DUPIN HYPERSURFACES

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0) Introduction

In [8] Cecil and Ryan propose to study the class of those hypersurfaces M in the sphere S^n with the property that each principal curvature of M is constant along the leaves of its corresponding principal foliation. Hypersurfaces with this property can be considered also in euclidean space or in hyperbolic space and we will call such a hypersurface a Dupin hypersurface. In $\mathbb{R}^3$ the only Dupin hypersurfaces are spheres, planes and the well-known cyclides of Dupin.

It is known that the class of Dupin hypersurfaces in S^n is invariant under conformal transformations of S^n. In this paper we will show that this class is invariant even under the transformations of Lie's sphere geometry. This geometry therefore provides a suitable setting for the study of Dupin hypersurfaces. We will give here a self-contained introduction to the differential geometric aspects of Lie geometry.

If, at each point, a Dupin hypersurface in $\mathbb{R}^n$ has only two distinct principal curvatures of multiplicities p,q $(p+q = n-1)$, it is called a cyclide of Dupin of characteristic (p,q). In [6,7] Cecil and Ryan classified all cyclides of Dupin that are complete with respect to the metric induced by $\mathbb{R}^n$. Even for $n = 3$ the condition of completeness excludes some interesting examples (see fig. 2). From the viewpoint of Lie-geometry it is easy

to decribe all cyclides of Dupin without any completeness assumption. It turns out that up to Lie-transformations there is only one cyclide of Dupin of a given characteristic.
Finally we discuss examples of Dupin hypersurfaces. Because of conformal invariance it does not matter whether we describe these examples as hypersurfaces of S^n or of any other space of constant curvature. The examples include the isoparametric hypersurfaces in S^n and the tubes around extrinsically symmetric submanifolds of $\mathbb{R}^n$, S^n and H^n. (For a classification of the latter see [2,9,16].

We shall indicate several inductive procedures that yield a Dupin hypersurface in $\mathbb{R}^{n+1}$ given one in $\mathbb{R}^n$. This can be used to show that for every n there are hypersurfaces of Dupin with any prescribed multiplicities of the principal curvatures.

1) Lie geometric hypersurfaces

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. Throughout this paper we will assume $n \geq 2$. A unit tangent vector to S^n at some point $\mathfrak{p}$ can be described as a pair $(\mathfrak{p},\mathfrak{n}) \in S^n \times S^n$ with $\mathfrak{p} \cdot \mathfrak{n} = 0$. Here "$\cdot$" denotes the scalar product in $\mathbb{R}^{n+1}$. We will identify the unit tangent bundle US^n of S^n with the set of all such pairs.

Let $p, n : US^n \longrightarrow S^n$ denote the restrictions to US^n of the canonical projections of $S^n \times S^n$ onto its factors. Then the equation $dp \cdot n = 0$ defines a codimension one distribution D on US^n. If $\mathfrak{p} : M^{n-1} \longrightarrow S^n$ is an immersion, $\mathfrak{n} : M^{n-1} \longrightarrow S^n$ a unit normal vector field for $\mathfrak{p}$ then $(\mathfrak{p},\mathfrak{n}) : M^{n-1} \longrightarrow US^n$ is an integral manifold of the distribution D. This motivates the following

Definition: A Lie geometric hypersurface in S^n is an $(n-1)$-dimensional integral manifold of the distribution D in US^n.

Remark: In the terminology of [1], appendix 4 D defines a "contact structure" on US^n. A Lie geometric hypersurface is a "Legendre submanifold" of this contact structure.

If $f : M^{n-1} \longrightarrow US^n$, $f = (\mathfrak{p}, \mathfrak{n})$ is a Lie geometric hypersurface then $\mathfrak{p} : M^{n-1} \longrightarrow S^n$ is a smooth mapping, but in general $\mathfrak{p}$ will not be an immersion. For example the unit normal bundle of any immersed submanifold N of S^n can be considered as a Lie geometric hypersurface, and in this case $\mathfrak{p}$ will not be an immersion unless the submanifold has codimension one. We will show however that in some sense a Lie geometric hypersurface is always "nearly" an immersed hypersurface: For each $\alpha \in [-\pi, \pi)$

$$(2) \qquad f_\alpha := (\cos\alpha\, \mathfrak{p} + \sin\alpha\, \mathfrak{n},\ -\sin\alpha\, \mathfrak{p} + \cos\alpha\, \mathfrak{n})$$

is again a Lie geometric hypersurface that will be called the parallel surface to f in the distance α.

Theorem 1: Let $f = (\mathfrak{p}, \mathfrak{n}) : M^{n-1} \longrightarrow US^n$ be a Lie geometric hypersurface, $f_\alpha = (\mathfrak{p}_\alpha, \mathfrak{n}_\alpha)$ the parallel surfaces of f. Then for each $q \in M^{n-1}$ the mapping $\mathfrak{p}_\alpha : M^{n-1} \longrightarrow S^n$ fails to be an immersion at q only for at most $n-1$ values of α in $[0, \pi)$.

Hence locally Lie geometric hypersurfaces are just parallel surfaces of ordinary hypersurfaces. The possible singularities of the mapping $\mathfrak{p}$ can always be "removed" by passing to a parallel surface.

Before we prove theorem 1 we first state two lemmas:

Lemma 1: Let M be differentiable manifold, W a real vectorspace endowed with a symmetric bilinear form $\langle , \rangle$, $q \in M$. Then for any two functions $\mathfrak{p},\mathfrak{n} : M \longrightarrow W$ such that $\langle \mathfrak{p},\mathfrak{n} \rangle =$ const and $\langle d\mathfrak{p},\mathfrak{n} \rangle = 0$ the bilinear form on T_qM

$$(3) \qquad (X,Y) \longmapsto \langle X\mathfrak{p}, Y\mathfrak{n} \rangle$$

is symmetric.

Proof: Let $X,Y \in T_qM$. Choose vectorfields $\tilde{X},\tilde{Y}$ in some neighborhood of q such that $\tilde{X}_q = X$, $\tilde{Y}_q = Y$ and $[X,Y] = 0$. Then differentiating the equation $\langle \mathfrak{p},\mathfrak{n} \rangle =$ const and observing $\langle d\mathfrak{p},\mathfrak{n} \rangle = 0$ we obtain

$$(4) \qquad \langle \mathfrak{p}, X\mathfrak{n} \rangle = 0 \ .$$

Differentiating again in the direction Y we arrive at

$$(5) \qquad \langle Y\mathfrak{p}, X\mathfrak{n} \rangle + \langle \mathfrak{p}, YX\mathfrak{n} \rangle = 0 \ .$$

Similarly we have

$$(6) \qquad \langle X\mathfrak{p}, Y\mathfrak{n} \rangle + \langle \mathfrak{p}, XY\mathfrak{n} \rangle = 0 \ .$$

Because $XY = YX$ equations (5) and (6) imply (3).

□

<u>Lemma 2</u>: Let V,W be real vectorspaces, $\dim V = \dim W = n$, $<,>$ a positive definite scalar product on W. Suppose $A,B : V \longrightarrow W$ are linear and satisfy the following conditions:

a) $<AX,BY> = <AY,BX>$ for all $X,Y \in V$.

b) $\text{kern } A \cap \text{kern } B = \{0\}$.

Then $\lambda A + B : V \longrightarrow W$ fails to be a bijection for at most $[n-1-\dim \text{kern } A]$ values of $\lambda \in \mathbb{R}$.

<u>Proof</u>: Set $\overline{V} := V/\text{kern } A$, $\overline{W} = \text{Im } A$. For $X \in V$ let $\overline{X}$ denote the image of X in $\overline{V}$ under the canonical projection. For $Y \in W$ the orthogonal projection of Y onto $\overline{W}$ will be denoted by $\overline{Y}$. We can define a mapping $\overline{B} : \overline{V} \longrightarrow \overline{W}$ by setting

$$\overline{B}(\overline{X}) = \overline{B(X)}, \tag{7}$$

because by property a) for $X \in \text{kern } A$ we have

$$< BX,AY > = 0, \quad \text{all } Y \in V \tag{8}$$

i.e. $B(\text{kern } A) \subset (\text{Im } A)^{\perp}$. Similarly we define a mapping $\overline{A} : \overline{V} \longrightarrow \overline{W}$ by setting $\overline{A}(\overline{X}) = \overline{A(X)}$. $\overline{A}$ is bijective. We can make $\overline{V}$ into a euclidean vectorspace by means of

$$< \overline{X},\overline{Y} > := < \overline{A}\,\overline{X}, \overline{A}\,\overline{Y} >. \tag{9}$$

If we define $L : \overline{V} \longrightarrow \overline{V}$ by $L = \overline{A}^{-1} \circ \overline{B}$ then for all $X,Y \in V$ we have

$$(10) \qquad \langle AX, BY\rangle = \langle AX, \overline{BY}\rangle = \langle \overline{A}\,\overline{X}, \overline{B}\,\overline{Y}\rangle = \langle \overline{X}, L\overline{Y}\rangle .$$

By (10) and property a) L is self-adjoint. Furthermore for all $X \in V$ we have

$$(11) \qquad \langle BX, BX\rangle \geq \langle \overline{BX}, \overline{BX}\rangle = \langle \overline{B}\,\overline{X}, \overline{B}\,\overline{X}\rangle = \langle L\overline{X}, L\overline{X}\rangle = \langle \overline{X}, L^2\overline{X}\rangle .$$

For $X \in V$, $\lambda \in \mathbb{R}$ we have $X \in$ kern $(\lambda A+B)$ if and only if

$$(12) \qquad \langle \lambda AX+BX, \lambda AX+BX\rangle = 0$$

By (11) the left hand term in (12) is greater or equal to

$$(13) \qquad \langle(\lambda I+L)\overline{X}, (\lambda I+L)\overline{X}\rangle \geq 0 .$$

Hence $X \in$ kern $(\lambda A+B)$ is equivalent to $\overline{X} \in$ kern $(\lambda I+L)$. Clearly $\lambda I+L$ fails to be bijective for at most ($n-1-\dim$ kern A) values of λ. We show that for these values of λ also $\lambda A+B$ is bijective: For such λ-values $X \in$ kern $(\lambda A+B)$ implies $\overline{X} = 0$ i.e. $X \in$ kern A , hence also $X \in$ kern B and by property b) $X = 0$.

□

<u>Proof of theorem 1</u>: By lemma 1 the mappings

$A = dp|_q : T_qM^{n-1} \longrightarrow [p(q), n(q)]^\perp$ and $B = dn|_q : T_qM^{n-1} \longrightarrow [p(q), n(q)]^\perp$

satisfy property a) of lemma 2. Property b) is satisfied because f is a submanifold of US^n. Therefore by lemma 2 $dp_\alpha|_q = \cos\alpha A + \sin\alpha B$ is bijective with the exception of at most $n-1$ values of α in $[0,\pi)$.

□

2) Lie transformations

Let $S^{n-1} \subset S^n$ be an umbillic hypersphere, $p : S^{n-1} \longrightarrow S^n$ the inclusion and $n : S^{n-1} \longrightarrow S^n$ a unit normal vector field on S^{n-1}. Then $(p,n) : S^{n-1} \longrightarrow US^n$ is a Lie geometric hypersurface that will be called an <u>oriented sphere</u>. For $q \in S^n$ let $i_q : U_qS^n \longrightarrow US^n$ denote the inclusion of the fibre of US^n over q into US^n. Then i_q is a Lie geometric hypersurface that may be considered as a limiting case (as S^{n-1} shrinks to a point) of an oriented sphere. It will be useful for our purpose to call also i_q an oriented sphere.

<u>Definition</u>: A diffeomorphism $\varphi : US^n \longrightarrow US^n$ is called a <u>Lie transformation</u> if it carries oriented spheres into oriented spheres.

It is easy to see that every Lie transformation is automatically a "contact transformation", that means it preserves the distribution D. Therefore a Lie transformation always carries Lie geometric hypersurfaces into Lie geometric hypersurfaces.

Examples of Lie transformations are

1) The "prolongation" to US^n of conformal transformations $\Psi : S^n \longrightarrow S^n$, defined as $\frac{d\Psi}{|d\Psi|}$.

2) The mappings $\varphi_\alpha : US^n \longrightarrow US^n$

$$(\mathbf{p},\mathbf{n}) \longmapsto (\cos\alpha\,\mathbf{p} + \sin\alpha\,\mathbf{n}), -\sin\alpha\,\mathbf{p} + \cos\alpha\,\mathbf{n})$$

already appearing in the last chapter.

We will prove below that the group G of all Lie transformations is a Lie group isomorphic to $O(n+1,2)/\{-I,I\}$; but first we have to give a convenient description of the manifold Q^{n+1} of all oriented spheres in S^n.

For $\mathbf{m} \in S^n$, $\rho \in \mathbb{R}$ those $(\mathbf{p},\mathbf{n}) \in US^n$ with

(14) $$\mathbf{m} = \cos\rho\,\mathbf{p} - \sin\rho\,\mathbf{n}$$

form an oriented sphere (fig. 1). $\mathbf{m}$ is one of the

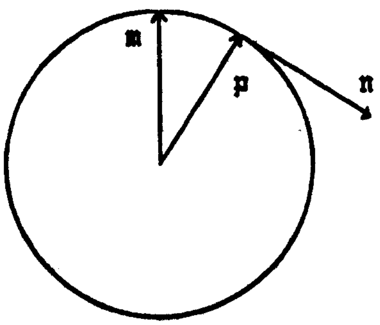

fig. 1

two "centers" of the sphere. If $\mathbf{k} = (\mathbf{x},x,y) \in \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}$ is any vector such that $\mathbf{k} \neq 0$, $\mathbf{x}\cdot\mathbf{x} - x^2 - y^2 = 0$ then there is a multiple

$\tilde{k} = \lambda k = (m, \tilde{x}, \tilde{y})$ of k such that $m \cdot m = \tilde{x}^2 + \tilde{y}^2 = 1$, that means

(15) $$\tilde{k} = (m, \cos \rho , \sin \rho)$$

for some $\rho \in \mathbb{R}$. $\tilde{k}$ is uniquely determined by k up to sign. Via (14) there corresponds to $\tilde{k}$ (hence also to k) an oriented sphere in S^n . Because all nonzero multiples of k determine the same oriented sphere we have a mapping of the quadric

(16) $$Q = \{[k] \in \mathbb{P}^{n+2} \,|< k , k > = 0\}$$

onto the set of all oriented spheres in S^n . Here $\mathbb{P}^{n+2}$ is the real projective space corresponding to the vectorspace $V_{n+3} := \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}$, $[k]$ is the projective point spanned by k and $<,>$ is the symmetric bilinear form on V_{n+3} with

(17) $$<(m, x, y), (m, x, y)> = m \cdot m - x^2 - y^2 \quad .$$

It is easy to verify that this correspondence between projective points on the quadric Q and oriented spheres in S^n is bijective. For the sake of brevity we will usually omit the word "oriented" and simply speak of "the sphere $[k]$ " or even "the sphere k ". Two oriented spheres are said to be in <u>oriented contact</u> if they intersect as subsets of US^n .

<u>Lemma 3</u>: Two oriented spheres $[k_1]$, $[k_2]$ are in oriented contact if and only if $<k_1, k_2> = 0$.

The proof is left to the reader. Lemma 3 implies that two spheres

$[k_1],[k_2]$ are in oriented contact if and only if the projective line in $\mathbb{P}^{n+2}$ spanned by $[k_1]$ and $[k_2]$ is entirely contained in Q.

We had defined a Lie transformation as a sphere-preserving diffeomorphism $US^n \longrightarrow US^n$. Every Lie transformation induces in the obvious way a line-preserving mapping $Q \longrightarrow Q$. Conversely every line-preserving diffeomorphism $Q \longrightarrow Q$ determines a Lie transformation.

Lemma 4: Every line-preserving diffeomorphism $\varphi : Q \longrightarrow Q$ is the restriction to Q of a projective transformation $\alpha : \mathbb{P}^{n+2} \longrightarrow \mathbb{P}^{n+2}$ leaving Q fixed.

Proof: Q carries in a natural way a conformal structure [4]. For each $p \in Q$ the light cone of this conformal structure in T_pQ consists just of the tangent vectors to the lines through p that lie on Q. Thus $\varphi : Q \longrightarrow Q$ must be conformal. The lemma now follows from lemma 6 of [4].

□

Corollary: The group of all Lie transformations is a Lie group isomorphic to $O(n+1,2)/\{-I,I\}$.

Remark: It can be proved that the group of all Lie transformations is generated by the special transformations of type 1) and 2) on page 8.

3) Curvature surfaces

In the rest of this paper we will study properties of Lie geometric hypersurfaces that are invariant under Lie transformations. The results of the last chapter show that the group G of all Lie transformations acts on the manifold Q of all oriented spheres in a way that is easy to describe algebraically. The action of G on US^n is visualized best if we identify US^n with the manifold of all projective lines on Q. It will thus be useful to have a description of Lie geometric hypersurfaces in terms of oriented spheres.

Let $(\mathfrak{p},\mathfrak{n}) : M \longrightarrow US^n$ be a Lie geometric hypersurface. Define $[\mathfrak{k}_1],[\mathfrak{k}_2] : M \longrightarrow Q$ by

(18) $$\mathfrak{k}_1 = (\mathfrak{p},1,0)$$
$$\mathfrak{k}_2 = (\mathfrak{n},0,1) \; .$$

Then $\mathfrak{k}_1$ and $\mathfrak{k}_2$ satisfy the following conditions: .

a) For all $p \in M$ the vectors $\mathfrak{k}_1(p)$ and $\mathfrak{k}_2(p)$ are linearly independent and we have

$$<\mathfrak{k}_i,\mathfrak{k}_j> = 0 \quad , \quad i,j = 1,2 \; .$$

b) There is no $p \in M,\ X \in T_pM$ such that simultaneously $d\mathfrak{k}_1(X)$ and $d\mathfrak{k}_2(X)$ are in span $(\mathfrak{k}_1(p),\ \mathfrak{k}_2(p))$.

c) $$<d\mathfrak{k}_1,\ \mathfrak{k}_2> = 0 \; .$$

The properties a), b), c) are conserved if we pass from $\mathfrak{k}_1$, $\mathfrak{k}_1$ to

(19) $$\tilde{k}_1 = \alpha\, k_1 + \beta k_2$$
$$\tilde{k}_2 = \gamma\, k_1 + \delta\, k_2$$

where $\alpha,\beta,\gamma,\delta : M \longrightarrow \mathbb{R}$ are functions such that $\alpha\delta - \beta\gamma$ is everywhere different from zero on M. Conversely, if we are given $k_1,k_2 : M \longrightarrow V_{n+3}$ satisfying a), b) and c) then it is easy to verify that there are functions $\alpha,\beta,\gamma,\delta : M \longrightarrow \mathbb{R}$, $p,n : M \longrightarrow S^n$ such that

(20) $$\alpha k_1 + \beta k_2 = (p,1,0)$$
$$\gamma k_1 + \delta k_2 = (n,0,1)$$

and that $(p,n) : M \longrightarrow US^n$ is a Lie geometric hypersurface. Thus every Lie geometric hypersurface in S^n can be described by an $(n-1)$-dimensional manifold M and a pair of functions $k_1,k_2 : M \longrightarrow V_{n+3}$ subject to conditions a),b),c). We will denote such a description by (M,k_1,k_2) . Geometrically this description amounts to the following: Let $(p,n) : M \longrightarrow US^n$ be a Lie geometric hypersurface such that p is an immersion. Then for any $q \in M$ we select in a smooth way two oriented hyperspheres $k_1(q),k_2(q)$ that are tangent to the hypersurface at $p(q)$ and that provide the right orientation at $p(q)$ (defined by the normal vector $n(q)$).

<u>Definition</u>: Let (M,k_1,k_2) be a Lie geometric hypersurface, $p \in M, \lambda,\mu \in \mathbb{R}$, $(\lambda,\mu) \neq (0,0)$. Then the sphere $k = \lambda k_1(p) + \mu k_2(p)$ is called a <u>curvature sphere</u> at p if there is a tangent vector $X \in T_pM , X \neq 0$ such that

(21) $\lambda d\mathfrak{k}_1(X)+\mu d\mathfrak{k}_2(X) \in \text{span}\ (\mathfrak{k}_1(p),\mathfrak{k}_2(p))$.

X is then called a "direction of curvature" corresponding to $\mathfrak{k}$.

Note that the above definition is invariant with respect to the transformations (19), i.e. the definition does not depend on the choice of $\mathfrak{k}_1,\mathfrak{k}_2$. Obviously the notion of a curvature sphere is invariant with respect to Lie transformations.

Let $(M,\mathfrak{k}_1,\mathfrak{k}_2)$ be a Lie geometric hypersurface in S^n , $x \in M$ and suppose that $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are given as in (18). By theorem 1 we can assume without loss of generality (applying if necessary a Lie transformation) that $\mathfrak{p}$ is an immersion in some neighborhood of x . It is then easy to verify that $Y \in T_xM$ is a direction of curvature with corresponding curvature sphere $\mathfrak{k} = \cos\rho\,\mathfrak{k}_1(x) + \sin\rho\,\mathfrak{k}_2(x)$ if and only if $\lambda = -\operatorname{arcot}(\rho)$ is a principal curvature of the hypersurface $\mathfrak{p}$ at x and $\mathfrak{m} = \cos \mathfrak{p} + \sin\rho\,\mathfrak{n}$ (one of the two center points of $\mathfrak{k}$) is a corresponding focal point. Thus we can apply well known results in the theory of hypersurfaces in S^n to obtain

<u>Proposition 1</u>: Let $(M,\mathfrak{k}_1,\mathfrak{k}_2)$ be a Lie geometric hypersurface in S^n . Then

a) At each point $p \in M$ there are at most $n-1$ different spheres of curvature $\tilde{\mathfrak{k}}_1,\dots,\tilde{\mathfrak{k}}_r$.

b) The directions of curvature corresponding to $\tilde{\mathfrak{k}}_\alpha$ form a linear subspace U_α of T_pM .

c) $T_pM = U_1 \oplus \dots \oplus U_r$.

d) If $\dim U_\alpha$ is constant on an open set $V \subset M$ then the distribution on V defined by U_α is integrable.

The dimension of U_α is called the multiplicity of the curvature sphere $\tilde{k}_\alpha$.

By a curvature surface of (M, k_1, k_2) we mean a connected submanifold N of M with the property that at each point p of N the tangent space T_pN is equal to one of the U_α . It follows from [14], proposition 1 that there is an open dense set $M_0 \subset M$ such that the multiplicities of the principal curvatures are locally constant on M_0 . If p is in M_0, $\tilde{k}_\alpha$ a curvature sphere at p and $U_\alpha \subset T_pM$ the corresponding subspace of directions of curvature then by d) of theorem 2 U_α is the tangent space at p of some curvature surface through p .

4) Dupin hypersurfaces

Definition: A Lie geometric hypersurface in S^n is called a Dupin hypersurface if along each curvature surface the corresponding principal curvature is constant. A Dupin hypersurface is called proper if all curvature spheres have constant multiplicities on M .

If a Lie geometric hypersurface $(\mathfrak{p}, \mathfrak{n}) : M \longrightarrow US^n$ is also a hypersurface in the usual sense (i.e. $\not{p} : M \longrightarrow S^n$ is an immersion) then we can talk about the principal curvatures of $(\not{p},\not{n})$. Using the representation (18) it is easy to see that in this case Dupin hypersurfaces are characterized by the fact that along each curvature surface the corresponding principal curvature is constant.

The next proposition shows that in order to check whether a given hypersurface is a Dupin hypersurface it is sufficient to consider the <u>lines of curvature</u>, i.e. the one-dimensional curvature surfaces.

<u>Proposition 2</u>: Let S be curvature surface in a Lie geometric hypersurface $(\not{p},\not{n}) : M \longrightarrow US^n$ and suppose $\dim S \geq 2$. Then along S the corresponding curvature sphere is constant.

<u>Proof</u>: It is sufficient to show that along S the corresponding curvature sphere is <u>locally</u> constant, hence we may assume that $\not{p} : M \longrightarrow S^n$ is an immersion. Then we have to show that along S the corresponding principal curvature is constant. To prove this the argument in the proof of proposition 2.3 in [15] can be applied with some obvious modifications.

□

Because the multiplicity of a continuous principal curvature function on a hypersurface of S^n is upper semicontinuous there is for every line of curvature S an open neighborhood U in M such that the multiplicity of the principal curvature corresponding to S is equal to one on U. Thus by the next proposition the lines of curvature of a Dupin hypersurface $(\mathfrak{p},\mathfrak{n}) : M \longrightarrow US^n$ are mapped by $\mathfrak{p}$ either onto points or onto pieces of circles.

<u>Proposition 3</u>: Let $(\mathfrak{p},\mathfrak{n}) : M \longrightarrow US^n$ be a Lie geometric hypersurface, $S \subset M$ a d-dimensional curvature surface, U an open neighborhood of S in M such that

(i) along S the corresponding curvature sphere is constant.

(ii) the multiplicity of the curvature sphere corresponding to S is constant on U .

Then $\mathfrak{p}(S)$ is either a single point or a d-dimensional umbillic submanifold of S^n .

<u>Proof</u>: Let $(\mathfrak{p}_\varphi,\mathfrak{n}_\varphi) : M \longrightarrow US^n$ be a parallel surface to $(\mathfrak{p},\mathfrak{n})$ such that $\mathfrak{p} : M \longrightarrow S^n$ is an immersion in some neighborhood U of a point $p \in S$ (cf. theorem 1). Then S is also a curvature surface of $(\mathfrak{p}_\varphi,\mathfrak{n}_\varphi)$ and by proposition 3.2 of [5] we know that $\mathfrak{p}_\varphi(S \cap U)$ is a d-dimensional umbillic submanifold of S^n . Now using property (i) it can be seen by elementary geometry that $\mathfrak{p}(U \cap S)$ is either a single point or also a d-dimensional umbillic submanifold of S^n . The same must then be true of course for $\mathfrak{p}(S)$.

□

<u>Theorem 2</u>: Let $(\mathfrak{p},\mathfrak{n}) : M \longrightarrow US^n$ be a Lie geometric hypersurface. Then the following statements are equivalent:

(i) $(\mathfrak{p},\mathfrak{n})$ is a Dupin hypersurface

(ii) Along every line of curvature (i.e. one-dimensional curvature surface) in M the corresponding curvature sphere is constant.

(iii) Every line of curvature in M is mapped by p either onto a single point or a piece of a circle in S^n .

<u>Proof</u>: According to the remark preceding proposition 3 we have (i) ⇒ (ii) . (ii) ⇒ (i) follows from proposition 2 and proposition 3.2 of [5] (general position can be achieved using our theorem 1).

□

5) Connection with euclidean and hyperbolic geometry

So far we have been concerned only with the geometry in S^n , but Lie geometric methods can also be applied to problems in euclidean and hyperbolic geometry. This is already clear from the fact that the group of all Lie transformations contains all conformal transformations of S^n , and from the conformal viewpoint all spaces of constant curvature are locally equivalent. For later use we supply some explicit formulas for the euclidean case:

A unit tangent vector to $\mathbb{R}^n$ can be described as a pair $(x,u) \in U\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$. An oriented sphere in $\mathbb{R}^n$ with center c and (signed) radius r is then the set of all $(x,u) \in U\mathbb{R}^n$ such that

(22) $$x = c + r u \ .$$

We now consider the stereographic projection

(23) $$\sigma : \mathbb{R}^n \longrightarrow S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$$

$$\sigma(x) = \left(\frac{2x}{1+x\cdot x}\ ,\ \frac{1-x\cdot x}{1+x\cdot x}\right) .$$

To each $(x,\mathfrak{u}) \in U\,\mathbb{R}^n$ we assign $(\mathfrak{p},\mathfrak{n}) \in US^n$ as

(24) $$(\mathfrak{p},\mathfrak{n}) = \left(\sigma(x)\ ,\ \frac{d\sigma(\mathfrak{u})}{\|d\sigma(\mathfrak{u})\|}\right).$$

Explicitly we have

(25) $$\mathfrak{n} = (\mathfrak{u},-x\cdot\mathfrak{u}) - (x\cdot\mathfrak{u})\,\mathfrak{p} \ .$$

Using (23), (24) and (25) a straightforward computation shows that (22) is equivalent to (14) if we define $\mathfrak{m}$ and ρ in such a way that $(\mathfrak{m}, \cos\rho, \sin\rho)$ is proportional to

(26) $$\mathfrak{k} \in \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

$$\mathfrak{k} = (2\mathfrak{c}, 1-\mathfrak{c}\cdot\mathfrak{c} + r^2\ ,\ 1 + \mathfrak{c}\cdot\mathfrak{c} - r^2\ ,\ 2r) \ .$$

Similarly an oriented hyperplane in $\mathbb{R}^n$ with normal vector $\mathfrak{v}$ and oriented distance d from the origin can be described as the set of all $(x,\mathfrak{v}) \in U\,\mathbb{R}^n$ such that

(27) $$x\cdot\mathfrak{v} = d$$

Under stereographic projection there corresponds to this hyperplane the oriented sphere in S^n

(28) $$k = (v,-d,d,1) .$$

Formulas similar to (26) and (28) are the starting point in the classical treatment of Lie geometry in Blaschke [3].

Thus Lie geometry also describes the geometry of oriented spheres (including points, hyperplanes and the point at infinity) in euclidean space. If we now call a hypersurface in $\mathbb{R}^n$ a "Dupin hypersurface" if it is the stereographic projection of a Dupin hypersurface in S^n then it is clear that also in euclidean space Dupin hypersurfaces are characterized by the property that along each curvature surface the corresponding principal curvature is constant.

6) Cyclides of Dupin

A Dupin hypersurface in S^n is called a cyclide of Dupin of characteristic (p,q) if at each point it has exactly two curvature spheres of muliplicities p,q respectively.

Theorem 3:

a) Every connected cyclide of Dupin is contained in a unique compact and connected cyclide of Dupin.

b) Any two cyclides of Dupin with the same characteristic are locally Lie equivalent.

<u>Proof</u>: Let (M,k_1,k_2) be a cyclide of Dupin of characteristic (p,q) . We may assume that k_1 and k_2 are the curvature spheres of M . At each point $p \in M$ there are two complementary subspaces U_1 and U_2 of T_pM corresponding to k_1 and k_2 . By theorem 1 the distributions on M defined by these subspaces are integrable. As a consequence M can locally be identified with an open set $W = U \times V \subset \mathbb{R}^p \times \mathbb{R}^q$ such that

(i) $k_1(u,v)$ depends only on v for $(u,v) \in W$
$k_2(u,v)$ depends only on u for $(u,v) \in W$

(ii) $k_1(W)$ and $k_2(W)$ are smooth submanifolds of $Q \subset \mathbb{P}^{n+2}$ of dimensions q,p respectively.

We claim that $k_1(W)$ is contained in a $(q+1)$-dimensional linear subspace of $\mathbb{P}^{n+2}$. Suppose on the contrary that there are $q+2$ linearly independent points $k_1(w_i)$, $w_1,\dots,w_{q+2} \in W$. Now $\langle k_1(u,v), k_2(u,v)\rangle = 0$ for $(u,v) \in W$ together with (i) implies $\langle k_1(w), k_1(\tilde{w})\rangle = 0$ for all $w,\tilde{w} \in W$. So $k_2(w)$ is contained in the subspace

$$(29) \qquad E = [k_1(w_1),\dots,k_2(w_{q+2})]^{\perp} \subset \mathbb{P}^{n+2}$$

which has dimension $(n+2)-(q+1)-1 = p$. This is not possible since $k_2(W) \subset E \cap Q$ contradicts (ii).

We know that $k_1(W)$ is an open subset of $E_1 \cap Q$, where E_1 is a $(q+1)$-dimensional linear subspace of $\mathbb{P}^{n+2}$. A similar argument as above shows that $k_2(W)$ is an open subset of $E_2 \cap Q$, where $E_2 = E_1^{\perp}$ is the $(p+1)$-dimensional polar subspace of E_1 . It is easy to show that if E_1 and $E_2 := E_1^{\perp}$ are subspaces of $\mathbb{P}^{n+2}$ such that $E_1 \cap Q$ and $E_2 \cap Q$ contain manifolds of dimensions p and q respectively, then the scalar product $<,>$ must have signature

$$(30) \qquad (\underbrace{-+\ldots+}_{q+1}) \text{ on } E_1\,, \qquad (\underbrace{-+\ldots+}_{p+1}) \text{ on } E_2\,.$$

It is now clear, that a maximal connected cyclide containing (M, k_1, k_2) is given by $(S^q \times S^p, \tilde{k}_1, \tilde{k}_2)$ where $S^q := E_1 \cap Q$, $S^p = E_2 \cap Q$ and $\tilde{k}_1$ for example makes the following diagramm commute:

$$(31) \qquad \begin{array}{ccc} & & p^{-1}(S^q) \\ & \overset{\tilde{k}_1}{\nearrow} & \Big\downarrow p \\ S^q \times S^p & \xrightarrow[\pi_1]{} & S^q \end{array}$$

Here p is the natural projection of V_{n+3} onto $\mathbb{P}^{n+2}$ and π_1 projection onto the first factor. Both assertions of the theorem are now clear.

□

From the viewpoint of Lie-geometry the geometric structure of a cyclide of Dupin is extremely simple: Any choice of two orthogonal complements E_1, E_2 of $\mathbb{P}^{n+2}$ with signature (30) will determine a cyclide of Dupin and vice-versa.

Fig. 2, which is taken from [10] illustrates two different appearances of the cyclide of Dupin in $\mathbb{R}^3$.

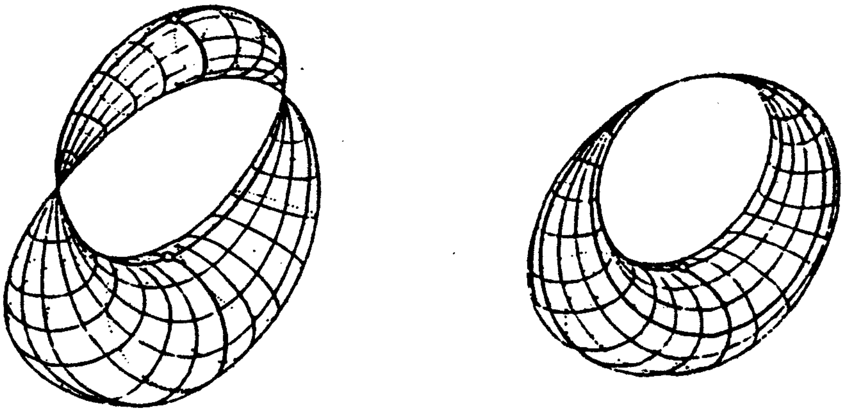

fig. 2

7) Reducible Dupin hypersurfaces

In [12] the Dupin hypersurfaces in S^4 are classified up to Lie transformations. In higher dimensions such a classification seems to be very involved, so one has to resort to some coarser type of classification. Here we will introduce a notion of reducibility for Dupin hypersurfaces. In order to know all Dupin hypersurfaces it will then be sufficient to know all the irreducible ones.

For the following considerations it is convenient to work in $\mathbb{R}^n$ rather than in S^n, so we will rely here on the concepts introduced in Chapter 5.

Suppose we are given a Dupin hypersurface $(x,\mathfrak{n}) : M \longrightarrow U\mathbb{R}^n$. Because the considerations to follow are of a local nature we can assume that $x: M \longrightarrow \mathbb{R}^n$ is an embedding and we may identify M with $x(M) \subset \mathbb{R}^n$. We consider $\mathbb{R}^n$ as the linear subspace $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. It is then easy to check that the following constructions lead to a Dupin hypersurface in $\mathbb{R}^{n+1}$:

(1) Let $\tilde{M}$ be the cylinder $M \times \mathbb{R} \subset \mathbb{R}^{n+1}$.

(2) Take an (n-1)-dimensional linear subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ and consider the rotations ϕ_t of $\mathbb{R}^{n+1}$ that leave $\mathbb{R}^{n-1}$ pointwise fixed. Let $\tilde{M}$ be the hypersurface of $\mathbb{R}^{n+1}$ generated by M under the rotations ϕ_t.

(3) Project M stereographically onto a hypersurface $\hat{M} \subset S^n \subset \mathbb{R}^{n+1}$. Let $\tilde{M}$ be the cone $\mathbb{R} \cdot \hat{M}$ over $\hat{M}$.

(4) Let $\tilde{M}$ be a tube in $\mathbb{R}^{n+1}$ around M.

<u>Definition</u>: A Dupin hypersurface $\tilde{M}$ in $\mathbb{R}^{n+1}$ that is the result of one of the above constructions (1)-(4) will be called reducible. More generally every Dupin hypersurface (M,k_1,k_2) that is locally Lie equivalent to such a hypersurface will also be called reducible.

For example the cyclides of Dupin described in section 6 are always reducible. It turns out that there is a simple criterion for reducibility:

<u>Theorem 4</u>: Let (M,k_1,k_2) be a proper Dupin hypersurface in $\mathbb{R}^n$, $\tilde{k}_1,\dots,\tilde{k}_r : M \longrightarrow \mathbb{P}^{n+2}$ its curvature spheres. Then the following properties are equivalent:

(i) (M,k_1,k_2) is reducible.

(ii) For some $i \in \{1,\dots,r\}$ $\tilde{k}_i(M)$ is contained in an n-dimensional linear subspace of $\mathbb{P}^{n+2}$.

<u>Proof</u>: Let us note first, that the following manifolds of spheres are hyperplane sections of the Lie-quadric:

a) the hyperplanes in $\mathbb{R}^n$

b) the spheres with a fixed radius r

c) the spheres that are orthogonal to a fixed sphere.

In the coordinates (26), (28) the hyperplanes are characterized by $x_{n+1} + x_{n+2} = 0$, the spheres with radius r by $r(x_{n+1}+x_{n+2}) = x_{n+3}$. This proves that a) and b) are hyperplane sections. As regards c) it can be assumed that the fixed sphere is a hyperplane H through the origin. A sphere is orthogonal to H if and only if its center lies in H . This clearly imposes a linear condition on the vector (26). The sets a), b), c) are of the form

$$(32) \qquad \{k \mid \langle k, a \rangle\} = 0$$

with $\langle a,a \rangle = 0,-1,1$ in case a), b), c) respectively.

It is now easy to see that every reducible hypersurface has a family of curvature spheres that is contained in two hyperplane sections of the Lie quadric. For example the tangent hyperplanes of a cylinder are curvature spheres and are orthogonal to a fixed hyperplane in $\mathbb{R}^n$. Hence (i) implies (ii).

Let (M,k_1,k_2) be a Dupin hypersurface, $\tilde{k} : M \longrightarrow \mathbb{P}^{n+2}$ a family of curvature spheres contained in an n-dimensional linear subspace E of $\mathbb{P}^{n+2}$. Then $\langle,\rangle$ must have signature $(++)$, $(0+)$ or $(-+)$ on the polar subspace $E^{\perp}$, because otherwise $E \cap Q$ would be empty or would consist of a single point.
If the signature is $(++)$ a Lie-transformation can achieve that $E \cap Q$ consists of all spheres that have their centers in a fixed (n-2)-dimensional linear subspace $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$. Since one family of curvature spheres of (M,k_1,k_2) lies in $E \cap Q$ and (M,k_1,k_2) is the envelope of these spheres, (M,k_1,k_2) must be a hypersurface of rotation with the "axis" $\mathbb{R}^{n-2}$.

If the signature of $E^{\perp}$ is $(0+)$ a Lie-transformation can achieve that the set of spheres $\tilde{k}(M)$ consists of hyperplanes orthogonal to a fixed hyperplane. In this case (M,k_1,k_2) is a cylinder.

If the signature of $E^{\perp}$ is $(-+)$ we have the choice to represent (M,k_1,k_2) either as a cone or as a tube.

□

8) Examples of Dupin hypersurfaces

Theorem 5: Given $\nu_1,\ldots,\nu_r \in \mathbb{N}$ with $\nu_1+\ldots+\nu_r = n-1$ there is a proper Dupin hypersurface in S^n whose principal curvatures are of multiplicities $\nu_1,\ldots\nu_r$.

Proof: The method of proof will be clear when we have shown how to construct Dupin hypersurfaces in S^5 with $(\nu_1,\ldots\nu_r)$

equal to (1,1,2) and (1,1,1,1) respectively.

Let M_2 be an open part of a cyclide of Dupin in $\mathbb{R}^3$ that does not contain parabolic points. The cylinder $M_3 := M_2 \times \mathbb{R} \subset \mathbb{R}^4$ is then a Dupin hypersurface in $\mathbb{R}^4$ with three distinct principal curvatures $\lambda_1, \lambda_2, \lambda_3$ at each point. One of these curvatures is zero. If we form again the cylinder $M_4 := M_3 \times \mathbb{R}$ the resulting hypersurface in $\mathbb{R}^5$ will have the desired properties with $(\nu_1, \ldots, \nu_r)$ equal to (1,1,2) . To obtain a Dupin hypersurface of type (1,1,1,1) we first invert M_3 in a suitable hypersphere of $\mathbb{R}^4$. Then we restrict attention to an open part $\tilde{M}_3$ of the image, on which all principal curvatures are distinct from zero. The image under stereographic projection into S^5 of the cylinder $M_4 := \hat{M}_3 \times \mathbb{R}$ will then have the desired properties.

□

Examples constructed as above are always reducible.

Every isoparametric hypersurface in S^n is a compact proper Dupin hypersurface. It would be interesting to know, which of these examples are irreducible. The tubes around totally geodesic submanifolds are reducible, while those isoparametric hypersurfaces whose symmetry group acts irreducibly on $\mathbb{R}^{n+1}$ are irreducible in the Lie geometric sense.

Another class of examples is provided by the extrinsically symmetric submanifolds of $\mathbb{R}^n$, S^n or H^n . A submanifold in a space of constant curvature is called "symmetric" if for each $p \in M$ there is an isometry σ of the surrounding space that leaves M invariant, leaves the normal space of M in p pointwise fixed and induces $-Id_{T_pM}$ in the tangent space T_pM [2,9,16]. We now show that every such submanifold M (considered as a Lie geometric hypersurface) is a Dupin hypersurface:

Let M_ε be a tube around M in the distance ε, where ε is small enough to ensure that M_ε is an immersed hypersurface. As a Lie geometric hypersurface M_ε is equivalent to M, hence it suffices to show that M_ε is a Dupin hypersurface.

For $p \in M_\varepsilon$ let $\pi(p)$ be its projection on M, σ as above the symmetry of the surrounding space that leaves $\pi(p)$ fixed. p is also fixed by σ and in T_pM_ε there is a subspace of dimension codim $(M)-1$ that is pointwise fixed by σ_*. We can decompose T_pM_ε as an orthogonal direct sum

$$(33) \qquad T_pM_\varepsilon = N \oplus V \oplus \sum_{\alpha=1}^{r} U_\alpha \quad ,$$

where $N \oplus V$ is the eigenspace corresponding to the principal curvature $1/\varepsilon$ and the U_α correspond to the other principal curvatures $\lambda_1,\ldots,\lambda_r$. By theorem 2 it is sufficient to show that along every line of curvature the corresponding principal curvature is constant. Let $t \mapsto \gamma(t)$ be a line of curvature, $t \mapsto \lambda(t)$ the corresponding principal curvature. We will show $\gamma'(t_o) = 0$ for all t_o. Let N, V, U_α be defined as in (33) with $p = \gamma(t_o)$. If $\gamma'(t_o)$ is in $N \oplus V$ then $\lambda(t) = 1/\varepsilon$ for all t and therefore $\lambda'(t_o) = 0$. If $\gamma'(t_o)$ is in U_α then $\lambda'(0) = 0$ because σ_* induces

$-\mathrm{Id}_{U_\alpha}$ in U_α , and by symmetry the curve γ is mapped onto itself by σ .

Thus we have proved that every extrinsically symmetric submanifold in $\mathbb{R}^n, S^n$ or H^n is a Dupin hypersurface. It would be interesting to know which of these examples are irreducible Dupin hypersurfaces, and which of them are proper.

9) Global properties

Recently some global properties of Dupin hypersurfaces have aroused interest. Call a Dupin hypersurface $(\mathfrak{p},\mathfrak{n}) : M \longrightarrow US^n$ regular if it is proper and $\mathfrak{p} : M \longrightarrow S^n$ is an immersion.
G. Thorbergsson [18] proved that every compact regular Dupin hypersurface in S^n is taut, so in particular $\mathfrak{p}$ is always an embedding in this case. Conversely it has been shown [13] that every taut submanifold of $\mathbb{R}^n$ or S^n (considered as a Lie geometric hypersurface) is a Dupin hypersurface.

R. Miyaoka [11] proved that every compact regular Dupin hypersurface in S^n with exactly three distinct principal curvatures at each point is Lie equivalent to an isoparametric hypersurface in S^n .

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