^\vee$ while or $\gamma = 0$ we have $X_\gamma^\vee = X_+^\vee$. Note that $X_\gamma^\vee$ is the fundamental domain for the action of the subgroup $W_\gamma \subset W$ preserving γ. An element $\gamma\infty + \zeta$ lies in Λ iff $\zeta \in X_\gamma^\vee$. For any $\xi \in X^\vee$ we denote by $\xi_\gamma \in X_\gamma^\vee$ the unique W_γ-translation of ξ lying in $X_\gamma^\vee$. Thus for $\gamma = 0$ $\xi_\gamma = \xi_+$ is the dominant representative.

As a left $\mathcal{H}_0$-module, each $\mathcal{H}'_\gamma$ has a very simple structure, namely

$$T_{\epsilon_m} T_{\gamma\infty+\zeta} = \sum_{w\in W/W_m} q^{\mathrm{lt}_m(w)} T_{(\gamma\infty+\zeta+w(\epsilon_m))_\gamma}, \tag{4.7.1}$$

similarly to (3.7.2). In particular, if γ is strictly dominant, then the subscript γ does not change anything, so $\mathcal{H}'_\gamma$ is isomorphic to $\mathcal{M} = \mathcal{M}_{G^{ad}}$, the stabilization of the Hecke algebra from (3.8).

Now we describe the right module structure of the simplest infinite graded components.

(4.7.2) Proposition. *We have the following equalities in $\mathcal{H}$:*

$$T_{\epsilon_j\infty+\zeta} T_{\epsilon_m} = \sum_{w\in W/W_m} q^{\mathrm{lt}_m(w)+\delta_j(w(\epsilon_m))} T_{(\epsilon_j\infty+\zeta+w(\epsilon_m))_{\epsilon_j}}.$$

Here δ_j is the sum of all the weights of N_j, the unipotent radical of the maximal parabolic subgroup P_j.

Proof: It is clear that $T_{\epsilon_j\infty+\zeta}T_{\epsilon_m}$ will have the same $|W/W_m|$ summands as the product in (4.7.1) but with different coefficients appearing from measure changes. These changes are found from Proposition 3.9.2.

(4.8) The bilinear forms Ψ and Φ and the Heisenberg algebras. We introduce a $\mathbf{Z}$-valued bilinear form Ψ on $Y^\vee$ by defining its values on pairs of basis vectors to be:

$$\Psi(\epsilon_j, \epsilon_m) = \delta_j(\epsilon_m) = \sum_{\substack{\alpha\in\Delta_+ \\ (\alpha,\epsilon_j)\neq 0}} (\alpha, \epsilon_m). \tag{4.8.1}$$

The form Ψ is, in general, neither symmetric nor W-invariant. However, there is a related form Φ possessing both these properties. It is given by

$$\Phi(\epsilon_j, \epsilon_m) = \sum_{\alpha\in\Delta_+} (\alpha, \epsilon_j)\cdot(\alpha, \epsilon_m), \tag{4.8.2}$$

so that for any $a, b \in Y^\vee$ we have

$$\Phi(a, b) = \sum_{\alpha\in\Delta_+} (\alpha, a)\cdot(\alpha, b) = \frac{1}{2}\sum_{\alpha\in\Delta} (\alpha, a)\cdot(\alpha, b). \tag{4.8.3}$$

The last form of writing Φ implies its W-invariance. In the case when each quasi-simple factor of G is of type A_n for some n (i.e., is a cover of $PGL(n+1)$), we have $\Psi = \Phi$ since (α, ϵ_j) is, in this case, always equal to either 0 or 1. However, in general it is the form Ψ which will appear in our description of the Hecke algebra.

Using the form Ψ we define the Heisenberg algebra $\tilde{\mathcal{A}}$ generated by monomials z^a, w^b for $a, b \in Y^\vee$ with relations

$$z^a z^b = z^{a+b}, \quad w^a w^b = w^{a+b}, \quad w^a z^b = q^{\Psi(a,b)} z^b w^a. \tag{4.8.4}$$

Thus we can think of z, w as points lying in $T^\vee_{ad}$, the dual torus for G^{ad}. An element from $\tilde{\mathcal{A}}$ can be written, uniquely, in the normal form

$$\sum_{b\in Y^\vee} \phi_b(z) w^b,$$

where each ϕ_b is a Laurent polynomial in z. Let also $\tilde{\mathcal{A}}(G) \subset \tilde{\mathcal{A}}$ be the subalgebra generated by the z^a, w^b with $a \in X^\vee \subset Y^\vee$. Thus $\mathcal{A} = \mathcal{A}(G^{ad})$. Let $\mathcal{A}(G) \subset \tilde{\mathcal{A}}(G)$ be the subalgebra consisting of polynomials of the form $\sum_{b\in X^\vee} \phi_b(z) w^b$, such that each ϕ_b is symmetric with respect to the action of W_b, the subgroup in the Weyl group preserving b. The algebra $\mathcal{A}(G^{ad})$ will be denoted simply by $\mathcal{A}$.

(4.9) Theorem. *The associated graded algebra* $\mathrm{gr}_F\mathcal{H}(G)$ *with respect to the filtration* F *described in (4.6.1), is naturally isomorphic to* $\mathcal{A}(G)$.

Proof: For the case $G = G^{ad}$ the statement follows from (4.7.2) and the fact that $T_\lambda T_\mu$ is a constant multiple of $T_{\lambda+\mu}$ plus lower order terms. The general case follows from this by identifying $\mathrm{gr}_F\mathcal{H}(G) \subset \mathrm{gr}_F\mathcal{H}(G^{ad})$ with the corresponding subalgebra in $\mathcal{A}(G^{ad})$.

(4.10) The rational subalgebra $\mathcal{H}_{rat}$. Let $\mathcal{H}(G)^{rat}_\gamma$ be the subspace in $\mathcal{H}(G)_\gamma$ consisting of elements $\sum_\zeta a_\zeta T_{\gamma\infty+\zeta}$ such that the formal Laurent series $\sum_\zeta a^\zeta z^\zeta$ represents a rational function on $T^\vee$. Let $\mathcal{H}(G)^{rat} = \bigoplus_\gamma \mathcal{H}(G)^{rat}_\gamma$.

(4.10.1) Proposition. *The subspace* $\mathcal{H}(G)^{rat} \subset \mathcal{H}(G)$ *is a subalgebra.*

Proof: Our statement will follow from the next lemma about the bigger space $\mathcal{H}^{rat} \subset \mathcal{H} = \mathcal{H}(G^{ad})$ which says that the product of any two generators $T_\lambda T_\mu$ lies in $\mathcal{H}(G)^{rat}$ but describes the corresponding rational function more precisely. We will say that a coweight γ is strictly I-dominant, where $I \subset \{1, ..., l\}$ is a subset, if $(\alpha_i, \gamma) > 0$ for $i \in I$ while $(\alpha_i, \gamma) = 0$ for $I \notin I$. It is clear that every dominant coweight is strictly I-dominant for a uniquely defined I.

(4.10.2) Lemma. *Let* $\gamma_\nu \in Y^\vee_+$, $\nu = 1, 2$ *be two dominant coweights and each* γ_ν *is strictly* I_ν*-dominant. Then for any* $\zeta_1, \zeta_2 \in Y^\vee$ *such that* $\gamma_\nu\infty + \zeta_\nu \in \lambda$ *we have*

$$T_{\gamma_1\infty+\zeta_1}T_{\gamma_2\infty+\zeta_2} = q^{\Psi(\zeta_1,\gamma_2)} \sum_{\lambda\in X^\vee_+} c^{(\lambda)}(I_1, I_2)q^{-(\gamma_1,\lambda)}T_{(\gamma_1+\gamma_2)\infty+(\zeta_1+\zeta_1-\lambda)},$$

where the numbers $c^{(\lambda)}(I_1, I_2)$ *are defined in Proposition 3.8.7.*

Proof: It is clear that the product will have the same summands as given in the statement of the lemma, and we just need to determine the coefficients. The coefficient at $T_{(\gamma_1+\gamma_2)\infty+(\zeta_1+\zeta_1-\lambda)}$ is the product gh of two factors. The first factor g can be defined as follows. Take two thin vertices w_1, w_3 of $\mathbf{B}$ on distance $(\gamma_1 + \gamma_2)\infty + (\zeta_1 + \zeta_1 - \lambda)$. Then u is the number of w_2 such that $d(w_1, w_2) = \gamma_1\infty + \zeta_1$ and $d(w_2, w_3) = \gamma_2\infty + \zeta_2$. The factor h is the change in the normalization of the measure produced by any such w_2. We claim that

$$g = c^{(\lambda)}(I_1, I_2), \quad h = q^{-(\gamma_1,\lambda)}. \tag{4.10.3}$$

To see the first of these equalities, note that the microscopic building where such a w_2 can lie, is uniquely defined, so the number of possibilities for w_2 is governed entirely by the geometry of this building. More precisely, the double apartments $\mathbf{A}(w_1, w_2)$ and $\mathbf{A}(w_2, w_3)$ can have some finite parts in common (cf. [Ko] for the discussion of PGL_3). But this geometry will be identical to what happens when we just multiply two finite Hecke operators $T_{\overline{\zeta_1}}T_{\overline{\zeta_2}}$ where each $\overline{\zeta_\nu}$ is far enough in the cone of structly I_ν-dominant coweights and see how many times $T_{\overline{\zeta_1}+\overline{\zeta_2}-\lambda}$ enters. This proves the first equality in (4.10.3). The second equality follows by repeatedly applying Proposition 3.9.2 to edge paths in the building $\mathcal{B}_F$ joining the relevant vertices.

Lemma (4.10.2) implies that for any fixed γ_1, γ_2 the product of any $T_{\gamma_1\infty+\zeta_1}$ with any $T_{\gamma_2\infty+\zeta_2}$ will have the coefficients at $T_{(\gamma_1+\gamma_2)\infty+(\zeta_1+\zeta_1-\lambda)}$ giving rise to a rational function of the form $q^{L(\zeta_1)}R_{\gamma_1,\gamma_2}(z)$ where R_{γ_1,γ_2} depends only on the γ_ν but not on the ζ_ν and L is a linear form depending on γ_2. This implies that $\mathcal{H}(G)^{rat}_{\gamma_1} \cdot \mathcal{H}(G)^{rat}_{\gamma_1} \subset \mathcal{H}(G)^{rat}_{\gamma_1+\gamma_2}$, proving Proposition 4.10.1.

(4.11) The rational Heisenberg algebra $\mathcal{A}(G)^{rat}$. For $a \in X^\vee$ let $\Psi(a) \in X$ be the image of a under Ψ considered as a map $X^\vee \to X^{\vee\vee} = X$. Note that X serves as the lattice of 1-parameter subgroups for the torus $X^\vee$. For $\xi \in X$ we will denote q^ξ the value of the corresponding 1-parameter subgroup on $q \in \mathbf{C}$. Consider the homomorphism $X^\vee \to T^\vee$ taking $a \mapsto q^{\Psi(a)}$. Note that the Heisenberg algebra $\tilde{\mathcal{A}}(G)$ can be written as the cross-product of the algebra of Laurent polynomials on $T^\vee$ with the group algebra of $X^\vee$, i.e., as algebra of polynomials $\sum_a \phi_a(z)w^a$ where each $\phi_a(z)$ is a Laurent polynomial, with the commutation law $w^a\phi(z) = \phi(q^{\Psi(a)}z)$. Let $\tilde{\mathcal{A}}(G)^{rat}$ be the extension of $\tilde{\mathcal{A}}(G)$ obtained by allowing each $\phi_a(z)$ to be an arbitrary rational function. Let $\mathcal{A}(G)^{rat}$ be the subalgebra in $\tilde{\mathcal{A}(\mathcal{G})}^{rat}$ obtained by requiring each $\phi_a(z)$ to be symmetric with respect to W_a, the subgroup in the Weyl group preserving a. Here we use the W-action on $T^\vee$ appearing in the Satake isomorphism. As before, we use the notations $\mathcal{A}^{rat}$ for the special case $G = G^{ad}$.

Now we can formulate the second main result of this paper.

(4.12) Theorem. *The algebra $\mathcal{H}^{rat}(G)$ is isomorphic to $\mathcal{A}^{rat}(G)$.*

The statement for a general G is deduced from the case $G = G^{ad}$. So we assume that this is the case. The is based on the following fact describing the space $\mathcal{H}'(G)_\gamma$ as a bimodule over the finite Hecke algebra $\mathcal{H}(G)_0$. Recall that $\mathrm{Spec}(\mathcal{H}(G)_0) = T^\vee/W$, so each bimodule gives a coherent sheaf on $(T^\vee/W) \times (T^\vee/W)$. Let also $p : T^\vee \to T^\vee/W$ be the projection.

(4.12.1) Proposition. *The coherent sheaf on $(T^\vee/W)\times(T^\vee/W)$ corresponding to $\mathcal{H}'(G)_\gamma$, is the structure sheaf of the subvariety $C(\gamma) = (p \times p)(\tilde{C}(\gamma))$ where $\tilde{C}(\gamma) \subset T^\vee \times T^\vee$ is the shifted diagonal $\{(z, q^{\Psi(\gamma)} \cdot z)\}$.*

As in §2, there is a natural algebra structure on $\bigoplus_{\gamma \in X^\vee_+} \Gamma((T^\vee/W) \times (T^\vee/W), \mathcal{O}_{C(\gamma)})$ and this algebra is nothing but $\mathcal{A}(G)$. But the multiplication in $\mathcal{H}(G)$ differs from that in $\mathcal{A}(G)$ because of the presense of lower order terms in $T_{\gamma_1\infty+\zeta_1}T_{\gamma-2\infty+\zeta_2}$. However, these lower order terms are explicitly controlled by Lemma 4.10.2. In particular, they follow a pattern depending only on the sets $I(\gamma_\nu)$ of simple roots such that $(\alpha_i, \gamma_\nu) = 0$. Recalling the rational functions $R_I(z)$ from (3.8.6), we find that the linear bijection $\chi : \mathcal{H}(G)^{rat} \to \mathcal{A}(G)^{rat}$ defined by

$$T_{\gamma\infty+\zeta} \to R^{-1}_{I(\gamma)} z^\zeta w^\gamma,$$

is an algebra isomorphism. Theorem 4.12 is proved.

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