On the free boundary problem for the Navier-Stokes equations governing the motion of a viscous incompressible fluid in a slowly rotating container

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1 Introduction

This article is a continuation of the series of papers [1-5] devoted to stationary free boundary problems for the Navier-Stokes equations with moving contact points. There were investigated problems governing a viscous flow in a capillary, a coating flow, and a piston problem. Here one more problem of this type is studied.

Let a heavy viscous incompressible liquid partially fill a circular container $V \subset R_2$ of the radius R_0 rotating about its center with a small angular velocity ω (see Fig.1). We suppose that the force of gravity is directed along the vector $-e_2 = (0, -1)$, and we denote by Ω a subdomain of V occupied with the liquid. The boundary of Ω consists of two parts: $\Sigma = \partial\Omega \cap \partial V$ (a part of a rigid wall $\partial\Omega$) and $\Gamma = \partial\Omega \setminus \partial V$ (a free boundary). The set $M = \bar{\Sigma} \cap \bar{\Gamma}$ is a union of two contact points: x_- and x_+. We are concerned with the following free boundary problem: find $\Omega \subset V$ (or, what is the same, a free boundary Γ), the velocity vector field $\vec{v}(x) = (v_1, v_2)$ and the pressure $p(x)$ satisfying in Ω the Navier-Stokes equations

$$-\nu\nabla^2\vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = 0, \quad x \in \Omega \tag{1.1}$$

and the boundary conditions

$$\vec{v}|_\Sigma = \vec{a}, \quad \vec{v} \cdot \vec{n}|_\Gamma = 0, \quad \vec{\tau} \cdot S(\vec{v})\vec{n}|_\Gamma = 0, \tag{1.2}$$

$$\sigma H - gx_2 - \vec{n} \cdot T(\vec{v}, p)\vec{n}|_\Gamma = -p_1 = Const. \tag{1.3}$$

Here $\vec{a} = \omega R_0 \vec{\tau_0}$, $\vec{\tau_0}$ is a tangential vector to Σ, $\vec{\tau}$ and $\vec{n}$ are a tangential and an exterior normal vectors to Γ, respectively, T and S are the stress and the deformation tensors, i.e.

$$T(\vec{v},p) = -pI + \nu S(\vec{v}), \quad S_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i},$$

and ν, σ, g are constant positive coeffitients of viscosity, of the surface tension, and the acceleration of gravity, respectively. In addition, we fix the volume of the liquid, i.e. the area of Ω:

$$|\Omega| = Q < \pi R_0^2$$

and we assume that the contact angle θ, i.e. the angle between Γ and Σ at the contact points, equals π. This means that Γ is tangential to $\partial\Omega$ at these points. For $\theta \in (0,\pi]$ our problem can not be solved in the class of vector fields $\vec{v}$ with a finite Dirichlet integral (see [1,3,4]).

Problem governing the motion of a viscous fluid in a rotating container was formulated and considered by a different approach in the paper [6] by C.Baiocchi and V.V.Pukhnachov who were able to reduce it to a certain variational inequality. However, it has required some modifications of the formulation of the problem, in particular, the prescription of Γ.

Let us recall the definition of weighted Hölder spaces in which we are going to work. For arbitrary non-integral $l, s > 0$, arbitrary domain $G \subset R^n$ and a closed set $F \subset \partial G$ we define the space $C^l_s(G,F)$ as the set of scalar- or vector-valued functions $u(x)$, $x \in G$, with the norm

$$|u|_{C^l_s(G,F)} = |u|_{C^s(G)} + \sum_{s<|j|<l} \sup_{x\in G} \rho^{|j|-s}(x)|D^j u(x)| + [u]_{C^l_s(G,F)},$$

where

$$[u]_{C^l_s(G,F)} = \sum_{|j|=[l]} \sup_{x\in G} \rho^{l-s}(x) \sup_{y\in K(x)} \frac{|D^j u(x) - D^j u(y)|}{|x-y|^{l-[l]}},$$

$\rho(x) = dist(x,F)$, $K(x) = \{y \in G : |x-y| \leq \rho(x)/2\}$ and

$$|u|_{C^s(G)} = \sum_{|j|<l} \sup_{x\in G} |D^j u(x)| + \sum_{|j|=[l]} \sup_{x,y\in G} \frac{|D^j u(x) - D^j u(y)|}{|x-y|^{s-[s]}}$$

is a usual Hölder norm in G.

The spaces $C^l_s(G,F)$ can be also introduced for $s < 0$, in which case the norm is given by

$$|u|_{C^l_s(G,F)} = \sum_{|j|<l} \sup_{x\in G} \rho^{|j|-s}(x)|D^j u(x)| + [u]_{C^l_s(G,F)}. \quad (1.4)$$

They can be defined for functions given on manifolds, in particular, on Γ. Finally, we say that $\Gamma \in C^l_s$ if this line may be given by the equations $\vec{x} = \vec{x}(s)$ where $s \in (0,d)$ is a parameter and $\vec{x} \in C^l_s(I,\partial I)$.

We prove the following theorem.

THEOREM 1. *Suppose that*

$$|\Omega| \in (\pi R_0^2/2 + b_1, \pi R_0^2 - b_2), \quad b_1, b_2 > 0, \tag{1.5}$$

and $g/\sigma > B_0 > 0$ *(see Proposition 1 in section 3). For arbitrary sufficiently small* ω *problem* (1.1)-(1.3) *has a unique solution with the following properties:*

1. $\Gamma \in V$ *is a curve of the class* $C_{1+\gamma}^{2+\alpha}$ $(\alpha \in (0,1),\ \gamma \in (1/2,1))$ *which is close to the curve* Γ_0 *corresponding to the rest state,*

2. $\vec{v} \in C_\beta^{2+\alpha}(\Omega, M)$, $p \in C_{\beta-1}^{1+\alpha}(\Omega, M)$ *with a positive* $\beta < 1/2$, *and*

$$|\vec{v}|_{C_\beta^{2+\alpha}(\Omega,M)} + |p|_{C_{\beta-1}^{1+\alpha}(\Omega,M)} \leq c_1|\omega|. \tag{1.6}$$

We shall construct the solution of (1.1)-(1.3) according to the scheme applied in [2-5] to other free boundary problems with moving contact points. We consider at first the rest state, then we construct a formal solution of (1.1)-(1.3) without paying attention to the property $\Gamma \subset V$ which can be established on the basis of the local analysis of the solution carried out in [2,4]. The main difficulties in this problem are connected with the formal construction of the solution, and it is at this point that we concentrate our attention. As for the asymptotics of the solution near the contact points, all the necessary information (i.e. the study of the behaviour of the solution both for receding and for advancing contact line with a contact angle π at the smooth rigid wall of arbitrary shape) is contained in the paper [4].

2 The rest state

In the rest state, when $\omega = 0$, $\vec{v} = 0$ and $p = p_0 = Const$, the free boundary Γ_0 is defined by the equation

$$\sigma H - g x_2 = -p_0. \tag{2.1}$$

We recall that the force of gravity is directed opposite to x_2-axis and we choose the origin in such a way that the contact points $x_\pm$ have coordinates $(\pm l_0, 0)$, $l_0 < R_0$. Under the condition (1.5) (which is purely technical) the curve Γ_0 can be given by the equation

$$x_2 = \varphi_0(x_1), \quad x_1 \in (-l_0, l_0).$$

where φ_0 is an even function and

$$\varphi_0(\pm l_0) = 0. \tag{2.2}$$

Equation (2.1) can be written in the form

$$\frac{d}{dx_1}\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} - B\varphi_0 = -\frac{p_0}{\sigma}, \quad x_1 \in (-l_0, l_0), \tag{2.3}$$

or

$$\frac{d}{dx_1}\sin\alpha - B\varphi_0 = -\frac{p_0}{\sigma} \tag{2.4}$$

where $B = g/\sigma$ and α is the angle between the tangential vector to Γ_0 and x_1-axis ($\tan\alpha(x_1) = \varphi_0'(x_1)$).

Let us consider φ_0 as a solution of equation (2.3) satisfying the boundary conditions

$$\varphi_0'(-l_0) = \tan\alpha_0, \quad \varphi_0'(l_0) = -\tan\alpha_0 \tag{2.5}$$

where $\alpha_0 = \alpha(-l_0)$. Choosing p_0 in an appropriate way, we can satisfy also the conditions (2.2). It is well known that for arbitrary $\alpha_0 \in (0, \pi/2)$ problem (2.3), (2.5) has a unique infinitely differentiable solution which is an even function of x_1 satisfying the inequality $\varphi_0(x) > \varphi(\pm l_0) = 0$. Let us verify that $\Gamma \subset V$. Differentiation of (2.4) gives

$$\frac{d^2}{dx_1^2}\sin\alpha = B\tan\alpha > 0 \quad (x_1 \in (-l_0, 0)).$$

In addition,

$$\sin\alpha(-l_0) = \sin\alpha^{(0)}(-l_0), \quad \sin\alpha(0) = \sin\alpha^{(0)}(0) = 0$$

where $\alpha^{(0)}(x_1)$ is the angle between x_1-axis and the tangential vector to the part of the circle ∂V located above this axis. Since $\sin\alpha^{(0)}(x_1)$ is a linear function of x_1, the above relations imply

$$\sin\alpha(x_1) < \sin\alpha^{(0)}(x_1), \quad x_1 \in (-l_0, 0)$$

which shows that Γ_0 lies between x_1-axis and the upper part of ∂V.

Next, we prove that the curves Γ_0 corresponding to different values of α_0 do not intersect each other, more exactly, the curve corresponding to the greater value of α_0 is located under the curve corresponding to the smaller value of this angle. We write equation (2.4) in the form

$$\frac{d}{dx_1}\sin\alpha(x_1) = By(x_1) \quad (y = \varphi - p_0/g)$$

and we suppose that there are given two functions $y_1(x_1)$ and $y_2(x_1)$ satisfying this equation and the conditions

$$y_i'(0) = 0 \quad (i.e. \alpha_i(0) = 0, \ i = 1, 2)$$

and

$$y_1(0) < y_2(0).$$

It follows that

$$\frac{d}{dx_1}\sin\alpha_1|_{x_1=0} < \frac{d}{dx_1}\sin\alpha_2|_{x_1=0},$$

hence, $\alpha_1(x_1) > \alpha_2(x_1)$ and $y_1(x_1) < y_2(x_2)$ for negative x_1.

Consider two curves $\Gamma_{0i}: \quad x_2 = \varphi_{0i}(x_1), \quad x_1 \in (-l_{0i}, l_{0i}), \quad i = 1, 2$, with $l_{0i} = R_0 \sin \alpha_{0i}, \quad \alpha_{01} > \alpha_{02}$. The function α_1 is less than the corresponding function for ∂V at the point $-l_2$, hence, $\alpha_1(-l_2) < \alpha_2(-l_2)$. As we have seen, this implies $\alpha_1(x_1) < \alpha_2(x_1)$ or

$$\varphi'_{01}(x_1) = \tan \alpha_1(x_1) < \tan \alpha_2(x_1) = \varphi'_{02}(x_1)$$

for $x_1 \in (-l_2, 0)$. Consequently,

$$\varphi_{01}(x_1) = \varphi_{01}(-l_2) + \int_{-l_2}^{x_1} \varphi'_{01}(\xi)d\xi < \varphi_{02}(-l_2) + \int_{-l_2}^{x_1} \varphi'_{02}(\xi)d\xi = \varphi_{02}(x_1),$$

q.e.d.

This shows that the area of Ω is a monotone decreasing function of the angle α_0. For every value of $|\Omega| \in (\pi R_0^2/2 + b_1, \pi R_0^2 - b_2)$ there exists exacty one corresponding value of $\alpha_0 \in (d_1, \pi/2 - d_2), \quad d_i > 0$, and Γ_0 can be found from (2.3),(2.5).

At the conclusion we compute the constant p_0. Integration of (2.3) over the interval $(-l_0, l_0)$ gives

$$-2 \sin \alpha_0 - BA = -\frac{2l_0 p_0}{\sigma} \tag{2.6}$$

where

$$A = \int_{-l_0}^{l_0} \varphi_0(x_1)dx_1 > 0$$

is the area of the domain between Γ_0 and x_1-axis. Hence,

$$p_0 = \frac{\sigma}{R_0} + \frac{BA\sigma}{2l_0} > \frac{\sigma}{R_0}. \tag{2.7}$$

3. Auxiliary propositions

Let us turn our attention to problem (1.1)-(1.3). The free boundary Γ will be found as a perturbation of Γ_0, and it will be given by the equation

$$x_2 = \varphi(x_1), \quad x_1 \in (-l_1, l_2) \tag{3.1}$$

where l_i are some unknown numbers close to l_0. The points $x_- = (-l_1, \varphi(-l_1))$ and $x_+ = (l_2, \varphi(l_2))$ should be located on ∂V and the line Γ should be tangential to ∂V at these points. Let $(0, y_c), \quad y_c = -R_0 \sin \alpha_0$, be coordinates of the center of V. The equations of the semi-circles $\{x \in \partial V, \quad x_2 > y_c\}$ and $\{x \in \partial V, \quad x > 0\}$ have the form

$$x_2 = k(x_1) \equiv y_c + \sqrt{R_0^2 - x_1^2}, \quad x_1 \in (-R_0, R_0)$$

and

$$x_1 = h(x_2) \equiv \sqrt{R_0^2 - (x_2 - y_c)^2}, \quad x_2 \in (y_c - R_0, y_c + R_0),$$

respectively, hence, the above conditions reduce to

$$-l_1 = -h(\varphi(-l_1)), \qquad l_2 = h(\varphi(l_2)),$$
$$\varphi'(-l_1) = k'(-l_0), \qquad \varphi'(l_2) = k'(l_2). \tag{3.2}$$

Equation (1.3) may be written in the form

$$\frac{d}{dx_1}\frac{\varphi'}{\sqrt{1+\varphi'^2}} - B\varphi = t(x_1) - q, \quad x_1 \in (-l_1, l_2) \tag{3.3}$$

with $q = p_1/\sigma$, $t(x_1) = \sigma^{-1}\vec{n}\cdot T\vec{n}|_{x_2=\varphi(x_1)}$. It is convenient to map the interval $(-l_0, l_0)$ onto $(-l_1.l_2)$ by means of a linear transformation

$$x_1 = \mu(\xi - \bar{\xi})$$

with

$$\mu = \frac{l_1+l_2}{2l_0}, \qquad \bar{\xi} = l_0\frac{l_1-l_2}{l_1+l_2},$$

and to introduce the function

$$\tilde{\varphi}(\xi) = \varphi(\mu(\xi - \bar{\xi})).$$

Then relations (3.3),(3.2) are transformed into

$$\frac{1}{\mu}\frac{d}{d\xi}\frac{\tilde{\varphi}'(\xi)}{\sqrt{\mu^2+\tilde{\varphi}'(\xi)}} - B\tilde{\varphi}(\xi) = \tilde{t}(\xi) - q, \tag{3.4}$$

$$\tilde{\varphi}'(l_0) = \mu k'(l_2), \qquad \tilde{\varphi}'(-l_0) = \mu k'(-l_1), \tag{3.5}$$
$$-l_1 = -h(\tilde{\varphi}(-l_0)), \qquad l_2 = h(\tilde{\varphi}(l_0))$$

with $\tilde{t}(\xi) = t(\mu(\xi - \bar{\xi}))$. The constant q may be found by the integration of (3.3) with respect to x_1 which gives

$$\frac{\varphi'(l_2)}{\sqrt{1+\varphi'^2(l_2)}} - \frac{\varphi'(-l_1)}{\sqrt{1+\varphi'^2(-l_1)}} - B\int_{-l_1}^{l_2}\varphi(x_1)dx_1 = \frac{1}{\sigma}\int_{-l_1}^{l_2}\vec{n}\cdot T\vec{n}dx_1 - (l_2+l_1)q. \tag{3.6}$$

Since

$$\int_{-l_1}^{l_2}\varphi dx_1 = \int_\Gamma x_2n_2dS = |\Omega| - \int_\Sigma x_2n_2dS,$$

the last relation is equivalent to

$$\frac{\tilde{\varphi}'(l_0)}{\sqrt{\mu^2+\tilde{\varphi}'^2(l_0)}} - \frac{\tilde{\varphi}'(-l_0)}{\sqrt{\mu^2+\tilde{\varphi}'^2(-l_0)}} - B|\Omega| + B\int_\Sigma x_2n_2dS = \frac{1}{\sigma}I_t - 2l_0\mu q \tag{3.7}$$

where $I_t = \int_{-l_1}^{l_2} \vec{n} \cdot T\vec{n} dx_1$. Similar equation holds for $q_0 = p_0/\sigma$ (see (2.6)):

$$\frac{\varphi_0'(l_0)}{\sqrt{1+\varphi_0'^2(l_0)}} - \frac{\varphi_0'(-l_0)}{\sqrt{1+\varphi_0'^2(-l_0)}} - B|\Omega_0| + B\int_{\Sigma_0} x_2 n_2 dS = -2l_0 q_0. \tag{3.8}$$

Let us write (3.4),(3.5) as a boundary value problem for the function

$$\psi(\xi) = \tilde{\varphi}(\xi) - \varphi_0(\xi).$$

We need to compute $l_2 - l_0,\ \ l_1 - l_0,\ \ \mu - 1,\ \ \psi'(\pm l_0),\ \ q - q_0$. Taking the conditions $\varphi_0(\pm l_0) = 0$ into account, we obtain

$$l_2 - l_0 = h(\tilde{\varphi}(l_0)) - h(\varphi_0(l_0)) = \psi(l_0)\int_0^1 h'(s\psi(l_0))ds =$$

$$= h'(0)\psi(l_0) + \psi(l_0)\int_0^1 [h'(s\psi(l_0)) - h'(0)]ds \equiv h'(0)\psi(l_0) + L_2,$$

$$l_1 - l_0 = h(\tilde{\varphi}(-l_0)) - h(\varphi_0(-l_0)) = \psi(-l_0)\int_0^1 h'(s\psi(-l_0))ds$$

$$= h'(0)\psi(-l_0) + \psi(-l_0)\int_0^1 [h'(s\psi(-l_0)) - h'(0)]ds \equiv h'(0)\psi(-l_0) + L_1. \tag{3.9}$$

These equations imply

$$\mu - 1 = \frac{1}{2l_0}(l_2 + l_1 - 2l_0) \equiv \delta\mu + M \tag{3.10}$$

where

$$\delta\mu = \frac{h'(0)}{2l_0}[\psi(l_0) + \psi(-l_0)]$$

is a linear part of the right-hand side with respect to ψ and

$$M = L_1 + L_2 = \frac{\psi(l_0)}{2l_0}\int_0^1 [h'(s\psi(l_0)) - h'(0)]ds + \frac{\psi(-l_0)}{2l_0}\int_0^1 [h'(s\psi(-l_0)) - h'(0)]ds$$

is the remainder consisting of higher order terms.

Further we have

$$\psi'(l_0) = \mu k'(l_2) - k'(l_0) = (\mu - 1)k'(l_0) + k'(l_2) - k'(l_0) +$$

$$+(\mu - 1)(k'(l_2) - k'(l_0)) = \delta\mu k'(l_0) + k''(l_0)h'(0)\psi(l_0) + M_+, \tag{3.11}$$

$$\psi'(-l_0) = \mu k'(-l_1) - k'(-l_0) = \delta\mu k'(-l_0) - k''(-l_0)h'(0)\psi(-l_0) + M_- \tag{3.12}$$

where

$$M_+ = Mk'(l_0) + (l_2 - l_0)\int_0^1 [k''(l_0 + s(l_2 - l_0)) - k''(l_0)]ds + k''(l_0)L_2 +$$

$$+(\mu-1)(k'(l_2)-k'(l_0)),$$

$$M_- = Mk'(-l_0)-(l_1-l_0)\int_0^1[k''(-l_0-s(l_1-l_0))-k''(-l_0)]ds-k''(-l_0)L_1+$$
$$+(\mu-1)(k'(-l_1)-k'(-l_0)).$$

For the computation of $q-q_0$ we subtract (3.8) from (3.7) which leads to

$$\mu q-q_0=-\frac{1}{2l_0}\left[\left(\frac{\tilde{\varphi}'(l_0)}{\sqrt{\mu^2+\tilde{\varphi}'^2(l_0)}}-\frac{\varphi_0'(l_0)}{\sqrt{1+\varphi'^2(l_0)}}\right)-\right.$$
$$\left.-\left(\frac{\tilde{\varphi}'(-l_0)}{\sqrt{\mu^2+\tilde{\varphi}'^2(-l_0)}}-\frac{\varphi_0'(-l_0)}{\sqrt{1+\varphi_0'^2(-l_0)}}\right)\right]-\frac{B}{2l_0}\left(\int_\Sigma x_2n_2dS-\right.$$
$$\left.-\int_{\Sigma_0}x_2n_2dS\right)+\frac{I_t}{2l_0}+\frac{B}{2l_0}(|\Omega|-|\Omega_0|). \tag{3.13}$$

We transform the right-hand side making use of the formula

$$\frac{\tilde{\varphi}'(\xi)}{\sqrt{\mu^2+\tilde{\varphi}'^2(\xi)}}-\frac{\varphi_0'(\xi)}{\sqrt{1+\varphi_0'^2(\xi)}}=\left(\frac{\tilde{\varphi}'(\xi)}{\sqrt{\mu^2+\tilde{\varphi}'^2(\xi)}}-\frac{\varphi_0'(\xi)}{\sqrt{\mu^2+\varphi_0'^2(\xi)}}\right)+$$
$$+\left(\frac{\varphi_0'(\xi)}{\sqrt{\mu^2+\varphi_0'^2(\xi)}}-\frac{\varphi_0'(\xi)}{\sqrt{1+\varphi_0'^2(\xi)}}\right)=\psi'(\xi)\int_0^1\frac{\mu^2ds}{[\mu^2+(\varphi_0'+s\psi')^2]^{3/2}}-$$
$$-\int_0^1\frac{(\mu-1)\varphi_0'(\xi)ds}{[(1+s(\mu-1))^2+\varphi_0'^2(\xi)]^{3/2}}=\frac{\psi'(\xi)-\delta\mu\varphi_0'(\xi)}{(1+\varphi'^2(\xi))^{3/2}}+\Phi(\xi) \tag{3.14}$$

where

$$\Phi(\xi)=\psi'(\xi)\int_0^1\left[\frac{\mu^2}{[\mu^2+(\varphi_0'+s\psi')]^{3/2}}-\frac{1}{(1+\varphi_0'^2)^{3/2}}\right]ds-\frac{M\varphi_0'(\xi)}{(1+\varphi_0'(\xi))^{3/2}}-$$
$$-(\mu-1)\varphi_0'(\xi)\int_0^1\left\{\frac{1}{[(1+s(\mu-1))^2+\varphi_0'^2(\xi)]^{3/2}}-\frac{1}{[1+\varphi_0'^2(\xi)]^{3/2}}\right\}ds$$

is the sum of all the terms in (3.14) which are at least quadratic with respect to ψ. If $|\Omega|=|\Omega_0|$, then the last term in (3.13) vanishes, and (3.14) implies

$$\mu q-q_0=-\frac{1}{2l_0}\left[\frac{\psi'(l_0)-\delta\mu\varphi_0'(l_0)}{(1+\varphi_0'^2(l_0))^{3/2}}-\frac{\psi'(-l_0)-\delta\mu\varphi_0'(-l_0))}{(1+\varphi_0'^2(-l_0))^{3/2}}+\right.$$
$$\left.+\Phi(l_0)-\Phi(-l_0)\right]-\frac{B}{2l_0}\left(\int_\Sigma x_2n_2dS-\int_{\Sigma_0}x_2n_2dS\right)+\frac{I_t}{2l_0}.$$

Next, we apply formulas (3.11),(3.12) and take account of the fact that φ_0 is an odd function of ξ. We obtain

$$\frac{\psi'(l_0)-\delta\mu\varphi_0'(l_0)}{(1+\varphi_0'^2(l_0))^{3/2}}-\frac{\psi'(-l_0)-\delta\mu\varphi_0'(-l_0)}{(1+\varphi_0'^2(-l_0))^{3/2}}=\frac{\psi'(l_0)-\psi'(-l_0)-2k'(l_0)\delta\mu}{(1+\varphi_0'^2(l_0))^{3/2}}=$$

$$=\frac{k''(l_0)h'(0)[\psi(l_0)+\psi(-l_0)]+M_+-M_-}{(1+k'^2(l_0))^{3/2}}=-\frac{h'(0)}{R_0}[\psi(l_0)+\psi(-l_0)]+$$

$$+\frac{M_+-M_-}{(1+k'^2(l_0))^{3/2}},$$

hence,

$$\mu q-q_0=\frac{\delta\mu}{R_0}-\frac{1}{2l_0}\left[\frac{M_+-M_-}{(1+k'^2(l_0))^{3/2}}+\Phi(l_0)-\Phi(-l_0)\right]-$$

$$-\frac{B}{2l_0}\left(\int_\Sigma x_2n_2dS-\int_{\Sigma_0}x_2n_2dS\right)+\frac{I_t}{2l_0},$$

$$q-q_0=-\frac{\mu-1}{\mu}q_0+\frac{\delta\mu}{\mu R_0}-\frac{1}{2l_0\mu}\left[\frac{M_+-M_-}{(1+k'^2(l_0))^{3/2}}+\Phi(l_0)-\Phi(-l_0)\right]-$$

$$-\frac{B}{2l_0\mu}\left(\int_\Sigma x_2n_2dS-\int_{\Sigma_0}x_2n_2dS\right)+\frac{I_t}{2l_0}.$$

or

$$q-q_0=\delta q+Q+\frac{I_t}{2l_0\mu}$$

where

$$\delta q=-\delta\mu q_0+\frac{\delta\mu}{R_0}$$

is a linear part of $q-q_0$ with respect to ψ, and

$$Q=\frac{\delta\mu(\mu-1)-M}{\mu}q_0+\frac{\delta\mu(1-\mu)}{R_0\mu}-\frac{1}{2l_0\mu}\left[\frac{M_+-M_-}{(1+k'^2(l_0))^{3/2}}+\right.$$

$$\left.+\Phi(l_0)-\Phi(-l_0)\right]-\frac{B}{2l_0\mu}\left(\int_\Sigma x_2n_2dS-\int_{\Sigma_0}x_2n_2dS\right)$$

is the sum of higher order terms.

It remains to write the differential equation for ψ. We subtract (2.3) from (3.4) and take account of (3.14) which leads to

$$\frac{d}{d\xi}\frac{\psi'(\xi)-\delta\mu\varphi_0'(\xi)}{(1+\varphi_0'^2(\xi))^{3/2}}-\delta\mu\frac{d}{d\xi}\frac{\varphi_0'}{(1+\varphi_0'^2)^{1/2}}-B\psi=\tilde{t}(\xi)-(q-q_0)+F_1(\xi),$$

or

$$\frac{d}{d\xi}\frac{\psi'(\xi)-\delta\mu\varphi_0'(\xi)}{(1+\varphi_0'^2(\xi))^{3/2}}-B(\psi+\delta\mu\varphi_0)+\frac{\delta\mu}{R_0}=\tilde{t}(\xi)-\frac{I_t}{2l_0\mu}+F_1(\xi)-Q \qquad (3.15)$$

with

$$F_1(\xi) = \frac{\mu-1}{\mu}\frac{d}{d\xi}\frac{\psi'(\xi) - \delta\mu\varphi_0'(\xi)}{(1+\varphi_0'^2(\xi))^{3/2}} - -\frac{1}{\mu}\frac{d\Phi}{d\xi} - \frac{\delta\mu(\mu-1) - M}{\mu}\frac{d}{d\xi}\frac{\varphi_0'(\xi)}{(1+\varphi_0'^2(\xi))^{1/2}}.$$

For given $\tilde{t}(\xi)$, we consider (3.15),(3.11),(3.12) as a boundary value problem for ψ. Let us study a linearized problem

$$L[\psi] \equiv \frac{d}{d\xi}\frac{\psi' - \delta\mu\varphi_0'}{(1+\varphi_0'^2)^{3/2}} - B(\psi + \delta\mu\varphi_0) + \frac{\delta\mu}{R_0} = f(\xi),$$

$$\psi'(l_0) - \delta\mu k'(l_0) - k''(l_0)h'(0)\psi(l_0) = a_+, \tag{3.16}$$

$$\psi'(-l_0) - \delta\mu k'(-l_0) + k''(-l_0)h'(0)\psi(-l_0) = a_-.$$

PROPOSITION 1 *There exists such $B_0 > 0$ that for $B > B_0$ problem (3.16) has a unique solution $\psi \in C^{3+\alpha}_{1+\beta}(I, \partial I)$ $(\alpha, \beta \in (0,1),\ \ I = (-l_0, l_0))$ for arbitrary $a_+, a_- \in R$, $f \in C^{1+\alpha}_{\beta-1}(I, \partial I)$. The solution satisfies the inequality*

$$|\psi|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \le c_1(|f|_{C^{1+\alpha}_{\beta-1}(I,\partial I)} + |a_+| + |a_-|). \tag{3.17}$$

PROOF It is convenient to introduce a new unknown function

$$\tilde{\psi}(\xi) = \psi(\xi) + \delta\mu\varphi_0(\xi).$$

Since $\varphi(\pm l_0) = 0$, we have

$$\delta\mu = \frac{h'(0)}{2l_0}[\tilde{\psi}(l_0) + \tilde{\psi}(-l_0)]$$

and we may express ψ in terms of $\tilde{\psi}$ by the formula

$$\psi(\xi) = \tilde{\psi}(\xi) - \delta\mu\varphi_0(\xi).$$

Problem (3.16) takes the form

$$\frac{d}{d\xi}\frac{\tilde{\psi}' - 2\delta\mu\varphi_0'}{(1+\varphi_0'^2)^{3/2}} - B\tilde{\psi} + \frac{\delta\mu}{R_0} = f(\xi),$$

$$\tilde{\psi}'(l_0) - \frac{\tilde{\psi}(l_0)}{l_0\cos^2\alpha_0} + 2\tan\alpha_0\delta\mu = a_+,$$

$$\tilde{\psi}'(-l_0) + \frac{\tilde{\psi}(-l_0)}{l_0\cos^2\alpha_0} - 2\tan\alpha_0\delta\mu = a_- :$$

A weak solution of this problem can be defined as a function $\tilde{\psi} \in W_2^1(I)$ satisfying the integral identity

$$L[\tilde{\psi}, \eta] \equiv \int_{-l_0}^{l_0} \left(\frac{\tilde{\psi}' - 2\delta\mu\varphi_0'}{(1+\varphi_0'^2)^{3/2}}\eta' + B\tilde{\psi}\eta - \frac{\delta\mu}{R_0}\eta \right) d\xi -$$

$$-\frac{1}{(1+k'^2(l_0))^{3/2}} \left[\left(\frac{\tilde{\psi}(l_0)}{l_0 \cos^2\alpha_0} - 2\tan\alpha_0\delta\mu \right) \eta(l_0) + \right.$$

$$\left. + \left(\frac{\tilde{\psi}(-l_0)}{l_0\cos^2\alpha_0} - 2\tan\alpha_0\delta\mu \right) \eta(-l_0) \right] +$$

$$+ \frac{2\delta\mu k'(l_0)}{(1+k'^2(l_0))^{3/2}}[\eta(l_0) + \eta(-l_0)] = \frac{a_+\eta(l_0) - a_-\eta(-l_0)}{(1+k'^2(l_0))^{3/2}} - \int_{-l_0}^{l_0} f\eta d\xi \tag{3.18}$$

for arbitrary $\eta \in W_2^1(I)$.

If the coefficient B is large enough, then the quadratic form $L[\tilde{\psi}, \tilde{\psi}]$ is positive definite:

$$L[\tilde{\psi}, \tilde{\psi}] \geq c_2 \int_{-l_0}^{l_0} (|\tilde{\psi}'|^2 + |\tilde{\psi}|^2) d\xi \tag{3.19}$$

Indeed, it is easy to see that

$$L[\tilde{\psi}, \tilde{\psi}] \geq c_3 \int_{-l_0}^{l_0} \tilde{\psi}'^2 d\xi + B\int_{-l_0}^{l_0} \tilde{\psi}^2 d\xi - \frac{|\delta\mu|}{R_0}\int_{-l_0}^{l_0} |\tilde{\psi}| d\xi - c_4(|\tilde{\psi}(l_0)|^2 + |\tilde{\psi}(-l_0)|^2) \tag{3.20}$$

with c_3, c_4 independent of B. For the estimate of $\tilde{\psi}$ we use the identity

$$\tilde{\psi}^2(l_0) + \tilde{\psi}^2(-l_0) = \frac{1}{l_0}\int_{-l_0}^{l_0} \tilde{\psi}^2(t)dt + \frac{2}{l_0}\int_{-l_0}^{l_0} t\tilde{\psi}(t)\tilde{\psi}'(t)dt$$

which implies

$$\tilde{\psi}^2(l_0) + \tilde{\psi}^2(-l_0) \leq \epsilon \int_{-l_0}^{l_0} \tilde{\psi}'^2(t)dt + \left(\frac{1}{\epsilon} + \frac{1}{l_0}\right)\int_{-l_0}^{l_0} \tilde{\psi}^2(t)dt, \qquad \forall\epsilon > 0.$$

Similar estimate holds for $|\delta\mu|^2$. It is clear now that (3.19) follows from (3.20) in the case of large B.

For arbitrary $f \in C_{\beta-1}^{1+\alpha}(I, \partial I)$, $\eta \in W_2^1(I)$ we have

$$|\int_{-l_0}^{l_0} f\eta d\xi| \leq \sup_I \rho^{1-\beta}(\xi)|f(\xi)| \int_{-l_0}^{l_0} \rho^{\beta-1}(\xi)|\eta(\xi)| d\xi \leq$$

$$\leq c_5 \sup_I |\eta(\xi)| \sup_I \rho^{1-\beta}(\xi)|f(\xi)| \leq c_6 \|\eta\|_{W_2^1(I)} |f|_{C_{\beta-1}^{1+\alpha}(I,\partial I)},$$

hence, the existence of a unique weak solution follows from the theorem of Lax-Milgram. Setting $\eta = \tilde{\psi}$ in (3.18) we easily obtain

$$\sup_I |\tilde{\psi}(\xi)| \leq c_7 \|\tilde{\psi}\|_{W_2^1(I)} \leq c_8(\sup_I \rho^{1-\beta}(\xi)|f(\xi)| + |a_+| + |a_-|), \tag{3.21}$$

and $\delta\mu$ can also be evaluated by the expression in the right-hand side. Now, we consider $\tilde{\psi}$ as a solution to the problem

$$\frac{d}{d\xi}\frac{\tilde{\psi}'}{(1+\varphi_0'^2)^{3/2}} - B\tilde{\psi} = f_1,$$

$$\tilde{\psi}'(l_0) = b_+, \qquad \tilde{\psi}'(-l_0) = b_-$$

where

$$f_1 = f + 2\delta\mu\frac{d}{d\xi}\frac{\varphi_0'}{(1+\varphi_0'^2)^{1/2}} - \frac{\delta\mu}{R_0},$$

$$b_+ = a_+ - 2\delta\mu\tan\alpha_0 + \frac{\tilde{\psi}(l_0)}{l_0\cos^2\alpha_0}, \qquad b_- = a_- + 2\delta\mu\tan\alpha_0 - \frac{\tilde{\psi}(-l_0)}{l_0\cos\alpha_0}.$$

This problem was studied in [7] where, in particular, the following estimate for the solution was established:

$$|\tilde{\psi}|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \le c_9(|f_1|_{C^{1+\alpha}_{\beta-1}(I,\partial I)} + |b_+| + |b_-|) \tag{3.22}$$

(the assumption $\beta = \alpha$ made in [7] is not essential). Estimate (3.17) is a consequence of (3.21) and (3.22). The proposition is proved.

Our second auxiliary proposition concerns the construction of a special mapping $Y : \Omega_0 \to \Omega$.

PROPOSITION 2 *Suppose that the line Γ is given by equations* (3.1) *on the interval* $(-l_0, l_0)$ *with φ satisfying conditions* (3.5),(3.6), *moreover, assume that $\tilde{\varphi}(\xi) = \varphi(\mu(\xi - \bar{\xi}))$ belongs to $C^{3+\alpha}_{1+\beta}(I, \partial I)$ and that*

$$|\tilde{\varphi} - \varphi_0|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \le \delta_1$$

with a small positive δ_1. Let $\Omega \subset V$ be a domain with $\partial\Omega = \Sigma\cup\Gamma\cup M$, $M = \{x_+, x_-\}$. There exists a mapping $Y : \Omega \to \Omega_0$ with the following properties: 1. *Y is invertible, continuous in $\bar{\Omega}$ and has bounded derivatives. Moreover, $Y|_{\Sigma_0} \in C^{3+\alpha}_{1+\beta}(\Sigma_0, M_0)$ and $Y|_{\Gamma_0} \in C^{3+\alpha}_{1+\beta}(\Gamma_0, M_0)$ where $M_0 = \bar{\Sigma}_0 \cap \bar{\Gamma}_0 = \{x_+^{(0)}, x_-^{(0)}\}$, $x_\pm^{(0)} = (\pm l_0, 0)$. For $\xi \in \Gamma_0$*

$$Y(z) = (\mu(z_1 - \bar{\xi}), \tilde{\varphi}(z_1)) \tag{3.23}$$

where

$$\mu = \frac{l_2 + l_0}{2l_0}, \qquad \bar{\xi} = l_0\frac{l_1 - l_2}{l_1 + l_2}.$$

2. *The Jacobian matrix J of the inverse transformation Y^{-1} satisfies the inequalities*

$$|J - I|_{C^{2+\alpha}_{\beta}(\Sigma_0.M)} + |J - I|_{C^{2+\alpha}_{\beta}(\Gamma_0.M)} \le c_{10}|\tilde{\varphi} - \varphi_0|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}, \tag{3.24}$$

$$\sup_{\Omega_0}|J(z) - I| + \sum_{|j|=1,2}\sup_{\Omega_0}\rho^{|j|}(z)|D^j J_0(z)| +$$

$$+\sum_{|j|=2}\sup_{z\in\Omega_0}\rho^{2+\alpha}(z)\sup_{y\in K(z)}|y-z|^{-\alpha}|D^jJ_0(z)-D^jJ_0(y)|\le$$

$$\le c_{11}|\tilde{\varphi}-\varphi_0|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \tag{3.25}$$

where $\rho(x) = dist(x, M_0)$.

PROOF We construct the mapping Y in the form

$$Y(z) = z + \Phi(z) = (z_1 + \Phi_1(z), z_2 + \Phi_2(z)) \tag{3.26}$$

Equation (3.23) determines Φ_i on Γ_0:

$$\Phi_1(z_1, \varphi_0(z_1)) = (\mu - 1)z_1 - \mu\bar{\xi},$$

$$\Phi_2(z_1, \varphi_0(z_1)) = \tilde{\varphi}(z_1) - \varphi_0(z_1) \equiv \psi(z_1).$$

In particular,

$$\Phi_1(-l_0, 0) = -(\mu - 1)l_0 - \mu l_0 \frac{l_1 - l_2}{l_1 + l_2} = -l_1 + l_0, \quad \Phi_1(l_0, 0) = l_2 - l_1,$$

$$\Phi_2(-l_0, 0) = \tilde{\varphi}(-l_0) = \varphi(-l_1), \quad \Phi_2(l_0, 0) = \varphi(l_2) \tag{3.27}$$

which implies

$$Y(x_\pm^{(0)}) = x_\pm^{(0)}. \tag{3.28}$$

Next, we extend $Y(z)$ onto Σ_0 in such a way that $Y\Sigma_0 = \Sigma$. Let (r, φ) be standard polar coordinates on R^2 with the center in $(0, y_c)$. A general form of automorphisms of ∂V is

$$x_1 = R_0 \cos(\varphi + h(\varphi)), \quad x_2 = R_0 \sin(\varphi + h(\varphi)).$$

Clearly, this transformation can be written in the form (3.26) with $z = (R_0 \cos\varphi, R_0 \sin\varphi)$ and

$$\Phi_1(z) = R_0[\cos(\varphi + h(\varphi)) - \cos\varphi] =$$

$$= R_0[\cos\varphi(\cos h(\varphi) - 1) - \sin\varphi \,\mathrm{sinh}(\varphi)],$$

$$\Phi_2(z) = R_0[\sin(\varphi + h(\varphi)) - \sin\varphi] =$$

$$= R_0[\sin\varphi(\cos h(\varphi) - 1) + \cos\varphi \sin h(\varphi)]. \tag{3.29}$$

These equations imply

$$R_0 \sin h(\varphi) = \Phi_2(z)\cos\varphi - \Phi_1(r)\sin\varphi. \tag{3.30}$$

We make the extension of Y by the construction of an appropriate function $h(\varphi)$ on Σ_0. We find the values of h at the points $x_\pm$ using relation (3.28). Because of this relation, the functions Φ_i computed at $x_\pm$ (they are given by (3.27)) satisfy (3.29) with certain $h_\pm$ which are determined by (3.30). It is elementary to construct a smooth (at least $C^{3+\alpha}$-smooth) function h on Σ_0 satisfying the conditions $h(x_\pm) = h_\pm$ and the inequality

$$|h|_{C^{3+\alpha}(\Sigma_0)} \leq c_{12}(|h_+| + |h_-|) \leq c_{13}(|\tilde{\varphi}(l_0) - \varphi_0(l_0)| + |\tilde{\varphi}(-l_0) - \varphi_0(-l_0)|). \quad (3.31)$$

The extensions of Φ_i are defined now by (3.29), and it is clear that their $C^{3+\alpha}(\Sigma_0)$-norms can also be evaluated by the right-hand side of (3.31). Now Φ_i are defined on $\partial\Omega_0$, and they can be extended farther into Ω_0. A special care should be taken in the neighbourhoods of $x_\pm$ (see also [5]). Let Φ_{i1} and Φ_{i2} be extensions of $\Phi_i|_{\Sigma_0}$ and $\Phi_i|_{\Gamma_0}$ made in such a way that

$$|\Phi_{i1}|_{C^{3+\alpha}_{1+\beta}(\Omega_0,M_0)} \leq c_{15}|\Phi_i|_{C^{3+\alpha}_{1+\beta}(\Sigma_0,M_0)},$$

$$|\Phi_{i2}|_{C^{3+\alpha}_{1+\beta}(\Omega_0,M_0)} \leq c_{16}|\Phi_i|_{C^{3+\alpha}_{1+\beta}(\Gamma_0,M_0)}.$$

We can define $\Phi_i(z)$ in the neighbourhood of x_-, for example, by the formula

$$\Phi_i(z) = \chi_1(z)(\Phi_{i1}(z) - \Phi_i(x_-)) + \chi_2(z)(\Phi_{i2}(z) - \Phi_i(x_-)) + \Phi_i(x_-)$$

where χ_i are functions defined hear x_- and possessing the following properties:

$$\chi_1|_{\Sigma_0} = 1, \quad \chi_1|_{\Gamma_0} = 0, \quad \frac{\partial\chi_1}{\partial n}|_{\partial\Omega_0} = 0,$$

$$\chi_2|_{\Sigma_0} = 0, \quad \chi_2|_{\Gamma_0} = 1, \quad \frac{\partial\chi_2}{\partial n}|_{\partial\Omega_0} = 0,$$

χ_i are smooth everywhere except the point x_-, $0 \leq \chi_i \leq 1$ and

$$|D^j\chi_i(z)| \leq c_{17}|z - x_-|^{-|j|}.$$

Inequatilies (3.24),(3.25) are easily verified. Away from $x_\pm$, the construction of extensions is quite standard.

The proposition is proved.

REMARK Let φ_1 and φ_2 be two functions satisfying the hypotheses of the proposition, and let Y_i be corresponding transformations. Since all the extensions operatops used in the proposition are linear, it is easily verified that the differences $J_1 - J_2$ satisfy the estimates (3.24),(3.25) with the norms of the differences $\tilde{\varphi}_1 - \tilde{\varphi}_2$ in the right-hand sides.

4 Proof of Theorem 1

The proof of Theorem 1 is based on the investigation of two auxiliary problems: of problem (1.1),(1.2) in a given domail Ω and of problem (3.15),(3.11),
(3.12).

THEOREM 2. 1.*Suppose that* Γ *is given by equation* (3.1) *with the function* φ *satisfying the hypotheses of Proposition* 2 *and that* ω *is sufficiently small:*

$$|\omega| < \epsilon \tag{4.1}$$

Then problem (1.1),(1.2) *possesses a unique solution* $\vec{v} \in C^{2+\alpha}_{\beta}(\Omega, M)$, $p \in C^{1+\alpha}_{\beta-1}(\Omega, M)$, *and*

$$|\vec{v}|_{C^{2+\alpha}_{\beta}(\Omega,M)} + |p|_{C^{1+\alpha}_{\beta-1}(\Omega,M)} \le c_1|\omega|. \tag{4.2}$$

2. *Let* φ_1 *and* φ_2 *be two functions satisfying hypotheses of Proposition* 2 *and defining the lines* Γ_1 *and* Γ_2, *and let* $\vec{v}_1, p_1$ *and* $\vec{v}_2, p_2$ *be solutions of* (1.1),(1.2) *in* Ω_1 *and* Ω_2, *respectively. The functions* $\tilde{t}_i(\xi) = \vec{n} \cdot T(\vec{v}_i, p_i)\vec{n}|_{x_2=\tilde{\varphi}_i(\xi)}$ *satisfy the inequatity*

$$|\tilde{t}_1 - \tilde{t}_2|_{C^{1+\alpha}_{\beta-1}(I,\partial I)} \le c_2|\omega||\tilde{\varphi}_1 - \tilde{\varphi}_2|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}. \tag{4.3}$$

The first part of the theorem is known. The linerized problem was studied in [8]. For small ω, the nonlinear problem can be solved by application of the contraction mapping principle, since the nonlinear term satisfies the inequality

$$|(\vec{v} \cdot \nabla)\vec{w}|_{C^{\alpha}_{\beta-2}(\Omega,M)} \le c_2|\vec{v}|_{C^{2+\alpha}_{\beta}(\Omega,M)}|\vec{w}|_{C^{2+\alpha}_{\beta}(\Omega,M)}$$

Inequality (4.3) is also established by a well known procedure. We map the domain Ω_1 onto Ω_2 by means of the transformation $y = Z(x)$ where $Z = Y_2 \circ Y_1^{-1}$ and $Y_i : \Omega_0 \to \Omega_i$ are mappings constructed in Proposition 2, and we write problem (1.1),(1.2) for $\vec{v}_1, p_1$ in new coordinates. It is easy to see that $\vec{v}_1 - \vec{v}_2 = \vec{w}, p_1 - p_2 = s$ can be considered as a solution of the linear problem

$$-\nabla^2\vec{w} + \nabla s = \vec{f}, \qquad \nabla \cdot \vec{w} = r,$$

$$\vec{w}|_{\Sigma_2} = \vec{w}_0, \quad \vec{w} \cdot \vec{n}|_{\Gamma_2} = b, \quad \vec{\tau} \cdot S(\vec{w})\vec{n}|_{\Gamma_2} = d \tag{4.4}$$

where $\vec{f}, r, \vec{v}_0, b, d$ are functions satisfying the inequality

$$|\vec{f}|_{C^{\alpha}_{\beta-2}(\Omega_2,M_2)} + |r|_{C^{1+\alpha}_{\beta-1}(\Omega_2,M_2)} + |\vec{w}_0|_{C^{2+\alpha}_{\beta}(\Sigma_2,M_2)} +$$

$$+|d|_{C^{1+\alpha}_{\beta-1}(\Gamma_2,M_2)} + |b|_{C^{2+\alpha}_{\beta}(\Gamma_2,M_2)} \le c_3|\omega||\tilde{\varphi}_1 - \tilde{\varphi}_2|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}. \tag{4.5}$$

This inequality follows from (4.2) and from the remark to Proposition 2 (see some details in [3], section 5). (4.3) is a consequence of (4.5) and of a coersive estimate of the solution of (4.5) in weighted Hölder norms (see [8]).

Let us consider problem (3.15),(3.11),(3.12).

THEOREM 3 *Suppose that condition* $B > B_0$ *is satisfied. For arbitrary* $\tilde{t} \in C^{1+\alpha}_{\beta-1}(I, \partial I)$ *with a small norm:*

$$|\tilde{t}|_{C^{1+\alpha}_{\beta-1}(I,\partial I)} \le \epsilon_1 \tag{4.6}$$

problem (3.15),(3.11),(3.12) *has a unique solution* $\psi \in C^{3+\alpha}_{\beta-1}(I,\partial I)$, *and for this solution the estimate*

$$|\psi|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \le c_4 |\tilde{t}|_{C^{1+\alpha}_{\beta-1}(I,\partial I)} \tag{4.7}$$

holds.

PROOF Consider $L_1, L_2, M, M_\pm$ etc. as nonlinear functionals depending on ψ. It is clear that

$$|L_1| + |L_2| + |M| + |M_+| + |M_-| \le c_5(|\psi(l_0)|^2 + |\psi(-l_0)|^2), \tag{4.8}$$

provided that $\psi(\xi)$is small enough, for instance,

$$|\psi|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \le \delta. \tag{4.9}$$

Moreover,

$$|\Phi|_{C^{2+\alpha}_{\beta}(I,\partial I)} \le c_6|\mu-1|(|\psi|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} + |\mu-1|) \le c_7|\psi|^2_{C^{3+\alpha}_{1+\beta}(I,\partial I)}, \tag{4.10}$$

$$|\Phi(l_0)| + |\Phi(-l_0)| \le c_8(|\psi(l_0)|^2 + |\psi(-l_0)|^2)$$

and, finally, since the endpoints of Σ_0 are located on x_1-axis, we have

$$|\int_\Sigma x_2 n_2 dS - \int_{\Sigma_0} x_2 n_2 dS| \le c_9(|\psi(l_0)|^2 + |\psi(-l_0)|^2),$$

hence,

$$|Q| \le c_{10}(|\psi(l_0)|^2 + |\psi(-l_0)|^2).$$

Let ψ_1 and ψ_2 be two functions from the ball (4.9) and let $L_i[\psi_j], M[\psi_j]$ etc. be corresponding functionals. It is easy to see that

$$\sum_{i=1}^{i=2} |L_i[\psi_1] - L_i[\psi_2]| \le c_{11}\delta(|\psi(l_0) - \psi(l_0)| + |\psi(-l_0) - \psi(-l_0)|),$$

$$|\Phi[\psi_1] - \Phi[\psi_2]|_{C^{2+\alpha}_{\beta}(I,\partial I)} \le c_{12}\delta|\psi_1 - \psi_2|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}, \tag{4.11}$$

$$|Q[\psi_1] - Q[\psi_2]| \le c_{13}\delta(|\psi(l_0) - \psi_2(l_0)| + |\psi_1(-l_0) - \psi_2(-l_0)|).$$

These inequalities make it possible to deduce the solvability of the problem (3.15), (3.11),(3.12) from the contraction mapping principle. We write it in the form of equation

$$\psi = \mathcal{A}[F_1 - Q, M_+, M_-] + \mathcal{A}[\tilde{t} - \frac{1}{2l_0\mu}I_t, 0, 0)] \equiv \mathcal{B}\psi \tag{4.12}$$

where $\mathcal{A}$ is a linear operator which makes correspond a solution of problem (3.16) to the data $[f, a_+, a_-]$. For arbitrary $\tilde{t}(\xi)$ satisfying condition (4.6) and arbitrary ψ, ψ_1, ψ_2 from the ball (4.9) we have

$$|\mathcal{B}\psi|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \le c_{14}(\delta^2 + \epsilon_1),$$

$$|\mathcal{B}\psi_1 - \mathcal{B}\psi_2|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \leq c_{15}(\delta + \epsilon_1)|\psi_1 - \psi_2|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}.$$

Hence, $\mathcal{B}$ is a nonlinear contraction operator in the ball (4.9), if

$$c_{14}(\delta^2 + \epsilon_1) < \delta, \quad c_{15}(\delta + \epsilon_1) < 1.$$

These inequalities are satisfied, if

$$\delta < \min(c_{14}^{-1}, c_{15}^{-1})/2, \quad \epsilon_1 < \min(\delta c_{14}^{-1}, c_{15}^{-1})/2.$$

Then the solvability of equation (4.12) follows from the contraction mapping principle. The theorem is proved.

The solution of problem (3.15),(3.11),(3.12) determines a curve Γ with $\partial\Gamma \in \partial V$ which is tangential tl ∂V at the endpoints, and the area of the corresponding domain Ω equals Q. Indeed, if we set

$$\tilde{\varphi}(\xi) = \varphi_0(\xi) + \psi(\xi),$$

$$l_1 = h(\tilde{\varphi}(-l_0)), \ \ l_2 = h(\tilde{\varphi}(l_0)), \ \ \mu = \frac{l_1 + l_2}{2l_0}$$

and define $\varphi(x_1)$ as in section 3, i.e. by equation

$$\varphi(\mu(\xi - \bar{\xi})) = \tilde{\varphi}(\xi),$$

then (3.11) is equivalent to

$$\psi'(l_0) = \mu k'(l_2) - k'(l_0),$$

or to

$$\tilde{\varphi}(l_0) = \mu k(l_2),$$

which immediately gives $\varphi'(l_2) = k'(l_2)$. Exactly in the same way the condition $\varphi'(-l_1) = k'(-l_1)$ can be verified. Finally, the addition of (3.15) and (2.3) leads to (3.4) (i.e. to (3.3)) with a constant q satisfying (3.13) without the last term. Integration of (3.3) gives

$$\frac{B}{2l_0}(|\Omega| - |\Omega_0|) = 0,$$

q.e.d.

We are ready now to carry out a formal construction of the solution of the free boundary problem (1.1)-(1.3). We use the following iterative procedure. Let $\vec{v}^{(0)} = 0, p^{(0)} = p_0, \Omega^{(0)} = \Omega_0$ and let $\vec{v}^{(1)}, p^{(1)}$ be a solution of the first auxiliary problem in $\Omega^{(0)}$. Further, we solve the second auxiliary problem with the function

$$\tilde{l}^{(1)}(\xi) = \frac{1}{\sigma}\vec{n} \cdot T(\vec{v}^{(1)}, p^{(1)})\vec{n})|_{x_2 = \varphi_0(\xi)}$$

in the right-hand side. This determines the curve $\Gamma^{(1)}$ and the domain $\Omega^{(1)}$. This procedure is repeated: we define $\psi^{(m+1)}$ as a solution of the second auxiliary problem with $\tilde{t}^{(m+1)}$ in the right-hand side, and $t^{(m+1)} = \sigma^{-1}\vec{n}\cdot T(\vec{v}^{(m+1)}, p^{(m+1)})\vec{n}$ where $\vec{v}^{(m+1)}, p^{(m+1)}$ is the solution of the first auxiliary problem in $\Omega^{(m)}$. Let us show that the sequence $\phi^{(m)}$ is convergent in $C^{3+\alpha}_{1+\beta}(I,\partial I)$. According to (4.12), we have

$$\psi^{(m+1)} = \mathcal{A}[F_1^{(m+1)} - Q^{(m+1)}, M_+^{(m+1)}, M_-^{(m+1)}] + \mathcal{A}[\tilde{t}^{(m+1)} - \frac{I_t^{(m+1)}}{2l_0\mu^{(m+1)}}, 0, 0],$$

hence,

$$\psi^{(m+1)} - \psi^{(m)} =$$
$$= \mathcal{A}[F_1^{(m+1)} - F_1^{(m)} - Q^{(m+1)} + Q^{(m)}, M_+^{(m+1)} - M_+^{(m)}, M_-^{(m+1)} - M_-^{(m)}] +$$
$$+\mathcal{A}[\tilde{t}^{(m+1)} - \tilde{t}^{(m)} - \frac{I_t^{(m+1)} - I_t^{(m)}}{2l_0\mu^{(m+1)}} + I_t^{(m)}[(2l_0\mu^{(m+1)})^{-1} - (2l_0\mu^{(m)})^{-1}], 0, 0]$$

Suppose that $\psi^{m)}$ satisfies the condition (4.6). In virtue of (4.7) and (4.2),

$$|\psi^{(m+1)}|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \leq c_{16}|\tilde{t}^{(m+1)}|_{C^{1+\alpha}_{\beta-1}(I,\partial I)} \leq c_{17}|\omega|. \tag{4.13}$$

For small ω, the right-hand side does not exceed δ. Hence, we see that all the approximations ψ^m satisfy (4.6).

Further, estimates (4.10) and (4.3) imply

$$|\psi^{(m+1)} - \psi^{(m)}|_{C^{3+\alpha}_{1+\beta}/I,\partial I)} \leq c_{19}[(\delta + |\omega|)|\psi^{(m+1)} - \psi^{(m)}|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} +$$
$$+|\omega||\psi^{(m)} - \psi^{(m-1)}|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}]$$

We see that if

$$c_{19}(\delta + |\omega|) < 1/2,$$

then

$$|\psi^{(m+1)} - \psi^{(m)}|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \leq 2c_{19}|\omega||\psi^{(m)} - \psi^{(m-1)}|_{C^{3+\alpha}_{1+\beta}(I,\partial I)}$$

which guarantees the convergence of $\{\psi^{(m)}\}$, since $2c_{19}|\omega| < 1$. It is evident that all the smallness conditions can be satisfied by the choice of small ω.

In virtue of (4.13) and (4.14), the limiting function ψ satisfies the inequality

$$|\psi|_{C^{3+\alpha}_{1+\beta}(I,\partial I)} \leq c_{20}|\omega|. \tag{4.14}$$

This functions defines the domain Ω, and $\vec{v}$ and p can be obtained as a solution of (1.1),(1,2).

Inequality (4.14) does not guarantee that $\Gamma \subset V$, since the space $C^{3+\alpha}_{1+\beta}(I,\partial I)$ is too wide and its elements may have singular second derivatives at the points $x_\pm$. As a consequence, the curves Γ corresponding to such elements may leave V. To show that this can not happen, we should study the asymptotics of solution near the contact points. We are not able to make it here and we refer the reader to the papers [1,2,4]. In particular, it is shown in [4] that the free boundary is more regular and it belongs, as a minimum, to the class $C^{3+\alpha}_{1+\gamma}$ with $\gamma \in (1/2, 1)$, and that it is contained in V, provided that

$$p_1 > \frac{\sigma}{R_0}.$$

This condition is guaranteed by (2.7) and by the smallness of $p_1 - p_0$, hence, the solution we have obtained is physically reasonable. The proof of Theorem 1 is now complete.

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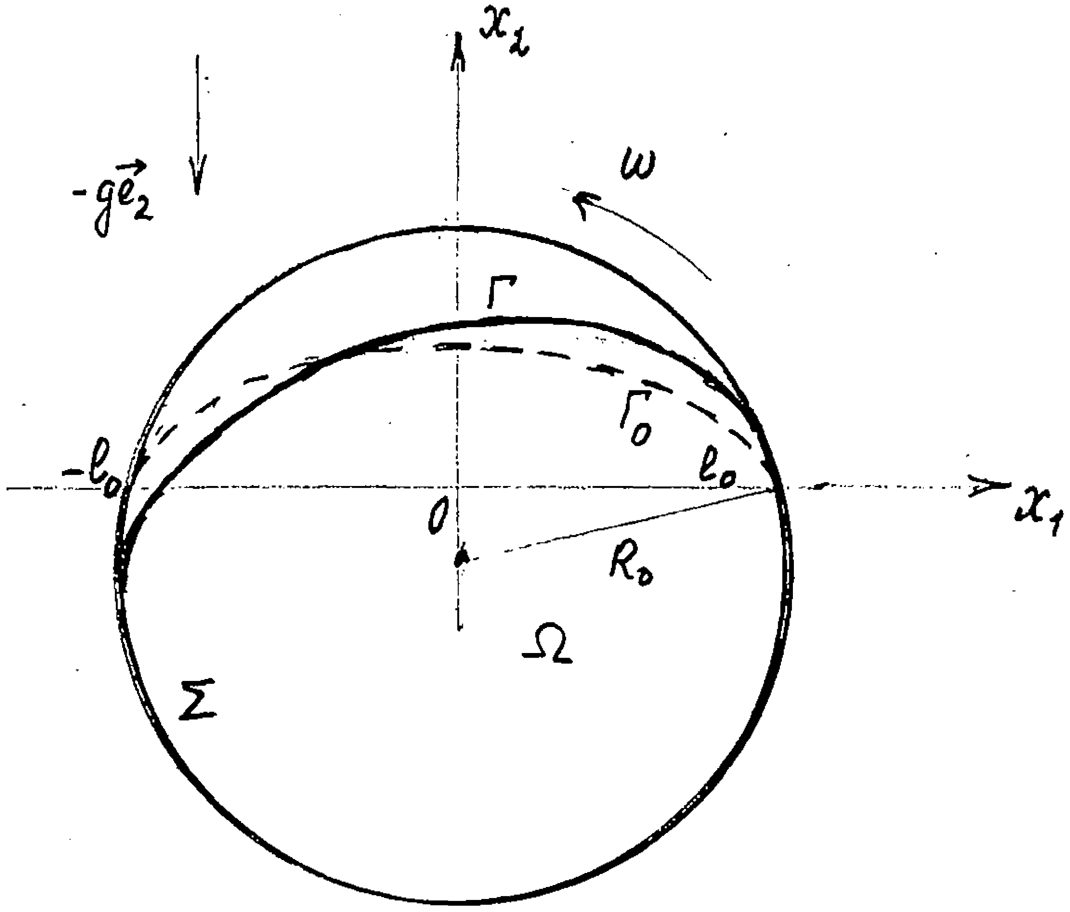

Fig 1