Holomorphic Automorphisms Of Quadrics Of Codimension 2

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HOLOMORPHIC AUTOMORPHISMS OF QUADRICS OF CODIMENSION 2

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ABSTRACT. In this paper we prove that the automorphisms of a nondegenerate quadric of codimension 2 in $\mathbb{C}^{n+2}$ is a rational map of degree ≤ 2 and give the explicit formulas for such automorphisms.

1. INTRODUCTION

H. Poincaré [9] proved in 1907 that any holomorphic automorphism of the sphere $S = \{(z,w) \in \mathbb{C}^2 : \operatorname{Im} w = \bar{z}z\}$ preserving the origin is a fractional linear transformation

$$
\begin{aligned}
z &\mapsto c(z+aw)(1-2i\bar{a}z-(r+i\bar{a}a))^{-1} \\
w &\mapsto \rho w(1-2i\bar{a}z-(r+i\bar{a}a))^{-1},
\end{aligned}
\tag{1}
$$

where $a, c \in \mathbb{C}$, $r \in \mathbf{R}$, and $\rho = |c|^2$.

N. Tanaka [11] proved in 1962 the analogous result for any nondegenerate hyperquadric.

Nondegenerate hyperquadrics are the quadratic models of CR surfaces with nondegenerate vector-valued Levi form in $\mathbb{C}^{n+k}$ [2]:

$$
Q = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^k : \operatorname{Im} w^\kappa = \langle z,z\rangle^\kappa,\ \kappa = 1,\dots,k\},
\tag{2}
$$

where $\langle z,z\rangle^\kappa$ are Hermitian forms in $\mathbb{C}^n$ with the properties:

i) $\langle z,b\rangle^\kappa = 0$ for all $\kappa = 1,\dots,k, z \in \mathbb{C}^n$ implies $b = 0$

ii) $\langle z,z\rangle^\kappa$ are linearly independent $\kappa = 1,\dots,k$.

Beloshapka proved that these properties are necassary and sufficient for having a finite dimensional automorphism group ([3]).

Since Q is a homogeneous manifold ($\operatorname{Aut} Q$ acts transitively via the transformations $z \mapsto p+z$, $w \mapsto q+w+2i\langle z,p\rangle$ with $(p,q) \in Q$) then $\operatorname{Aut} Q \cong Q \times \operatorname{Aut}_0 Q$, where $\operatorname{Aut}_0 Q$ is the isotropy group of a fixed point, say the origin.

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Using the reflection principle G.Henkin and A.Tumanov [8] proved that $\operatorname{Aut}_0 Q$ consists of rational transformations.

V. Beloshapka [4] gave a description of the Lie algebra of the infinitesimal automorphisms of Q and he proved also that the quadrics of codimension $k > 2$ in general position are rigid, i.e. their isotropy groups consist of trivial automorphisms $z \mapsto cz$, $w \mapsto |c|^2 w$ for some complex number c (see [5]).

In the cases $n = k = 2$ and $n = 3, k = 2$ any quadric is equivalent to one of a finite number of standard quadrics. The autors obtained in these cases the complete description of the automorphisms [6, 7].

For $k = 2$ A.Abrosimov [1] discovered a sufficient condition for $\operatorname{Aut}_0 Q$ to consist of linear transformations: if in some coordinates the operator $(H^1)^{-1}H^2$ (H^j - the Hermitian matrix related to $\langle z, z\rangle^j$) has more than two different eigenvalues.

Recently, S. Shevchenko [10] has obtained a classification for quadrics of codimension 2 with respect to the the linear action of $G_{n,2} = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{R})$: $z \mapsto Cz, w \mapsto \rho w$, $z \in \mathbb{C}^n$, $w \in \mathbb{C}^2$, $(C, \rho) \in G_{2,n}$.

Using this result we complete the description of the automorphisms of nondegenerate quadrics of codimension 2.

Since any autimorphism $\Phi \in \operatorname{Aut}_0 Q$ can be represented as $\Phi = \Phi_{(C,\rho)} \circ \Phi_{id}$, where $\Phi_{(C,\rho)} \in G_{n,2}$ is a linear automorphism and Φ_{id} has an identical projection of the differential at 0 on the complex tangent space, it is sufficient to describe the subgroup $\operatorname{Aut}_{0,id}$ of automorphisms Φ preserving 0 and with $d\Phi|_{T_{\mathbb{C}}M} = \mathrm{id}$.

We show below that any $\Phi \in \operatorname{Aut}_{0,id}(Q)$ can be represented by a matrix analogue of the Poincaré formula (1) or it is fractional linear.

2. A Matrix Poincaré formula for $\operatorname{Aut}_{0,id}(Q)$

According to the result of S. Shevchenko cited above, nondegenerate quadrics of codimension two have nonlinear automorphisms only in the following four cases:

$$
\begin{aligned}
(3) \qquad H^1 &= \sum_{i=1}^{r} \epsilon_i |z^i|^2, \\
H^2 &= \sum_{i=r+1}^{n} \epsilon_i |z^i|^2 \text{ (hyperbolic case)}, \\
(4) \qquad H^1 &= \sum_{i=1}^{s} \epsilon_i |z^{2i}|^2, \\
H^2 &= \sum_{i=1}^{s} 2\,\mathrm{Re}\, z^{2i-1}\bar{z}^{2i} + \sum_{i=2s+1}^{n} \epsilon_i |z^i|^2 \text{ (parabolic case)}, \\
(5) \qquad H^1 &= \sum_{i=1}^{n/2} \mathrm{Re}\, z^{2i-1}\bar{z}^{2i}, \\
H^2 &= \sum_{i=1}^{n/2} \mathrm{Im}\, z^{2i-1}\bar{z}^{2i} \text{ (elliptic case)}, \\
(6) \qquad H^1 &= \sum_{i=1}^{s} \mathrm{Re}\, z^{3i-2}\bar{z}^{3i-1} + \tilde{H}^1(z'), \\
H^2 &= \sum_{i=1}^{s} \mathrm{Re}\, z^{3i-2}\bar{z}^{3i} + \tilde{H}^2(z') \text{ (null-case)},
\end{aligned}
$$

where $\epsilon_i \in \{-1, 1\}$, $z' = (z^{3s+1}, \dots, z^n)$, and $\det(\lambda_1\tilde{H}^1 + \lambda_2\tilde{H}^2) \not\equiv 0$.

Moreover, the dimension of the Lie algebras corresponding to $\mathrm{Aut}_{0,id}$ eaquals $2n+2$ in the hyperbolic, elliptic and parabolic cases, and $2s$ in the null-case.

In the elliptic, hyperbolic and parabolic cases the automorphisms are fractional quadratic transformations, given by a matrix Poincaré formula in the following way:

in the elliptic case we set:

$$
\begin{aligned}
Z &= \begin{pmatrix} z^1 & -z^2 \\ z^2 & z^1 \\ \dots & \dots \\ z^{n/2-1} & -z^{n/2} \\ z^{n/2} & z^{n/2-1} \end{pmatrix} \\
\bar{Z} &= \begin{pmatrix} \bar{z}^1 & -\bar{z}^2 & \dots & \bar{z}^{n/2-1} & -\bar{z}^{n/2} \\ \bar{z}^2 & \bar{z}^1 & \dots & \bar{z}^{n/2} & \bar{z}^{n/2-1} \end{pmatrix} \\
W &= \begin{pmatrix} w^1 & -w^2 \\ w^2 & w^1 \end{pmatrix} \\
\bar{W} &= \begin{pmatrix} \bar{w}^1 & -\bar{w}^2 \\ \bar{w}^2 & \bar{w}^1 \end{pmatrix},
\end{aligned}
$$

in the hyperbolic case:

$$
\begin{aligned}
Z &= \begin{pmatrix} z^1 & 0 \\ \dots & \dots \\ z^s & 0 \\ 0 & z^{s+1} \\ \dots & \dots \\ 0 & z^n \end{pmatrix} \\
\bar{Z} &= \begin{pmatrix} \epsilon_1 \bar{z}^1 & \dots & \epsilon_s \bar{z}^s & 0 & \dots & 0 \\ 0 & \dots & 0 & \epsilon_{s+1} \bar{z}^{s+1} & \dots & \epsilon_n \bar{z}^n \end{pmatrix} \\
W &= \begin{pmatrix} w^1 & 0 \\ 0 & w^2 \end{pmatrix} \\
\bar{W} &= \begin{pmatrix} \bar{w}^1 & 0 \\ 0 & \bar{w}^2 \end{pmatrix},
\end{aligned}
$$

and, in the parabolic case:

$$
\begin{aligned}
Z &= \begin{pmatrix} z^1 & 0 \\ z^2 & z^1 \\ \dots & \dots \\ z^{2s-1} & 0 \\ z^{2s} & z^{2s-1} \\ z^{2s+1} & 0 \\ \dots & \dots \\ z^n & 0 \end{pmatrix} \\
\bar Z &= \begin{pmatrix} \epsilon_1 \bar z^1 & 0 & \dots & \epsilon_s \bar z^{2s-1} & 0 & 0 & \dots & 0 \\ \bar z^2 & \bar z^1 & \dots & \bar z^{2s} & \bar z^{2s-1} & \epsilon_{2s+1}\bar z^{2s+1} & \dots & \epsilon_n \bar z^n \end{pmatrix} \\
W &= \begin{pmatrix} w^1 & 0 \\ w^2 & w^1 \end{pmatrix} \\
\bar W &= \begin{pmatrix} \bar w^1 & 0 \\ \bar w^2 & \bar w^1 \end{pmatrix}.
\end{aligned}
$$

Then the equation of Q can be written

$$
\frac{W - \bar W}{2i} = \bar Z Z \tag{7}
$$

A complex n-vector a will be be represented as a $2 \times n$ matrix like the corresponding z, and a real 2-vector r as a 2×2 matrix like the corresponding w.

Then the Poincaré formula

$$
\begin{aligned}
Z &\mapsto (Z + AW)(\mathrm{id} - 2i\bar A Z - (R + i\bar A A)W)^{-1} \\
W &\mapsto W(\mathrm{id} - 2i\bar A Z - (R + i\bar A A)W)^{-1}
\end{aligned}
$$

describes $\mathrm{Aut}_{0,id}$.

In the null-case $\mathrm{Aut}_{0,id}$ consists of fractional linear transformations:

$$\begin{aligned}
z^{3k-2} &\mapsto \frac{z^{3k-2}}{1-2i\sum_{j=1}^{s}\bar{a}^j z^{3j-2}} \\
z^{3k-1} &\mapsto \frac{z^{3k-1}+a^k w^1}{1-2i\sum_{j=1}^{s}\bar{a}^j z^{3j-2}} \\
z^{3k} &\mapsto \frac{z^{3k}+a^k w^2}{1-2i\sum_{j=1}^{s}\bar{a}^j z^{3j-2}} \text{ for } k=1,\dots,s \\
z^{k} &\mapsto \frac{z^{k}}{1-2i\sum_{j=1}^{s}\bar{a}^j z^{3j-2}} \text{ for } k=3s+1,\dots,n \\
w^{1} &\mapsto \frac{w^{1}}{1-2i\sum_{j=1}^{s}\bar{a}^j z^{3j-2}} \\
w^{2} &\mapsto \frac{w^{2}}{1-2i\sum_{j=1}^{s}\bar{a}^j z^{3j-2}}.
\end{aligned}$$

3. Linear representation of the automorphism groups

The construction from [6] and [7] can be used to give a linear representation of the automorphism groups.

Let $\mathfrak{A}$ be the 2-dimensional commutative algebra of 2×2 matrices of type W. Then $\mathbb{C}^{n+4}$ can be equipped with a structure of an $\mathfrak{A}$-module: Let $\Theta \in \mathfrak{A}$ and $(\theta_0, \theta_1, \theta_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^n$, and, let $(\Theta_0, \Theta_1, \Theta_2)$ be the 2×2 resp. $n \times 2$ matrices corresponding to $(\theta_0, \theta_1, \theta_2)$. Then Θ acts on $(\Theta_0, \Theta_1, \Theta_2)$ by matrix multiplication (from the right).

By $\mathfrak{A}^*$ we denote the group of invertible elements of $\mathfrak{A}$ and by $\widehat{\mathbb{C}^{n+2}}$ the factor space under the action of $\mathfrak{A}^*$. $\widehat{\mathbb{C}^{n+2}}$ is a compact variety which can be considered as a compactification of $\mathbb{C}^{n+4}$ by the embedding

$$(Z, W) \mapsto (\mathrm{id}, Z, W),$$

where Z, W are the matrices corresponding z, w.

Now, any automorphism of the quadrics from above can be represented as a linear transformation of $\mathbb{C}^{n+4}$ in the following way:

$$\begin{aligned}
\theta_0 &\mapsto \theta_0 - 2i\bar{A}\theta_1 - (R + iA\bar{A})\theta_2, \\
\theta_1 &\mapsto C\theta_1 + CA\theta_2, \\
\theta_2 &\mapsto \rho\theta_2,
\end{aligned}$$

where A is the the $n \times 2$ matrix corresponding to the complex n-vector a, R is the the 2×2 matrix corresponding to the real 2-vector r and $(C, \rho) \in G_{n,2}$.

References

1. A.B. Abrosimov, *On local automorphisms of certain quadrics of codimension two (Russian)*, Mat. Zametki **52** (1992), no. 1, 9–14.
2. S. Baouendi, H. Jacobowitz, and F. Trèves, *On the analyticity of CR mappings*, Annals of Math. **122** (1985), no. 2, 365–4.
3. V.K. Beloshapka, *Finite-dimensionality of the group of automorphisms of a real analytic surface*, USSR Izvestiya **32** (1989), no. 2, 437–442.
4. ———, *A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space*, Math. Notes **47** (1990), 239–242.
5. ———, *Automorphisms of real quadrics of high codimension and normal forms of CR manifolds (Russian)*, Ph.D. thesis, Steklov Mathematical Institute Moscow, 1991.
6. V.V. Ežov and G. Schmalz, *Biholomorphic automorphisms of Siegel domains in* $\mathbb{C}^4$, preprint, Max-Planck-Institut für Mathematik Bonn, 1992.
7. ———, *Holomorphic automorphisms of quadrics of codimension 2 in* $\mathbb{C}^5$, preprint, Max-Planck-Institut für Mathematik Bonn, 1993.
8. G.M. Henkin and A.E. Tumanov, *Local characterization of holomorphic automorphisms of Siegel domains*, Funkt. Analysis **17** (1983), no. 4, 49–61.
9. H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Math. Palermo (1907), 185–220.
10. S. Shevchenko, *Description of the algebra of infinitesimal automorphisms of a CR quadric of codimension two (Russian)*, Mat. Sbornik **to appear** (1993).
11. N.J. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J.Math.Soc.Japan **14** (1962), 397–429.

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