Saddle Point Method and Resurgent Analysis

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Abstract

The Stokes phenomenon for Laplace-type integrals in the complex domain is investigated. It is shown that this problem is a special case of the general problem of investigating the Stokes phenomenon in the framework of the resurgent analysis. The investigation method worked out in the paper is illustrated on classical examples: functions of Airy and Weber type etc.

In this paper we present a method of investigating of (parametric) Laplace integrals by means of the resurgent analysis. This approach is very natural. Actually, one can show (and we do it below) that the investigation of integralsof the Laplace type is none more than some (rather special) chapter of the resurgent functions theory.

Thus, let us consider an integral of the form

$$I(x,k) = \int\limits_{\gamma(x)} e^{kS(x,y)} a(x,y)\, dy \tag{1}$$

(the so-called Laplace integrals). Here k is a large (real) parameter, $x = (x^1, \dots, x^n) \in \mathbf{C}^n$ are complex parameters which are supposed to belong to a compact set in the space $\mathbf{C}^n$,

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$y = (y^1, \ldots, y^m) \in \mathbf{C}^m$, and the integration is performed along some m-dimensional contour γ in the complex space $\mathbf{C}^m$ which can depend on the parameter x in a regular way[1].

We shall show that the theory of integrals of the Laplace type is a particular case of the theory of the Borel-Laplace transform

$$\mathcal{L}[F(s,x)] = \int_{\Gamma} e^{-ks} F(s,x)\, ds. \tag{2}$$

More exactly, we shall show that integrals of the type (1) (under some additional requirements) determine resurgent functions of the parameter k depending on parameters $x \in \mathbf{C}^n$ (see [1] – [5]. Thus, the resurgent analysis can be applied to the asymptotic investigation of such integrals; the examples of such an investigation is presented in the last Section of this paper.

The representation of a function given by an integral of the type (1) in the form of resurgent function allows to construct the theory of asymptotic expansions of such integrals up to exponentially decreasing terms. In this connection we mention the paper [6] by M. Berry and C. J. Howls where asymptotic expansions of Laplace integrals in one-dimensional case was constructed up to the terms which have lower exponential type than the integral itself (for obtaining these asymptotics the authors use a kind of resurgent equations for coefficients of asymptotic expansions corresponding to different saddle points).

In this paper the authors used some idea of B. Malgrange [7] and F. Pham [8]. The mentioned idea is the representation of integral (1) in the form of the Laplace transform of function $F(s,x)$ obtained from the amplitude function $a(x,y)$ of integral (1) by integration over vanishing cycles of the surface $\{s = S(x,y)\}$. In [8] this idea was used for obtaining asymptotic expansions of integrals of the type (1) over special contours $\gamma(x)$ (the so-called Lefschetz thimbles; see [9]). However, (as far as we know) this representation have not been yet applied for investigation of the Stokes phenomenon for integrals (1).

1 Statement of the problem

We shall suppose for simplicity that the contour γ involved in the definition of the integral (1) is chosen in such a way that

$$\operatorname{Re} S > -c_1 |y|$$

with some positive constant c_1 for sufficiently large $|y|$ and that the function $a(x,y)$ is of exponential type in y:

$$|a(x,y)| \le C e^{c_2|y|}$$

[1]More exactly, the contour $\gamma(x)$ is a representative of some ramifying homology class $h(x)$ (see below).

with some constants $C > 0$ and c_2. Under these conditions integral (1) converges for sufficiently large values of k. We suppose also that all stationary points of the function $S(x,y)$ are nondegenerated ones.

Let us calculate first the exponential type of integral (1). Evidently, we have

$$|I(x,k)| \leq C_1 e^{M(x)k}$$

where

$$M(x) = \sup_{y \in \gamma(x)} \operatorname{Re} S(x,y). \tag{3}$$

However, the latter estimate is a rough one due to the fact that one can change the integration contour $\gamma(x)$ in one of the same homology class without changing the integral $I(x,k)$ itself. In fact, the integration in (1) is performed over a ramifying homology class $h(x)$ (see [10]) which goes to infinity along the directions of decrease of the function $\operatorname{Re} S(x,y)$ rather than over the contour $\gamma(x)$ itself (clearly, $\gamma(x) \in h(x)$). Thus, the estimate (3) can be improved as follows:

$$M(x) = \inf_{\gamma'(x) \in h(x)} \left(\sup_{y \in \gamma'(x)} \operatorname{Re} S(x,y) \right)$$

where the infimum is taken over all m-dimensional contours homological to the initial contour $\gamma(x)$.

From the above considerations it is clear that for investigation of the asymptotic properties of the integral $I(x,k)$ it is convenient to choose the representative $\gamma(x)$ in the homology class $h(x)$ in such a way that the function $\operatorname{Re} S(x,y)$ has on this contour as small values as possible. To formulate the exact requirements on the choice of the contour $\gamma(x)$ we shall introduce the notion of a *steepest descent contour.*

Denote by $X = \operatorname{grad} \operatorname{Re} S(x,y)$ the (real) gradient of the function $\operatorname{Re} S(x,y)$ on the complex space $\mathbf{C}^m$. Then the steepest descent contour is, roughly speekeing, a contour which containes at least one saddle point of the function $S(x,y)$ and which is tangent to the vector field X, so that the function $\operatorname{Re} S(x,y)$ decreases along the trajectories of the vector field X. To refine this definition we must take into account the structure of the contour near saddle points since these points are singular points of the field X.

Since we suppose that all stationary points of the function $S(x,y)$ are nondegenerated (due to the Cauchy-Riemann conditions these points evidently coincide with saddle points of the function $\operatorname{Re} S(x,y)$), one can easily check that each saddle point y_0 of the function $\operatorname{Re} S(x,y)$ is a hyperbolic point and its repelling subspace L_- has dimension m. Now a *steepest descent contour* is defined as an m-dimensional contour which is invariant with respect to the vector field X, contains at least one saddle point y_0, and is tangent at this point to the space L_-.

In generic position each steepest descent contour contains exactly one saddle point of the function $\mathrm{Re}\, S(x,y)$. However, as the parameter x is changed, for some values of x contours of steepest descent can change their topological structure; as we shall see below, this is exactly the Stokes phenomenon for integrals of the type (1).

Our considerations become more clear when we consider the one-dimensional case (that is, $m = 1$). In this case we can imagine that the contour γ is a cord put on the surface of the graph of the function $\mathrm{Re}\, S(x,y)$. We deform this cord (without changing a value of integral (1)) in such a way that it will be posited as low as possible on the considered surface; as a result the cord will hang by a saddle point of the function $\mathrm{Re}\, S(x,y)$ and will leave this point in the directions of the gradient X of the function $\mathrm{Re}\, S(x,y)$. The further location of the cord will be uniquely determined unless the integral curve of the field X along which the cord is located meets another saddle point of the function $\mathrm{Re}\, S(x,y)$. In this case the further part of the steepest descent contour can have two different directions corresponding the two repelling directions of the field X and further construction of the contour is ambiguous. The values of the parameter x for which such a situation takes place are exactly points of the topological rebuilding of steepest descent contours (the Stokes lines of the corresponding integral).

We remark that in the one-dimensional case the imaginary part of the function $S(x,y)$ is constant along any steepest descent contour. Inversely, if the imaginary part of S is constant along some contour γ, then this contour is tangent to the vector field X introduced above.

The above considerations show that the saddle point method of constructing asymptotic expansions to integrals of the type (1) consists of the two distinct parts:

1. The topological part of this theory which is aimed at the investigation of integration contours and reducing the considered integral to the sum of integrals along contours of the steepest descent.

2. The analytic part which deals with the investigation of integrals along such contours and constructing the asymptotic expansion itself.

The second part of this theory is worked out in detail and the reader can find it in the numerous textbooks on the saddle point method (see, for example, [11]). What concernes the first (topological) part of this theory, it can be in turn divided into two steps in the case (considered here) when the integral in question depends on some additional parameters x. The first step is the decomposition of the integration contour into the sum of contours of the steepest descent for some fixed value of the parameter, and the second one is the continuation of this decomposition (and, as the subsequence, the continuation of the asymptotic expansion of the considered integral) to all the rest values of the parameter x.

Below we concentrate on the second step of the topological part of the theory. Namely, we suppose that the decomposition of the integration contour is given for some value of x

and investigate such decomposition for all the rest values of x. We remark that in this investigation the methods of the resurgent analysis are of use.

Since the theory of one-dimensional integrals (with $m = 1$) is much simpler than the multidimensional one, we consider first this simple case postponing the investigation of the general (multidimensional) case until Section 4.

2 One-dimensional case

Consider the integral (1) with $y \in \mathbf{C}$ (that is, $m = 1$). In this case the integral is taken over a one-dimensional contour γ. We suppose that the following conditions are fulfilled[2].

1. The function $S(x,y)$ is a polynomial in y with holomorphic coefficients in $D \subset \mathbf{C}^n$ where D is some domain in the complex space $\mathbf{C}^n$.

2. For sufficiently large $|y|$ the estimate

$$\operatorname{Re} S(x,y) \leq -c\,|y|$$

is fulfilled on the contour γ with some positive constant c.

3. The function $a(x,y)$ is an entire function of exponential type in y with order 1 for $x \in D \subset \mathbf{C}^n$, that is,

$$|a(x,y)| \leq Ce^{c|y|}$$

with some constants c and $C > 0$.

Clearly, if Conditions 1 - 3 are fulfilled then the integral (1) converges for sufficiently large k for any $x \in D$ and determines a function $I(x,k)$ of exponential growth in k.

To reduce the integral (1) to the form of the Laplace transform (2) of some hyperfunction, we perform the variable change

$$y = y(s,x)\,, \tag{4}$$

where the function $y(s,x)$ is a solution to the equation

$$s = -S(x,y)\,.$$

Since, due to condition 2 above the function $S(x,y)$ is not identically constant in y for each fixed $x \in D$, we have $S'(x,y) \not\equiv 0$ and, hence, the function $y(s,x)$ is defined as an endlessly continuous[3] ramifying function with finite order of ramification at each its singular point.

[2]These requirements are not the most general ones for the theory of resurgent functions to be applicable to the investigation of the Laplace integral. However, for simplicity we consider the most simple situation.

[3]The definition of an endlessly continuable function see, for example, [2].

Besides, this function is bounded near each point of ramification. Performing variable change (4) in integral (1) we reduce the expression for the function $I(x,k)$ to the form

$$I(x,k) = \int_{\gamma^*} e^{-ks} b(s,x)\, ds \tag{5}$$

where

$$b(s,x) = a(x, y(x,s)) \frac{\partial y(x,s)}{\partial s}. \tag{6}$$

The following properties of representation (5) are quite evident.

a) The function $b(s,x)$ determined by relation (6) is an endlessly continuable function with integrable singularity at each its point of ramification.

b) The intersection of the contour γ^* with a half-plane $\operatorname{Re} s < A$ is compact for any value of the constant A. In other words, $\operatorname{Re} s$ tends to $+\infty$ along the contour γ^*.

Now one can deform the contour γ^* moving it to the right in the complex plane $\mathbf{C}_s$. The result of such a deformation is

$$I(x,k) = \sum_j \int_{\Gamma_j} e^{-s} b(s,x)\, ds$$

up to rapidly decreasing terms where each contour Γ_j is a standard contour encircling some point of singularity of the function $b(s,x)$ counter-clockwise and going to infinity along the direction of the positive real axis (see Figure 1). The sum on the right in the latter expression is taken over all singular points of the function $b(s,x)$ 'visible' from points of the contour γ^* along rays directed along the positive real axis.

Thus, the following affirmation is proved.

Theorem 1 *Under Conditions* 1 – 3 *above, the function* $I(x,k)$ *given by integral* (1) *is a resurgent function such that the corresponding hyperfunction* $b(s,x)$ *is regular and have algebraic ramification at each its singular point.*

This Theorem allows one to apply the methods of the resurgent analysis to the investigation of integrals of the Laplace type.

There is one more feature of resurgent functions corresponding to integrals of the type (1) which is of use for investigation of these integrals. Namely, from the definition of $I(x,k)$ and from Conditions 1 – 3 it follows that the function $I(x,k)$ is a univalued function of the variable x in the domain D. Hence, the corresponding resurgent function must satisfy the *resurgent equations* near each its focal point (see [1], [2]).

Theorem 1 together with the above remark allows one to achieve a certain success in the investigation of the topological part of the investigation of the integrals of the Laplace type,

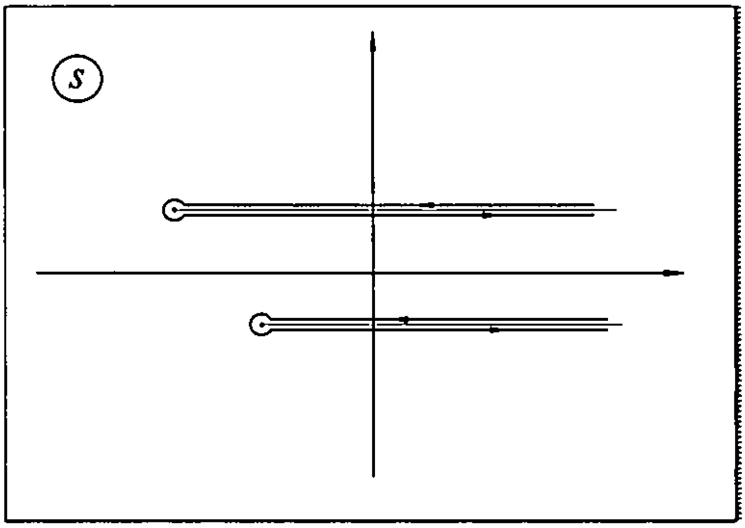

Figure 1:

at least in the case when these integrals depend on some additional parameters x. Namely, if the decomposition of the integration contour γ into the sum of steepest descent contours is known for one value of the parameters, the resurgent analysis allows one to construct such decomposition for *all* the rest values of the parameters. Let us describe the corresponding procedure.

First, let us fix some value of the parameters x and consider a canonical contour Γ corresponding to some singular point s^* of the function $b(s,x)$ given by (6). This contour consists of the two rays emanated from the point s^* in the direction of the positive real axis which are passed in different directions and lye on different sheets of the Riemannian surface of the function $b(s,x)$ (such contour can be used due to regularity of the hyperfunction $b(s,x)$, that is, due to the fact that this function is integrable at its singular point s^*). Since the imaginary part of s is constant along this contour, it is clear that under the action of the variable change $s = -s(x,y)$ (see formula (4) above) it will be transformed to the contour of the steepest descent coming through the saddle point corresponding to s^* due to the variable change in question. Thus, we obtain the decomposition of the (arbitrary) contour γ^* in (1) into the sum of contours of the steepest descent.

Second, it is well-known that for some values of the parameters x (on the corresponding Stokes surface) a bifurcation of the structure of contours of the steepest descent takes place. More exactly, a contour of the steepest descent can be transformed into a sum of such contours when the point x intersects the corresponding Stokes line. This takes place due to the fact that the contour of steepest descent going through one saddle point can come to another saddle point (this phenomenon takes place exactly when the point x belongs to the Stokes surface) and then there are exactly two (for a non-degenerated saddle point) possibilities of continuing the contour of steepest descent: one can continue it to one of the two valleys of the graph of the function $\operatorname{Re} S(x, y)$. Therefore, the contour of steepest descent changes by jump when the parameter point x intersects the Stokes surface, and so does the constructed decomposition.

Due to the one-to-one correspondence between contours of the steepest descent in the y-plane and the canonical contours in the s-plane, the described bifurcation of the steepest descent contour corresponds to a bifurcation of the corresponding canonical contour in the s-plane. This bifurcation in the resurgent functions theory is exactly the bifurcation described by the connection homomorphism (see [2]). This allows to apply the technique of the connection homomorphism and the resurgent equations to the investigation of the bifurcations of contours of the steepest descent in the theory of the Laplace integrals. The example of such an investigation will be considered in the last Section.

In conclusion of this Section we remark that the class of resurgent functions obtained from integrals of the Laplace type can be described as the class of such resurgent functions $f(x)$ whose Borel transform $F(s, x)$ can be uniformized with the help of some holomorphic substitution $s = S(x, y)$ (we recall that a function $F(s, x)$ is *uniformized* by a substitution $s = S(x, y)$ if the function $F(S(x, y), x)$ is a regular holomorphic function of the variables (x, y)).

3 Multidimensional case

The theory of bifurcations of contours of steepest descent can be also worked out in the multidimensional case similar to the above constructed theory for one-dimensional Laplace integrals, though in this case the correspondence between canonical contours and contours of the steepest descent is a little bit more complicated. To establish the mentioned correspondence, we recall some facts from the theory of multidimensional Laplace integrals (see, for example, [8], [11]).

Consider an integral of the type (1). As above, we suppose (for simplicity) that the following three conditions are valid.

1. The function $S(x, y)$ is a polynomial in the variable $y \in \mathbf{C}^m$ and $a(x, y)$ is an entire function for each fixed $x \in D \subset \mathbf{C}^n$ where D is some domain in the complex space $\mathbf{C}^n$.

2. For sufficiently large $|y|$ the estimate

$$\operatorname{Re} S(x, y) \leq -c|y|$$

is valid on the (m-dimensional) contour γ with some positive constant c.

3. The function $a(x,y)$ is a function of exponential type with order 1, that is,

$$|a(x,y)| \leq Ce^{c|y|}$$

with some constants c and $C > 0$.

Let x be a value of the parameter such that exactly one saddle point of the function $\operatorname{Re} S(x,y)$ lyes on each contour of the steepest descent (this means that the point x does not lye on the Stokes surface corresponding to the integral in question) and that all saddle points of the function $\operatorname{Re} S(x,y)$ are non-degenerated.

Let us investigate in more detail the structure of the contour of the steepest descent near the corresponding saddle point.

We remark that, due to the Cauchy-Riemann conditions any saddle point of the function $\operatorname{Re} S(x,y)$ is a stationary point of the function $S(x,y)$. Since, by assumption, all such points are non-degenerated, then due to the Morse lemma there exist a holomorphic variable change $y = y(z)$ such that

$$S(x, y(z)) = S_0 + \sum_{j=0}^{m} \left(z^j\right)^2$$

where S_0 is the value of the function $S(x,y)$ at the saddle point which corresponds to the origin in the z-plane. Then the steepest descent contour will be given in the z-coordinates by

$$z^j = u^j \in \mathbf{R}, \; j = 1, \ldots, m \tag{7}$$

near the considered saddle point. Now we remark that, due to Condition 2, the function $\operatorname{Re} S(x,y)$ is bounded from above on the integration contour γ. Denote by A the upper bound of this function on γ:

$$A \stackrel{\text{def}}{=} \sup_{y\in\gamma} \operatorname{Re} S(x,y).$$

Hence, the part $\gamma_{A'}$ of the contour γ lying in the domain

$$\Omega_{A'} = \{y \,:\, \operatorname{Re} S(x,y) \geq A'\}$$

for any $A' < A$ determines an element of compact relative homology group

$$\gamma_{A'} \in H_{m,c}\left(\Omega_{A'},\; \operatorname{Re} S(x,y) = A'\right).$$

As it is proved in [11] the element $\gamma_{A'}$ admits a decomposition

$$\gamma_{A'} \simeq \sum_j \overline{\gamma}_j$$

where each contour $\overline{\gamma}_j$ corresponds to some saddle point y_j of the function $\operatorname{Re} S(x,y)$ and coincides with the contour of the steepest descent (7) in a neighborhood of this point. Hence, modulo functions with exponential type A' we have

$$I(x,k) = \sum_j I_j(x,k) \tag{8}$$

where

$$I_j(x,k) = \int_{\overline{\gamma}_j} e^{kS(x,y)} a(x,y)\, dy.$$

Remark 1 *Decomposition* (8) *can be obtained also from the results of F. Pham* [8].

Now we shall examine the structure of integral $I_j(x,k)$ near the corresponding saddle point. Consider the part of this integral over the contour $\overline{\gamma}_j^r = \overline{\gamma}_j \cap \{|z| \le r\}$ where r is small enough. Then one can suppose that the contour $\overline{\gamma}_j^r$ is described in the coordinates z by equations (7). We have

$$\begin{aligned}\int_{\overline{\gamma}_j^r} e^{kS(x,y)} a(x,y)\, dy &= \int_{|u|\le r} e^{kS(x,y(u))} a(x,y(u)) \frac{Dy(u)}{Du}\, du \\ &= \int_0^r d\rho^2 \int_{|u|=\rho} e^{kS(x,y(u))} a(x,y(u)) \frac{Dy(u)}{Du}\, \omega,\end{aligned} \tag{9}$$

where $\rho = |u|$ and ω is the form determined by the relation $du = d\rho^2 \wedge \omega$. Since on the contour $\overline{\gamma}_j^r$ we have $S = S_0^j + \rho^2$ where S_0^j is the value of the function $S(x,y)$ at the saddle point y_j. The inner integral on the right in the latter formula can be treated as the integral over the vanishing cycle $h_j(s,x)$ on the manifold

$$\Sigma_{s,x} = \{y : S(x,y) = -s\}$$

for $s = S_0^j + \rho^2$. Therefore, the integral on the right in (9) can be represented in the form

$$\int_{\overline{\gamma}_j^r} e^{kS(x,y)} a(x,y)\, dy = \int_{\gamma_j^r} e^{-ks} \left\{ \int_{h_j(s,x)} a(x,y)\, \omega \right\} ds,$$

where γ_j^r is a segment of length r of the ray emanated from the point $s = -S_0^j$ along the direction of the positive real axis and the form ω is determined by the relation

$$dy = ds \wedge \omega.$$

We remark that the point $s = -S_0^j$ is one of the singular points of the function

$$\mathcal{F}_S[a](s,x) \stackrel{\text{def}}{=} \int\limits_{h_j(s,x)} a(x,y)\,\omega$$

which will be called the *S-transform* of the function $a(x,y)$ (cf. F. Pham [8]). We remark that values of the right-hand side of the latter formula for different values of j are branches of one and the same ramified function $\mathcal{F}_S[a](s,x)$. We remark also that, under Conditions 1 – 3 above this function is an endlessly continuable one.

Consider now the integrals

$$I'_j(x,k) = \int\limits_{\gamma_j} e^{-ks}\mathcal{F}_S[a](s,x)\,ds \tag{10}$$

for any saddle point y_j involved into decomposition (8). Here γ_j are rays emanated from the critical values S_0^j corresponding to the saddle points y_j and directed along the positive real axis[4]. The above considerations show that each of integrals (10) can be written down as an integral of the form (1) taken over the contour which coincide in a neighborhood of the corresponding saddle point with the contour of the steepest descent. Taking into account decomposition (8), we see that up to functions of exponential type A' one has

$$I(x,k) = \sum_j I'_j(x,k).$$

This is exactly a resurgent representation of the function $I(x,k)$ since integral (10) is the Laplace transform of an endlessly continuable microfunction represented by its variation $\mathcal{F}_S[a](s,x)$.

Let us summarize the above obtained results.

Theorem 2 *Under Conditions* 1 – 3 *above integral* (1) *is a resurgent function. The corresponding hyperfunction is given by the S-transform* $\mathcal{F}_S[a](s,x)$ *of the amplitude function* $a(x,y)$ *of integral* (1) *This hyperfunction is regular and is given by the corresponding variation.*

Now the investigation of the topological part of the theory of multidimensional Laplace integrals goes quite similar to that in the one-dimensional case and we leave it to the reader.

[4]We remark that the convergence of these integrals is not needed since they may be meantin the following sence. Let $I'_{j,A}$ be integrals of the same form taken over contours truncated at $\operatorname{Re} s = A$ for any real A. Then, by the Borel lemma (see, for example, [2]) there exists a function $I'_j(x,k)$ which coincides with $I'_{j,A}$ modulo $O(e^{-Ak})$. This function is, by definition, the value of the considered integral. We remark also that integrals (10) can be written down as integrals of the type (1) over the so-called *Lefschetz thimbles*, see [8], [9].

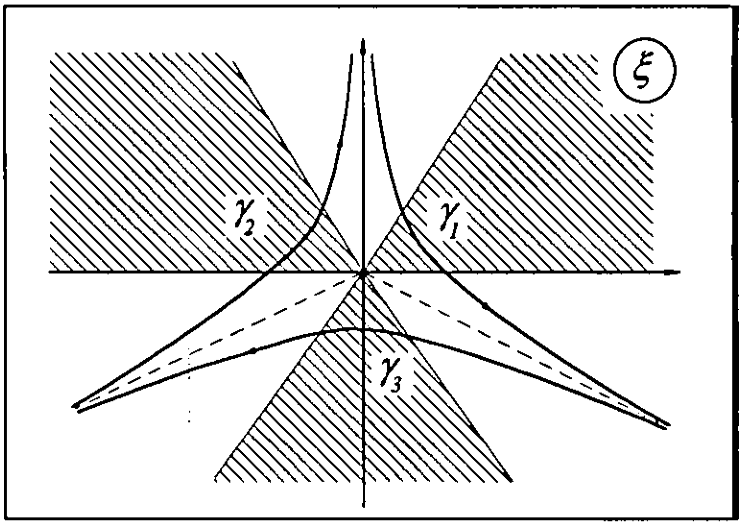

Figure 2:

4 Examples

4.1 Integrals of the Airy type

Here we shall consider the integrals of the Airy type, that is, the Laplace integrals of the form

$$u(x,k) = \int_\gamma e^{kS(x,\xi)} a(x,\xi)\, d\xi \tag{11}$$

(with a contour γ satisfying the above requirements) under the assumption that the function $S(x,\xi)$ has two saddle points in the variable ξ with stationary values given by

$$s = s(x) = \frac{2}{3} x^{\frac{3}{2}}. \tag{12}$$

One of examples of the integrals of the Airy type is the Airy function

$$Ai(x,k) = k^{\frac{1}{2}} \int_\gamma e^{k\left(\xi x - \frac{\xi^3}{3}\right)}\, d\xi, \tag{13}$$

where the contour γ can be chosen as one of the contours shown on Figure 2. As it is well-known, the Airy function is a solution to the following ordinary differential equation

$$k^{-2}\frac{d^2u}{dx^2} - xu = 0. \tag{14}$$

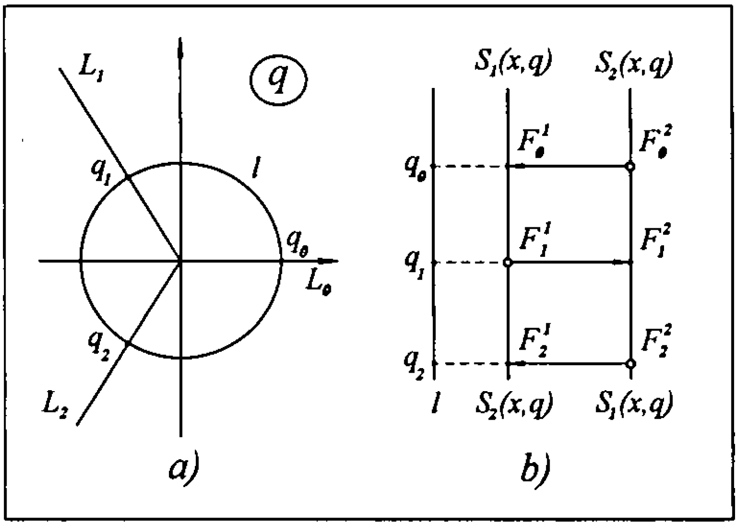

Figure 3:

Performing the variable change $\xi = \xi\,(s, x)$ given by

$$S(x, \xi) = -s, \tag{15}$$

we reduce integral (11) to the form

$$u\,(x, k) = \int_{\gamma'} e^{-ks} F\,(s, x)\; ds,\;\; F\,(s, x) = a\,(x, \xi\,(s, x))\,\frac{d\xi\,(s, x)}{ds}. \tag{16}$$

We remark that, due to the above assumptions on the stationary values of the function $S\,(x, \xi)$, the solution $\xi = \xi\,(s, x, k)$ to equation (15) with respect to ξ is a ramifying function of the variable s with singularitie at points given by formula (12); the two values of the function $s\,(x)$ will be denoted by $s_j\,(x)\,,\, j = 1, 2$. Clearly, the only focal point for function (11) in the x-plane is the origin $x = 0$ (we recall that focal points of a resurgent function are exactly points of ramification of the function $s\,(x)$ describing singularities of the integrand in the representation of the type (16)). Hence, the only resurgent equation expressing the univaluedness of (11) can be written down along the unit circle

$$l = \left\{x = e^{i\varphi}\right\},\, \varphi \in [0, 2\pi]\,.$$

Let us describe the corresponding illumination diagram (see [12]).

An illumination diagram (see, for example, Figure 3 b)) contains rows corresponding to points x_1, x_2, x_3 of intersection of the considered path with the Stokes surface of the function in question. For each such point one of points of singularities of the integrand in (16) lyes on the ray emanated from the other point of singularity along the direction of the positive real axis; we shall say that the latter point *illuminates* the former one. This fact is shown on the diagram with the help of an arrow coming from the illuminating to the illuminated point. We denote by F_i^j the microfunction determined by the function $F(s, x_i)$ at the point $s = s_j(x_i)$.

Evidently, one of the two points (12) illuminates the other iff $\varphi = 0$, $\varphi = \frac{2\pi}{3}$, or $\varphi = \frac{4\pi}{3}$ (we shall carry out our considerations for real positive values of k). The corresponding Stokes lines are drawn on Figure 3 a). One can easily check that the corresponding illumination diagram has the form shown on Figure 3 b). Tracing along the loop l, the points $s_1(x)$ and $s_2(x)$ change their places three times.

It is not hard to see that the corresponding system of resurgent equations for the illumination diagram of this form (written for the microfunctions F_0^1, F_1^2, and F_2^1 corresponding to the illuminated points of this diagram) is[5]

$$\begin{cases} F_0^1 = \mathcal{A}^2 F_1^2 - \Delta(\mathcal{A} F_2^1), \\ F_1^2 = \mathcal{A}^2 F_2^1 - \Delta(\mathcal{A} F_0^1), \\ F_2^1 = \mathcal{A}^2 F_0^1 - \Delta(\mathcal{A} F_1^2). \end{cases} \tag{17}$$

where Δ is an alient derivative (see [1], [2]) and $\mathcal{A}$ is the operator of analytic continuation of the corresponding microfunction along the loop l from one point of intersection of l with the Stokes surface to another.

We emphasize that for any given hyperfunction $F(s, x)$ with (12) as ramification points one can choose a set of microfunctions F_0^1, F_1^2, and F_2^1 determined by singular points of the function $F(s, x)$ in question in such a way that the corresponding resurgent function $u(x, k)$ is univalued in a neighborhood of the origin and, hence, the resurgent equations (17) are valid. To do this, it suffices to define the function $u(x, k)$ as the integral of the form (16) with the integration contour Γ encircling both ramification points as it is shown on Figure 4. Then the decomposition of the obtained function will give us the required microfunctions which satisfy system (17). However, we know that the Laplace transform of any microfunction determined by the function $F(s, x)$ at some point x (say, $x = x_0$) is a univalued function.

To be definite, let us consider the system of resurgent equations for the microfunction corresponding to the recessive component of $u(x, k)$ at the point x_0. To do this, we set $F_0^2 = 0$ in (17). Then we have $F_1^2 = \mathcal{A} F_0^2 = 0$ and, due to the first equation in (17) we have $F_0^1 = \mathcal{A}^2 F_1^2$. Now, excluding the microfunction F_1^2, we arrive to the following system of resurgent equations:

[5]In what follows we shall suppose for brevity that all alient derivatives except for those included into the resurgent system in the explicit way vanish identically.

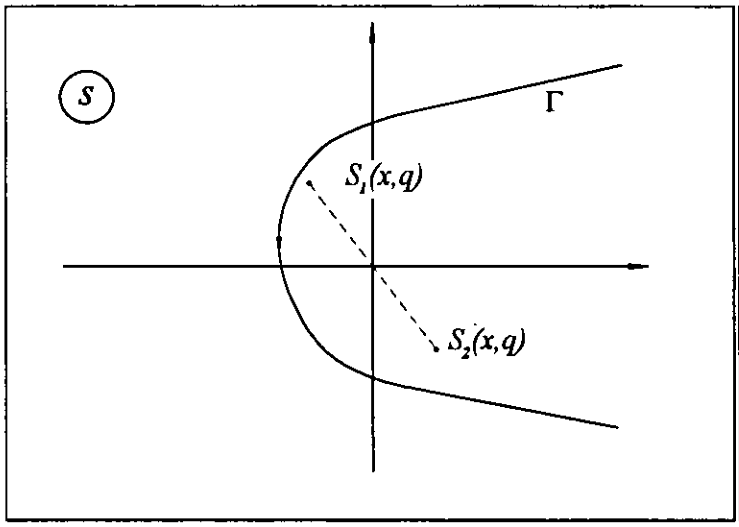

Figure 4:

$$\begin{cases} \Delta\,(\mathcal{A}F_0^1) = -\mathcal{A}^{-2}F_0^1, \\ \Delta\,(\mathcal{A}^{-1}F_0^1) = \mathcal{A}^2F_0^1. \end{cases}$$

Denoting by F_d and F_r the dominant and the recessive components at the point $q = q_1$, correspondingly:

$$F_d = \mathcal{A}F_0^1 = F_1^1,\ F_r = \mathcal{A}^{-2}F_0^1 = F_1^2,$$

we rewrite the latter system in the form

$$\begin{cases} \Delta F_d = -F_r, \\ (\mathcal{A}^{-1}\Delta\mathcal{A})\,F_r = F_d. \end{cases}$$

Now we notice that the operator $\mathcal{A}^{-1}\Delta\mathcal{A}$ in the second equation is none more than the alient derivative of the microfunction F_r at the point $s_2\,(x,q)$ (as well as the operator Δ in the first equation can be treated as the alient derivative of the microfunction F_d at $s_1\,(x,q)$). Thus, the latter system can be considered as the system of resurgent equations

$$\begin{cases} \Delta_{s_1}F_d = -F_r, \\ \Delta_{s_2}F_r = F_d \end{cases} \tag{18}$$

at one and the same value x of the parameter.

To present the general solution to the above derived resurgent system we need some facts from the general theory of resurgent equations. Let us briefly recall these facts (see

[13], [2]). Consider the following system of linear alient differential equations with resurgent coefficients

$$\begin{cases} \Delta_{s_1} F = A_1(x) F + B_1(x) G, \\ \Delta_{s_2} G = A_2(x) F + B_2(x) G, \\ \Delta_s F = 0 \text{ for } s \neq s_1, \text{ and } \Delta_s G = 0 \text{ for } s \neq s_2. \end{cases} \tag{19}$$

The following statement is valid.

Theorem 3 *Let (F_1, G_1) and (F_2, G_2) are two solutions to (19) such that* [6]

$$D \stackrel{\text{def}}{=} \begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix} = F_1 G_2 - F_2 G_1 \tag{20}$$

is an invertible element in $\mathcal{M}_{0,\text{cont}}$. Then the general solution to (19) is given by

$$\begin{pmatrix} F \\ G \end{pmatrix} = C_1 \begin{pmatrix} F_1 \\ G_1 \end{pmatrix} + C_2 \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} \tag{21}$$

where C_1 and C_2 are constants of resurgence, that is elements $C_1, C_2 \in \mathcal{M}_{0,\text{cont}}$ such that $\Delta_s C_j = 0$ for any $s \in \mathbf{C}$.

The next step in the investigation of the obtained system of resurgent equations is to show that the classical Airy function (13) is a resurgent function satisfying system (18). Performing the variable change

$$\left(\xi x - \frac{\xi^3}{3}\right) = -s, \tag{22}$$

we reduce the definition of the function $Ai(x,k)$ to the form

$$Ai(x,k) = \int_{\gamma'} e^{-ks} \frac{d\xi(s,x)}{ds} \, ds$$

where $\xi = \xi(s,x)$ is an (in general, ramifying) solution to equation (22) with respect to ξ. It is easy to see that the singular points of the function $\xi(s,x)$ are posited at

$$s = \frac{2}{3} x^{\frac{3}{2}}$$

which coinside with formula (12). Further, directly from the definition of the function $u(x,k)$ it follows that this function is univalued in a neighborhood of the corresponding focal points (in fact, the function defined by formula (13) is an entire function of the variable

[6]All products below are products in the convolutive algebra $\mathcal{M}_{0,\text{cont}}$ of infinitely continuable microfunctions.

x). Therefore, the two microfunctions $F_d^{(1)}$ and $F_r^{(1)}$ determined by the function $d\xi(s,x)/ds$ form a solution of resurgent system (18).

Since alient derivatives commute with usual ones, one can easily construct another solution $\left(F_d^{(2)}, F_r^{(2)}\right)$ to (18) where $F_d^{(2)}$ and $F_r^{(2)}$ are correspondingly the dominant and the recessive components of the function $\partial u(x,k)/\partial x$. The corresponding determinant

$$D = \begin{vmatrix} F_d^{(1)} & F_d^{(2)} \\ F_r^{(1)} & F_r^{(2)} \end{vmatrix} \tag{23}$$

is the Borel transform of the Wronskian

$$J = \begin{vmatrix} u_d(x,k) & \dfrac{\partial u_d(x,k)}{\partial x} \\ u_r(x,k) & \dfrac{\partial u_r(x,k)}{\partial x} \end{vmatrix},$$

where $u_d(x,q)$ and $u_r(x,q)$ are the Laplace transforms of the microfunctions $F_d^{(1)}$ and $F_r^{(1)}$. Since the functions u_d and u_r are two linearly independent solutions to equation (14), the function J is a nonvanishing constant and, therefore, the microfunction D given by (23) is an invertible element of $M_{0,\mathrm{cont}}$. Now, using Theorem 3, we can write down the general solution to resurgent system (18):

$$\begin{pmatrix} F_d \\ F_r \end{pmatrix} = C_1 \begin{pmatrix} F_d^{(1)} \\ F_r^{(1)} \end{pmatrix} + C_2 \begin{pmatrix} F_d^{(2)} \\ F_r^{(2)} \end{pmatrix}$$

with arbitrary constants of resurgence C_1 and C_2.

The latter formula allows to investigate the Stokes phenomenon for any solution (F_d, F_r) to resurgent system (18). Actually, it is easy to see that the Riemannian surface of all of the functions $F_d^{(j)}$, $F_r^{(j)}$, $j = 1,2$ is such as it is shown on Figure 5. Then one can easily compute alient derivatives (and, hence, the connection homomorphisms) from these functions. Then the latter formulas allow one to compute alient derivatives and the connection homomorphisms from an arbitrary solution to resurgent equations (18).

4.2 Cylinder-parabolic functions

In this Subsection we shall investigate Laplace integrals with resurgent structure coinciding with that of the cylinder-parabolic functions[7]. We recall that these functions are determined as solutions to the Weber equation

$$k^{-2}\frac{d^2u}{dx^2} - x^2u = 0 \tag{24}$$

[7]Such functions are also called the Weber functions.

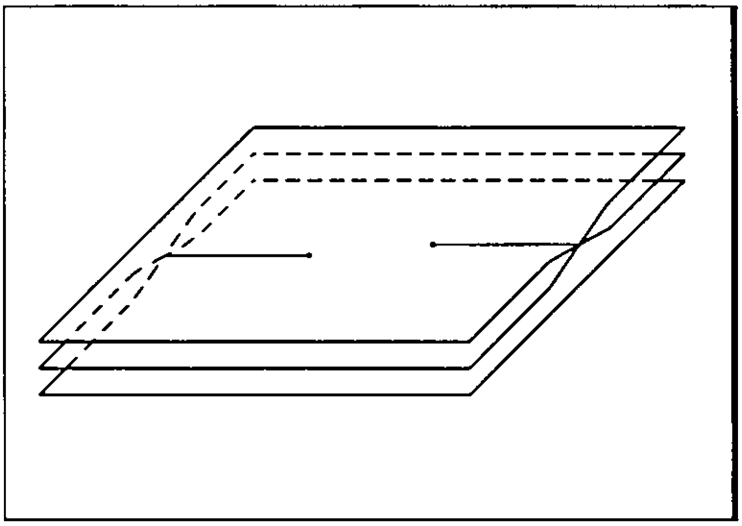

Figure 5:

and have the following integral representation

$$u(x,k) = k^{\frac{1}{2}} \int\limits_{\gamma} e^{k\left(2p^2x+\frac{x^2}{2}+p^4\right)}\, dp,$$

where the contour γ is chosen in such a way that the exponential under the integral sign on the right in the latter expression decreases along this contour when $|p| \to \infty$. As above, the variable change

$$, \; s = -\left(2p^2x + \frac{x^2}{2} + p^4\right)$$

leads us to the formula

$$u(x,q) = k^{\frac{1}{2}} \int\limits_{\gamma} e^{-ks} \frac{dp}{ds}(s,x)\, ds. \tag{25}$$

The function $p = p(s,x)$ can be computed in the explicit way:

$$p(s,x) = \sqrt{-x + \sqrt{s + \frac{x^2}{2}}}$$

and has the two ramification points $s = \pm x^2/2$. The Riemannian surface of this function is drawn on Figure 6. One can easily check that the two microfunctions corresponding to the singular point $s = -x^2/2$ differ from each other only by the sign and, hence, formula (25) gives us the two linearly independent solutions to equation (24) if we choose the two

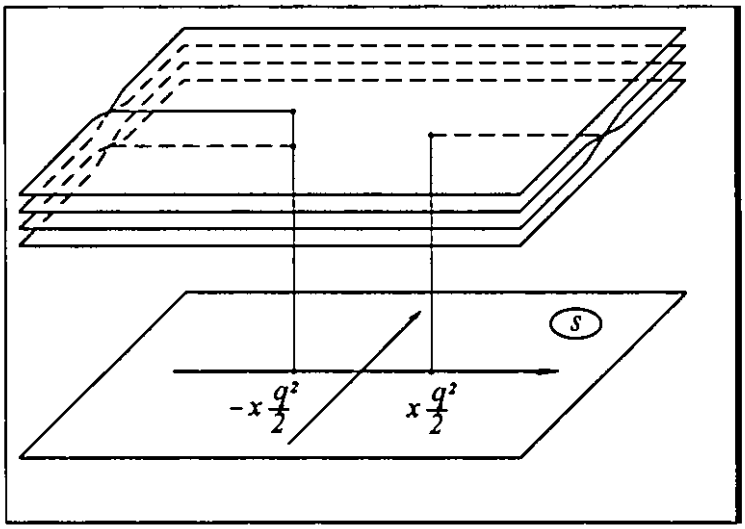

Figure 6:

contours γ in such a way that (25) is the Laplace transform of the microfunctions determined by $p(s,x)$ at points $s = \pm x^2/2$. Thus, we have constructed the two linearly independent resurgent solutions to the Weber equation with the resurgent structure given by

$$s = \pm x^2/2. \tag{26}$$

Now we shall investigate the general form of the Laplace integral

$$u(x,k) = \int_\gamma e^{kS(x,\xi)} a(x,\xi)\, d\xi$$

with the resurgent structure given by (26). The illumination diagram is shown on Figure 7. The corresponding system of resurgent equations is

$$\begin{cases} F_1^1 = \mathcal{A}^2 F_3^1 - \Delta(\mathcal{A}F_4^2), \\ F_3^1 = \mathcal{A}^2 F_1^1 - \Delta(\mathcal{A}F_2^2), \\ F_2^2 = \mathcal{A}^2 F_4^2 - \Delta(\mathcal{A}F_1^1), \\ F_4^2 = \mathcal{A}^2 F_2^2 - \Delta(\mathcal{A}F_3^1). \end{cases}$$

To write down the corresponding system of resurgent equations for the dominant (at the point x_1) microfunction, we put $F_1^1 = 0$ and obtain the following system of resurgent equations:

$$\begin{cases} \mathcal{A}^{-1}\Delta(\mathcal{A}F_3^1) = \mathcal{A}^3 F_4^2 - \mathcal{A}^{-1}F_4^2, \\ \mathcal{A}^{-2}\Delta(\mathcal{A}F_4^2) = -F_3^1, \\ \Delta(\mathcal{A}^3 F_4^2) = F_3^1. \end{cases}$$

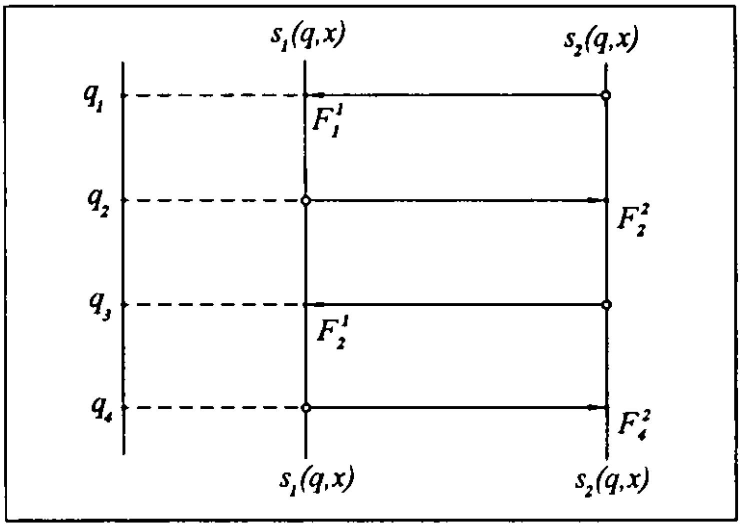

Figure 7:

Denoting $G_1 = F_3^1$, $G_2' = \mathcal{A}^3 F_4^2$, and $G_2'' = \mathcal{A}^{-1} F_4^2$ we reduce the latter system to the form

$$\begin{cases} \Delta_{s_1} G_1 = G_2' - G_2'', \\ \Delta_{s_2} G_2' = G_1, \\ \Delta_{s_2} G_2'' = -G_1. \end{cases} \tag{27}$$

One can see that the obtained system contains, unlike the systems constructed in the previous example, the two microfunctions G_2' and G_2'' at one and the same point of singularity. The reason for this phenomenon the reader can see from Figure 6, where the Riemannian surface of the typical solution to the constructed system of resurgent equations is drawn. Actually, it is clear that the function with such a Riemannian surface determines at one of its singular points two (different, in general) microfunctions.

Now let us try to construct the general solution to system (27). First of all, adding the second equation of this system to the third one, we obtain the relation

$$\Delta_{s_2} (G_2' + G_2'') = 0$$

and, hence, the function $G_2' + G_2''$ is a constant of resurgence:

$$G_2' + G_2'' = C_1.$$

Excluding the microfunction G_2'' from system (27) with the help of the latter relation, we obtain

$$\begin{cases} \Delta_{s_1} G_1 = 2G_2' - C_1, \\ \Delta_{s_2} G_2' = G_1. \end{cases}$$

Using the substitution $G_2' = G_2''' + C_1/2$, we reduce the considered system to the form

$$\begin{cases} \Delta_{s_1} G_1 = 2G_2''', \\ \Delta_{s_2} G_2''' = G_1. \end{cases} \tag{28}$$

It is not hard to verify that the two microfunctions $H_1^{(1)}$ and $H_2^{(1)}$ determined by function (25) as well as the two microfunctions $H_1^{(2)}$ and $H_2^{(2)}$ determined by the derivative of (25) satisfy the latter system. Arguing similar to the preceding Subsection, we shall see that the general solution to (28) has the form

$$\begin{pmatrix} G_1 \\ G_2''' \end{pmatrix} = C_2 \begin{pmatrix} H_1^{(1)} \\ H_2^{(1)} \end{pmatrix} + C_3 \begin{pmatrix} H_1^{(2)} \\ H_2^{(2)} \end{pmatrix}.$$

Thus, the general solution to (27) is given by

$$\begin{cases} G_1 = C_2 H_1^{(1)} + C_3 H_1^{(2)}, \\ G_2' = \frac{C_1}{2} + C_2 H_2^{(1)} + C_3 H_2^{(2)}, \\ G_2'' = \frac{C_1}{2} - C_2 H_2^{(1)} - C_3 H_2^{(2)}, \end{cases}$$

where C_j, $j = 1, 2, 3$ are arbitrary constants of resurgence. Certainly, to describe the general form of a function with resurgent structure of the Weber type, one should add to the obtained general solution the similar one corresponding to the case when the singular points $s = s_1(x, q)$ and $s = s_2(x, q)$ are interchanged. We leave the corresponding computations to the reader. The only thing we shall mention in conclusion to this Subsection is that, unlike the previous example, the resurgent system obtained here admits a resurgent constant as the solution. The reason for this is that the monodromy of singular points corresponding to resurgent functions of the Weber type is trivial and, hence, we obtain the corresponding trivial solution to the obtained resurgent system.

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