MODULI SPACES OF STABLE 2–BUNDLES AND POLARIZATIONS

by

Kai–Cheong MONG

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

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Introduction

The purpose of this note is to discuss, and in some cases explicitly describe, the differences between moduli spaces of stable 2–bundles with $c_1 = 0$ over a simply–connected Kähler surface as the Kähler metric varies. Just by using simple arguments we obtain two clear–cut results. To be more precise, let Y be a compact simply–connected Kähler surface and $\tilde{\Omega}_Y$ the associated Kähler cone. For a Kähler form ω on Y, we denote $M_k^s(\omega)$ the moduli space of ω–stable 2–bundles E over Y which satisfy $(c_1(E), c_2(E)) = (0,k)$. Note that the integer k is necessarily positive and we assume $k > 0$ for convenience. Let $\tilde{\mathscr{C}}_Y^k$ be the set of connected components of $\tilde{\Omega}_Y \setminus \bigcup_{1 \leq \ell \leq k} \tilde{W}_\ell$ where

$$\tilde{W}_\ell = \bigcup \{ \langle e \rangle^\perp \subset H^2(Y;\mathbb{R}) \mid e \cdot e = -\ell \,;\, e \in H^{1,1}(Y;\mathbb{Z}) \} .$$

We call an element $\tilde{C} \in \mathscr{C}_Y^k$ a chamber.

Proposition 1. The moduli space $M_k^s(\omega)$ does not change as long as the Kähler form ω varies inside a chamber $\tilde{C} \in \mathscr{C}_Y^k$.

In other words, stability condition is uniformly defined on each chamber $\tilde{C}$ and for any two given Kähler forms ω_{-1}, ω_1 on Y we always have

$$M_k^s(\omega_{-1}) = M_k^s(\omega_1)$$

assuming that ω_{-1} and ω_1 both lie in a common chamber $\tilde{C} \in \mathscr{C}_Y^k$. This result contains, as a special case, remark (2.2) in [F].

Now we wish to compare $M_k^s(\omega_{-1})$ with $M_k^s(\omega_1)$ the Kähler forms ω_{-1}, ω_1 lie in two different chambers $\tilde{C}_{-1}$, $\tilde{C}_1$ in the divided Kähler cone $\tilde{\Omega}_Y$. The following situation is most explicit to describe.

Proposition 2. Let $\{\omega_t \mid t \in [-1,1]\}$ be a path of Kähler forms on Y meeting only a single wall $\langle e \rangle^\perp$ of the system $\bigcup_{1 \le \ell \le k} \tilde{W}_\ell$. Assuming

$$\omega_{-1} \cdot e < 0 = \omega_0 \cdot e < \omega_1 \cdot e \quad \text{and} \quad e \cdot e = -k,$$

we have that

$$M_k^s(\omega_{-1}) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L_e^2) \backslash \{0\}) \quad \text{and}$$

$$M_k^s(\omega_1) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L_e^{-2}) \backslash \{0\})$$

where L_e denotes the holomorphic line bundle over Y determined by e.

What this proposition describes is the following. For $t < 0$, the moduli spaces $M_k^s(\omega_t)$ always contain a copy of projective space $\mathbb{P}(H^1(L_e^2)\setminus\{0\})$ parametrizing non–trivial extensions E in the exact sequence

$$0 \longrightarrow L_e \longrightarrow E \longrightarrow L_e^{-1} \longrightarrow 0 .$$

However, as t increases to zero, this copy of projective space degenerates (into the reduction $L_e \oplus L_e^{-1}$ in fact). As t passes through zero and become positive, there emerges another copy of projective space $\mathbb{P}(H^1(L_e^{-2})\setminus\{0\})$ in $M_k^s(\omega_t)$ parametrizing non–trivial extensions E in the exact sequence

$$0 \longrightarrow L_e^{-1} \longrightarrow E \longrightarrow L_e \longrightarrow 0 .$$

Apart from such projective spaces, the moduli spaces $M_k^s(\omega_t)$ do not change.

This proposition is less satisfactory in that we compare the moduli spaces only as sets while quite possibly a corresponding model in deformation theory could have been given. Actually this can be done but we find the argument a bit diverge. For this reason we leave this story to some other occasion.

Our discussion here, despite in the field of algebraic geometry, is in fact resulted from the consideration of extending the Γ–invariant and the polynomial invariants introduced by Donaldson to formulate further differential invariants for smooth 4–manifolds with $b_2^+ = 1$. It turns out that these two propositions are closely related to calculations for such differential invariants. We shall explain this in a future article.

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§ 1. The uniformity of moduli spaces of stable 2–bundles

Here we give the proof of proposition 1 but recall first some preliminary material required in the argument. For a given Kähler form ω on Y we define

$$\deg_\omega E = \int_Y c_1(\det E) \wedge \omega$$

for each holomorphic bundle $E \longrightarrow Y$. Note that the degree $\deg_\omega E$ of E depends upon the Kähler form ω.

(1.1) Definition. A 2–bundle $E \longrightarrow Y$ is ω–stable if for every non–trivial holomorphic bundle map $\varphi : \mathscr{L} \longrightarrow E$ from a holomorphic line bundle $\mathscr{L}$ into E we have

$$\deg_\omega \mathscr{L} < \tfrac{1}{2} \deg_\omega E \ .$$

For simplicity we write $\omega \cdot \mathscr{L}$ in place of $\deg_\omega \mathscr{L}$ if $\mathscr{L}$ is a line bundle over Y.

(1.2) <u>Remark</u>. Regarding φ an element of $H^0(\mathscr{L}^{-1} \otimes E)$, we can actually require in the definition (1.1) that the inequality holds only for those non–zero $\varphi \in H^0(\mathscr{L}^{-1} \otimes E)$ with isolated zeros on Y. Indeed, if $\varphi \in H^0(\mathscr{L}^{-1} \otimes E)$ vanishes along an effective divisor $D \geq 0$, we may find a non–zero bundle map $\tilde{\varphi}$ fitting into an exact sequence

$$0 \longrightarrow \mathscr{L} \otimes D \xrightarrow{\tilde{\varphi}} E \longrightarrow \mathscr{L}^{-1} \otimes D^{-1} \otimes I \longrightarrow 0$$

where $\tilde{\varphi} \in H^0(\mathscr{L}^{-1} \otimes D^{-1} \otimes E)$ has isolated zeros defining an ideal sheaf I. Now the requirement

$$\deg_\omega(\mathscr{L} \otimes D) < \tfrac{1}{2} \deg_\omega E$$

for such situations certainly implies the inequality for $\mathscr{L}$ in (1.1) as $\deg_\omega D \geq 0$.

It follows from this remark to test the ω–stability of a 2–bundle $E \longrightarrow Y$ with $c_1(E) = 0$ it suffices to check the inequality $\omega \cdot \mathscr{L} < 0$ holds for all possible exact sequences

$$(1.3) \qquad 0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0$$

induced from non–zero elements φ in $H^0(\mathscr{L}^{-1} \otimes E)$ with isolated zero. This is our main tool of studying how the moduli space $M^s_k(\omega)$ of ω–stable 2–bundles $E \longrightarrow Y$ with $c_1(E) = 0$, $c_2(E) = k > 0$ changes as the Kähler form ω varies.

Proof of proposition 1. Note first the assumption ω_{-1}, ω_1 lie in a common connected component of the divided Kähler cone $\tilde{\Omega}_Y$ is equivalent to that

$$\text{sign}(\omega_1 \cdot e) = \text{sign}(\omega_{-1} \cdot e) \neq 0 \tag{1.4}$$

for all lattice points $e \in H^{1,1}(Y;\mathbb{Z})$ with $e \cdot e = -1, \ldots, -k$. Now suppose on the contrary $M_k^s(\omega_{-1}) \neq M_k^s(\omega_1)$ and so we may assume without loss in such cases there is an element $[E] \in M_k^s(\omega_{-1}) \backslash M_k^s(\omega_1)$. Then, as E is not ω_1–stable, we can find an ω_1–destabilizing line bundle $\mathcal{L}$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow E \longrightarrow \mathcal{L}^{-1} \otimes I \longrightarrow 0$$

as in (1.3) with $\omega_1 \cdot \mathcal{L} \geq 0$. Note that $\omega_{-1} \cdot \mathcal{L} < 0$ in this situation as E is ω_{-1}–stable. Our aim here is to check that

$$\mathcal{L} \cdot \mathcal{L} = -1, \ldots, -k \ .$$

Granted this one infers from (1.4)

$$\text{sign}(\omega_1 \cdot \mathcal{L}) = \text{sign}(\omega_{-1} \cdot \mathcal{L}) < 0$$

which however contradicts $\omega_1 \cdot \mathcal{L} \geq 0$ and the lemma will then follow.

To check $\mathcal{L} \cdot \mathcal{L} = -1, \ldots, -k$, one observes first

$$0 \leq c_2(E \otimes \mathcal{L}^{-1}) = c_2(E) + \mathcal{L} \cdot \mathcal{L}$$

which gives $\mathscr{L} \cdot \mathscr{L} \geq -k$ as $c_2(E) = k$. To show on the other hand $\mathscr{L} \cdot \mathscr{L} < 0$ we apply the Hodge index theorem to a Kähler form ω_0 on Y attaining $\omega_0 \cdot \mathscr{L} = 0$. Such ω_0 exists somewhere in the path

$$\{(1-t)\omega_{-1} + t\omega_1 \mid t \in [0,1]\}$$

of Kähler forms as $\omega_1 \cdot \mathscr{L} \geq 0 > \omega_{-1} \cdot \mathscr{L}$. This proves the proposition.

<u>Remark</u>. There might have interest to know using similar argument one can prove the semi–stability condition and the stability condition are actually equivalent on each chamber $\tilde{C} \in \tilde{\mathscr{C}}^k_Y$. (Following the present context, a 2–bundle $E \longrightarrow Y$ is ω–semi–stable if it satisfies weaker requirements that $\omega \cdot \mathscr{L} \leq 0$ in definition (1.1).) Thus, in some sense, "essential" ω–semi–stable bundles can possibly occur only when the polarization ω lives in $<e>^{\perp}$ for some lattice $e \in H^{1,1}(Y;\mathbb{Z})$ with $-1 \leq e \cdot e \leq -k$.

§ 2. <u>The changes of moduli spaces of stable 2–bundles</u>

Now we wish to compare $M^s_k(\omega_{-1})$ with $M^s_k(\omega_1)$ when ω_{-1}, ω_1 lie in two different chambers $\tilde{C}_{-1}$, $\tilde{C}_1$ in the divided Kähler cone $\tilde{\Omega}_Y$ as described in proposition 2. For convenience, we write $\{\omega_t \mid t \in [-1,1]\}$, or simply $\{\omega_t\}$, to denote the path of Kähler forms on Y joining ω_{-1}, ω_1 in a usual way. Assume also that $\{\omega_t\}$ meets the system of walls

$$\{<e>^{\perp} \subset H^2(Y;\mathbb{R}) \mid e \in H^{1,1}(Y;\mathbb{Z}),\ e \cdot e = -1, \ldots, -k\}$$

only at $t = 0$ in such a way that

$$\omega_{-1} \cdot L_e < 0 = \omega_0 \cdot L_e < \omega_1 \cdot L_e \quad \text{and} \quad L_e \cdot L_e = -k .$$

To show proposition 2 it is required to establish that

$$M_k^s(\omega_{-1}) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L_e^2)\setminus\{0\}) \quad \text{and} \tag{2.1}$$

$$M_k^s(\omega_1) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L_e^{-2})\setminus\{0\}) . \tag{2.2}$$

We shall however only prove (2.1) as the argument for (2.2) is completely similar. Denote t_- for an element $t \in [-1,0]$ in the following argument.

Proof of proposition 2. It is not difficult to see that

$$M_k^s(\omega_0) \subset M_k^s(\omega_{-1}) .$$

Indeed, if on the contrary there is an element $[E] \in M_k^s(\omega_0)\setminus M_k^s(\omega_{-1})$, then we can find as before an exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0$$

having the properties that

(i) $\omega_{-1} \cdot \mathscr{L} \geq 0 > \omega_0 \cdot \mathscr{L}$ and

(ii) $-1 \leq \mathscr{L} \cdot \mathscr{L} \leq -k$.

Since ω_{t_-} , ω_{-1} lie in a common chamber, one infers from (ii) that

$$\text{sign}(\omega_{t_-} \cdot \mathscr{L}) = \text{sign}(\omega_{-1} \cdot \mathscr{L}) \tag{2.3}$$

which moreover has to be (strictly) positive as $\omega_{-1} \cdot \mathscr{L} > 0$ in this situation by (i). It follows then

$$\omega_{t_-} \cdot \mathscr{L} > 0 > \omega_0 \cdot \mathscr{L} . \tag{2.4}$$

Now set $t_- \longrightarrow 0$ and one sees immediately a contradiction in (2.4) as $\omega_{t_-} \cdot \mathscr{L} \longrightarrow \omega_0 \cdot \mathscr{L}$ when t_- approaches zero.

To identify $M_k^s(\omega_{-1}) \backslash M_k^s(\omega_0)$, one argues similarly that every element $[E] \in M_k(\omega_{-1}) \backslash M_k(\omega_0)$ can be obtained by an extension

$$0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0 \tag{2.5}$$

with $\omega_0 \cdot \mathscr{L} \geq 0 > \omega_{-1} \cdot \mathscr{L}$ and that $-1 \leq \mathscr{L} \cdot \mathscr{L} \leq -k$. Again, using (2.3) we have

$$\omega_0 \cdot \mathscr{L} \geq 0 > \omega_{t_-} \cdot \mathscr{L}$$

and from which one infers $\omega_0 \cdot \mathscr{L} = 0$ by setting $t\_ \longrightarrow 0$. Assuming $\{\omega_t\}$ meets only the wall $< L_e >^{\perp}$, we conclude $\mathscr{L} = L_e^{\pm 1}$. Furthermore, the assumption $\omega_{-1} \cdot L_e < 0$ determines that $\mathscr{L} = L_e$ as $\mathscr{L} \cdot \omega_{-1} < 0$ by the ω_{-1}–stability of E. In the case when $L_e \cdot L_e = \mathscr{L} \cdot \mathscr{L} = -k$, we have $|I| = 0$ and therefore (2.5) reads

(2.6) $$0 \longrightarrow L_e \longrightarrow E \longrightarrow L_e^{-1} \longrightarrow 0 \ .$$

Now assertion (2.1) follows should one prove that non–trivial extensions of (2.6) define ω_{-1}–stable bundles. To see this is the case, consider the following potential destabilizing model for E, where $\mathscr{L}_1 \longrightarrow Y$ is a line bundle with $\omega_{-1} \cdot \mathscr{L}_1 \geq 0$.

(2.7) Diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{L}_1 & & & & \\
 & & & & \downarrow \beta & & & & \\
0 & \longrightarrow & L_e & \longrightarrow & E & \xrightarrow{\alpha} & L_e^{-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{L}_1^{-1} \otimes I_1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

We are to prove the map β has to be zero if the bundle E does not split.

The case when the composition map $\alpha \circ \beta$ is identically zero never causes problem since then the map β factors through L_e and from this one infers

$$\omega_{-1} \cdot \mathscr{L}_1 \leq \omega_{-1} \cdot L_e < 0 \ ,$$

a contradiction to the assumption that $\omega_{-1} \cdot \mathscr{L}_1 \geq 0$. In the case when $\alpha \circ \beta \not\equiv 0$, we have either

(i) $\alpha \circ \beta$ vanishes somewhere, or

(ii) $\alpha \circ \beta$ is nowhere vanishing.

In the former case it is easy to observe

$$\omega_0 \cdot (\mathscr{L}_1^{-1} \otimes L_e^{-1}) = \omega_0 \cdot \mathscr{L}_1^{-1} > 0$$

as $\omega_0 \cdot L_e = 0$. It follows then

$$\omega_0 \cdot \mathscr{L}_1 < 0 \leq \omega_{-1} \cdot \mathscr{L}_1 \tag{2.8}$$

and one deduces as before that $-1 \leq \mathscr{L}_1 \cdot \mathscr{L}_1 \leq -k$. Thus, as ω_{t_-} and ω_{-1} lie in a common chamber, we have

$$\text{sign}(\omega_{t_-} \cdot \mathscr{L}_1) = \text{sign}(\omega_{-1} \cdot \mathscr{L}_1) > 0$$

and by setting $t_- \longrightarrow 0$ we obtain $\omega_0 \cdot \mathscr{L}_1 \geq 0$. This however contradicts (2.8) and we may then exclude the possibility of (i).

Now if $\alpha \circ \beta$ is nowhere vanishing, one finds $\mathscr{L} = L_e^{-1}$ and so to prove $\beta \in H^0(E \otimes L_e)$ is zero in this situation it suffices to check non–trivial extensions

$$0 \longrightarrow L_e \longrightarrow E \longrightarrow L_e^{-1} \longrightarrow 0$$

give simple bundles, i.e. $h^0(E^* \otimes E) = 1$. Indeed, as $h^0(E \otimes L_e^{-1}) > 0$, the simplicity of E ensures $h^0(E \otimes L_e) = 0$ and hence that $\beta \in H^0(E \otimes L_e)$ is zero.

The simplicity of E follows conveniently from a lemma of Oda which asserts as a special case that 2–bundles $E \longrightarrow Y$ obtained from non–trivial extensions of two line bundles, say L_e and L_e^{-1} in our situation, are simple if $h^0(L_e^{\pm 2}) = 0$ (cf. [O]). This is certainly the case here as $\omega_0 \cdot L_e^{\pm 2} = 0$ and ω_0 is a Kähler form on Y . Thus we exclude the possibility of (ii) and complete the proof of this lemma.

These arguments applies equally well to the cases when $L_e \cdot L_e = -1, \dots ,-k+1$. In such situations, one proves as before

$$M_k^s(\omega_0) \subset M_k^s(\omega_{-1}) , \quad M_k^s(\omega_0) \subset M_k^s(\omega_1)$$

and that $M_k^s(\omega_{-1}) \backslash M_k^s(\omega_0)$, $M_k^s(\omega_1) \backslash M_k^s(\omega_0)$ contain locally free extensions

$$0 \longrightarrow L_e \longrightarrow E \longrightarrow L_e^{-1} \otimes I \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow L_e^{-1} \longrightarrow E \longrightarrow L_e \otimes I \longrightarrow 0$$

respectively. It is not clear to me though what the description of $M_k(\omega_{-1}) \backslash M_k(\omega_0)$ or $M_k(\omega_1) \backslash M_k(\omega_0)$ in general would be.

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