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GEOMETRIZATION OF PRINCIPAL SERIES REPRESENTATIONS OF REDUCTIVE GROUPS

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Abstract. In geometric representation theory, one often wishes to describe representations realized on spaces of invariant functions as trace functions of equivariant perverse sheaves. In the case of principal series representations of a connected split reductive group G over a local field, there is a description of families of these representations realized on spaces of functions on G invariant under the translation action of the Iwahori subgroup, or a suitable smaller compact open subgroup, studied by Howe, Bushnell and Kutzko, Roche, and others. In this paper, we construct categories of perverse sheaves whose traces recover the families associated to regular characters of $T(\mathbb{F}_q[\![t]\!])$, and prove conjectures of Drinfeld on their structure. We also propose conjectures on the geometrization of families associated to more general characters.

Contents

1. Introduction: main results and conjectures

To every complex of constructible sheaves on a variety over a finite field, Grothendieck attached the trace of Frobenius function on its rational points. He then initiated a program to study geometric (or sheaf-theoretic) analogues of various classical constructions on rational points. In this article, we study geometric analogues of principal series representations of $G(\mathbb{F}_q(\!(t)\!))$, where G is a connected split reductive group over $\mathbb{F}_q[\![t]\!]$. Our main theorems, stated in §1.3.1, concern geometrizing the principal series representations that are associated to characters of the torus $T(\mathbb{F}_q(\!(t)\!))$ whose restrictions to $T(\mathbb{F}_q[\![t]\!])$ are *regular*, i.e., have trivial stabilizers under the Weyl group action. In §1.4 we present conjectures concerning the geometrization of more general families of principal series representations.

The geometric objects we study are certain (twisted equivariant) perverse sheaves on quotients of the loop group of G. Before getting into details, let us mention two notable features of this work. First, the quotients we consider are not, in general, proper. As far as we know, considering perverse sheaves on non-proper quotients of the loop group is not standard in geometric representation

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theory. Second, some of the perverse sheaves that arise in the regular setting turn out to be *clean* (see Theorem 7). This means that these perverse sheaves are the extensions by zero of shifted local systems on certain locally closed subvarieties. This fact is a reflection of the particularly simple Hecke algebras that arise in the regular case.

1.1. **Motivation.** Let $\mathbb{F}_q$ be a finite field of order q, $F = \mathbb{F}_q(\!(t)\!)$, $\mathcal{O} = \mathbb{F}_q[\![t]\!]$, and $\mathfrak{p} = t\mathcal{O} \subseteq \mathcal{O}$. Let G be a connected split reductive group over $\mathcal{O}$ and T be a maximal split torus. Choose a Borel subgroup B containing T. A principal series representation of $G(F)$ is a representation obtained by parabolic induction of a character of $T(F)$. Note that a principal series representation is, roughly speaking, realized on a space of twisted functions on $G(F)/B(F)$. It is well known that $G(F)$ and $B(F)$ are the sets of $\mathbb{F}_q$-points of group ind-schemes $\mathbf{G}$ and $\mathbf{B}$ over $\mathbb{F}_q$. Therefore, the naive geometric analogue of principal series representations should be perverse sheaves on $\mathbf{G}/\mathbf{B}$. The problem is that the latter ind-scheme is not an inductive limit of schemes of finite type (we will henceforth call this property *ind-finite type*). For this reason, the category of perverse sheaves on $\mathbf{G}/\mathbf{B}$ is not well understood.[1] This issue is the source of much difficulty in geometric representation theory.

1.1.1. *A family of unramified representations.* One way to overcome this difficulty is to geometrize representations in families. As an example, let us consider the geometrization of unramified (principal series) representations. Let $\mathscr{W}^c = \operatorname{ind}_{G(\mathcal{O})}^{G(F)} \mathbf{1}$, where $\mathbf{1}$ denotes the trivial character. We think of $\mathscr{W}^c$ as a family of unramified (principal series) representations. The endomorphism ring of this family identifies with the spherical Hecke algebra $\mathscr{H}^c = \mathscr{H}(G(F), G(\mathcal{O}))$. The Satake isomorphism states that $\mathscr{H}^c \xrightarrow{\sim} \mathrm{K}_0(\mathrm{Rep}(\check{G}))$, where the latter is the Grothendieck group of the category of finite dimensional rational representations of the dual (complex reductive) group of G.

1.1.2. *Geometrizing the unramified family.* It is known that $G(\mathcal{O})$ (resp. $G(F)$) is the group of $\mathbb{F}_q$-points of a proalgebraic group $\mathbf{G}_{\mathcal{O}}$ (resp. a group ind-scheme $\mathbf{G}$) over $\mathbb{F}_q$. Fix a prime ℓ not dividing q. Let $\mathbf{Gr} := \mathbf{G}/\mathbf{G}_{\mathcal{O}}$ denote the affine Grassmannian. Let

$$\mathscr{W}^c_{\mathrm{geom}} = \mathscr{P}(\mathbf{Gr}), \quad \mathscr{W}^{c,\mathrm{der}}_{\mathrm{geom}} = \mathscr{D}(\mathbf{Gr}), \quad \mathscr{H}^c_{\mathrm{geom}} = \mathscr{P}_{\mathbf{G}_{\mathcal{O}}}(\mathbf{Gr}), \quad \mathscr{H}^{c,\mathrm{der}}_{\mathrm{geom}} = \mathscr{D}_{\mathbf{G}_{\mathcal{O}}}(\mathbf{Gr}).$$

Here $\mathscr{D}$ denotes the bounded constructible derived category of $\overline{\mathbb{Q}}_\ell$-sheaves and $\mathscr{P}$ denotes the subcategory of perverse sheaves (see Appendix §B). There is a convolution functor

$$\star : \mathscr{W}^c_{\mathrm{geom}} \times \mathscr{H}^c_{\mathrm{geom}} \to \mathscr{W}^{c,\mathrm{der}}_{\mathrm{geom}}.$$

This functor restricts to a functor $\star : \mathscr{H}^c_{\mathrm{geom}} \times \mathscr{H}^c_{\mathrm{geom}} \to \mathscr{H}^{c,\mathrm{der}}_{\mathrm{geom}}$. Let $\mathsf{LocSys}(\operatorname{Spec}\mathbb{F}_q)$ denote the monoidal category of ℓ-adic local systems on $\operatorname{Spec}(\mathbb{F}_q)$. Note that this category is equivalent to the category of finite dimensional continuous ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The following theorem is known as the geometric Satake isomorphism, and is due to Lusztig [Lus83], Ginzburg [Gin99], Mirković and Vilonen [MV07], and Beilinson and Drinfeld [BDa, §5].

Theorem 1. (i) *The category $\mathscr{H}^c_{\mathrm{geom}}$ is closed under convolution.*
(ii) *There is an equivalence of monoidal abelian categories* $\mathrm{Rep}(\check{G}) \boxtimes \mathsf{LocSys}(\operatorname{Spec}\mathbb{F}_q) \xrightarrow{\sim} \mathscr{H}^c_{\mathrm{geom}}$.[2]

The nontrivial part of Theorem 1.(i) is that the convolution of two objects of $\mathscr{H}^c_{\mathrm{geom}}$ is perverse (and not merely an object of the equivariant derived category $\mathscr{H}^{c,\mathrm{der}}_{\mathrm{geom}}$). The next theorem states that this remains true if one of the perverse sheaves is allowed to lie in the larger category $\mathscr{W}^c_{\mathrm{geom}}$.

[1]For some results regarding perverse sheaves on $\mathbf{G}/\mathbf{B}$ see, e.g., [FM99] and [FFKM99].

[2]Most references work with the loop group over an algebraically closed field (e.g., $\mathbb{C}$ or $\overline{\mathbb{F}}_q$). In this case, $\mathsf{LocSys}(\operatorname{Spec}\mathbb{F}_q)$ disappears. On the other hand, Gaitsgory [Gai01] works over $\mathbb{F}_q$; see the proof of Proposition 1 in *op. cit.*.

Theorem 2 ([Gai01])**.** *$\mathscr{H}^c_{\mathrm{geom}}$ acts on $\mathscr{W}^c_{\mathrm{geom}}$ by convolution.*

1.1.3. *The subject of this paper.* In this paper, we consider the problem of geometrizing families of principal series representations associated to nontrivial characters $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$. The families we consider have been studied by Howe, Bushnell and Kutzko, Roche, and others. In particular, they explain how to realize these families as representations induced from characters of compact open subgroups. From the geometric point of view, the advantage of inducing from a compact open subgroup J is that the corresponding quotient of varieties, $\mathbf{G}/\mathbf{J}$, turns out to be of ind-finite type.

Our main theorems, in the case of $G = \mathrm{GL}_N$, were conjectured by Drinfeld in June 2005. Two of our main theorems are analogues of Theorems 1 and 2 in the regular setting. The other theorem concerns a phenomenon unique to the regular setting: namely, that the irreducible objects of $\mathscr{H}_{\mathrm{geom}}$ turn out to be clean.

1.1.4. *Connections to local geometric Langlands.* Frenkel and Gaitsgory have outlined a program for geometrizing (or categorifying) the local Langlands correspondence; see [FG06] and [Fre07]. Theorems 1 and 2 play important roles in their description of the unramified and tamely ramified part of this correspondence. We expect that our main results will have applications in the wildly ramified part of the Frenkel-Gaitsgory program. In particular, in future work, we hope to construct the geometric analogue of an irreducible principal series representation as a category on which $G((t))$ acts. We expect that this category is related to the category of representations of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ at the critical level via an infinite dimensional analogue of Bernstein-Beilinson localization, as conjectured in [FG06].

1.2. **Principal series representations via compact open subgroups: recollections.** We continue using the notation introduced at the beginning of previous section. Henceforth, we assume that q is restricted as in §3.1.2. Fix a character $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$.

1.2.1. *A Family of principal series representations.* Let

$$\Pi := \iota_{B(F)}^{G(F)} \left(\mathrm{ind}_{T(\mathcal{O})}^{T(F)} \bar{\mu} \right). \tag{1.1}$$

Here "ind" denotes the compact induction and ι denotes the (unnormalized) parabolic induction. We think of Π as the family of principal series representations of $G(F)$ associated to characters of $T(F)$ whose restriction to $T(\mathcal{O})$ is $\bar{\mu}$. In the language of [Ber84], Π is a projective generator of the Bernstein block of representations of $G(F)$ corresponding to $(T(\mathcal{O}), \bar{\mu})$. Inducing endomorphisms, we obtain a canonical homomorphism

$$\mathrm{End}_{T(F)} \left(\mathrm{ind}_{T(\mathcal{O})}^{T(F)} \bar{\mu} \right) \to \mathrm{End}_{G(F)}(\Pi). \tag{1.2}$$

In the case that $\bar{\mu}$ is regular, one can show that this is an isomorphism; see, e.g., [Roc09, §1.9].

1.2.2. *Realization of Π via compact open subgroups.* The following theorem, in its full generality, is due to Roche. Previous results in this direction (for GL_N) were obtained by Howe [How73] and Bushnell and Kutzko [BK98, BK99].

Theorem 3 ([Roc98])**.** *There exists a compact open subgroup $J \subset G(F)$ containing $T(\mathcal{O})$ such that*

(i) *$\bar{\mu}$ extends to a character $\mu : J \to \overline{\mathbb{Q}}_\ell^\times$.*

(ii) *There exists an isomorphism of $G(F)$-modules $\mathscr{W} := \mathrm{ind}_J^{G(F)} \mu \cong \Pi$.*

In the language of [BK98], (J, μ) is a *type* for the Bernstein block defined by $(T(\mathcal{O}), \bar{\mu})$.

Example 4. Suppose the character $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$ factors through a character ν of $T(\mathbb{F}_q)$. Then one can (and Roche does) take J to be the Iwahori subgroup $I \subset G(F)$. Let $T_1 < T(\mathcal{O})$ be the subgroup generated by the image of $1 + \mathfrak{p}$ under all coweights, cf. §2.1.1 below. Then, the character μ is defined to be the composition

$$I \to I/I_u \cong T(\mathbb{F}_q) \cong T(\mathcal{O})/T_1 \xrightarrow{\nu} \overline{\mathbb{Q}}_\ell^\times,$$

where I_u is the prounipotent radical of I. More generally, Roche's subgroup J equals the Iwahori subgroup if and only if $(\bar{\mu} \circ \alpha^\vee) : \mathbb{G}_m(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$ factors through a character of $\mathbb{G}_m(\mathbb{F}_q)$ for all coroots $\alpha^\vee : \mathbb{G}_m \to T$ of G.

Example 5. Suppose $G = \mathrm{GL}_N$. Identify $T(\mathcal{O})$ with $(\mathcal{O}^\times)^N$ and write $\bar{\mu} = (\bar{\mu}_1, ..., \bar{\mu}_N)$. Suppose the conductor $\mathrm{cond}(\bar{\mu}_i/\bar{\mu}_j)$ equals a fixed integer n for all $1 \le i, j \le N$. (The conductor of a character $\chi : \mathcal{O}^\times \to \overline{\mathbb{Q}}_\ell^\times$ is the smallest positive integer c for which $\chi(1 + \mathfrak{p}^c) = \{1\}$.) Then

$$J = \begin{pmatrix} \mathcal{O}^\times & \mathfrak{p}^{[\frac{n}{2}]} & \cdots & \mathfrak{p}^{[\frac{n}{2}]} \\ \mathfrak{p}^{[\frac{n+1}{2}]} & \mathcal{O}^\times & \cdots & \mathfrak{p}^{[\frac{n}{2}]} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p}^{[\frac{n+1}{2}]} & \mathfrak{p}^{[\frac{n+1}{2}]} & \cdots & \mathcal{O}^\times \end{pmatrix}.$$

1.2.3. *Endomorphism algebras.* Given a group K, a character χ of K, and a space X on which K acts (by a left action), we say a function f is (K, χ)-invariant if

$$f(k \cdot x) = \chi(k) f(x), \qquad \forall\, k \in K,\ \forall\, x \in X.$$

In this language, $\mathscr{W} = \mathrm{ind}_J^{G(F)} \mu$ is the space of compactly supported (J, μ^{-1})-invariant functions on $G(F)$ with respect to right multiplication (the inverse is here since our convention is to use left actions: so this says that $f(j \cdot_R g) = f(g)\mu(j)^{-1}$, where $j \cdot_R g := gj^{-1}$). Let $\mathscr{H} := \mathscr{H}(G(F), J, \mu)$ denote the space of compactly supported $(J \times J, \mu \times \mu^{-1})$-invariant functions. So $\mathscr{H}$ is the space of functions $f : G(F) \to \overline{\mathbb{Q}}_\ell$ satisfying

$$f(jgj') = \mu(j) f(g) \mu(j'), \qquad \forall j \in J,\ g \in G(F).$$

Convolution defines a right action $\star : \mathscr{W} \times \mathscr{H} \to \mathscr{W}$. It is a standard fact that this action identifies $\mathscr{H}$ with $\mathrm{End}_{G(F)}(\mathscr{W})$. The isomorphism of $G(F)$-modules $\mathscr{W} \cong \Pi$ defines an isomorphism of algebras $\mathscr{H} \cong \mathrm{End}_{G(F)}(\Pi)$.

1.2.4. *Description of $\mathscr{H}$ in the regular case.* In view of §1.2.1, in the case that $\bar{\mu}$ is regular, we obtain an isomorphism

$$\Psi : \mathrm{K}_0(\mathrm{Rep}(\check{T})) \xrightarrow{\sim} \mathscr{H} \tag{1.3}$$

Since these algebras are commutative, one can show that Ψ is *canonical*; i.e., it does not depend on the choice of isomorphism $\Pi \xrightarrow{\sim} \mathscr{W}$. Using general results of Bushnell and Kutzko on types, Roche [Roc98, §6] has proved that Ψ sends each irreducible character of $\check{T}$ to an element of $\mathscr{H}$ supported on a corresponding double coset of J, which determines the image uniquely up to a nonzero constant. In this paper, we compute these constants explicitly; see Theorem 34.

1.3. **Geometrization in the regular case: main theorems.** In this subsection, we geometrize Roche's family $\mathscr{W}$, the Hecke algebra $\mathscr{H}$, and the action of $\mathscr{H}$ on $\mathscr{W}$ by convolution, *in the case that the family $\mathscr{W}$ is associated to a regular character of $T(\mathcal{O})$.*

The combinatorial description of J makes it obvious that it is equal to the group of $\mathbb{F}_q$-points of a connected proalgebraic group $\mathbf{J}$ over $\mathbb{F}_q$ (see §4.2.1 for an alternative explanation). Using standard constructions from the sheaf-function dictionary, we construct a one-dimensional character sheaf $\mathcal{M}$ on $\mathbf{J}$ whose trace of Frobenius function equals the character μ. The ind-scheme $\mathbf{G}/\mathbf{J}$ is of ind-finite type; thus, we can consider the category of (twisted perverse) sheaves and the bounded constructible derived category of sheaves on this quotient. We observe that $\mathbf{G}/\mathbf{J}$ *is not proper* unless J is the Iwahori subgroup.

We define $\mathscr{W}^{\mathrm{der}}_{\mathrm{geom}}$ to be the bounded $(\mathbf{J}, \mathcal{M}^{-1})$-equivariant constructible derived category of sheaves on $\mathbf{G}$. Since $\mathbf{J}$ acts freely on $\mathbf{G}$, this derived category can be defined naively; that is, it equals the category of $(\mathbf{J}, \mathcal{M}^{-1})$-equivariant complexes of sheaves on $\mathbf{G}$ with bounded constructible cohomology. Let $\mathscr{W}_{\mathrm{geom}}$ denote the perverse heart of this triangulated category, which should be thought of as the category of $\mathcal{M}^{-1}$-twisted perverse sheaves on $\mathbf{G}/\mathbf{J}$ (we will define this more precisely in §4; see also the remark below).

Next, define $\mathscr{H}_{\mathrm{geom}}$ as the category of $(\mathbf{J}\times\mathbf{J}, \mathcal{M}\boxtimes\mathcal{M}^{-1})$-equivariant perverse sheaves on $\mathbf{G}$ (again, we will define this more precisely in §4). We define the convolution with compact support as a functor $\star : \mathscr{W}^{\mathrm{der}}_{\mathrm{geom}} \times \mathscr{H}_{\mathrm{geom}} \to \mathscr{W}^{\mathrm{der}}_{\mathrm{geom}}$. There is also a convolution without compact support defining a functor with the same source and target. A priori, they need not coincide, since $\mathbf{G}/\mathbf{J}$ is not proper (unless J is the Iwahori subgroup, as mentioned above). However, it turns out that the two notions of convolution nonetheless coincide (Corollary 51).

Remark 6. One can probably define $(\mathbf{J}, \mathcal{M}^{-1})$-equivariant perverse sheaves on $\mathbf{G}$ in (at least) three equivalent, but philosophically distinct, ways:

(i) Twist the category of perverse sheaves on $\mathbf{G}/\mathbf{J}$ by a certain gerbe associated to the pair $(\mathbf{J}, \mathcal{M}^{-1})$.

(ii) The local system $\mathcal{M}$ becomes trivial after pulling back to a certain finite central cover $\tilde{\mathbf{J}} \to \mathbf{J}$. One then defines $(\mathbf{J}, \mathcal{M}^{-1})$-equivariant perverse sheaves on $\mathbf{G}$ as a certain full abelian subcategory of perverse sheaves on the ind-Deligne-Mumford stack $\mathbf{G}/\tilde{\mathbf{J}}$.

(iii) There exists a proalgebraic normal subgroup $\mathbf{J}' < \mathbf{J}$ such that $\mathcal{M}$ is trivial on $\mathbf{J}'$ and $\mathbf{J}/\mathbf{J}'$ is a commutative algebraic group (Lemma 37). Thus, $\mathcal{M}$ is the pullback of a local system $\mathcal{M}_0$ on the quotient algebraic group $\mathbf{A} := \mathbf{J}/\mathbf{J}'$. One then defines a $(\mathbf{J}, \mathcal{M}^{-1})$-equivariant perverse sheaf on $\mathbf{G}$ to be an $(\mathbf{A}, \mathcal{M}_0^{-1})$-equivariant perverse sheaf on $\mathbf{G}/\mathbf{J}'$.

We only consider the last approach in the present text; see §4. For more details regarding the first two approaches, see §B.5.

1.3.1. *Statements of the main results.* We continue to use the notation employed above. So $\mathscr{W}$ is the family of principal series representation associated to a regular character of $T(\mathcal{O})$ and $\mathscr{H}$ is the endomorphism ring of this family. The abelian categories of perverse sheaves $\mathscr{W}_{\mathrm{geom}}$ and $\mathscr{H}_{\mathrm{geom}}$ are the geometric analogues of these spaces.

Theorem 7. *The simple objects of $\mathscr{H}_{\mathrm{geom}}$ are clean.*

For a precise definition of clean, see Definition 62.

Theorem 8. (i) *The category $\mathscr{H}_{\mathrm{geom}}$ is closed under convolution.*

(ii) *There exists an equivalence of monoidal abelian categories*

$$\Psi_{\mathrm{geom}} : \mathrm{Rep}\,\check{T} \boxtimes \mathsf{LocSys}(\mathrm{Spec}\,\mathbb{F}_q) \xrightarrow{\sim} \mathscr{H}_{\mathrm{geom}}.$$

Theorem 9. *Convolution defines a monoidal abelian action $\star : \mathscr{W}_{\mathrm{geom}} \times \mathscr{H}_{\mathrm{geom}} \to \mathscr{W}_{\mathrm{geom}}$.*

Theorems 8 and 9 are the analogues, in the regular setting, of Theorems 1 and 2, respectively. Observe that, taking the trace of Frobenius, we obtain the isomorphism (1.3) and the action of $\mathscr{H}$ on $\mathscr{W}$. For more precise statements of the above theorems and their proofs, see §5.

Remark 10. The abelian category $\mathscr{H}_{\mathrm{geom}}$ should be the perverse heart of the bounded $(\mathbf{J}\times\mathbf{J}, \mathcal{M}\boxtimes\mathcal{M}^{-1})$-equivariant constructible derived category of sheaves on $\mathbf{G}$. For a discussion of definition of this twisted equivariant derived category see §1.4.2. In view of cleanness (Theorem 7), it is reasonable to ask whether the correct bounded $(\mathbf{J}\times\mathbf{J},\mathcal{M}\boxtimes\mathcal{M}^{-1})$-equivariant derived category in the regular setting is merely the bounded derived category of $\mathscr{H}_{\mathrm{geom}}$; we do not address this issue here (note that this statement is, however, a special case of Conjecture 13).

Remark 11. As mentioned in Example 4, if $\bar{\mu}$ factors through $T(\mathbb{F}_q)$, then the corresponding subgroup equals the Iwahori subgroup I. The category of perverse sheaves on the affine flag variety $\mathbf{G}/\mathbf{I}$ and the corresponding bounded derived category have been studied extensively; see, for instance, [AB09], [Bez09], [Bez04], [BO04], and [BFO09]. Therefore, in this case, it might be possible to extract our results from the aforementioned (or related) references. We have not attempted to do this. On the other hand, we are not aware of any references where perverse sheaves on $\mathbf{G}/\mathbf{J}$ are studied, where J is one of Roche's subgroups other than the Iwahori.

Remark 12. There is some similarity between our setup and that of [FGV01]. In *op. cit.*, the authors define and study a category of perverse sheaves which geometrize $(U(F),\chi)$-invariant functions on $G(F)/G(\mathcal{O})$, where $U<B$ is the unipotent radical, and $\chi: U(F)\to\mathbb{F}_q$ is a generic character, which together with a fixed character $\psi:\mathbb{F}_q\to\overline{\mathbb{Q}}_\ell^\times$ yields a character $\psi\circ\chi: U(F)\to\overline{\mathbb{Q}}_\ell^\times$. The irreducible objects of their category are in bijection with dominant coweights. Their main result states that, like in our situation, these irreducible perverse sheaves are clean, and that their category is semisimple.[3] Note that, unlike our categories, their categories of perverse sheaves are defined in global terms, using Drinfeld's compactification of the moduli space of bundles on curves;[4] also, their geometrization of the character $\chi: U(F)\to\mathbb{F}_q$ is a homomorphism $\mathbf{U}\to\mathbb{G}_a$ (rather than a character sheaf).

1.4. **Geometrization in the non-regular case: conjectures.** The discussions of this subsection are not used anywhere else in the paper; in particular, a reader who is only interested in the regular case or not interested in conjectures can skip this subsection and go to §1.5.

Let $\bar{\mu}: T(\mathcal{O})\to\overline{\mathbb{Q}}_\ell^\times$ be an arbitrary character. We would like to geometrize the family $\Pi=\Pi_{\bar{\mu}}$ of principal series representations induced from characters of $T(F)$ whose restriction to $T(\mathcal{O})$ is $\bar{\mu}$ (see §1.2.1 for the precise definition of Π). By Theorem 3, $\Pi\cong\mathscr{W}=\mathrm{ind}_J^{G(F)}\mu$, and its endomorphism ring identifies with $\mathscr{H}=\mathscr{H}(G(F),J,\mu)$. To geometrize, we would like to replace (J,μ^{-1})- and $(J\times J,\mu\times\mu^{-1})$-invariant functions on $G(F)$ by $(\mathbf{J},\mathcal{M}^{-1})$- and $(\mathbf{J}\times\mathbf{J},\mathcal{M}\times\mathcal{M}^{-1})$-equivariant perverse sheaves on $\mathbf{G}$. However, it is known that, in general, the category of $(\mathbf{J}\times\mathbf{J},\mathcal{M}\times\mathcal{M}^{-1})$-equivariant perverse sheaves on $\mathbf{G}$ is not closed under convolution (this fails already in the unramified setting: the convolution of two $\mathbf{I}$-equivariant perverse sheaves on $\mathbf{G}/\mathbf{I}$ is not necessarily perverse).

In what follows, we propose two remedies:

(I) geometrize $\mathscr{W}$ and $\mathscr{H}$ using the equivariant derived categories;
(II) provided $\bar{\mu}$ is of a special form (which we call parabolic), geometrize a closely related family of representations and its endomorphism ring using perverse sheaves.

1.4.1. *Roche's description of $\mathscr{H}$.* According to [Roc98, §6], for arbitrary $\bar{\mu}$, there exists a (possibly disconnected) split reductive group L over F such that

$$\mathscr{H}=\mathscr{H}(G(F),J,\mu)\cong\mathscr{H}(L(F),I_{L^\circ}),\tag{1.4}$$

where L° is the connected component of the identity of L and I_{L° is an Iwahori subgroup of L°.[5] Note that in the unramified setting (i.e., $\bar{\mu}=\mathbf{1}$), we have $L=L^\circ=I$; see Example 4. On the other hand, in the regular setting, $L=L^\circ=T$; see §1.2.4.

[3] Although our category $\mathscr{H}_{\mathrm{geom}}$ is not semisimple, that is an artifact of working over $\mathbb{F}_q$ instead of over $\overline{\mathbb{F}}_q$ as is done in [FGV01]; note that, after base change to $\overline{\mathbb{F}}_q$, $\mathscr{H}_{\mathrm{geom}}$ becomes semisimple in our setting.

[4] As explained in, e.g., [AB09, §1.1.1], this category can probably be alternatively defined in a purely local way as a certain subcategory of perverse sheaves on the affine flag variety $\mathbf{G}/\mathbf{I}$.

[5] Note that according to §8 of *op. cit.*, L° is an endoscopic group of G (but not necessarily a subgroup).

1.4.2. *Geometrization via the derived category.* Let $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$ denote the (bounded) equivariant derived category $\mathscr{D}_{(\mathbf{J}\times\mathbf{J},\mathcal{M}\boxtimes\mathcal{M}^{-1})}(\mathbf{G})$. Since the action of $\mathbf{J}\times\mathbf{J}$ is not free, the latter is not necessarily the same as the derived category of $(\mathbf{J}\times\mathbf{J},\mathcal{M}\boxtimes\mathcal{M}^{-1})$-equivariant complexes of sheaves with bounded constructible cohomology. We do not know of a reference that gives a proper definition. The correct definition can probably be obtained by modifying the approach of Bernstein and Lunts [BL94] (they consider the case where $\mathcal{M}$ is trivial). Alternatively, one can try to twist triangulated categories following [Rei10, §I.2]. Henceforth, we assume that one has the correct definition of this twisted equivariant derived category. Convolution of sheaves should then endow $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$ with a structure of monoidal triangulated category. The following is a conjectural geometrization of (1.4).

Conjecture 13. *There exists an equivalence of monoidal triangulated categories* $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}\cong\mathscr{D}_{\mathbf{I}_{\mathbf{L}^\circ}}(\mathbf{L}/\mathbf{I}_{\mathbf{L}^\circ})$.

For applications to the Langlands program, we would like to have a description of $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$ in Langlands dual terms. In the case that L is connected (i.e. $L=L^\circ$), there is an answer explained to us by Bezrukavnikov (see the very similar [Bez06, Theorem 4.2.(a)]):

Theorem 14. *There exists a canonical equivalence of triangulated categories*

$$\mathscr{D}_{\mathbf{I}_{\mathbf{L}}}(\mathbf{L}/\mathbf{I}_{\mathbf{L}})\xrightarrow{\sim}\mathscr{D}\,\mathrm{Coh}_{\check{L}}(\tilde{\mathcal{N}}\times^R_{\check{\mathfrak{l}}}\tilde{\mathcal{N}}).\tag{1.5}$$

where $\tilde{\mathcal{N}}$ is the Springer resolution of the cone of nilpotent elements in $\check{\mathfrak{l}}$ and the $\times^R_{\check{\mathfrak{l}}}$ means one must take a derived (dg-algebra) fibered product.[6]

The above conjecture (together with Bezrukavnikov's theorem) gives a geometrization of $\mathscr{H}$, in the sense that $\mathrm{K}_0(\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}})\cong\mathscr{H}$. Moreover, the monoidal category $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$ acts on $\mathscr{W}^{\mathrm{der}}_{\mathrm{geom}}$ geometrizing the action of $\mathscr{H}$ on $\mathscr{W}$.

1.4.3. *Parabolic characters.* To geometrize using only perverse sheaves (as opposed to the whole equivariant derived category) we need to consider not the family $\mathscr{W}$, but a closely related family which we will call $\mathscr{W}^c$. In the case $\bar{\mu}=\mathbf{1}$, we defined this family in §1.1. In general, we only know how to define this family when the character $\bar{\mu}$ is parabolic (and do not know if it should exist more generally).

Definition 15. A character $\bar{\mu}:T(\mathcal{O})\to\overline{\mathbb{Q}}_\ell^\times$ is *parabolic* if the stabilizer $\mathrm{Stab}_W(\bar{\mu})\subseteq W$ of $\bar{\mu}$ is a parabolic subgroup of W, i.e., there exists a parabolic subgroup $P\subseteq G$ whose associated Weyl group is $\mathrm{Stab}_W(\bar{\mu})$.[7]

Let $\bar{\mu}$ be a parabolic character. Let L denote the Levi of the parabolic associated to $\bar{\mu}$. Thus, L is a connected split reductive subgroup of G. It is easy to see that $\bar{\mu}$ extends to a character of $L(\mathcal{O})$ which, by an abuse of notation, will also be denoted by $\bar{\mu}$. Define a new family of principal series representations by

$$\Pi^c:=\iota_{P(F)}^{G(F)}(\mathrm{ind}_{L(\mathcal{O})}^{L(F)}\bar{\mu}).\tag{1.6}$$

We think of Π^c also as a family of principal series representations associated to $\bar{\mu}$. We believe that Π^c can also be realized by inducing a character of a compact open subgroup of $G(F)$. To this end, let $J^c:=J.L(\mathcal{O})$. One can show that $\bar{\mu}:L(\mathcal{O})\to\overline{\mathbb{Q}}_\ell^\times$ extends to a character $\mu^c:J^c\to\overline{\mathbb{Q}}_\ell^\times$. Let $\mathscr{W}^c:=\mathrm{ind}_{J^c}^{G(F)}\mu^c$ and $\mathscr{H}^c:=\mathrm{End}_{G(F)}(\mathscr{W}^c)=\mathscr{H}(G(F),J^c,\mu^c)$.

[6]According to Bezrukavnikov, the reason the fibered product is derived has to do with the fact that $\tilde{\mathcal{N}}\times_{\check{\mathfrak{l}}}\tilde{\mathcal{N}}$ is not a complete intersection in $\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}$. To avoid this one can work instead with the monodromic equivariant category $\mathscr{D}_{\mathbf{I}_{\mathbf{L},u}}(\mathbf{L}/\mathbf{I}_{\mathbf{L}})\xrightarrow{\sim}\mathscr{D}\,\mathrm{Coh}_{\check{L}}(\tilde{\mathfrak{l}}\times_{\check{\mathfrak{l}}}\tilde{\mathcal{N}})$, where $\tilde{\mathfrak{l}}\twoheadrightarrow\check{\mathfrak{l}}$ is the Grothendieck-Springer resolution, and $\mathbf{I}_{\mathbf{L},u}$ is the prounipotent radical of $\mathbf{I}_{\mathbf{L}}$. Perhaps, in our situation, one might similarly prefer to define $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$ as a certain monodromic equivariant category; if correctly defined, we could ask for an analogue of Conjecture 13 that states that the result is equivalent to $\mathscr{D}_{\mathbf{I}_{u,\mathbf{L}}}(\mathbf{L}/\mathbf{I}_{\mathbf{L}})$.

[7]Equivalently, the group L associated to $\bar{\mu}$ of §1.4.1 is the Levi of a parabolic subgroup of G.

Conjecture 16. (i) *There exists an isomorphism of $G(F)$-modules $\Pi^c \cong \mathscr{W}^c$.*
(ii) *There exists a "Satake-type" isomorphism $\mathscr{H}^c \cong \mathrm{K}_0(\mathrm{Rep}(\check{L}))$.*

Remark 17. It should not be difficult to prove part (ii): first, $\mathscr{H}^c$ is a subalgebra of $\mathscr{H}$, which by (1.4) is isomorphic to $\mathscr{H}(L(F), I_L)$. The latter is the affine Hecke algebra for L, whose basis consists of functions supported on I_L-double cosets labeled by the affine Weyl group of L. Then, the J^c-double cosets of $G(F)$ which support functions of $\mathscr{H}^c$ should correspond to the I_L-double cosets of $L(F)$ labeled by $\mathrm{Stab}_W(\bar{\mu})$-invariant coweights of $\check{T}$, cf. §2.3. These, in turn, are also a basis for $\mathrm{K}_0(\mathrm{Rep}(\check{L}))$, which should yield (ii).

Example 18. We now list some cases where the above conjecture is known to be true:

(i) Every regular character $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$ is parabolic. In this case, the $G(F)$-modules $\mathscr{W}^c$ and $\mathscr{W}$ (and, therefore, the algebras $\mathscr{H}^c$ and $\mathscr{H}$) coincide.
(ii) The trivial character (corresponding to the unramified case) is parabolic. In this case, $J = I$ and $J^c = G(\mathcal{O})$.
(iii) If $G = \mathrm{GL}_N$ then every character of $T(\mathcal{O})$ is parabolic. In this case, the above conjecture was proved in [How73] (with a somewhat different choice of J^c).

1.4.4. *Geometrization of families associated to parabolic characters.* Let $\bar{\mu}$ be a parabolic character and let $J^c, \mu^c, \mathscr{W}^c$ and $\mathscr{H}^c$ be as above. One can show that J^c is the group of points of a proalgebraic group $\mathbf{J}^c$ and that μ^c is the trace of Frobenius function of a one-dimensional character sheaf $\mathcal{M}^c$ on $\mathbf{J}^c$. Let $\mathscr{H}^c_{\mathrm{geom}}$ and $\mathscr{W}^c_{\mathrm{geom}}$ be the abelian categories of twisted equivariant perverse sheaves corresponding to $\mathscr{H}^c$ and $\mathscr{W}^c$. Explicitly, $\mathscr{H}^c_{\mathrm{geom}} = \mathscr{P}_{(\mathbf{J}^c \times \mathbf{J}^c, \mathcal{M}^c \times (\mathcal{M}^c)^{-1})}(\mathbf{G})$ and $\mathscr{W}^c_{\mathrm{geom}} = \mathscr{P}_{(\mathbf{J}^c, (\mathcal{M}^c)^{-1})}(\mathbf{G})$.

Conjecture 19. (i) *$\mathscr{H}^c_{\mathrm{geom}}$ is closed under convolution.*
(ii) *There exists an equivalence of monoidal abelian categories $\mathscr{H}^c_{\mathrm{geom}} \cong \mathrm{Rep}\, \check{L} \boxtimes \mathsf{LocSys}(\mathrm{Spec}\, \mathbb{F}_q)$.*

Conjecture 20. *$\mathscr{H}^c_{\mathrm{geom}}$ acts on $\mathscr{W}^c_{\mathrm{geom}}$ by convolution.*

Note that the above conjectures specialize to Theorems 1 and 2 in the unramified case and Theorems 8 and 9 in the regular case. We think of the equivalence of Conjecture 19 as a geometric Satake isomorphism for $\bar{\mu}$.

1.4.5. *Central functor.* One can ask if there is a categorification of the inclusion $\mathscr{H}^c \cong Z(\mathscr{H}) \hookrightarrow \mathscr{H}$. In the unramified setting this was done by Gaitsgory [Gai01]:[8] there exists a functor

$$\mathscr{Z} : \mathscr{H}^c_{\mathrm{geom}} = \mathscr{P}_{\mathbf{G}_\mathcal{O}}(\mathbf{Gr}) \to \mathscr{H}_{\mathrm{geom}} = \mathscr{P}_{\mathbf{I}}(\mathbf{G}/\mathbf{I}),$$

such that $\mathscr{Z}(-) \star -$ and $- \star \mathscr{Z}(-)$ yield well-defined, isomorphic functors $\mathscr{P}_{\mathbf{G}_\mathcal{O}}(\mathbf{Gr}) \times \mathscr{P}_{\mathbf{I}}(\mathbf{G}/\mathbf{I}) \to \mathscr{P}_{\mathbf{I}}(\mathbf{G}/\mathbf{I})$. Considered as a functor to $\mathscr{D}_{\mathbf{I}}(\mathbf{G}/\mathbf{I})$, $\mathscr{Z}$ is monoidal. Moreover, according to [Zhu10], $\mathscr{Z}$ can be upgraded to a monoidal functor $\mathscr{D}_{\mathbf{G}_\mathcal{O}}(\mathbf{Gr}) \to \mathscr{D}_{\mathbf{I}}(\mathbf{G}/\mathbf{I})$, also with the property that $\mathscr{Z}(\mathcal{F}) \star -$ is isomorphic to $- \star \mathscr{Z}(\mathcal{F})$, and which induces an isomorphism from the K-theory of the source to the center of the K-theory of the target.

Conjecture 21. (i) *There exists a functor $\mathscr{Z} : \mathscr{H}^c_{\mathrm{geom}} \to \mathscr{H}_{\mathrm{geom}}$ such that $\mathscr{Z}(-) \star -$ and $- \star \mathscr{Z}(-)$ yield well-defined, isomorphic actions of $\mathscr{H}^c_{\mathrm{geom}}$ on $\mathscr{W}_{\mathrm{geom}}$.*
(ii) *The functor $\mathscr{Z}$ can be upgraded to a monoidal functor $\mathscr{H}^{c,\mathrm{der}}_{\mathrm{geom}} \to \mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$, such that $\mathscr{Z}(-) \star -$ is isomorphic to $- \star \mathscr{Z}(-)$, and which induces an isomorphism from the K-theory ring of the source to the center of the K-theory ring of the target.*

[8]This was, in a sense, the first step behind the description in Theorem 14 of $\mathscr{D}_{\mathbf{I}}(\mathbf{G}/\mathbf{I})$.

Roughly speaking, the above conjectures says that we can think of $\mathscr{H}^{c,\mathrm{der}}_{\mathrm{geom}}$ as a "monoidal center" of the triangulated category $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$. The monoidal abelian category $\mathscr{H}^{c}_{\mathrm{geom}}$ is the perverse heart of this monoidal center.

Remark 22. Let $\bar{\mu}$ be an arbitrary character of $T(\mathcal{O})$. Possibly, one could still geometrize $\mathscr{H}^c$ as the perverse heart $\mathscr{H}^{c}_{\mathrm{geom}}$ of some "central" subcategory of $\mathscr{H}^{\mathrm{der}}_{\mathrm{geom}}$. If so, we would hope that $\mathscr{H}^{c}_{\mathrm{geom}}$ is closed under convolution and has a description in Langlands dual terms. Moreover, one can ask if Conjectures 19 and 21 still hold, at least when the group L from §1.4.1 is connected.

1.5. **Restrictions on the field, group, and coefficients.** In this paper, we work over $\mathbb{F}_q((t))$, assuming that q is mildly restricted depending on which reductive group we are considering; see §3.1.2 for details. This restriction is necessary because it is required by [Roc98] for Theorem 3.(ii) (we actually relax the conditions of that theorem in view of [Yu01]: see §3.1.2). We note that the main results of this paper and their proofs (§§4 - 5) also hold for $k((t))$, where k is an algebraically closed field of characteristic restricted as in §3.1.2; the main caveat is that one does not necessarily have a $\bar{\mu}$, but only a choice of Roche's group $\mathbf{J}$ and an appropriate character sheaf $\mathcal{M}$ and subgroup $\mathbf{J}' < \mathbf{J}$ on which $\mathcal{M}$ is trivial; see Remark 39 for details. We can also work over algebraically closed fields of characteristic zero, at the price of making J be necessarily the Iwahori subgroup; see Remark 40.

In the unequal characteristic setting, one also has group ind-schemes (resp. group schemes) $\mathbf{G}$ (resp. $\mathbf{G}_{\mathcal{O}}$) such that $\mathbf{G}(\mathbb{F}_q) = G(F)$ (resp. $\mathbf{G}_{\mathcal{O}}(\mathbb{F}_q) = G(\mathcal{O})$). However, as far as we know, there is no realization of the affine Grassmannian as an ind-scheme of ind-finite type; see, for instance, [Kre10]. Therefore, we don't know how to make sense of sheaves on the affine Grassmannian in the unequal characteristic.

We consider connected split reductive groups G over $\mathcal{O}$. One can ask if there are analogues of our results for non-split groups. In this regard, we note that Haines and Rostami have proved a version of Satake isomorphism for non-split groups [HR10]. Furthermore, X. Zhu has proved an analogue of Theorem 2 for non-split groups [Zhu10]. In particular, for quasisplit groups, where principal series representations still make sense, we expect that our results should admit a generalization.

We work with $\overline{\mathbb{Q}}_\ell$-sheaves on the étale topology. Over $\mathbb{C}$, one can also work with sheaves in the complex topology. In particular, in [MV07], Mirković and Vilonen prove the geometric Satake isomorphism for sheaves of R-modules on the complex topology of the loop group, where R is a commutative Noetherian unital ring of finite global dimension. Our main results and proofs (§§4 - 5) also extend to this setting, at least when R is a field; see Remark 41 for details.

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2. Roche's compact open subgroups

2.1. **Conventions.** In the present section, as well as §3, F and $\mathcal{O}$ need not be $\mathbb{F}_q((t))$ and $\mathbb{F}_q[[t]]$ as we assume in the rest of the paper; it suffices for F to be a local field with ring of integers $\mathcal{O}$, unique maximal ideal $\mathfrak{p}$, residue field $\mathbb{F}_q$, and uniformizer t.

2.1.1. *Reductive groups.* Let G be a connected split reductive group over $\mathcal{O}$. Fix a split maximal torus $T < G$. Let $\Delta \subset \mathrm{Hom}(T, \mathbb{G}_m)$ denote the set of roots of G with respect to T. Let $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ be the coweight lattice. To an element $\lambda \in \Lambda$, we associate $t^\lambda = \lambda(t) \in T(F)$.

For every $\alpha \in \Delta$, let $u_\alpha : \mathbb{G}_a \to G$ be the corresponding one-parameter subgroup, where $\mathbb{G}_a$ is the additive group. Let $U_\alpha < G$ be the image. For all $i \in \mathbb{Z}$, let $U_{\alpha,i} = u_\alpha(\mathfrak{p}^i) < G(F)$. Moreover, for $i \geq 1$, let T_i be the subgroup of $T(\mathcal{O})$ generated by the image of $1 + \mathfrak{p}^i$ under all coweights, i.e., the image of $1 + \mathfrak{p}^i$ under the natural isomorphism of topological groups $\Lambda \otimes_{\mathbb{Z}} F^\times \xrightarrow{\sim} T(F)$. In particular, for $i \geq 1$, T_i and $U_{\alpha,i}$ are the kernels of $T(\mathcal{O}) \to T(\mathcal{O}/\mathfrak{p}^i)$ and $U_\alpha(\mathcal{O}) \to U_\alpha(\mathcal{O}/\mathfrak{p}^i)$.

Fix a partition $\Delta = \Delta_+ \sqcup \Delta_-$ into positive and negative roots. Let $B = B^+$ denote the Borel subgroup defined by Δ_+ and B^- denote the Borel defined by Δ_-. Let $U = U^+$ denote the unipotent radical of B and let U^- denote the unipotent radical of B^-. Let $\Lambda_+ \subseteq \Lambda$ denote the subset of dominant coweights; that is,

$$\Lambda_+ := \{\lambda \in \Lambda \mid \alpha(\lambda) \geq 0 \quad \forall\, \alpha \in \Delta_+\}.$$

Then $-\Lambda_+$ is the set of antidominant coweights. (Note that, by our conventions, $\Lambda_+ \cap -\Lambda_+$ is the sublattice of coweights whose image is in the center of G.)

2.1.2. *Representation theory over $\mathbb{C}$ vs. $\overline{\mathbb{Q}}_\ell$.* We fix, once and for all, a prime number ℓ not a factor of q, and an isomorphism of fields $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Using the isomorphism, we carry over results regarding complex coefficients to the $\overline{\mathbb{Q}}_\ell$ case.

2.2. **Roche's compact open subgroup.** Suppose that $f : \Delta \to \mathbb{Z}$ is a function satisfying the properties

(a) $f(\alpha) + f(\beta) \geq f(\alpha + \beta)$, whenever $\alpha, \beta, \alpha + \beta \in \Delta$;
(b) $f(\alpha) + f(-\alpha) \geq 1$.

Define the following subgroups of $G(F)$:

$$\begin{aligned} U_f &:= \langle U_{\alpha, f(\alpha)} \mid \alpha \in \Delta \rangle; \\ U_{f,\alpha} &:= U_f \cap U_\alpha(F); \\ J_f &:= \langle U_f, T(\mathcal{O}) \rangle; \\ T_f &:= \prod_{\alpha \in \Delta} \alpha^\vee(1 + \mathfrak{p}^{f(\alpha) + f(-\alpha)}) < T(F). \end{aligned}$$

Then Roche proved (based on results from [BT72])

Lemma 23. [Roc98, Lemma 3.2]

(i) *$U_{f,\alpha} = U_{\alpha,f(\alpha)}$ for all $\alpha \in \Delta$;*
(ii) *The product map $\prod_{\alpha \in \Delta_\pm} U_{\alpha,f(\alpha)} \to U_f^\pm$ is bijective for any ordering of the factors in the product and any choice of sign $\pm$;*
(iii) *U_f has the direct product decomposition $U_f = U_f^- T_f U_f^+$;*
(iv) *J_f has the direct product decomposition $J_f = U_f^- T(\mathcal{O}) U_f^+$.*

Henceforth, we assume that f is fixed and write $J = J_f$ for the corresponding compact open subgroup of $G(F)$. The above lemma implies that, under a suitable ordering of Δ (i.e., any ordering of Δ_+ followed by any ordering of Δ_-, or vice versa), there are direct product decompositions

$$t^\lambda J t^{-\lambda} = T(\mathcal{O}) \times \prod_{\alpha \in \Delta} U_{\alpha, f(\alpha) + \alpha(\lambda)}; \tag{2.1}$$

$$J \cap t^\lambda J t^{-\lambda} = T(\mathcal{O}) \times \prod_{\alpha \in \Delta} U_{\alpha, \max\{f(\alpha), f(\alpha)+\alpha(\lambda)\}}; \tag{2.2}$$

$$(t^{-\lambda}U_f^+t^\lambda)\cap(t^{-\nu}U_f^+t^\nu)=\prod_{\alpha\in\Delta_+}U_{\alpha,f(\alpha)-\min\{\alpha(\lambda),\alpha(\nu)\}}. \tag{2.3}$$

We will make frequent use of the decomposition of Lemma 23.(iv). For convenience, let $J^- := U_f^-$, $J^0 := T(\mathcal{O})$, and $J^+ := U_f^+$, so that $J = J^-J^0J^+$, which is a direct product decomposition. We refer to this as the Iwahori decomposition of J; we will also use that the decomposition remains valid if the three factors J^-, J^0, and J^+ are rearranged in any order. Note that if f is defined to be 0 on the positive roots and 1 on the negative roots, then J coincides with the Iwahori subgroup of $G(F)$ defined by Δ_+, and the above product is the usual Iwahori decomposition.

2.3. **Relevant double cosets.** We are particularly interested in the following special double cosets of J, since, by [Roc98, Theorem 4.15], for a regular character $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$, they are the only ones that support $(J\times J,\mu\times\mu^{-1})$-invariant functions (cf. §1.2.3), for $J = J_{f_{\bar{\mu}}}$ and $\mu : J \to \overline{\mathbb{Q}}_\ell^\times$ as defined in *op. cit.*, and recalled in §3.1.1 below.

Definition 24. A *relevant double coset* is a double coset of J in $G(F)$ of the form $Jt^\lambda J$, for $\lambda\in\Lambda$.

We now establish some elementary properties of relevant double cosets.

Lemma 25. *Suppose* $\lambda\in\Lambda_+$*.*
(i) $t^\lambda J^+t^{-\lambda}\subseteq J^+$*.*
(ii) $t^{-\lambda}J^-t^\lambda\subseteq J^-$*.*
(iii) $t^\lambda J^0t^{-\lambda}=J^0$*.*

Proof. (i) and (ii) follow immediately from (2.1) and definition of dominant and antidominant coweights. (iii) is obvious. □

The following proposition will be crucial for us. Particularly note the "miracle" of part (c), which will be a key ingredient of the proof of Proposition 46.

Proposition 26. (a) *For all* $\lambda\in\Lambda_+$*,* $t^{-\lambda}Jt^\lambda J = t^{-\lambda}J^+t^\lambda\times J^0\times J^-$*.*
(b) *For all* $\lambda,\nu\in\Lambda_+$*,* $Jt^\lambda Jt^\nu J = Jt^{\lambda+\nu}J$*.*
(c) *Suppose* $\lambda,\nu,\kappa\in\Lambda$*. Then* $Jt^\kappa J\cap Jt^\lambda Jt^\nu J$ *is empty unless* $\kappa=\lambda+\nu$*.*
(d) *For all* $\lambda,\nu\in\Lambda_+$*,* $Jt^\nu Jt^{-\nu}J\bigcap Jt^{-\lambda}Jt^\lambda J = J$*.*

Proof. Lemma 25 implies that $t^{-\lambda}Jt^\lambda J = (t^{-\lambda}J^+t^\lambda)J^0J^-$. Part (a) follows by observing that $t^{-\lambda}J^+t^\lambda\subseteq U^+(F)$, and that $G(F)=U^+(F)T(F)U^-(F)$ is a direct product decomposition.

For (b), note that

$$Jt^\nu Jt^\lambda J = Jt^\nu(J^+J^0J^-)t^\lambda J = J(t^\nu J^+t^{-\nu})t^\nu J^0t^\lambda(t^{-\lambda}J^-t^\lambda)J\subseteq Jt^\nu J^0t^\lambda J = Jt^{\nu+\lambda}J.$$

The reverse inclusion is obvious.

Next, for (c), first note that, up to the choice of positive roots $\Delta_+\subseteq\Delta$ (which does not change J and therefore does not change the statement), we can assume that λ is dominant. Then,

$$Jt^\kappa J\cap Jt^\lambda Jt^\nu J\subseteq Jt^\kappa J\cap Jt^\lambda J^-t^\nu J = J[J^+J^0t^\kappa J^+J^0\cap J^-t^\lambda J^-t^\nu J^-]J.$$

The last equality is easily established using the Iwahori decomposition (and that $J^0J^+=J^+J^0$ and similarly $J^0J^-=J^-J^0$). Now, $J^+J^0t^\kappa J^+J^0\subset t^\kappa J^0U^+(F)$ and $J^-t^\lambda J^-t^\nu J^-\subset t^{\lambda+\nu}U^-(F)$. Their intersection evidently is $\{t^{\lambda+\nu}\}$ if $\kappa=\lambda+\nu$ and is empty otherwise.

Finally, for (d), the containment $\supseteq$ is obvious. Then,

$$\begin{aligned}Jt^\nu Jt^{-\nu}J\cap Jt^{-\lambda}Jt^\lambda J &\subseteq Jt^\nu J^-t^{-\nu}J\cap Jt^{-\lambda}J^+t^\lambda J\\ &= J[J^-t^\nu J^-t^{-\nu}J^-\cap J^+J^0t^{-\lambda}J^+t^\lambda J^+J^0]J\subseteq J[t^\nu J^-t^{-\nu}\cap t^{-\lambda}J^0J^+t^\lambda]J = J.\end{aligned}$$ □

The main application of parts (b) and (d) is

Corollary 27. *Let $\lambda, \nu \in \Lambda_+$ or $\lambda, \nu \in -\Lambda_+$. Then, the multiplication map defines a bijection $(Jt^\lambda J) \times_J (Jt^\nu J) \to Jt^{\lambda+\nu} J$.*

Here, for any subsets $S_1, S_2 \subseteq G(F)$ invariant under right and left multiplication by J, respectively, $S_1 \times_J S_2 := S_1 \times S_2/((s_1, s_2) \sim (s_1 j^{-1}, js_2), \forall j \in J)$ (i.e., it is the quotient of $S_1 \times S_2$ by the inner adjoint action of J).

Proof. First of all, note that, by Proposition 26.(b), the multiplication map is surjective. To prove it is injective, suppose that $xy = x'y'$, with $x, x' \in Jt^\lambda J$ and $y, y' \in Jt^\nu J$. Then $x^{-1}x' \in Jt^{-\lambda} Jt^\lambda J$, whereas $y(y')^{-1} \in Jt^\nu Jt^{-\nu} J$. Since $x^{-1}x' = y(y')^{-1}$, Proposition 26.(d) implies that $x^{-1}x' \in J$. ☐

2.4. **Volume of relevant double cosets and semismallness.** In this section we prove some results we need about volumes of double cosets. Fix a left-invariant Haar measure, vol, on $G(F)$ such that J has measure 1.

Lemma 28. *For all $\lambda \in \Lambda$, $\mathsf{vol}(Jt^\lambda J) = q^{\sum_{\alpha\in\Delta_+} |\alpha(\lambda)|}$.*

Proof. By left-invariance, $\mathsf{vol}(Jt^\lambda J)$ is the number of left cosets of J in $Jt^\lambda J$. Since J acts transitively by left multiplication on the set of left cosets in $Jt^\lambda J$ with stabilizer of $t^\lambda J$ equal to $J \cap (t^\lambda J t^{-\lambda})$, we obtain that $\mathsf{vol}(Jt^\lambda J) = |J/(J \cap (t^\lambda Jt^{-\lambda}))|$. The result then follows from (2.2). ☐

Corollary 29. (i) *For all $\lambda \in \Lambda$, $\mathsf{vol}(Jt^\lambda J) = \mathsf{vol}(Jt^{-\lambda} J)$.*
(ii) *For all $\lambda, \nu \in \Lambda_+$, $\log_q \mathsf{vol}(Jt^\lambda Jt^\nu J) = \log_q \mathsf{vol}(Jt^\lambda J) + \log_q \mathsf{vol}(Jt^\nu J)$.*
(iii) *For $\lambda, \nu \in \Lambda_+$, $\log_q \mathsf{vol}\big(t^{-\lambda} Jt^\lambda J \cap t^{-\nu} Jt^\nu J\big) = \frac{1}{2}[\log_q \mathsf{vol}(Jt^\nu J) + \log_q \mathsf{vol}(Jt^\lambda J) - \log_q \mathsf{vol}(Jt^{\nu-\lambda} J)]$.*

Proof. (i) follows immediately from Lemma 28. (ii) follows from Lemma 28 and Corollary 27. For (iii), note that by Proposition 26.(a), $t^{-\lambda} Jt^\lambda J \cap t^{-\nu} Jt^\nu J = \big((t^{-\lambda} J^+ t^\lambda) \cap (t^{-\nu} J^+ t^\nu)\big) J$. Thus, by (2.3),

$$\log_q \mathsf{vol}\Big(\big((t^{-\lambda} J^+ t^\lambda) \cap (t^{-\nu} J^+ t^\nu)\big) J\Big) = \sum_{\alpha\in\Delta_+} \min\{\alpha(\lambda), \alpha(\nu)\} = \sum_{\alpha\in\Delta_+} \frac{1}{2}\big(\alpha(\lambda) + \alpha(\nu) - |\alpha(\lambda) - \alpha(\nu)|\big) =$$

$$= \frac{1}{2}\Big(\sum_{\alpha\in\Delta_+} |\alpha(\lambda)| + |\alpha(\nu)| - |\alpha(\lambda - \nu)|\Big) = \frac{1}{2}[\log_q \mathsf{vol}(Jt^\nu J) + \log_q \mathsf{vol}(Jt^\lambda J) - \log_q \mathsf{vol}(Jt^{\nu-\lambda} J)]. \quad ☐$$

2.4.1. *Semismallness.* Abusively, we will let vol also denote the product Haar measure on $G(F) \times G(F)$.

Proposition 30. *Let λ be dominant and ν be antidominant coweights. Let $p^{\lambda,\nu} : (Jt^\lambda J) \times (Jt^\nu J) \to G(F)$ denote the restriction of the multiplication map. For every $x \in Jt^{\lambda+\nu} J$,*[9]

$$\log_q \mathsf{vol}((p^{\lambda,\nu})^{-1}(x)) = \frac{1}{2}\big(\log_q \mathsf{vol}(Jt^\lambda J) + \log_q \mathsf{vol}(Jt^\nu J) - \log_q \mathsf{vol}(Jt^{\lambda+\nu} J)\big).$$

Let $P(x) := (p^{\lambda,\nu})^{-1}(x)$. Let $\pi_1 : (Jt^\lambda J) \times (Jt^\nu J) \to Jt^\lambda J$ denote the projection onto the first factor. Let π denote the restriction of π_1 to $P(x)$. The proposition follows from Corollary 29.(iii) and the following lemma.

Lemma 31. *π is injective and its image is canonically identified with $t^{-\lambda} Jt^\lambda J \cap t^\nu Jt^{-\nu} J$.*

Proof. The fact that π is injective is evident. The image of π identifies with $y \in Jt^\lambda J$ such that $y^{-1}x \in Jt^\nu J$, i.e., $y \in xJt^{-\nu} J$. Write $x = jt^{\lambda+\nu} j'$, for $j, j' \in J$. Then, the image of π equals

$$Jt^\lambda J \cap xJt^{-\nu} J = Jt^\lambda J \cap jt^{\lambda+\nu} Jt^{-\nu} J \xrightarrow{\sim} Jt^\lambda J \cap t^{\lambda+\nu} Jt^{-\nu} J \xrightarrow{\sim} t^{-\lambda} Jt^\lambda J \cap t^\nu Jt^{-\nu} J. \quad ☐$$

[9] Note that according to Proposition 26.(c), the only relevant double coset in the image of $p^{\lambda,\nu}$ is $Jt^{\lambda+\nu} J$.

3. Representations via compact open subgroups

3.1. **Families of principal series representations.** Fix a (continuous) character $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$. In §1.2.1, we defined a family of principal series representations Π associated to $\bar{\mu}$. We now give an alternative definition of this family. Let $B^0 := U(F)T(\mathcal{O})$. Abusively, let $\bar{\mu}$ also denote the extension of $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$ to B^0 such that $\bar{\mu}|_{U(F)} = 1$. Then, it follows from the definition that

$$\Pi \cong \operatorname{ind}_{B^0}^{G(F)} \bar{\mu} := \{f : G(F) \to \overline{\mathbb{Q}}_\ell^\times \mid f(gb) = f(g)\bar{\mu}(b), \forall\, b \in B^0\},$$

and the action is the left regular one; i.e., $g \cdot f(x) = f(g^{-1}x)$.

3.1.1. *Realization via compact open subgroups.* For every $\alpha \in \Delta$, let $c_\alpha := \operatorname{cond}(\bar{\mu} \circ \alpha^\vee)$ denote the conductor of $\bar{\mu} \circ \alpha^\vee$; that is, the smallest positive integer c for which $\bar{\mu}(\alpha^\vee(1 + \mathfrak{p}^c)) = \{1\}$. Let

$$f_{\bar{\mu}}(\alpha) = \begin{cases} \lfloor c_\alpha/2 \rfloor, & \text{if } \alpha > 0, \\ \lceil c_\alpha/2 \rceil, & \text{if } \alpha < 0. \end{cases} \tag{3.1}$$

Lemma 32. [Roc98, Lemma 3.4] *Suppose that $2 \nmid q$ if Δ has an irreducible factor of the form B_n, C_n, or F_4, and $3 \nmid q$ if Δ has an irreducible factor of the form G_2. Then, $f_{\bar{\mu}}$ satisfies conditions (a) and (b) of §2.2.*

The conditions in the lemma are designed so that the characteristic does not divide a ratio of square-lengths of two roots; see *op. cit.* To avoid the above restrictions for certain characters, see Remark 33.

In particular, in view of Lemma 23, we obtain an associated compact open subgroup $J = J_{f_{\bar{\mu}}}$ and the subgroup $T_{\bar{\mu}} = T_{f_{\bar{\mu}}}$. By construction, $T_{\bar{\mu}} \subseteq \ker \bar{\mu}$. Hence $\bar{\mu}$ defines a character of $T(\mathcal{O})/T_{\bar{\mu}}$ and so can be lifted to a character $\mu : J \to \overline{\mathbb{Q}}_\ell^\times$. Recall from the introduction that we set $\mathscr{W} := \operatorname{ind}_J^{G(F)} \mu$.

3.1.2. *The isomorphism $\mathscr{W} \cong \Pi$ and residue characteristic restrictions.* Theorem 3.(ii) states that there is an isomorphism of $G(F)$-modules $\mathscr{W} \cong \Pi$, provided that the characteristic of $\mathbb{F}_q$ is restricted: in particular, the characteristic should not be a torsion prime for the Langlands dual group to any semistandard Levi subgroup of G (i.e., connected subgroup containing the maximal torus). Additionally, in order for J to be defined, we assume the characteristic obeys the conditions of Lemma 32 (but see Remark 33). In Roche's paper, to prove [Roc98, Theorem 4.15], additionally the characteristic is further restricted so as to obtain a nondegenerate bilinear form on the Lie algebra (in particular, restricted to be greater than $n + 1$ in the A_n case, or alternatively to have certain more technical conditions satisfied), but this restriction can be lifted by considering elements of the dual to the Lie algebra, as in [Yu01], and not associating to them elements of the Lie algebra itself using a pairing; cf. Appendix A.2.

Put together, for every irreducible direct factor of the root system of the split reductive group, $\operatorname{char}(\mathbb{F}_q)$ should not be one of the primes

(3.2)

Root system	Excluded primes
B_n, C_n, D_n	$\{2\}$
F_4, G_2, E_6, E_7	$\{2,3\}$
E_8	$\{2, 3, 5\}$

In particular, our results hold unconditionally for type A_n groups, including GL_N. Additionally, in the case that J equals the Iwahori subgroup (cf. Remark 4), then no restriction on the characteristic is needed, in view of Proposition 36.(i), since all double cosets of the Iwahori contain an element of $N(T(F))$ (cf. [Roc98, §4]).

Remark 33. We could avoid the restrictions of Lemma 32 if we impose restrictions on the conductors c_α of μ; this would potentially allow characteristic 2 in the B_n case and characteristic 3 in the G_2 case. The important thing is to ensure condition (a) of §2.2. In view of the proof of [Roc98, Lemma 3.4], the problem arises where $p\alpha^\vee = q(\alpha+\beta)^\vee - r\beta^\vee$ for some q, r (p is the characteristic). So, to ensure the condition, whenever $\langle \alpha, \beta^\vee \rangle = -p$ for roots α and β, we should ask that either $c_{\alpha+\beta} > 1$ or c_β is odd. (Then, similarly, one gets that either $c_\beta > 1$ or $c_{\alpha+\beta}$ is odd.) In particular, if short roots β all satisfy $c_\beta \geq 2$, or if c_β is odd for all short roots β, then the condition would appear to be satisfied.

3.1.3. *Explicit (iso)morphism* $\mathscr{W} \to \Pi$. Next, following a suggestion of Drinfeld, we give an explicit description of a morphism $\mathscr{W} \to \Pi$. Define $p_0 : G(F) \to \overline{\mathbb{Q}}_\ell$ by

$$\begin{cases} p_0(g) = 0 & \text{if } g \notin JB^0. \\ p_0(jb) = \mu(j)\bar{\mu}(b), & \forall\, j \in J,\ b \in B^0. \end{cases}$$

One can show that p_0 is a well-defined J-invariant function in Π (we omit the easy proof). It follows that $1 \mapsto p_0$ is a homomorphism of J-modules $\mu \to \mathrm{Res}_J^{G(F)} \Pi$. By compact Frobenius reciprocity, we obtain a morphism $\Phi : \mathscr{W} \to \Pi$. One can probably show that Φ is an isomorphism. We neither prove nor use this fact; we will only use Φ in the proof of Theorem 34, and we only need to know that it is a morphism of $G(F)$-modules.

3.2. **Endomorphism rings.** Henceforth, we assume that $\bar{\mu}$ is regular. In this setting we have an explicit canonical isomorphism $\Psi : \mathrm{K}_0(\mathrm{Rep}(\check{T})) \xrightarrow{\sim} \mathrm{End}_{G(F)}(\mathscr{W})$ (1.3) given as follows: (see, e.g., [Roc09, §1.9]): For every $\lambda \in \Lambda$, let Θ_λ denote the corresponding character of $\check{T}$. Then, the action of $\mathrm{K}_0(\mathrm{Rep}(\check{T}))$ on $\mathscr{W}$ is

$$([\Theta_\lambda] f)(x) = f(xt^{-\lambda}), \qquad f \in \Pi, x \in G(F), \lambda \in \Lambda. \tag{3.3}$$

On the other hand, $\mathrm{End}_{G(F)}(\mathscr{W})$ is identified with the Hecke algebra $\mathscr{H} = \mathscr{H}(G(F), J, \mu)$. We will abusively let Ψ also denote the obtained isomorphism $\mathrm{K}_0(\mathrm{Rep}(\check{T})) \xrightarrow{\sim} \mathscr{H}$. Let

$$f_\lambda : \mathscr{H} \to \overline{\mathbb{Q}}_\ell, \qquad f_\lambda(jt^\lambda j') := \mu(jj'), \quad \forall j, j' \in J; \quad f_\lambda|_{G(F)\setminus Jt^\lambda J} = 0.$$

According to [Roc98, Theorem 4.15], these functions are well-defined and form a basis for $\mathscr{H}$. We now express Ψ explicitly in terms of this basis:

Theorem 34. *For all* $\lambda \in \Lambda$, $\Psi([\Theta_\lambda]) = b_\lambda f_\lambda$ *where* $b_\lambda = q^{-\sum_{\alpha \in \Delta_+} \max\{\alpha(\lambda), 0\}}$.

Remark 35. In view of Lemma 28, $b_\lambda b_{-\lambda} = \mathsf{vol}(Jt^\lambda J)^{-1}$.

The fact that Ψ sends $[\Theta_\lambda]$ to a multiple of f_λ is known as "preservation of support" and is proved by Roche [Roc98, §6] using methods of Bushnell and Kutzko [BK98]. The computation of the scalars b_λ appears to be new. The first step in the computation is to show that $b_\lambda^{-1} = \Phi(f_\lambda)(t^\lambda)$. Then one explicitly computes $\Phi(f_\lambda)$ in terms of p_0. For details of the proof, see §A.1.

4. Geometrization of the vector spaces $\mathscr{H}$ and $\mathscr{W}$

For the rest of this paper, we assume that the character $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$ is regular. Our goal is to geometrize the vector spaces $\mathscr{H}$ and $\mathscr{W}$, the convolution product of $\mathscr{H}$ and the action of $\mathscr{H}$ on $\mathscr{W}$ by convolution.

We will deal with both set-theoretic and scheme-theoretic points of varieties (set-theoretic points of $\mathrm{Spec}\, B$ are, by definition, the prime ideals of B). When a point is scheme-theoretic, we will specify it as an R-point for some $\mathbb{F}_q$-algebra R; otherwise, we will be referring to a set-theoretic point. For our conventions (and some recollections) regarding perverse sheaves and bounded ℓ-adic derived categories see Appendix §B. We only mention here that if $f : X \to Y$ is a morphism of algebraic varieties, then the pushforwards $f_!$ and f_* are always derived, and accordingly we omit any prefix of R (for right derived).

4.1. **Recollections on the affine Grassmannian.** It is well known that there exists a group ind-scheme $\mathbf{G}$ over $\mathbb{F}_q$ such that $\mathbf{G}(\mathbb{F}_q) = G(F)$. Moreover, there exists a proalgebraic group $\mathbf{G}_{\mathcal{O}}$ over $\mathbb{F}_q$ such that $\mathbf{G}_{\mathcal{O}}(\mathbb{F}_q) = G(\mathcal{O})$. The affine Grassmannian $\mathbf{Gr}$ is the fpqc quotient $\mathbf{G}/\mathbf{G}_{\mathcal{O}}$. There exist proper schemes of finite type over $\mathbb{F}_q$,

$$\mathbf{Gr}_1 \subset \mathbf{Gr}_2 \subset \cdots ,$$

which are fixed under the action of $\mathbf{G}_{\mathcal{O}}$ (which factors through finite dimensional quotients of $\mathbf{G}_{\mathcal{O}}$), and whose union is $\mathbf{Gr}$; see, for instance, [Lus83, §11] and [Gin99, Proposition 1.2.2]. Thus, $\mathbf{Gr}$ is an ind-proper scheme of ind-finite type. According to [Gai99, §5.2.1], this ind-scheme may be non-reduced. This will not affect us, however, since we are only interested in perverse sheaves on $\mathbf{Gr}$ (or on related ind-schemes).

Next suppose that $\mathbf{K}$ is a closed subgroup of $\mathbf{G}_{\mathcal{O}}$ such that $\mathbf{G}_{\mathcal{O}}/\mathbf{K}$ is finite dimensional. Let $\mathbf{Y} := \mathbf{G}/\mathbf{K}$ and let $\pi : \mathbf{Y} \to \mathbf{Gr}$ denote the canonical morphism. Then $\mathbf{Y}_i := \pi^{-1}(\mathbf{Gr}_i)$ is a scheme of finite type and $\mathbf{Y}$ is the union of the $\mathbf{Y}_i$. Therefore, $\mathbf{Y}$ is an ind-scheme of ind-finite type.

4.2. **Geometrization of $\mathscr{W}$ and $\mathscr{H}$.**

4.2.1. *Geometrization of J and stabilizers.* Let J be a group of the form defined in §2.2. There exists a proalgebraic subgroup $\mathbf{J} < \mathbf{G}_{\mathcal{O}}$ such that $\mathbf{J}(\mathbb{F}_q) = J$. Indeed, J has a combinatorial description in terms of $\mathfrak{p}^n$ for various n, and it is easy to deduce that it is the group of points of a proalgebraic group. Moreover, the Iwahori decomposition of J implies that $\mathbf{J}$ is connected.

Alternatively, $\mathbf{J}$ can be constructed abstractly as follows: by Bruhat-Tits theory [BT72], there exists a canonical affine smooth group scheme $\underline{J}$ over $\mathcal{O}$ such that $\underline{J}(\mathcal{O}) = J$ (characterized by additional properties). Applying the Greenberg functor [Gre61], we obtain for every $n \geq 1$, a connected algebraic group $\mathbf{J}^{(n)}$ over $\mathbb{F}_q$ such that $\mathbf{J}^{(n)}(\mathbb{F}_q) = \underline{J}(\mathcal{O}/\mathfrak{p}^n)$. It follows that $\mathbf{J} := \varprojlim \mathbf{J}^{(n)}$ is a proalgebraic group over $\mathbb{F}_q$ and

$$\mathbf{J}(\mathbb{F}_q) = \varprojlim \mathbf{J}^{(n)}(\mathbb{F}_q) = \varprojlim \underline{J}(\mathcal{O}/\mathfrak{p}^n) = \underline{J}(\mathcal{O}).$$

Next, observe that $\mathbf{J}$ acts on $\mathbf{G}$ by left and right multiplication. For every $x \in \mathbf{G}(\mathbb{F}_q) = G(F)$, one can consider the stabilizer $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(x)$. This is a proalgebraic subgroup of $\mathbf{J} \times \mathbf{J}$. Projection onto the first factor defines an isomorphism of proalgebraic groups $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(x) \cong \mathbf{J} \cap x\mathbf{J}x^{-1}$. When $G = \mathrm{GL}_N$, one can show that $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(x)$ is connected for all $x \in \mathrm{GL}_N(F)$, using the fact that GL_N is the set of invertible elements of the algebra of $N \times N$ matrices. For arbitrary (connected split reductive) G, we don't know if this stabilizer group is connected. However, the following will suffice for our purposes. Let $N(T(F))$ denote the normalizer of $T(F)$.

Proposition 36. (i) *For all $n \in N(T(F))$, $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(n)$ is connected.*
(ii) *If $x \in G(F) = \mathbf{G}(\mathbb{F}_q)$ is not in $\mathbf{J}t^\lambda\mathbf{J}$ for any $\lambda \in \Lambda$, then $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(x)^\circ(\mathbb{F}_q) \not\subseteq \ker(\mu \times \mu^{-1})$.*

We think of (i) as saying that the stabilizer of an element of the affine Weyl group $W_{\mathrm{aff}} = N(T(F))/T(F)$, considered as an element of $G(F)$ by any section of the quotient, is connected. Note that the stabilizer does not depend on the choice of section, because, for all $t \in T(F)$, $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(nt) \cong \mathbf{J} \cap nt\mathbf{J}t^{-1}n^{-1} = \mathbf{J} \cap n\mathbf{J}n^{-1}$, so the stabilizer of nt is independent of t up to isomorphism.

Proof. First we prove (i). Let $n \in N(T)$ have image $w \in W_{\mathrm{aff}}$ in the affine Weyl group. Furthermore, let $w_0 \in W \cong W_{\mathrm{aff}}/\Lambda$ be the image in the finite Weyl group. Note that $nU_{\alpha,i}n^{-1} = U_{w(\alpha,i)}$, where W_{aff} acts on $\Delta \times \mathbb{Z}$ by the usual action. By Lemma 23.(ii), the direct product $T(\mathcal{O}) \cdot \prod_{\alpha\in\Delta} U_{\alpha,f(\alpha)}$ equals J for any choice of ordering of Δ such that Δ_+ appears first, followed by Δ_-. Moreover, we can replace $\Delta_\pm$ by $w_0^{-1}(\Delta_\pm)$, and infer that it is also acceptable to have $w_0^{-1}(\Delta_+)$ appear first, followed by $w_0^{-1}(\Delta_-)$. Hence, $J \cap nJn^{-1} = T(\mathcal{O}) \cdot \prod_{\alpha\in\Delta} U_{\alpha,\max\{f(\alpha),w(w_0^{-1}(\alpha),f(w_0^{-1}(\alpha)))_2\}}$, where the subscript of 2 denotes the second component of a pair in $\Delta \times \mathbb{Z}$, and we take an ordering where Δ_+ appears first followed by Δ_-. We conclude that this intersection is connected.

Part (ii) is a strengthening of [Roc98, Theorem 4.15], which can be extracted from *op. cit.* along with [AR00] and [Adl98]. For details of the proof, see §A.2. □

4.2.2. *Geometrization of* μ. Let $\bar{\mu} : T(\mathcal{O}) \to \overline{\mathbb{Q}}_\ell^\times$ be a (regular) character and let $J = J_{\bar{\mu}}$ be the corresponding compact open subgroup of $G(\mathcal{O})$ (§3.1.1). To geometrize μ (and later $\mathscr{H}$ and $\mathscr{W}$) it is convenient to define two auxiliary groups. Fix, once and for all, a positive integer c such that $\bar{\mu}|_{T_c}$ is trivial. Define

$$J' := \langle U_{f_{\bar{\mu}}}, T_c\rangle, \qquad A := J/J'. \tag{4.1}$$

For example, if $\bar{\mu}$ factors through a character of $T(\mathbb{F}_q)$, we can take J to be the Iwahori group, and $c = 1$. In this case, J' is the prounipotent radical of J.

Note that J' is well known as the subgroup $J' = J_{f'_{\bar{\mu}}}$ where $f'_{\bar{\mu}} : \Delta \cup \{0\} \to \mathbb{Z}_{\geq 0}$ is the concave function defined by $f'_{\bar{\mu}}|_\Delta = f_{\bar{\mu}}$ and $f'_{\bar{\mu}}(0) = c$. According to [Yu02], there exists a canonical smooth group scheme $\underline{J}'$ over $\mathcal{O}$ such that $\underline{J}'(\mathcal{O}) = J'$ (characterized by additional properties). As in §4.2.1, using the Greenberg functor [Gre61], we can construct a proalgebraic group $\mathbf{J}'$ such that $\mathbf{J}'(\mathbb{F}_q) = J'$. We now give a proof of this fact independent of the results of Greenberg and Yu.

Lemma 37. *The groups J' and A are the sets of $\mathbb{F}_q$-points of connected proalgebraic and connected commutative algebraic groups $\mathbf{J}'$ and $\mathbf{A}$ over $\mathbb{F}_q$, respectively.*

Proof. Let $\mathbf{T}_\mathcal{O} < \mathbf{G}_\mathcal{O}$ be the obvious proalgebraic subgroup whose $\mathbb{F}_q$-points is $T(\mathcal{O})$. In view of the Iwahori decomposition $J = J^- J^0 J^+$, we see that $J' = J^-(J' \cap J^0)J^+$, which is a direct product decomposition. It suffices to show that $T' := J' \cap J^0 = \langle T_c, T_{f_{\bar{\mu}}}\rangle$ is the group of $\mathbb{F}_q$-points of a proalgebraic subgroup $\mathbf{T}' < \mathbf{T}_\mathcal{O}$, and that the quotient $\mathbf{T}_\mathcal{O}/\mathbf{T}' \cong \mathbf{J}/\mathbf{J}' = \mathbf{A}$ has finite type. This is relatively easy to see, but for the convenience of the reader (and independent interest), we explicitly describe T' and $T_{f_{\bar{\mu}}}$ in §A.3. □

It is clear from the assumptions that μ is trivial on J'. In other words, μ is the pullback of a character $\mu_0 : A \to \overline{\mathbb{Q}}_\ell^\times$ along the canonical morphism $J \to A$. We now apply the construction of §B.7.1 to obtain a one-dimensional character sheaf $\mathcal{M}_0$ on $\mathbf{A}$ whose trace function is μ_0. Let $\mathcal{M}$ be the pullback of $\mathcal{M}_0$ via the natural morphism $\mathbf{J} \to \mathbf{A}$. The local system $\mathcal{M}$ is our geometrization of $\mu : J \to \overline{\mathbb{Q}}_\ell^\times$. The following result is an immediate consequence of Proposition 36.(ii).

Corollary 38. *Let $x \in G(F)$ be a point which is not contained in any relevant double coset. Then the restriction of $\mathcal{M}$ to $\mathrm{Stab}_{\mathbf{J}\times\mathbf{J}}(x)^\circ$ is nontrivial.*

Remark 39. The results of this and subsequent sections (and in particular the main theorems in §5) can be generalized in a manner that allows one to replace $\mathbb{F}_q$ with any algebraically closed field of characteristic restricted as in §3.1.2. To do so, one must eliminate $\bar{\mu}$ and begin with $\mathbf{J}, \mathbf{J}'$, and $\mathcal{M}_0$. In more detail, instead of beginning with $\bar{\mu}$, one begins with a choice of Roche's subgroup $\mathbf{J} = \mathbf{J}_f$, cf. §2.2, where f is determined by coefficients $c_\alpha \geq 1$ as in (3.1), i.e., for all $\alpha \in \Delta_+$, either $f(\alpha) = f(-\alpha)$ or $f(\alpha) = f(-\alpha) - 1$. One must then pick a c such that $c \geq c_\alpha$ for all α, and define the corresponding $\mathbf{J}' < \mathbf{J}$. Next, one can allow $\mathcal{M}_0$ to be a one-dimensional character sheaf on $\mathbf{A}$ such that: (i) $\mathcal{M}_0$ is regular: its pullback to $\mathbf{T}_\mathcal{O}$ under the projection $\mathbf{T}_\mathcal{O} \twoheadrightarrow \mathbf{T}_\mathcal{O}/\mathbf{T}' \cong \mathbf{J}/\mathbf{J}' = \mathbf{A}$ has trivial stabilizer under the Weyl group action (i.e., the corresponding local system on $\mathbf{T}_\mathcal{O}$ is not isomorphic to its pullback under the action of any element of the Weyl group); and (ii) the restriction of $\mathcal{M}_0$ to the one-parameter subgroup of $\mathbf{T}_\mathcal{O}$ corresponding to each coroot $\alpha^\vee$ is nontrivial on $\alpha^\vee(1+\mathfrak{p}^{c_\alpha-1})$. As before, $\mathcal{M}$ is defined to be the pullback of $\mathcal{M}_0$ to $\mathbf{J}$. Provided the centralizers of restrictions of $\mathcal{M}_0$ as in the proof of Proposition 36.(ii) (Appendix A.2) remain semisimple (which will be true, for instance, if $\mathcal{M}_0$ is obtained from one of the $\mathcal{M}_0$ in the case of $\mathbb{F}_q$ by base change), all the statements and proofs go through with this generalization. We note that, in the algebraically closed setting, the Tate twists should be suppressed.

Remark 40. One can also consider analogues of our results over an algebraically closed field of characteristic zero. However, since the affine line is simply connected in this case, $\mathbb{G}_a$ admits no nontrivial character sheaves. Thus, the restriction of $\mathcal{M}_0$ to $\alpha^\vee(1+\mathfrak{p}^{c_\alpha-1})$ is always trivial if $c_\alpha > 1$, and we are reduced to the case $c_\alpha = 1$ for all α, with J the Iwahori subgroup.

Remark 41. If we work over $\operatorname{Spec}\mathbb{C}$, we can also consider sheaves in the complex topology, and then, as in [MV07], we can use coefficients in an arbitrary commutative Noetherian ring of finite global dimension. In the case that R is a field, all of the results and proofs here go through without modification (see, e.g., [Dim04] for the necessary facts about perverse sheaves in this context), with $c_\alpha = 1$ for all α, in accordance with the previous remark. We believe (but have not carefully checked) that our main results also extend to the case of commutative Noetherian rings of finite global dimension, provided one replaces the mention of simple or irreducible objects (e.g., Corollary 45) by the statement that all objects of $\mathscr{H}_{\mathrm{geom}}$ are finite direct sums of objects of the form $j_!^\lambda \otimes_R L$, for L a finitely-generated R-module.

4.2.3. *Geometrization of $\mathscr{W}$ and $\mathscr{H}$.* Recall that $\mathscr{W}$ is defined to be the vector space of functions on $G(F)$ which satisfy $f(gj) = f(g)\mu(j)$ for all $g \in G(F)$ and $j \in J$. Equivalently, $\mathscr{W}$ is the vector space of functions on $G(F)/J'$ satisfying $f(ga) = f(g)\mu_0(a)$ for all $g \in G(F)/J'$ and $a \in A$. Similarly, $\mathscr{H}$ is the vector space of functions on $G(F)/J'$ satisfying $f(jga) = \mu_0(j)f(g)\mu_0(a)$ for all $j \in J$ and $a \in A$. Using this observation, geometrizing $\mathscr{W}$ and $\mathscr{H}$ becomes straightforward.

Let $\mathbf{X} := \mathbf{G}/\mathbf{J}'$. By the discussion in §4.1, $\mathbf{X}$ is a union of schemes $\mathbf{X}_i$ of finite type over $\mathbb{F}_q$. The bounded constructible derived category $\mathscr{D}(\mathbf{X})$ of sheaves on $\mathbf{X}$ is, by definition, the inductive limit of $\mathscr{D}(\mathbf{X}_i)$ (see Appendix §B for our conventions regarding the derived category and perverse sheaves).

The connected algebraic group $\mathbf{A}$ acts freely on $\mathbf{X}$ by the right multiplication action $r : \mathbf{A}\times\mathbf{X} \to \mathbf{X}$, $r(a,x) := xa^{-1}$. Similarly, $\mathbf{J}$ acts on $\mathbf{X}$ by the left multiplication action $l : \mathbf{J}\times\mathbf{X} \to \mathbf{X}$. The scheme $\mathbf{X}_i$ is invariant under the action of $\mathbf{J}\times\mathbf{A}$. Indeed,

$$\pi(\mathbf{J}\mathbf{X}_i\mathbf{A}) \subseteq \pi(\mathbf{G}_\mathcal{O}\mathbf{X}_i\mathbf{A}) \subseteq \mathbf{G}_\mathcal{O}\mathbf{Gr}_i = \mathbf{Gr}_i.$$

The left action of $\mathbf{J}$ on each $\mathbf{X}_i$ clearly factors through a finite dimensional quotient. Let $\mathbf{J}_i$ be such a quotient for each i which factors the quotient $\mathbf{J} \twoheadrightarrow \mathbf{A}$, and let $l_i : \mathbf{J}_i \times \mathbf{X}_i \to \mathbf{X}_i$ be the resulting map descending from l. Furthermore, let $\mathcal{M}^i$ be the pullback of $\mathcal{M}_0$ to $\mathbf{J}_i$ under the quotient $\mathbf{J}_i \twoheadrightarrow \mathbf{A}$. Each $\mathcal{M}^i$ is a multiplicative local system on $\mathbf{J}_i$. Let $\mathscr{W}^i_{\mathrm{geom}}$ be the full subcategory of perverse sheaves on $\mathbf{X}_i$ satisfying $r^*\mathcal{F} \cong \mathcal{M}_0^{-1}\boxtimes\mathcal{F}$, and let $\mathscr{H}^i_{\mathrm{geom}}$ be the full subcategory of perverse sheaves on $\mathbf{X}_i$ satisfying $(l_i\times r)^*\mathcal{F} \cong \mathcal{M}^i \boxtimes \mathcal{M}_0^{-1} \boxtimes \mathcal{F}$. Let $\mathscr{W}_{\mathrm{geom}}$ denote the direct limit of the abelian categories $\mathscr{W}^i_{\mathrm{geom}}$, and let $\mathscr{H}_{\mathrm{geom}}$ denote the direct limit of the abelian categories $\mathscr{H}^i_{\mathrm{geom}}$. We consider the objects of $\mathscr{W}_{\mathrm{geom}}$ the $(\mathbf{A}, \mathcal{M}_0^{-1})$-equivariant perverse sheaves on $\mathbf{X}$, and the objects of $\mathscr{H}_{\mathrm{geom}}$ the $(\mathbf{J}\times\mathbf{A}, \mathcal{M}\times\mathcal{M}_0^{-1})$-equivariant perverse sheaves on $\mathbf{X}$.

Taking trace of Frobenius of elements of the abelian categories $\mathscr{W}_{\mathrm{geom}}$ and $\mathscr{H}_{\mathrm{geom}}$, we recover the vector spaces $\mathscr{W}$ and $\mathscr{H}$.

4.3. **Objects of $\mathscr{H}_{\mathrm{geom}}$.**

4.3.1. *Relevant orbits.* Recall that we called double cosets of the form $Jt^\lambda J$ relevant. We call the schemes $\mathbf{J}^\lambda := \mathbf{J}(t^\lambda\mathbf{J}')\mathbf{A} \subseteq \mathbf{X}$ *relevant orbits.* All the facts that we proved about relevant double cosets in §2.3 hold for relevant orbits. This is because our arguments, which concerned $\mathbb{F}_q$-points, carry over to R-points for any $\mathbb{F}_q$-algebra R. Similarly, the facts about volume proved in §2.4 compute the dimension of the associated schemes, under the correspondence

$$\log_q(\mathsf{vol}\, Y) = \dim \mathbf{Y} - \dim \mathbf{A},$$

where Y is one of the subsets obtained from cosets used in §2.4, and $\mathbf{Y} \subseteq \mathbf{X}$ is the associated subscheme of $\mathbf{X}$ satisfying $\mathbf{Y}(\mathbb{F}_q) = Y/J'$.

Below, we will often use the following basic fact: each stratum $\mathbf{X}_i$ contains only finitely many relevant orbits. This is true because $\mathbf{Gr}_i$ contains only finitely many cosets of the form $t^\lambda \mathbf{G}_\mathcal{O}$ for $\lambda \in \Lambda$ (which follows, for example, by taking a standard choice of $\mathbf{Gr}_i$; see, e.g., [Gin99]).

4.3.2. *Geometrization of f_λ.* Recall that $f_\lambda : Jt^\lambda J \to \overline{\mathbb{Q}}_\ell$ is defined by $f_\lambda(jt^\lambda j') = \mu(j)\mu(j')$. Our goal is to geometrize the following statements:

(i) The restriction of $\mu \times \mu^{-1}$ to $\mathrm{Stab}_{J\times J}(x)$ is trivial for all $x \in Jt^\lambda J$;
(ii) f_λ is the unique function on $Jt^\lambda J$ whose pullback to $J \times J$ under the map $(j_1, j_2) \mapsto j_1 t^\lambda j_2$ equals $\mu \times \mu^{-1}$;
(iii) f_λ is the unique function on $Jt^\lambda J$, up to a scalar multiple, which is $(J\times J, \mu\times\mu^{-1})$-invariant.

Here and below, $J \times J$ on $G(F)$ acts by left and right multiplication, i.e., $(j_1, j_2)\cdot g = j_1 g j_2^{-1}$.

Lemma 42. (i) *For every $\lambda \in \Lambda$ and $x \in \mathbf{J}^\lambda$, the restriction of $\mathcal{M} \boxtimes \mathcal{M}_0^{-1}$ to* $\mathrm{Stab}_{\mathbf{J}\times\mathbf{A}}(x)$ *is trivial.*
(ii) *There exists a unique, up to isomorphism, local system $\mathcal{F}'_\lambda$ on $\mathbf{J}^\lambda$ such that $(\pi^\lambda)^*\mathcal{F}'_\lambda \cong \mathcal{M} \boxtimes \mathcal{M}_0^{-1}$, where $\pi^\lambda : \mathbf{J} \times \mathbf{A} \to \mathbf{J}^\lambda$ denotes the map $(j, a) \mapsto jt^\lambda a^{-1}\mathbf{J}'$.*
(iii) *Suppose $\mathcal{G}$ is a $(\mathbf{J} \times \mathbf{A}, \mathcal{M} \boxtimes \mathcal{M}_0^{-1})$-equivariant local system on $\mathbf{J}^\lambda$. Then $\mathcal{G} \cong \mathcal{F}'_\lambda \otimes \mathcal{L}$ where $\mathcal{L}$ is the pullback, via $\mathbf{J}^\lambda \to \mathrm{Spec}(\mathbb{F}_q)$, of a local system on* $\mathrm{Spec}(\mathbb{F}_q)$.

Proof. Note that $\mathcal{M} \boxtimes \mathcal{M}_0^{-1}$ is pulled back from $\mathcal{M}_0 \boxtimes \mathcal{M}_0^{-1}$ on $\mathbf{A} \times \mathbf{A}$. To prove (i), it is enough to show that the restriction of $\mathcal{M}_0 \boxtimes \mathcal{M}_0^{-1}$ to the image of $\mathrm{Stab}_{\mathbf{J}\times\mathbf{A}}(t^\lambda \mathbf{J}')$ in $\mathbf{A} \times \mathbf{A}$ is trivial. The latter is contained in the diagonal, $\{(a, a) \mid a \in \mathbf{A}\} \subseteq \mathbf{A} \times \mathbf{A}$, and the restriction of $\mathcal{M}_0 \boxtimes \mathcal{M}_0^{-1}$ to this locus is a tensor product of two inverse local systems, which is trivial.

(ii) follows immediately from (i) by equivariant descent, since π^λ is the quotient by the (free) action of $\mathrm{Stab}_{\mathbf{J}\times\mathbf{A}}(t^\lambda)$.

For (iii) consider the local system $\mathcal{L} := (\mathcal{F}'_\lambda)^{-1} \otimes \mathcal{G}$ on $\mathbf{J}^\lambda$. Then, $\mathcal{L}$ is a $(\mathbf{J} \times \mathbf{A})$-equivariant local system on $\mathbf{J}^\lambda$. Hence, $(l \times r)^*(\mathcal{L})$ is a local system on $(\mathbf{J} \times \mathbf{A}) \times \mathbf{J}^\lambda$ which is trivial in the $(\mathbf{J} \times \mathbf{A})$ direction. If we restrict to $(\mathbf{J} \times \mathbf{A}) \times \{t^\lambda \mathbf{J}'\}$, we obtain that $(\pi^\lambda)^*(\mathcal{L})$ (with π^λ as in (ii)) is pulled back from the local system $\mathcal{L}|_{t^\lambda \mathbf{J}'}$ on $t^\lambda \mathbf{J}' \cong \mathrm{Spec}\,\mathbb{F}_q$. Thus, $\mathcal{L}$ is also pulled back from a local system on $\mathrm{Spec}\,\mathbb{F}_q$. □

4.3.3. *The local systems $\mathcal{F}_\lambda$ and their extensions.* Let $\mathcal{F}_\lambda := \mathcal{F}'_\lambda[\dim \mathbf{J}^\lambda](-\log_q b_\lambda)$, where (m) denotes the Tate twist by m. Then $\mathcal{F}_\lambda$ is a $\mathbf{J} \times \mathbf{A}$-equivariant perverse sheaf on $\mathbf{J}^\lambda$ and

$$\mathrm{Tr}(\mathbb{F}_q, \mathcal{F}_\lambda) = (-1)^{\dim \mathbf{J}^\lambda} b_\lambda f_\lambda. \tag{4.2}$$

(see §B.7 for our conventions regarding the trace of Frobenius). Let $j^\lambda : \mathbf{J}^\lambda \hookrightarrow \mathbf{X}$ denote the natural inclusion. To these are associated derived functors $j^\lambda_!$ and j^λ_* from $\mathscr{D}(\mathbf{J}^\lambda)$ to $\mathscr{D}(\mathbf{X})$. We use the abusive abbreviations

$$j^\lambda_! := j^\lambda_! \mathcal{F}_\lambda, \qquad j^\lambda_{!*} := j^\lambda_{!*}\mathcal{F}_\lambda, \qquad j^\lambda_* := j^\lambda_* \mathcal{F}_\lambda; \tag{4.3}$$

We will eventually see that $j^\lambda_! \cong j^\lambda_{!*} \cong j^\lambda_*$ (Theorem 48).

Lemma 43. *$j^\lambda_!$, $j^\lambda_{!*}$, and j^λ_* belong to $\mathscr{H}_{\mathrm{geom}}$.*

Proof. By definition, $j^\lambda_{!*}$ is a perverse sheaf. Since $\mathbf{J}\times\mathbf{A}$ is solvable, all of its orbits are affine. Thus, j^λ is an open affine embedding. Therefore, $j^\lambda_!$ and j^λ_* are perverse sheaves as well. To prove the result, it remains to show that these perverse sheaves are $(\mathbf{J}\times\mathbf{A}, \mathcal{M}\boxtimes\mathcal{M}_0^{-1})$-equivariant. Using the projection formula (twice),

$$l^*(j^\lambda_!\mathcal{F}_\lambda) \cong (\mathrm{Id}\times j^\lambda)_!(l^*\mathcal{F}_\lambda) \cong (\mathrm{Id}\times j^\lambda)_!(\mathcal{M}\boxtimes\mathcal{F}) \cong \mathcal{M}\boxtimes j^\lambda_!\mathcal{F}_\lambda.$$

Thus, $j^\lambda_!$ is $(\mathbf{J},\mathcal{M})$-equivariant. Similarly, one shows that j^λ_* is $(\mathbf{J},\mathcal{M})$-equivariant for the left multiplication action. Now $j^\lambda_{!*}$ is the image of the canonical morphism $j^\lambda_!\to j^\lambda_*$. Since the canonical morphism is functorial, it is easy to see that $l^*(j^\lambda_{!*})\cong\mathcal{M}\boxtimes j^\lambda_{!*}$. One proves in an analogous manner that $j^\lambda_!, j^\lambda_*$ and $j^\lambda_{!*}$ are $(\mathbf{A},\mathcal{M}_0^{-1})$-equivariant for the right multiplication action. □

4.3.4. *Restriction to irrelevant points.* Let an *irrelevant point* $x\in\mathbf{X}$ denote a point which does not lie in any relevant orbit. Recall that if $f\in\mathscr{H}$, then $f(x)=0$ for all irrelevant points x. In this section, we prove a geometric analogue of this statement.

Proposition 44. *Let $y\in\mathbf{X}$ be an irrelevant (set-theoretic) point.*

(i) *The stalk of every $\mathcal{F}\in\mathscr{H}_{\mathrm{geom}}$ at y is zero.*
(ii) *The stalk at y of every $(\mathbf{J}\times\mathbf{A},\mathcal{M}\boxtimes\mathcal{M}_0^{-1})$-equivariant complex $\mathcal{G}\in\mathscr{W}^{\mathrm{der}}_{\mathrm{geom}}$ is zero.*

Proof. It is clear that (i) is a consequence of (ii), so we only prove (ii).

First, we claim that it suffices to assume that y is closed. Indeed, otherwise, since finitely many relevant orbits lie in each stratum $\mathbf{X}_i$, only finitely many can intersect the closure $\bar{y}$ of y. Now the complement of these relevant orbits in $\bar{y}$ is a dense open subvariety U of $\bar{y}$. If the stalks at the (necessarily irrelevant) closed points in U vanish, then the restriction of $\mathcal{G}$ to U vanishes. Hence, the restriction of $\mathcal{G}$ to $y\in U$ also vanishes.

So, assume that y is closed. By Lemma 70, it suffices to show that $\mathcal{M}\boxtimes\mathcal{M}_0^{-1}$ is nontrivial on the connected component of the identity of the stabilizer of y; then it would follow that all cohomology sheaves of $\mathcal{G}$ have zero stalk at y, and hence also that the stalk of $\mathcal{G}$ at y is zero. For $\mathbb{F}_q$-points the result follows from Corollary 38. For $\mathbb{F}_{q^n}$-points one can use the norm maps (Remark 74). In more detail, we can replace $\bar{\mu}$ by the corresponding character of $T(\mathbb{F}_{q^n}[\![t]\!])$ and work over the field $\mathbb{F}_{q^n}$. This implies the result for all set-theoretic points y. □

Corollary 45. *The irreducible objects in $\mathscr{H}_{\mathrm{geom}}$ are of the form $j^\lambda_{!*}\otimes\mathcal{L}$, where $\mathcal{L}$ is the pullback, via $\mathbf{J}^\lambda\to\operatorname{Spec}\mathbb{F}_q$, of a one-dimensional local system on $\operatorname{Spec}\mathbb{F}_q$.*

Proof. Note that $j^\lambda_{!*}$ is irreducible for every $\lambda\in\Lambda$, since $\mathcal{F}_\lambda$ is irreducible (in fact, one-dimensional). For the converse, let $\mathcal{F}$ be an irreducible object of $\mathscr{H}_{\mathrm{geom}}$. Then there must exist a $\mathbf{J}\times\mathbf{A}$-invariant locally closed subscheme $\mathbf{Y}\subset\mathbf{X}_i$ of one of the strata $\mathbf{X}_i$ of $\mathbf{X}$, and an irreducible $(\mathbf{J}\times\mathbf{A},\mathcal{M}\boxtimes\mathcal{M}_0^{-1})$-equivariant local system $\mathcal{G}$ on $\mathbf{Y}$ such that $\mathcal{F}=j^{\mathbf{Y}}_{!*}\mathcal{G}[\dim\mathbf{Y}]$, where $j^{\mathbf{Y}}:\mathbf{Y}\hookrightarrow\mathbf{X}$ is the inclusion. Hence, $\mathbf{Y}$ must lie in the union of the finitely many relevant orbits in $\mathbf{X}_i$. Since it is $\mathbf{J}\times\mathbf{A}$-invariant, $\mathbf{Y}$ equals a finite union of relevant orbits. As $\mathcal{F}$ is irreducible, $\mathbf{Y}$ is also irreducible. Therefore, there must exist $\lambda\in\Lambda$ such that $\mathbf{J}^\lambda\cap\mathbf{Y}$ is open and dense in $\mathbf{Y}$. Since $\mathbf{Y}$ is $\mathbf{J}\times\mathbf{A}$-invariant, in fact $\mathbf{J}^\lambda\subseteq\mathbf{Y}$. Thus, we conclude that $\mathcal{F}\cong j^\lambda_{!*}(\mathcal{F}|_{\mathbf{J}^\lambda})$. Hence, Lemma 42 implies that it has the desired form (note that all irreducible local systems on $\operatorname{Spec}\mathbb{F}_q$ are one-dimensional). □

5. Convolution product and main results

5.1. **Definition of convolution.** Let $p:\mathbf{G}\times_{\mathbf{J}}\mathbf{X}\to\mathbf{X}$ denote the product map (where $\times_{\mathbf{J}}$, as in the setting of $\mathbb{F}_q$-points in §2.3, denotes the quotient of the product by the inner adjoint action of $\mathbf{J}$, $j\cdot(g,x)=(gj^{-1},jx)$). The convolution with compact support is the functor defined by

$$\star_!:\mathscr{W}_{\mathrm{geom}}\times\mathscr{H}_{\mathrm{geom}}\to\mathscr{W}^{\mathrm{der}}_{\mathrm{geom}},\qquad(\mathcal{F},\mathcal{G})\mapsto p_!(\mathcal{F}\widetilde{\boxtimes}\mathcal{G}).\tag{5.1}$$

Here $\mathcal{F}\widetilde{\boxtimes}\mathcal{G}$ is the the twisted external product of twisted equivariant sheaves (§B.6). We usually write $\star$ for $\star_!$. There are associativity isomorphisms

$$\mathcal{F}\star(\mathcal{G}\star\mathcal{G}') \xrightarrow{\sim} (\mathcal{F}\star\mathcal{G})\star\mathcal{G}', \qquad \forall\,\mathcal{G},\mathcal{G}'\in\mathscr{H}_{\mathrm{geom}}, \quad \mathcal{F}\in\mathscr{W}^{\mathrm{der}}_{\mathrm{geom}} \tag{5.2}$$

satisfying natural properties; see, for instance, [BDa, §7] or [Gai01]. One can easily check that

$$\mathrm{Tr}(\mathrm{Fr}_{q^n},\mathcal{F})\star\mathrm{Tr}(\mathrm{Fr}_{q^n},\mathcal{G}) = (-1)^{\dim\mathbf{A}}\,\mathrm{Tr}(\mathrm{Fr}_{q^n},\mathcal{F}\star\mathcal{G}), \qquad \forall\,\mathcal{F}\in\mathscr{W}_{\mathrm{geom}},\ \mathcal{G}\in\mathscr{H}_{\mathrm{geom}}. \tag{5.3}$$

Thus, up to sign, this geometrizes the usual convolution product $\mathscr{W}\star\mathscr{H}\to\mathscr{W}$ (5.1).

5.2. **Convolution of $j_!^\lambda$.** Using Lemma 28, Theorem 34, (4.2), and (5.3), one can easily show that

$$\mathrm{Tr}(\mathrm{Fr}_{q^n}, j_!^\lambda\star j_!^\nu) = \mathrm{Tr}(\mathrm{Fr}_{q^n}, j_!^{\lambda+\nu}). \tag{5.4}$$

The following is the geometric analogue of (5.4). It is the key result for proving our main theorems, and was suggested to us by D. Gaitsgory.

Proposition 46. *For all $\lambda,\nu\in\Lambda$, $j_!^\lambda\star j_!^\nu\cong j_!^{\lambda+\nu}$.*

In the case that λ and ν are both dominant or antidominant, the proposition follows easily from the isomorphism $\mathbf{J}^\lambda\times_{\mathbf{J}}\mathbf{J}^\nu\cong\mathbf{J}^{\lambda+\nu}$ (Corollary 27). In the case that λ is dominant and ν is antidominant, we combine the fact that the only relevant orbit in the closure of $\mathbf{J}^\lambda\times_{\mathbf{J}}\mathbf{J}^\nu$ is $\mathbf{J}^{\lambda+\nu}$ (Proposition 26.(c)) and the semismallness result proved in §2.4.1 to show that $j_!^\lambda\star j_!^\nu$ is perverse. It is then easy to show that it must be isomorphic to $j_!^{\lambda+\nu}$. For details of the proof see §A.4.

Corollary 47. *For all $\lambda,\nu\in\Lambda$ and all local systems $\mathcal{L},\mathcal{K}$ on* $\mathrm{Spec}\,\mathbb{F}_q$,

$$\mathrm{Ext}^\bullet(j_!^\lambda\otimes\mathcal{L}, j_!^\nu\otimes\mathcal{K}) = \begin{cases}\mathrm{Ext}^\bullet_{\mathrm{Spec}\,\mathbb{F}_q}(\mathcal{L},\mathcal{K}), & \textit{if } \lambda=\nu,\\ 0, & \textit{otherwise.}\end{cases}$$

The above corollary is proved using the monoidal property of $j_!^\lambda$ established in Proposition 46; see §A.5 for details.

5.3. **Proof of Theorem 7: cleanness of irreducible objects.** The following theorem clearly implies Theorem 7:

Theorem 48. *For every $\lambda\in\Lambda$, $j_!^\lambda\cong j_{!*}^\lambda$.*

Proof. By Corollary 45, all objects are obtained by iterated extensions of objects of the form $j_{!*}^\lambda\otimes\mathcal{L}$, for $\mathcal{L}$ a local system on $\mathrm{Spec}\,\mathbb{F}_q$. We first claim that, for fixed λ, $j_!^\lambda$ is obtained by iterated extensions of objects of the form $j_{!*}^\lambda\otimes\mathcal{L}$ (the point is that we use the *same* λ). Inductively, it suffices to show that, if $j_{!*}^\nu\otimes\mathcal{L}\hookrightarrow j_!^\lambda$ is an injection, then $\nu=\lambda$. If we precompose the inclusion $j_{!*}^\nu\otimes\mathcal{L}\hookrightarrow j_!^\lambda$ with the defining surjection $j_!^\nu\otimes\mathcal{L}\twoheadrightarrow j_{!*}^\nu\otimes\mathcal{L}$, one obtains a nonzero map $j_!^\nu\otimes\mathcal{L}\to j_!^\lambda$. By Corollary 47, this implies $\nu=\lambda$.

Now, suppose that $j_{!*}^\lambda\otimes\mathcal{L}\hookrightarrow j_!^\lambda$ is an injection. Applying Corollary 47 again, the composition $j_!^\lambda\otimes\mathcal{L}\twoheadrightarrow j_{!*}^\lambda\otimes\mathcal{L}\hookrightarrow j_!^\lambda$ is obtained from a map $\mathcal{L}\to\overline{\mathbb{Q}}_\ell$. This map is injective, since the map $\mathcal{L}\cong j_!^\lambda\otimes\mathcal{L}|_{t^\lambda\mathbf{J}'}\cong j_{!*}^\lambda\otimes\mathcal{L}|_{t^\lambda\mathbf{J}'}\to j_!^\lambda|_{t^\lambda\mathbf{J}'}$ is injective. Hence the canonical map $j_!^\lambda\otimes\mathcal{L}\twoheadrightarrow j_{!*}^\lambda\otimes\mathcal{L}$ is an isomorphism. Inductively, $j_!^\lambda$ is obtained by iterated extensions of objects of the form $j_!^\lambda\otimes\mathcal{L}$ by maps of the form $\mathrm{Id}\otimes\phi$ for ϕ a map of local systems over $\mathrm{Spec}\,\mathbb{F}_q$. Moreover, every object $j_!^\lambda\otimes\mathcal{L}$ that appears is isomorphic to $j_{!*}^\lambda\otimes\mathcal{L}$ by the canonical map. We conclude that $j_!^\lambda\cong j_{!*}^\lambda$ (by the canonical map). $\square$

5.4. **Monoidal equivalence.** The following theorem clearly implies Theorem 8:

Theorem 49. *The category $\mathscr{H}_{\mathrm{geom}}$ is closed under convolution; moreover, the functor*

$$\Psi_{\mathrm{geom}} : \operatorname{Rep} \check{T} \boxtimes \mathsf{LocSys}(\operatorname{Spec} \mathbb{F}_q) \to \mathscr{H}_{\mathrm{geom}}, \qquad \Theta_\lambda \otimes \mathcal{L} \mapsto j_!^\lambda \otimes \mathcal{L}$$

is an equivalence of monoidal abelian categories.

Proof. Corollary 47 and Theorem 48, together with Corollary 45, imply that all objects of $\mathscr{H}_{\mathrm{geom}}$ are finite direct sums of objects of the form $j_!^\lambda \otimes \mathcal{L}$, where $\mathcal{L}$ is a local system on $\operatorname{Spec} \mathbb{F}_q$. Therefore, to prove $\mathscr{H}_{\mathrm{geom}}$ is closed under convolution, it is enough to show that, for all coweights λ and ν and all local systems $\mathcal{L}$ and $\mathcal{L}'$ on $\operatorname{Spec} \mathbb{F}_q$, $(j_!^\lambda \otimes \mathcal{L}) \star (j_!^\nu \otimes \mathcal{L}') \in \mathscr{H}_{\mathrm{geom}}$. This follows at once from Proposition 46. Proposition 46 also implies that Ψ is monoidal. In view of Corollary 47, we obtain the desired equivalence of abelian categories. □

5.5. **Convolutions with and without compact support are isomorphic.** Recall that we have two, a priori different, monoidal actions of $\mathscr{H}_{\mathrm{geom}}$ on $\mathscr{W}_{\mathrm{geom}}^{\mathrm{der}}$ given by $\star = \star_!$ and $\star_*$. We now prove that these two actions are isomorphic by proving that their adjoints are isomorphic.

Proposition 50. *For every $\mathcal{F}, \mathcal{G} \in \mathscr{W}_{\mathrm{geom}}^{\mathrm{der}}$, there is a canonical functorial isomorphism*

$$\operatorname{Hom}_{\mathscr{W}_{\mathrm{geom}}^{\mathrm{der}}}(\mathcal{F} \star_! j_!^\lambda, \mathcal{G}) \cong \operatorname{Hom}_{\mathscr{W}_{\mathrm{geom}}^{\mathrm{der}}}(\mathcal{F}, \mathcal{G} \star_* j_*^{-\lambda}).$$

The above proposition is probably known, at least in the untwisted setting. For completeness, we include a proof in §A.6.

Corollary 51. *There exists an isomorphism between the actions of $\mathscr{H}_{\mathrm{geom}}$ on $\mathscr{W}_{\mathrm{geom}}^{\mathrm{der}}$ given by $\star_!$ and $\star_*$.*

Proof. By Theorem 49 $\mathscr{H}_{\mathrm{geom}}$ is a rigid category (in fact, it is a Picard category). Therefore, the adjoint functor to $- \star_! j_!^\lambda$ is isomorphic to $- \star_! j_!^{-\lambda}$. On the other hand, by the above proposition, this adjoint functor is also isomorphic to $- \star_* j_*^{-\lambda}$. We conclude that the functors $- \star_! j_!^\lambda$ and $- \star_* j_*^\lambda$ are isomorphic. Using cleanness (Theorem 48), we conclude that the functors $- \star_! j_!^\lambda$ and $- \star_* j_!^\lambda$ are isomorphic. For any $\mathcal{L} \in \mathsf{LocSys}(\operatorname{Spec} \mathbb{F}_q)$, it follows also that $- \star_! (j_!^\lambda \otimes \mathcal{L})$ is isomorphic to $- \star_* (j_!^\lambda \otimes \mathcal{L})$. The result then follows from Theorem 49. □

As explained in the next subsection, one can (probably) show that the canonical morphism $\star_! \to \star_*$ (obtained from the general functorial maps $f_! \to f_*$, which we call "forgetting compact support") is an isomorphism between the two actions of $\mathscr{H}_{\mathrm{geom}}$ on $\mathscr{W}_{\mathrm{geom}}^{\mathrm{der}}$. (However, we do not need this fact). We note that the idea of showing $\star_!$ and $\star_*$ are isomorphic by proving that their adjoints are isomorphic was suggested to us by D. Gaitsgory. The same idea is employed in [BD06, §G] and [BD08, §6.7]

5.5.1. *Aside: relationship to Grothendieck-Verdier duality.* In [BDb, Appendix A] and [BDc], Boyarchenko and Drinfeld explain the following picture. A monoidal category $(\mathcal{C}, \otimes_1, \mathbf{1})$ is a called an *r-category* if for every $Y \in \mathcal{C}$ the functor $\operatorname{Hom}(- \otimes_1 Y, \mathbf{1})$ is representable by some object DY and the contravariant functor $D : \mathcal{C} \to \mathcal{C}$ is an antiequivalence. D is called the *duality functor*. In every r-category, there is a second tensor product defined by

$$X \otimes_2 Y := D^{-1}(DY \otimes_1 DX).$$

Moreover, there is a monoidal natural transformation $X \otimes_1 Y \to X \otimes_2 Y$, which is an isomorphism if and only if $\mathcal{C}$ is rigid.

As an example, let G be a connected algebraic group over a field k. Let $\iota : G \to G$ be the map $\iota(g) = g^{-1}$. Let $\mathbb{D} : \mathscr{D}(G) \to \mathscr{D}(G)$ denote the Verdier duality functor and $\overline{\mathbb{D}} : \mathscr{D}(G) \to \mathscr{D}(G)$ denote $\mathbb{D} \circ \iota^* = \iota^* \circ \mathbb{D}$. Then $(\mathscr{D}(G), \star_!)$ is an r-category with duality functor $\overline{\mathbb{D}}$. The second tensor product is convolution without compact support, $\star_*$. According to [BDb, Appendix A], the natural transformation $\star_! \to \star_*$ defined above should coincide with the canonical map of "forgetting compact support."

Now we apply the above considerations to our situation. The argument in [BDb, Appendix A] for proving $(\mathscr{D}(G), \star_!, \overline{\mathbb{D}})$ is an r-category applies verbatim to show that $(\mathscr{H}_{\mathrm{geom}}, \star_!, \overline{\mathbb{D}})$ is an r-category. (This amounts to a special case of Proposition 50 where one takes $\mathcal{F}, \mathcal{G} \in \mathscr{H}_{\mathrm{geom}}$.) The second tensor product in $\mathscr{H}_{\mathrm{geom}}$ is $\star_*$. In analogy with $\mathscr{D}(G)$, the natural transformation $\star_! \to \star_*$ should coincide with the canonical map coming from forgetting the support (but we have not checked this). As $\mathscr{H}_{\mathrm{geom}}$ is rigid, we see that the canonical morphism $\star_! \to \star_*$ is an isomorphism.

Next, we have two monoidal actions of $(\mathscr{H}_{\mathrm{geom}}, \star_!)$ on $\mathscr{W}^{\mathrm{der}}_{\mathrm{geom}}$: one given by $\star_!$ and the other one given by $\star_*$, via the monoidal equivalence $(\mathscr{H}_{\mathrm{geom}}, \star_!) \xrightarrow{\sim} (\mathscr{H}_{\mathrm{geom}}, \star_*)$. One can show that the canonical "forgetting compact support" maps define a monoidal natural transformation going from the first action to the second one. Moreover, it follows from the following general lemma that this is an isomorphism.

Lemma 52. *Let $\mathcal{C}$ be a rigid monoidal category with unit object $\mathbf{1}$, $\mathcal{D}$ a monoidal category, $F, G : \mathcal{C} \to \mathcal{D}$ two monoidal functors, and $\eta \in \mathrm{Hom}(F, G)$ a monoidal natural transformation such that $\eta_{\mathbf{1}} : F(\mathbf{1}) \to G(\mathbf{1})$ is an isomorphism. Then, η is an isomorphism.*

A version of the above lemma appears in [SR72, Proposition 5.2.3].

5.6. **Action of $\mathscr{H}_{\mathrm{geom}}$ on $\mathscr{W}_{\mathrm{geom}}$.** The following implies Theorem 9.

Theorem 53. *Let $\mathcal{F} \in \mathscr{H}_{\mathrm{geom}}$ and $\mathcal{G} \in \mathscr{W}_{\mathrm{geom}}$. Then $\mathcal{G} \star_! \mathcal{F} \in \mathscr{W}_{\mathrm{geom}}$.*

Proof. It is enough to show that $\mathcal{G} \star_! j^\lambda_!$ is perverse for every $\lambda \in \Lambda$. Let $p^\lambda = p \circ (\mathrm{Id} \times j^\lambda)$. Then $p^\lambda_!(\mathcal{G} \tilde{\boxtimes} \mathcal{F}_\lambda) \cong \mathcal{G} \star_! j^\lambda_!$. Since p^λ is an affine morphism, by Artin's Theorem 61, $\mathcal{G} \star_! j^\lambda_! \in {}^p\mathscr{D}^{\geq 0}_{(\mathbf{A}, \mathcal{M}_0^{-1})}(\mathbf{X})$. By Corollary 51 and cleanness, we obtain an isomorphism $\mathcal{G} \star_! j^\lambda_! \cong \mathcal{G} \star_* j^\lambda_*$. Applying Artin's Theorem again, $\mathcal{G} \star_* j^\lambda_* \in {}^p\mathscr{D}^{\leq 0}_{(\mathbf{A}, \mathcal{M}_0^{-1})}(\mathbf{X})$. Hence, $\mathcal{G} \star_! j^\lambda_! \in {}^p\mathscr{D}^{\leq 0}_{(\mathbf{A}, \mathcal{M}_0^{-1})}(\mathbf{X})$ as well. □

Appendix A. Postponed proofs

A.1. **Proof of Theorem 34.** First we describe the morphism $\Phi : \mathscr{W} \to \Pi$ explicitly. Let $f_0 \in \mathscr{W}$ be the function $f_0(j) = \mu(j)$ for $j \in J$ and $f_0(g) = 0$ for $g \notin J$. Then $\mathscr{W}$ has a basis consisting of functions $g_i \cdot f_0$ where $g_i \in G(F)$ ranges over a set of representatives for $G(F)/J$; see, e.g. [BH06, §1.2.5]. Define a morphism of $G(F)$-modules $\Phi : \mathscr{W} \to \Pi$ by $f_0 \mapsto p_0$, where p_0 was defined in §3.1.3. Next, suppose that $\Omega : \mathscr{W} \to \Pi$ is any morphism of $G(F)$-modules. Using the fact that $\mathrm{End}_{G(F)}(\Pi)$ is commutative, one can easily show that

$$\Omega(f \star \Psi(\phi)) = \phi(\Omega(f)), \qquad \forall f \in \mathscr{W}, \phi \in \mathrm{End}_{G(F)}(\Pi).$$

Lemma 54. *$b_\lambda \Phi(f_\lambda)(t^\lambda) = 1$.*

Proof. By the above, $\Phi(f \star (b_\lambda f_\lambda)) = [\Theta_\lambda](\Phi(f))$ for all $f \in \mathscr{W}$. Hence,

$$\Phi(b_\lambda f_\lambda) = \Phi(f_0 \star (b_\lambda f_\lambda)) = [\Theta_\lambda](\Phi(f_0)) = [\Theta_\lambda](p_0).$$

The result follows by evaluating both sides at t^λ. □

Hence, to compute b_λ, it suffices to compute $\Phi(f_\lambda)(t^\lambda)$. First we need two lemmas.

Lemma 55. *For all $j \in J$,*

$$p_0(t^\lambda j t^{-\lambda}) = \begin{cases} \mu(j), & \text{if } t^\lambda j t^{-\lambda} \in JB^0, \\ 0, & \text{otherwise.} \end{cases} \tag{A.1}$$

Proof. Write $j = j^- j^0 j^+$, with $j^- \in J^-, j^0 \in J^0$, and $j^+ \in J^+$. Then, $t^\lambda j t^{-\lambda} = (t^\lambda j^- t^{-\lambda})(t^\lambda j^0 j^+ t^{-\lambda})$. So, first of all,

$$t^\lambda j t^{-\lambda} \in JB^0 \iff t^\lambda j^- t^{-\lambda} \in J^-, \tag{A.2}$$

since we know that $t^\lambda j^- t^{-\lambda} \in U^-(F)$, the group of unipotent lower-triangular matrices. So, we find that

$$p_0(t^\lambda j t^{-\lambda}) = \mu(j^0) p_0(t^\lambda j^- t^{-\lambda}),$$

which yields (A.1). $\square$

Lemma 56. $\Phi(f_\lambda) = \sum_i \mu(j_i) j_i t^\lambda \cdot p_0$*, where* j_i *is a set of representatives of the finite quotient* $J/(J \cap t^\lambda J t^{-\lambda})$*.*

Proof. By definition, $\Phi(f_0) = p_0$. Next, for arbitrary λ, consider the left cosets of J in $Jt^\lambda J$. J acts transitively on these by left multiplication, so these cosets have the form $\{j_i t^\lambda J\}$, where j_i is a set of representatives of the finite quotient $J/(J \cap t^\lambda J t^{-\lambda})$. Thus, $f_\lambda = \sum_i \mu(j_i) j_i t^\lambda \cdot f_0$, where each $j_i t^\lambda \cdot f_0$ is the unique function in $\mathscr{W}$ supported on the left coset $j_i t^\lambda J$ whose value at $j_i t^\lambda$ is 1. Applying Φ yields the result. $\square$

To prove the desired equality, note that

$$\Phi(f_\lambda)(t^\lambda) = \sum_i \mu(j_i) p_0(t^{-\lambda} j_i^{-1} t^\lambda). \tag{A.3}$$

For each j_i, write $j_i = j_i^+ j_i^0 j_i^-$ for $j_i^+ \in J^+, j_i^0 \in J^0$, and $j_i^- \in J^-$. Substituting (A.1) and (A.2) into (A.3), we obtain

$$\Phi(f_\lambda)(t^\lambda) = |\{j_i : t^{-\lambda}(j_i^-)^{-1} t^\lambda \in J^-\}|. \tag{A.4}$$

To conclude, recall that j_i are representatives of $J/(J \cap t^\lambda J t^{-\lambda})$. Note that the RHS of (A.4) identifies with $|K/(J \cap t^\lambda J t^{-\lambda})|$, where K has the same form as J except with $f(\alpha)$ replaced by $f(\alpha) + \max\{\alpha(\lambda), 0\}$ for $\alpha \in \Delta_-$ (leaving $f(\alpha)$ the same when $\alpha \in \Delta_+$). Hence, $\log_q \Phi(f_\lambda)(t^\lambda) = \sum_{\alpha \in \Delta_+} \max\{\alpha(\lambda), 0\}$. In view of Lemma 54, this implies the desired formula. $\square$

A.2. **Proof of Proposition 36.(ii).** We will follow to some extent the arguments of [Roc98, Theorem 4.15], with an innovation from [Yu01] to reduce restrictions on the residue characteristic. Note that [Roc98] works in the mixed-characteristic setting where F has characteristic zero (and $\mathcal{O}/\mathfrak{p} = \mathbb{F}_q$), unlike us. However, as pointed out there, those arguments extend to our equal-characteristic setting by replacing Proposition 4.11 there by the more general [AR00, Theorem 7.1], which is for arbitrary local fields F with residue field $\mathbb{F}_q$ (see Theorem 57 below), proved similarly.

The proof is by induction on the semisimple rank of G. If G is a torus, then the assumption $x \notin \mathbf{J}t^\lambda\mathbf{J}$ is vacuous, so the result follows. So we assume G has positive semisimple rank and that the result holds follows for all connected split reductive groups of strictly smaller rank (for all characters, using Roche's corresponding subgroup).

Let $\ell = \mathrm{cond}(\mu) \geq 1$. If $\ell = 1$, then J is the Iwahori subgroup, in which case the Bruhat decomposition implies that $x \in JnJ$ for some $n \in N(T(F))$. In this case, we can assume $x = n$, and the result follows from part (i) of the proposition. Henceforth, we assume $\ell \geq 2$.

A.2.1. *Review of some notation used in* [Roc98]. Following [Roc98, §4], define the groups

$$L := \langle T_{\ell-1}, U_{\alpha,\ell-1}, U_{\beta,f(\beta)} \mid c_\alpha < \ell, c_\beta = \ell \rangle,$$
$$K_i := \langle T_i, U_{\alpha,i} \mid \alpha \in \Delta \rangle \quad (\forall i \geq 1),$$
$$\widetilde{K}_\ell := \begin{cases} K_\ell, & \text{if } \ell \text{ is even,} \\ \langle K_\ell, U_{\alpha,\ell-1} \mid \alpha > 0, c_\alpha = \ell \rangle, & \text{if } \ell \text{ is odd.} \end{cases}$$

Note that $L \subseteq J$. Moreover, $K_{\lfloor \ell/2 \rfloor} \supseteq L \supseteq K_{\ell-1} \supsetneq \widetilde{K}_\ell$. Finally, K_i/K_{2i} is abelian for all $i \geq 1$.

Next, we recall the Lie algebras of the above subgroups and bijections $\mathfrak{K}_i/\mathfrak{K}_{2i} \xrightarrow{\sim} K_i/K_{2i}$ from *op. cit.* Let $\mathfrak{g}$, $\mathfrak{t}$, and $\mathfrak{u}_\alpha$ be the Lie algebras of G, T, and U_α over F. Let $X = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ be the lattice of characters of $\mathbf{T}$ and $X^\vee = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ the lattice of coweights. There is a natural map $X^\vee \otimes_{\mathbb{Z}} F \xrightarrow{\sim} \mathfrak{t}$. Let $\mathfrak{t}_i \subset \mathfrak{t}$ be the $\mathcal{O}$-sublattice which is the image of $\mathfrak{p}^i \otimes_{\mathbb{Z}} X^\vee$ (note that $\mathfrak{t}_i$ is the Lie algebra of T_i). Similarly, let $\mathfrak{u}_{\alpha,i} \subset \mathfrak{u}_\alpha$ be the $\mathcal{O}$-sublattice which is the image of $\mathfrak{p}^i$ under the isomorphism $F \xrightarrow{\sim} \mathfrak{u}_\alpha$ defined by the map $\mathrm{Lie}(u_\alpha)$ (note that $\mathfrak{u}_{\alpha,i}$ is the Lie algebra of $U_{\alpha,i}$). Define the following $\mathcal{O}$-sublattices of $\mathfrak{g}$, which are the Lie algebras of the groups L, K_i, and $\widetilde{K}_\ell$:

$$\mathfrak{L} := \mathfrak{t}_{\ell-1} \oplus \bigoplus_{\alpha : c_\alpha < \ell} \mathfrak{u}_{\alpha,\ell-1} \oplus \bigoplus_{\alpha : c_\alpha = \ell} \mathfrak{u}_{\alpha,f(\alpha)},$$
$$\mathfrak{K}_i := \mathfrak{t}_i \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{u}_{\alpha,i},$$
$$\widetilde{\mathfrak{K}}_\ell := \begin{cases} \mathfrak{K}_\ell, & \text{if } \ell \text{ is even,} \\ \mathfrak{t}_\ell \oplus \bigoplus_{\alpha > 0, c_\alpha = \ell} \mathfrak{u}_{\alpha,\ell-1} \oplus \bigoplus_{\alpha : c_\alpha < \ell \text{ or } \alpha < 0} \mathfrak{u}_{\alpha,\ell}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Next, for $i \geq 1$, the bijections $\mathrm{Lie}(u_\alpha) : \mathfrak{p}^i \xrightarrow{\sim} \mathfrak{u}_{\alpha,i}$ and $u_\alpha : \mathfrak{p}^i \xrightarrow{\sim} U_{\alpha,i}$ induce a bijection of sets $\mathfrak{u}_{\alpha,i} \xrightarrow{\sim} U_{\alpha,i}$. Similarly, the bijections $X^\vee \otimes_{\mathbb{Z}} \mathfrak{p}^i \xrightarrow{\sim} \mathfrak{t}_i$ and $X^\vee \otimes_{\mathbb{Z}} (1 + \mathfrak{p}^i) \xrightarrow{\sim} T_i$, together with the bijection $\mathfrak{p}^i \xrightarrow{\sim} (1 + \mathfrak{p}^i), b \mapsto 1 + b$, induce a bijection of sets $\mathfrak{t}_i \xrightarrow{\sim} T_i$. Using these and the direct product and sum decompositions $\mathfrak{K}_i = \mathfrak{t}_i \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{u}_{\alpha,i}$ and $K_i = T_i \cdot \prod_{\alpha \in \Delta} U_{\alpha,i}$ (for some choice of ordering of the roots), one obtains a *noncanonical* bijection

$$\varphi_i : \mathfrak{K}_i \xrightarrow{\sim} K_i,$$

depending on the choice of ordering of $\alpha \in \Delta$. In fact, $\varphi_i = \varphi_1|_{\mathfrak{K}_i}$ for all $i \geq 1$. We also have

$$\varphi_L := \varphi_1|_{\mathfrak{L}} : \mathfrak{L} \xrightarrow{\sim} L.$$

Since K_i/K_{2i} is abelian, the resulting isomorphism $\overline{\varphi_i} : \mathfrak{K}_i/\mathfrak{K}_{2i} \xrightarrow{\sim} K_i/K_{2i}$ is independent of the ordering of roots and hence *canonical*. Similarly, the isomorphism $\overline{\varphi_L} : \mathfrak{L}/\widetilde{\mathfrak{K}}_\ell \xrightarrow{\sim} L/\widetilde{K}_\ell$ is independent of the ordering of roots and hence canonical.

Let $\psi : F \to \overline{\mathbb{Q}}_\ell^\times$ be an additive character such that $\mathfrak{p} \subseteq \ker(\psi)$ and $\mathcal{O} \nsubseteq \ker(\psi)$. Then, $a := \mu \circ \overline{\varphi_L}$ is a character of $\mathfrak{L}/\widetilde{\mathfrak{K}}_\ell$, and can be viewed as an element of $(\mathfrak{t}_{\ell-1}/\mathfrak{t}_\ell)^*$ (an element acting trivially on the off-diagonal part $\bigoplus_{\alpha : c_\alpha < \ell} \mathfrak{u}_{\alpha,\ell-1} \oplus \bigoplus_{\alpha : c_\alpha = \ell} \mathfrak{u}_{\alpha,f(\alpha)}$ of $\mathfrak{L}$).

Following [Roc98], define

$$\mathcal{I}(\mu|H) := \{g \in G(F) : \mu(h) = \mu(g^{-1}hg), \forall h \in H \cap gHg^{-1}\}.$$

The relationship to our objects of study is: $g \in \mathcal{I}(\mu|H)$ if and only if, for all pairs $(h, g^{-1}hg) \in \mathrm{Stab}_{H \times H}(g)$, $\mu(h)\mu(g^{-1}hg)^{-1} = 1$. That is,

$$g \in \mathcal{I}(\mu|H) \iff \mathrm{Stab}_{H \times H}(g) \subseteq \ker(\mu \times \mu^{-1}). \tag{A.5}$$

A.2.2. *Proof of the proposition in the case $x \in \mathcal{I}(\mu|L)$.* We will need the following result, which follows from [Roc98, Proposition 4.11] and [AR00, Theorem 7.1] (slightly modifying the proof to use the dual Lie algebra as in [Yu01]; for instance, [Yu01, Lemma 5.1] replaces [Adl98, Lemma 1.8.1] with the same proof).

Theorem 57. $\mathcal{I}(\mu|L) = LC_{G(F)}(a)L$.

Here, $C_{G(F)}(a)$ is the centralizer in $G(F)$ of a. According to [Yu01, Proposition 7.3], under our restrictions on residue characteristic, this is the group of F-points of a semistandard Levi subgroup, call it $C_G(a)$, of G. As explained in the proof of [Roc98, Theorem 4.15], up to multiplying μ by a suitable character of G (which leaves J unchanged, since such characters are trivial on $[G, G]$ and hence the c_α are unchanged), we can assume that $C_G(a) \neq G$, and $C_G(a)$ is a connected split reductive group of strictly lower semisimple rank than G. Then, the subgroup $J_{\mu,C_{G(F)}(a)} < C_{G(F)}(a)$ associated to μ is nothing but the intersection $J_{\mu,C_{G(F)}(a)} = C_{G(F)}(a) \cap J$. By induction on the semisimple rank of G, we can therefore assume that, if the element x in the statement of the proposition is in $C_{G(F)}(a)$, then $\mathrm{Stab}_{\mathbf{J}_{\mu,C_{G(F)}(a)} \times \mathbf{J}_{\mu,C_{G(F)}(a)}}(x)^\circ(\mathbb{F}_q) \not\subseteq \ker(\mu \times \mu^{-1})$. Therefore, the proposition follows for x. Hence, it also follows if $x \in JC_{G(F)}(a)J$, and hence if $x \in LC_{G(F)}(a)L$.

A.2.3. *Proof of the proposition in the case $x \notin \mathcal{I}(\mu|L)$.* By (A.5), $\mathrm{Stab}_{\mathbf{L}\times\mathbf{L}}(x)(\mathbb{F}_q) \not\subseteq \ker(\mu \times \mu^{-1})$. Our goal is to show

$$\mathrm{Stab}_{\mathbf{L}\times\mathbf{L}}(x)^\circ(\mathbb{F}_q) \not\subseteq \ker(\mu \times \mu^{-1})$$

Note that $\mathrm{Stab}_{\mathbf{L}\times\mathbf{L}}(x)(\mathbb{F}_q) = \{(g, x^{-1}gx) : g \in L \cap xLx^{-1}\} \cong L \cap xLx^{-1}$. We will need to recall the following observation of [Adl98]. For all $x \in G(F)$, let $\mathfrak{K}_{x,r} := \mathfrak{K}_r \cap \mathrm{Ad}(x)\mathfrak{K}_r$. Similarly define $K_{x,r}$ as well as $\widetilde{\mathfrak{K}}_{x,\ell}$ and $\widetilde{K}_{x,\ell}$. Then, as observed in [Adl98, (1.5.2)], for all $x \in G(F)$ and all $r \geq 1$, $\mathfrak{K}_{x,r}/\mathfrak{K}_{x,2r}$ is abelian, and φ_r restricts to an isomorphism $\overline{\varphi_{x,r}} : \mathfrak{K}_{x,r}/\mathfrak{K}_{x,2r} \xrightarrow{\sim} K_{x,r}/K_{x,2r}$, which is independent of the ordering of the roots.

It follows from the definition of φ_r that $\overline{\varphi_{x,r}}$ is the map on $\mathbb{F}_q$-points of an isomorphism of commutative algebraic groups. In the case that ℓ is even, we deduce by restriction that one also has a canonical isomorphism

$$(\mathfrak{L} \cap \mathrm{Ad}(x)\mathfrak{L})/\widetilde{\mathfrak{K}}_{x,\ell} \xrightarrow{\sim} (L \cap \mathrm{Ad}(x)L)/\widetilde{K}_{x,\ell},$$

which is the map on $\mathbb{F}_q$-points of an isomorphism of commutative algebraic groups. It is easy to generalize to the case where ℓ is odd.

Now, since $(\mathfrak{L} \cap \mathrm{Ad}(x)\mathfrak{L})/\widetilde{\mathfrak{K}}_{x,\ell}$ is the group of $\mathbb{F}_q$-points of a product of finitely many copies of $\mathbb{G}_a$, the same is true for $(L \cap \mathrm{Ad}(x)L)/\widetilde{K}_{x,\ell}$. So, this quotient is connected. Now, since $\{(g, x^{-1}gx) : g \in \widetilde{K}_{x,\ell}(\mathbb{F}_q)\} \subseteq \ker(\mu \times \mu^{-1})$, we conclude the desired statement. ☐

A.3. **Completion of the proof of Lemma 37.** Let $c_\alpha := f(\alpha) + f(-\alpha)$ (this is consistent with our other definition $c_\alpha = \mathrm{cond}(\bar{\mu} \circ \alpha^\vee)$ when $f = f_{\bar{\mu}}$). For all $m \geq 1$, let $T_{f,m}$ be the subtorus of T generated by all coroots $\alpha^\vee$ such that $c_\alpha \leq m$. Clearly, $T_{f,i} \leq T_{f,j}$ for $i \leq j$. Furthermore, $T \cap [G, G] = T_{f,m}$ for $m \geq \max_{\alpha\in\Delta} c_\alpha$. Observe that T_f is the product (not direct) of $T_{f,m}(1 + \mathfrak{p}^m)$ for all m, where for any algebraic subtorus $S < T$, $S(1 + \mathfrak{p}^m)$ denotes the subgroup of $S(\mathcal{O})$ generated by the coweights of S evaluated at $1 + \mathfrak{p}^m$. It follows easily that one has an isomorphism of groups

$$T_f \cong T_{f,1}(1 + \mathfrak{p}) \times \prod_{m\geq 2} (T_{f,m}/T_{f,m-1})(1 + \mathfrak{p}^m). \tag{A.6}$$

From this one sees that T_f is the $\mathbb{F}_q$-points of a canonical proalgebraic (and prounipotent) subgroup of $\mathbf{T}_\mathcal{O}$. Similarly, for $m \geq \max_{\alpha\in\Delta} c_\alpha$,

$$\langle T_f, T_m\rangle \cong T_f \times (T/T_{f,m})(1 + \mathfrak{p}^m), \tag{A.7}$$

and one concludes also that this is the $\mathbb{F}_q$-points of a canonical proalgebraic subgroup of $\mathbf{T}_{\mathcal{O}}$ (depending on c as well as f). Finally, since $A = T(\mathcal{O})/T'$ is a quotient of $T(\mathcal{O})/T_m$, which is finite, so is A, and the geometric version of this statement is that $\mathbf{A}$ is an algebraic group (of finite type).

Applying the above to $f = f_{\bar{\mu}}$, one sees that $T_{f_{\bar{\mu}}}$ and T' are the $\mathbb{F}_q$-points of canonical proalgebraic subgroups of $\mathbf{T}_{\mathcal{O}}$, and that $\mathbf{A} \cong \mathbf{T}_{\mathcal{O}}/\mathbf{T}'$ is an algebraic group.

A.4. **Proof of Proposition 46.** Let $\mathbf{J}^{\lambda,\nu} := \mathbf{J}t^{\lambda}\mathbf{J} \times_{\mathbf{J}} \mathbf{J}^{\nu} \subseteq \mathbf{G} \times_{\mathbf{J}} \mathbf{X}$ and let $p^{\lambda,\nu} : \mathbf{J}^{\lambda,\nu} \to \mathbf{X}$ denote the restriction of p to $\mathbf{J}^{\lambda,\nu}$. Note that there is a natural action of $\mathbf{J} \times \mathbf{A}$ on $\mathbf{J}^{\lambda,\nu}$ given by left multiplication by $\mathbf{J}$ (on the first factor, $\mathbf{J}t^{\lambda}\mathbf{J}$) and right multiplication by $\mathbf{A}$ (on the second factor, $\mathbf{J}^{\nu}$), where, by convention, we use the right multiplication action $a \cdot_R j := ja^{-1}$ (even though $\mathbf{A}$ is commutative).

Lemma 58. *For all $\lambda, \nu \in \Lambda$,*

(i) *$\mathcal{F}'_{\lambda}\widetilde{\boxtimes}\mathcal{F}'_{\nu}$ is a $(\mathbf{J} \times \mathbf{A}, \mathcal{M} \boxtimes \mathcal{M}_0^{-1})$-equivariant local system on $\mathbf{J}^{\lambda,\nu}$.*
(ii) *$j_!^{\lambda} \star j_!^{\nu} \cong p_!^{\lambda,\nu}(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu})$.*
(iii) *$j_!^{\lambda} \star j_!^{\nu} \in {}^p\mathscr{D}^{\geq 0}(\mathbf{X})$.*
(iv) *$j_!^{\lambda} \star j_!^{\nu}$ is $(\mathbf{J} \times \mathbf{A}, \mathcal{M} \boxtimes \mathcal{M}_0^{-1})$-equivariant.*

Proof. By Lemma 72, $\mathcal{F}'_{\lambda}\widetilde{\boxtimes}\mathcal{F}'_{\nu}$ is a local system on $\mathbf{J}^{\lambda,\nu}$. Since $\mathcal{F}'_{\lambda}$ and $\mathcal{F}'_{\nu}$ are both equivariant, so is $\mathcal{F}'_{\lambda}\widetilde{\boxtimes}\mathcal{F}'_{\nu}$; thus, (i) is established. For (ii) see Lemma 73. Since $\mathbf{J}^{\lambda,\nu}$ (and therefore $p^{\lambda,\nu}$) is affine, Theorem 61 implies (iii). Finally, for (iv) we apply the projection formula to compute

$$l^*(j_!^{\lambda} \star j_!^{\nu}) = l^* p_!^{\lambda,\nu}(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu}) \cong (\mathrm{Id}_{\mathbf{J}} \times p^{\lambda,\nu})_! l^*(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu}) \cong$$
$$(\mathrm{Id}_{\mathbf{J}} \times p^{\lambda,\nu})_!(\mathcal{M} \boxtimes (\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu})) \cong \mathcal{M} \boxtimes p_!^{\lambda,\nu}(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu}) \cong \mathcal{M} \boxtimes (j_!^{\lambda} \star j_!^{\nu}). \quad \square$$

Next, we prove that $j_!^{\lambda} \star j_!^{\nu} \cong j_!^{\lambda+\nu}$ in three steps:

Step I: λ and ν are both dominant (or both antidominant): In this case, Corollary 27 implies that $\mathbf{J}t^{\lambda}\mathbf{J} \times_{\mathbf{J}} \mathbf{J}^{\nu} \cong \mathbf{J}^{\lambda+\nu}$ by the multiplication map, and making this identification, $p^{\lambda,\nu}$ becomes the inclusion $j^{\lambda+\nu}$. It is clear that $\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu}$ is a $(\mathbf{J} \times \mathbf{A}, \mathcal{M} \times \mathcal{M}_0^{-1})$-equivariant rank-one local system. It follows from the definitions that $(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu})|_{t^{\lambda+\nu}\mathbf{J}'} \cong \mathcal{F}_{\lambda+\nu}|_{t^{\lambda+\nu}\mathbf{J}'}$. We deduce from Lemma 42.(iii) that $\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu} \cong \mathcal{F}_{\lambda+\nu}$.

Step II: λ is dominant, ν is antidominant: Let $x \in \mathbf{X}$. We claim that $(j_!^{\lambda} \star j_!^{\nu})_x = 0$ if $x \notin \mathbf{J}^{\lambda+\nu}$. Indeed, by Lemma 58.(ii), $j_!^{\lambda} \star j_!^{\nu} \cong p_!^{\lambda,\nu}(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu})$. By Proposition 44 the stalk of this complex at $x \in \mathbf{X}$ is nonzero only if x is a relevant point in the image of $p^{\lambda,\nu}$. By Proposition 26.(c), the only relevant orbit inside this image is $\mathbf{J}^{\lambda+\nu}$.

Now we claim that we are in a position to apply Theorem 65 to prove that $j_!^{\lambda} \star j_!^{\nu}$ is perverse. It is clear that $p^{\lambda,\nu}$ is an affine morphism. Next, let $\mathcal{P}$ denote the partition of the closure of the image of $p^{\lambda,\nu}$ consisting of three locally closed subschemes: $\mathbf{J}^{\lambda+\nu}, \overline{\mathbf{J}^{\lambda+\nu}} \setminus \mathbf{J}^{\lambda+\nu}$, and the complement of $\overline{\mathbf{J}^{\lambda+\nu}}$. Note that, as the closure of the image of $p^{\lambda,\nu}$ is irreducible, one of these locally closed subschemes must, in fact, be open and dense. By Proposition 30, for every closed point $x \in \mathbf{J}^{\lambda+\nu}$,

$$\dim((p^{\lambda,\nu})^{-1}(x)) = \frac{1}{2}[\dim(\mathbf{J}^{\lambda}) + \dim(\mathbf{J}^{\nu}) - \dim(\mathbf{J}^{\lambda+\nu})].$$

From this, it follows that $p^{\lambda,\nu}$ is semismall at every $x \in \mathbf{J}^{\lambda+\nu}$ (at non-closed points y, the LHS should be replaced by the dimension of the generic fiber at closed points in the closure of y, cf. (B.3), and the result follows from the one for closed points). Since the stalk of $j_!^{\lambda} \star j_!^{\nu}$ at every point outside $\mathbf{J}^{\lambda+\nu}$ vanishes, Theorem 65 shows that $j_!^{\lambda} \star j_!^{\nu} \cong p_!^{\lambda,\nu}(\mathcal{F}_{\lambda}\widetilde{\boxtimes}\mathcal{F}_{\nu})$ is perverse.

Since $j_!^\lambda \star j_!^\nu$ is perverse and its stalks vanish outside of $\mathbf{J}^{\lambda+\nu}$, it must be isomorphic to $j_! j^*(j_!^\lambda \star j_!^\nu)$, where $j = j^{\lambda+\nu}$. Let $\mathcal{F}$ be the restriction of $j_!^\lambda \star j_!^\nu$ to $\mathbf{J}^{\lambda+\nu}$. This is a perverse sheaf, hence it must be a local system on an open subvariety. Since it is $(\mathbf{J}\times\mathbf{A}, \mathcal{M}\boxtimes\mathcal{M}_0^{-1})$-equivariant, we conclude that it is, in fact, a local system. Lemma 42.(iii) implies that $\mathcal{F}$ is isomorphic to $\mathcal{F}_{\lambda+\nu}$.

Step III: λ and ν arbitrary: Write $\lambda = \lambda_+ - \lambda_-$ and $\nu = \nu_+ - \nu_-$ for $\lambda_+, \lambda_-, \nu_+, \nu_- \in \Lambda_+$. Moreover, we can arrange this so that $\lambda_- = \nu_+$. Then

$$j_!^\lambda \star j_!^\nu = (j_!^{\lambda_+} \star j_!^{-\lambda_-}) \star (j_!^{\nu_+} \star j_!^{-\nu_-}) = j_!^{\lambda_+} \star (j_!^{-\lambda_-} \star j_!^{\lambda_-}) \star j_!^{-\nu_-} = j_!^{\lambda_+} \star j_!^0 \star j_!^{-\nu_-} = j_!^{\lambda_+ - \nu_-} = j_!^{\lambda+\nu}. \quad \square$$

A.5. **Proof of Corollary 47.** Note that J^0 is closed. Let $\mathbf{1} := j_!^0 \cong j_*^0$. Then for all $\mathcal{F}\star\mathbf{1} \cong \mathbf{1}\star\mathcal{F} \cong \mathcal{F}$ for all $\mathcal{F}\in\mathscr{H}_{\mathrm{geom}}$. Let $\mathcal{F},\mathcal{G},\mathcal{H}\in\mathscr{H}_{\mathrm{geom}}$ and assume that $\mathcal{H}\star\mathcal{F}$ and $\mathcal{H}\star\mathcal{G}$ are in $\mathscr{H}_{\mathrm{geom}}$. Then $\mathcal{H}\star -$ defines a homomorphism $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{H}\star\mathcal{F},\mathcal{H}\star\mathcal{G})$. Now assume there exists $\mathcal{H}'\in\mathscr{H}_{\mathrm{geom}}$ such that $\mathcal{H}'\star\mathcal{H} = \mathbf{1}$. Then the composition

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{H}\star\mathcal{F},\mathcal{H}\star\mathcal{G}) \to \mathrm{Hom}(\mathcal{H}'\star\mathcal{H}\star\mathcal{F},\mathcal{H}'\star\mathcal{H}\star\mathcal{G}) = \mathrm{Hom}(\mathbf{1}\star\mathcal{F},\mathbf{1}\star\mathcal{G}) = \mathrm{Hom}(\mathcal{F},\mathcal{G})$$

is the identity. Similarly, the composition

$$\mathrm{Hom}(\mathcal{H}\star\mathcal{F},\mathcal{H}\star\mathcal{G}) \to \mathrm{Hom}(\mathcal{H}'\star\mathcal{H}\star\mathcal{F},\mathcal{H}'\star\mathcal{H}\star\mathcal{G}) = \mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(\mathcal{H}\star\mathcal{F},\mathcal{H}\star\mathcal{G})$$

is the identity. Hence $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{H}\star\mathcal{F},\mathcal{H}\star\mathcal{G})$. The same holds when we replace Hom by $\mathrm{Ext}^\bullet$ or take convolution on the right instead of the left (with $\mathcal{H}$ having a right inverse).

Applying the above considerations and Proposition 46 we conclude

$$\begin{aligned}\mathrm{Ext}^\bullet(j_!^\lambda\otimes\mathcal{L}, j_!^\nu\otimes\mathcal{K}) &= \mathrm{Ext}^\bullet(j_!^\lambda\star j_!^{-\nu}\otimes\mathcal{L}, j_!^\nu\star j_!^{-\nu}\otimes\mathcal{K}) = \mathrm{Ext}^\bullet(j_!^{\lambda-\nu}\otimes\mathcal{L}, j_!^0\otimes\mathcal{K}) = \\ &= \mathrm{Ext}^\bullet(j_!^{\lambda-\nu}\otimes\mathcal{L}, j_*^0\otimes\mathcal{K}) = \mathrm{Ext}^\bullet_{\mathrm{Spec}\,\mathbb{F}_q}((j^0)^* j_!^{\lambda-\nu}\otimes\mathcal{L}, \mathcal{F}_0\otimes\mathcal{K}).\end{aligned}$$

This is zero unless $\lambda=\nu$, in which case it is $\mathrm{Ext}^\bullet_{\mathrm{Spec}\,\mathbb{F}_q}(\mathcal{F}_0\otimes\mathcal{L},\mathcal{F}_0\otimes\mathcal{K}) = \mathrm{Ext}^\bullet_{\mathrm{Spec}\,\mathbb{F}_q}(\mathcal{L},\mathcal{K})$. $\square$

A.6. **Proof of Proposition 50.** Let p_λ denote the multiplication morphism $\mathbf{G}\times_{\mathbf{J}}\mathbf{J}^\lambda \to \mathbf{X}$. Let $d = \dim(\mathbf{J}^\lambda)$. Then,

$$\begin{aligned}\mathrm{Hom}(\mathcal{F}\star_! j_!^\lambda, \mathcal{G}) &= \mathrm{Hom}((p_\lambda)_!(\mathcal{F}\widetilde{\boxtimes}\mathcal{F}_\lambda), \mathcal{G}) \\ &\cong \mathrm{Hom}(\mathcal{F}\widetilde{\boxtimes}\mathcal{F}_\lambda, p_\lambda^!\mathcal{G}) \\ &\cong \mathrm{Hom}(\mathcal{F}\widetilde{\boxtimes}\mathcal{F}_\lambda, p_\lambda^*\mathcal{G}[2d](d)).\end{aligned}$$

In the last isomorphism, we used that $p_\lambda^! = p_\lambda^*[2d](d)$, since p_λ is a smooth morphism of relative dimension d.

Claim 59. $\mathrm{Hom}(\mathcal{F}\widetilde{\boxtimes}\mathcal{F}_\lambda, p_\lambda^*\mathcal{G}[2d](d)) \cong \mathrm{Hom}(p_{-\lambda}^*\mathcal{F}, \mathcal{G}\widetilde{\boxtimes}\mathcal{F}_{-\lambda})$.

Using the claim, we can easily complete the proof similarly to the above:

$$\begin{aligned}\mathrm{Hom}(p_{-\lambda}^*\mathcal{F}, \mathcal{G}\widetilde{\boxtimes}\mathcal{F}_{-\lambda}) &\cong \mathrm{Hom}(\mathcal{F}, (p_{-\lambda})_*(\mathcal{G}\widetilde{\boxtimes}\mathcal{F}_{-\lambda})) \\ &\cong \mathrm{Hom}(\mathcal{F}, \mathcal{G}\star_* j_*^{-\lambda}).\end{aligned}$$

It remains to prove Claim 59. The proof relies on converting between the functors $-\widetilde{\boxtimes}\mathcal{F}_\lambda$ and p_λ^*-. We first explain how to do this in a simpler (and probably standard) situation, where $\mathbf{G}$ is an algebraic group (i.e., of finite type), $\mathbf{H}\subseteq\mathbf{G}$ is a subvariety, we replace $\mathcal{F}_\lambda$ by $\overline{\mathbb{Q}}_\ell|_{\mathbf{H}}$, and eliminate the twists and twisted products. The analogous claim in this simpler situation can be formulated as follows. Let $\tilde{p}:\mathbf{G}\times\mathbf{G}\to\mathbf{G}$ denote the multiplication map (we use tildes to distinguish from the maps we will define eventually on the level of $\mathbf{G}\times_{\mathbf{J}}\mathbf{X}$ and in the twisted setting.) Let $\mathcal{F},\mathcal{G}\in\mathscr{D}(\mathbf{G})$. Let $\mathbf{H}^{-1}$ be the image of $\mathbf{H}$ under the inversion automorphism of $\mathbf{G}$.

Claim 60. $\mathrm{Hom}_{\mathbf{G}\times\mathbf{H}}(\mathcal{F}\boxtimes\overline{\mathbb{Q}}_\ell, \tilde{p}^*\mathcal{G}|_{\mathbf{G}\times\mathbf{H}}) \cong \mathrm{Hom}_{\mathbf{G}\times\mathbf{H}^{-1}}(\tilde{p}^*\mathcal{F}|_{\mathbf{G}\times\mathbf{H}^{-1}}, \mathcal{G}\boxtimes\overline{\mathbb{Q}}_\ell)$.

Proof. Let $\tilde{\Gamma} : \mathbf{G}\times\mathbf{G} \to \mathbf{G}\times\mathbf{G}$ denote the isomorphism $\tilde{\Gamma}(g,x) = (gx,x)$. Then there is a commutative diagram

$$\begin{tikzcd} \mathbf{G}\times\mathbf{G} \arrow[r, "\tilde{\Gamma}"] \arrow[d, "\tilde{p}"] & \mathbf{G}\times\mathbf{G} \arrow[d, "\tilde{\pi}_1"] \\ \mathbf{G} \arrow[r, equal] & \mathbf{G}. \end{tikzcd} \tag{A.8}$$

Therefore, $\tilde{p} = \tilde{\pi}_1\tilde{\Gamma}$, and hence $\tilde{\Gamma}^*(\mathcal{F}\boxtimes\overline{\mathbb{Q}}_\ell) \cong \tilde{\Gamma}^*\pi_1^*\mathcal{F} \cong \tilde{p}^*\mathcal{F}$.

Next, let $\tilde{\iota}_2 : \mathbf{G}\times\mathbf{G} \to \mathbf{G}\times\mathbf{G}$ denote the isomorphism $\tilde{\iota}_2(g,x) = (g,x^{-1})$. We need the identity

$$(\tilde{\Gamma}\tilde{\iota}_2)^2 = \mathrm{Id}_{\mathbf{G}\times\mathbf{G}} = (\tilde{\iota}_2\tilde{\Gamma})^2. \tag{A.9}$$

Finally,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{G}\times\mathbf{H}}(\mathcal{F}\boxtimes\overline{\mathbb{Q}}_\ell, \tilde{p}^*\mathcal{G}|_{\mathbf{G}\times\mathbf{H}}) &\cong \mathrm{Hom}_{\mathbf{G}\times\mathbf{H}^{-1}}(\mathcal{F}\boxtimes\overline{\mathbb{Q}}_\ell, \tilde{\iota}_2^*\tilde{p}^*\mathcal{G}|_{\mathbf{G}\times\mathbf{H}^{-1}}) \\ &\cong \mathrm{Hom}_{\mathbf{G}\times\mathbf{H}^{-1}}(\tilde{p}^*\mathcal{F}|_{\mathbf{G}\times\mathbf{H}^{-1}}, \tilde{\Gamma}^*\tilde{\iota}_2^*\tilde{p}^*\mathcal{G}|_{\mathbf{G}\times\mathbf{H}^{-1}}) \\ &\cong \mathrm{Hom}_{\mathbf{G}\times\mathbf{H}^{-1}}(\tilde{p}^*\mathcal{F}|_{\mathbf{G}\times\mathbf{H}^{-1}}, ((\tilde{\iota}_2)^{-1})^*(\tilde{\Gamma}^{-1})^*\tilde{p}^*\mathcal{G}|_{\mathbf{G}\times\mathbf{H}^{-1}}) \\ &\cong \mathrm{Hom}_{\mathbf{G}\times\mathbf{H}^{-1}}(\tilde{p}^*\mathcal{F}|_{\mathbf{G}\times\mathbf{H}^{-1}}, \mathcal{G}\boxtimes\overline{\mathbb{Q}}_\ell), \end{aligned}$$

as desired. $\square$

Proof of Claim 59. Let $\mathbf{K}^\lambda = \mathbf{J}t^\lambda\mathbf{J} \subset \mathbf{G}$. We want to imitate the proof of the previous claim with $\mathbf{H}$ replaced by $\mathbf{K}^\lambda$. Since we have difficulty working with objects over $\mathbf{G}\times\mathbf{K}^\lambda$, as it is not ind-finite, we instead descend $\tilde{\Gamma}|_{\mathbf{G}\times\mathbf{K}^\lambda}$ to an isomorphism

$$\Gamma_\lambda : \mathbf{G}\times_{\mathbf{J}}\mathbf{J}^\lambda \xrightarrow{\sim} (\mathbf{G}\times\mathbf{J}\backslash\mathbf{G})/\mathbf{J}', \quad (g,x)\mapsto(gx,x), \tag{A.10}$$

where $\mathbf{J}'$ acts diagonally by $(g,x)\cdot j := (gj, xj)$.

Similarly, the inversion map in the second component, $\tilde{\iota}_2$, descends to

$$\iota_2 : \mathbf{G}\times_{\mathbf{J}}\mathbf{X} \xrightarrow{\sim} (\mathbf{G}\times\mathbf{J}'\backslash\mathbf{G})/\mathbf{J}, \quad (g,x)\mapsto(g,x^{-1}), \tag{A.11}$$

where again $\mathbf{J}$ acts diagonally by $(g,x)\cdot j := (gj,xj)$.

We will need the equivalence

$$\tau_\lambda : \mathscr{P}_{(\mathbf{A},\mathcal{M}_0^{-1})}((\mathbf{G}\times\mathbf{J}'\backslash\mathbf{K}^\lambda)/\mathbf{J}) \xrightarrow{\sim} \mathscr{P}_{(\mathbf{A},\mathcal{M}_0)}((\mathbf{G}\times\mathbf{J}\backslash\mathbf{K}^\lambda)/\mathbf{J}'), \quad \mathcal{G}\mapsto\mathcal{G}\otimes\overline{(\overline{\mathbb{Q}}_\ell\boxtimes(\mathcal{F}'_\lambda)^{-1})}. \tag{A.12}$$

Here $\overline{(\overline{\mathbb{Q}}_\ell\boxtimes(\mathcal{F}'_\lambda)^{-1})}$ is the local system $\mathbf{J}'\backslash\mathbf{K}^\lambda)/\mathbf{J}'$ obtained by equivariant descent from the local system $\overline{\mathbb{Q}}_\ell\boxtimes(\mathcal{F}'_\lambda)^{-1}$ on $\mathbf{G}\times\mathbf{K}^\lambda$, and we view both categories in (A.12) as categories of twisted-equivariant perverse sheaves on $(\mathbf{G}\times\mathbf{J}'\backslash\mathbf{K}^\lambda)/\mathbf{J}'$ (lifting from a quotient by $\mathbf{J}$ to ordinary $\mathbf{A}$-equivariant objects on the quotient by $\mathbf{J}'$).

The identity analogous to (A.9) in this twisted setting is

$$(\Gamma_\lambda^*\tau_\lambda(\iota_2^{-1})^*)^{-1} \cong \Gamma_{-\lambda}^*\tau_{-\lambda}(\iota_2^{-1})^*. \tag{A.13}$$

From now on, an overlined quantity means the object living over the appropriate base (indicated by the subscript of Hom) obtained by equivariant descent. For instance, $\mathcal{F}\widetilde{\boxtimes}\mathcal{G} = \overline{\mathcal{F}\boxtimes\mathcal{G}}$. Also, note that $\mathcal{F}'_\lambda = \overline{\mathcal{M}\times\mathcal{M}^{-1}}$ for all λ, working over the base $\mathbf{J}^\lambda$ (which is a quotient of $\mathbf{J}\times\mathbf{J}$). Then,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{G}\times_{\mathbf{J}}\mathbf{J}^\lambda}(\mathcal{F}\widetilde{\boxtimes}\mathcal{F}'_\lambda, p_\lambda^*\mathcal{G}) &\cong \mathrm{Hom}_{(\mathbf{G}\times\mathbf{J}'\backslash\mathbf{K}^{-\lambda})/\mathbf{J}}((\iota_2^{-1})^*(\mathcal{F}\widetilde{\boxtimes}\mathcal{F}'_\lambda), (\iota_2^{-1})^*p_\lambda^*\mathcal{G}) \\
&\cong \mathrm{Hom}_{(\mathbf{G}\times\mathbf{J}'\backslash\mathbf{K}^{-\lambda})/\mathbf{J}}(\overline{\mathcal{F}\boxtimes(\mathcal{M}\times\mathcal{M}^{-1})}, (\iota_2^{-1})^*\Gamma_\lambda^*(\mathcal{G}\widetilde{\boxtimes}\overline{\mathbb{Q}}_\ell)) \\
&\overset{\tau_{-\lambda}}{\cong} \mathrm{Hom}_{(\mathbf{G}\times\mathbf{J}\backslash\mathbf{K}^{-\lambda})/\mathbf{J}'}(\overline{\mathcal{F}\boxtimes\overline{\mathbb{Q}}_\ell}, \tau_{-\lambda}(\iota_2^{-1})^*\Gamma_\lambda^*(\mathcal{G}\widetilde{\boxtimes}\overline{\mathbb{Q}}_\ell)) \\
&\overset{\Gamma_{-\lambda}^*}{\cong} \mathrm{Hom}_{\mathbf{G}\times_{\mathbf{J}}\mathbf{J}^{-\lambda}}(p_{-\lambda}^*\mathcal{F}, \Gamma_{-\lambda}^*\tau_{-\lambda}(\iota_2^{-1})^*\Gamma_\lambda^*(\mathcal{G}\widetilde{\boxtimes}\overline{\mathbb{Q}}_\ell)) \\
&\overset{(A.13)}{\cong} \mathrm{Hom}_{\mathbf{G}\times_{\mathbf{J}}\mathbf{J}^{-\lambda}}(p_{-\lambda}^*\mathcal{F}, \iota_2^*\tau_\lambda^{-1}(\mathcal{G}\widetilde{\boxtimes}\overline{\mathbb{Q}}_\ell)) \\
&\cong \mathrm{Hom}_{\mathbf{G}\times_{\mathbf{J}}\mathbf{J}^{-\lambda}}(p_{-\lambda}^*\mathcal{F}, \mathcal{G}\widetilde{\boxtimes}\mathcal{F}'_{-\lambda}).
\end{aligned}$$

Now, the same computation with the appropriate shifts and Tate twists (using Remark 35) yields the desired result. □

Appendix B. Recollections on perverse sheaves

B.1. **Definition of perverse sheaves.** Let X be a connected scheme of finite type over a field k, which we assume to be finite or algebraically closed. Fix a prime ℓ invertible in k. Let $\mathscr{D}(X)$ denote the derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on X with bounded constructible cohomology [Del80, §1.1.2-1.1.3]. Let ${}^p\mathscr{D}^{\le 0}(X)\subseteq\mathscr{D}(X)$ denote the full subcategory consisting of all complexes $\mathcal{K}$ such that

$$\dim\operatorname{supp}\mathrm{H}^i(\mathcal{K})\le -i,\qquad \forall\, i\in\mathbb{Z}. \tag{B.1}$$

Equivalently, for all (not necessarily closed) points $x\in X$,

$$\mathrm{H}^n(\mathcal{K}_x)=0\qquad\forall\, n>-\dim(x). \tag{B.2}$$

Using Verdier duality (or the notion of cosupport), one similarly defines ${}^p\mathscr{D}^{\ge 0}(X)$ (see, e.g., [BBD82]). The category of perverse sheaves is defined by

$$\mathscr{P}(X) := {}^p\mathscr{D}^{\ge 0}(X)\cap{}^p\mathscr{D}^{\le 0}(X).$$

The following theorem is essentially due to Artin; see [BBD82, Theorem 4.1.1].

Theorem 61. *If $f: X\to Y$ is an affine morphism of separated schemes of finite type over k, the functor $f_*:\mathscr{D}(X)\to\mathscr{D}(Y)$ takes ${}^p\mathscr{D}^{\le 0}(X)$ into ${}^p\mathscr{D}^{\le 0}(Y)$. By Verdier duality, this is equivalent to saying that the functor $f_!$ takes ${}^p\mathscr{D}^{\ge 0}(X)$ into ${}^p\mathscr{D}^{\ge 0}(Y)$.*

B.2. **Intermediate extensions and cleanness.** Let $j: Y\hookrightarrow X$ be an embedding of a locally closed subvariety Y. Recall the intermediate extension function $j_{!*}:\mathscr{P}(Y)\to\mathscr{P}(X)$, which has the properties $j^*j_{!*}\mathcal{F}\cong\mathcal{F}$, and which takes irreducible perverse sheaves to irreducible perverse sheaves.

Definition 62. Let $\mathcal{F}\in\mathscr{P}(Y)$. The intermediate extension $j_{!*}\mathcal{F}$ is called *clean* (or a "clean extension") if $j_{!*}\mathcal{F}\cong j_!\mathcal{F}$.

B.3. **Semismall morphisms.** The standard reference for semismall morphisms and their relationship to perverse sheaves is [GM83, §6.2]. Here we follow the treatment of [KW01, §III.7], since this reference does not assume properness.

A partition $\mathcal{P}$ of Y is a collection $\{Y_\alpha\}$ of disjoint locally closed subschemes of Y such that

(i) $Y=\bigsqcup Y_\alpha$,
(ii) one of these subschemes is open and dense.

Definition 63. Let Y be a separated scheme of finite type over k. Let $\mathcal{P}$ be a partition of Y. Let $y \in Y_\alpha \subseteq Y$ be a possibly non-closed point. A morphism of separated schemes $f : X \to Y$ is called *semismall at $y \in Y_\alpha \subseteq Y$ with respect to $\mathcal{P}$* if

$$\dim(f^{-1}(y)) - \dim(y) \leq \frac{1}{2}[\dim(X) - \dim(Y_\alpha)]. \tag{B.3}$$

For the following result, see [KW01, Lemma 7.4].

Lemma 64. *Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on X. Suppose $f : X \to Y$ is a morphism of separated schemes of finite type over k. Let $\mathcal{P}$ be a finite partition of Y. If f is semismall at every point $y \in Y$ with respect to $\mathcal{P}$, then $f_!(\mathcal{F}[\dim(X)]) \in {}^p\mathscr{D}^{\leq 0}(Y)$.*

Theorem 65. *Let $f : X \to Y$ be an affine morphism of separated schemes of finite type over k. Let $\mathcal{L}$ be a local system on X and set $\mathcal{K} := f_!(\mathcal{L}[\dim X])$. Let $\mathcal{P}$ be a finite partition of Y. Assume that for every $y \in Y$ either*

(i) *$\mathcal{K}_y = 0$, or*
(ii) *f is semismall at y with respect to $\mathcal{P}$.*

Then $\mathcal{K} \in \mathscr{P}(Y)$.

Proof. By Theorem 61, $\mathcal{K} \in {}^p\mathscr{D}^{\geq 0}(X)$. It remains to show that $\mathcal{K} \in {}^p\mathscr{D}^{\leq 0}(X)$. According to (B.2), it is sufficient to check the required vanishing at each stalk $y \in Y$. If y is not in the image of f, then $\mathcal{K}_y = 0$ and the condition is automatically satisfied. This is also the case if y is in the image of f and $\mathcal{K}_y = 0$. If we are in neither situation, then the result follows from Lemma 64. ☐

B.4. **Twisted equivariant sheaves.** Let G be a connected algebraic group over k. Recall that this means that G is a smooth connected group scheme of finite type over k. Let $m : G \times G \to G$ denote the multiplication.

Definition 66. A *one-dimensional character sheaf* on G is a local system $\mathcal{L}$ satisfying $m^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$.

Remark 67. Another name for a one-dimensional character sheaf is a *multiplicative local system.* We note that character sheaves are usually defined to be irreducible perverse sheaves on a group over an algebraically closed field. It is, therefore, more appropriate to call $\mathcal{L} \otimes_k \bar{k}[\dim(G)]$ a one-dimensional character sheaf. Working over an arbitrary field and ignoring the shift is, however, more convenient for our purposes.

Let $\mathcal{L}$ be a one-dimensional character sheaf on G. Let X be a separated scheme of finite type over k equipped with an action $a : G \times X \to X$.

Definition 68. The category $\mathscr{P}_{(G,\mathcal{L})}(X)$ of $(G, \mathcal{L})$-equivariant perverse sheaves on X is the full subcategory of $\mathscr{P}(X)$ consisting of perverse sheaves $\mathcal{F}$ satisfying $a^*\mathcal{F} \cong \mathcal{L} \boxtimes \mathcal{F}$.

If $\mathcal{L}$ is trivial, we recover the usual notion of equivariant perverse sheaves; see [Lus84, §0].

Remark 69. Let G' be a connected subgroup of G and let $A := G/G'$. Let $\mathcal{L}_0$ be a one-dimensional character sheaf on A, and let $\mathcal{L}$ be its pullback to G. Suppose G (and therefore G') acts freely on X. Let $X' = G' \backslash X$. Note that A acts freely on X' and $X = A \backslash X'$. Let $r : X \to X'$ denote the canonical projection. Then

$$r^*[\dim(G)] : \mathscr{P}_{(A,\mathcal{L}_0)}(X') \to \mathscr{P}_{(G,\mathcal{L})}(X)$$

is an equivalence of categories.

B.4.1. *Support of twisted equivariant sheaves.* Given an algebra R over k, an R-point x of X is a morphism $x : \operatorname{Spec} R \to X$. Now the stabilizer G_x is the sub-group scheme of G fixing the map x. If $x \in X$ is a set-theoretic point, we can think of it as a point in the above sense in the standard manner, by letting R be the algebra of functions on x (an extension field of k).

Let G be an algebraic group and $\mathcal{L}$ be a nontrivial one-dimensional character sheaf on G. Suppose G acts on a variety X. Let x be a set-theoretic point of X and let G_x denote the stabilizer of X. Then G_x is a subgroup of G. Let G_x° denote the connected component of the identity of G_x. Let $\mathcal{F}$ be a $(G, \mathcal{L})$-equivariant sheaf on X.

Lemma 70. *If the restriction $\mathcal{L}|_{G_x^\circ}$ is nontrivial, then the restriction $\mathcal{F}|_x := x^*\mathcal{F}$ is zero.*

Proof. The restriction $x^*\mathcal{F}$ is an $(G_x^\circ, \mathcal{L}|_{G_x^\circ})$-equivariant local system on x, with respect to the trivial action. Let $\pi : \{x\} \times G_x^\circ \twoheadrightarrow \{x\}$ denote the projection, which is also the action map. Since $x^*\mathcal{F}$ is equivariant, $x^*\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell = \pi^* x^* \mathcal{F} \cong x^*\mathcal{F} \boxtimes \mathcal{L}|_{G_x^\circ}$. However, by assumption, the first local system is constant in the G_x° direction, but if $x^*\mathcal{F}$ is nonzero, the second is not. Hence, $x^*\mathcal{F}$ is zero. $\square$

B.5. **Alternative definitions of twisted sheaves.** The purpose of this subsection is to expand on Remark 6. The discussions of this subsection are not used anywhere else in the paper.

B.5.1. *Central extensions and one-dimensional character sheaves.* Let $\mathcal{L}$ be a one-dimensional character sheaf on G. Let $\pi_1(G) = \pi_1(G, e)$ denote the algebraic fundamental group of G. It is well known that the local system $\mathcal{L}$ defines a homomorphism $\pi_1(G) \to \overline{\mathbb{Q}}_\ell^\times$. Let us assume that this homomorphism factors through $\chi_{\mathcal{L}} : B_{\mathcal{L}} \to \overline{\mathbb{Q}}_\ell^\times$, where $B_{\mathcal{L}}$ is a finite quotient of $\pi_1(G)$. The local systems we consider in this article satisfy this property. In this situation, the epimorphism $\pi_1(G) \twoheadrightarrow B_{\mathcal{L}}$ defines a finite covering $\tilde{G} \to G$. Using the fact that $\mathcal{L}$ is multiplicative, one can show that $\tilde{G}$ is a central extension of G; see the introduction of [Kam09]. Thus, we obtain a central extension

$$1 \to B_{\mathcal{L}} \to \tilde{G} \to G \to 1 \tag{B.4}$$

in the category of algebraic groups over k.

Remark 71. If k has positive characteristic, there exist étale covers of G which cannot be endowed with the structure of a central extension of G; see [Kam09, §2.4 and §B.4].

B.5.2. *Twisted sheaves via gerbes.* Let Y denote the quotient stack $G\backslash X$. Note that Y is an Artin stack.[10] The central extension (B.4) gives rise to a homomorphism $H^1(Y, G) \to H^2(Y, B_{\mathcal{L}})$; see [Gir71]. Composing with the morphism $H^2(Y, B_{\mathcal{L}}) \to H^2(Y, \overline{\mathbb{Q}}_\ell^\times)$ defined by $\chi_{\mathcal{L}} : B_{\mathcal{L}} \to \overline{\mathbb{Q}}_\ell^\times$, we obtain a morphism

$$H^1(Y, G) \to H^2(Y, \overline{\mathbb{Q}}_\ell^\times). \tag{B.5}$$

The scheme X is a G-torsor on Y; therefore, it defines an element in $H^1(Y, G)$. Let $\mathscr{L}$ denote the $\overline{\mathbb{Q}}_\ell^\times$-gerbe on Y defined by the image of this element under the morphism (B.5). Then the notion of $(G, \mathcal{L})$-twisted equivariant (perverse) sheaf coincides with the notion of $\mathscr{L}$-twisted perverse sheaf on Y. We note that the idea of twisting sheaves by gerbes goes back to [Gir71]. In [Rei10], Reich applies twisting to the constructible derived category and the category of perverse sheaves.

[10]For ℓ-adic sheaves on an Artin stacks see [LO08].

B.5.3. *Twisted sheaves via equivariant sheaves.* The character sheaf $\mathcal{L}$ pulls back to a trivial local system on $\tilde{G}$. Therefore, $(G, \mathcal{L})$-equivariant perverse sheaves on X are automatically $\tilde{G}$-equivariant, where $\tilde{G}$ acts on X via the natural map $\tilde{G} \twoheadrightarrow G$. Moreover, one can show that we have an equivalence of categories between $(G, \mathcal{L})$-equivariant perverse sheaves on X and the full abelian subcategory of perverse sheaves on the algebraic stack $\tilde{G}\backslash X$ whose pullback to X are $(B_{\mathcal{L}}, \chi_{\mathcal{L}})$-equivariant (i.e., $B_{\mathcal{L}}$ acts on the fibers by $\chi_{\mathcal{L}}$).

B.6. **Twisted external product $\tilde{\boxtimes}$.** The notion of twisted external product of perverse sheaves has been used widely (e.g., in [FGV01, §1.4], [Gai01, §0.2], [MV07, §4] and [Nad05, §2.2]). In this subsection, we given an overview of this construction and apply it to twisted equivariant sheaves.

Let Y and Z be separated schemes of finite type over a field k. Let H be a connected algebraic group over k and let $p : X \to Y$ be a right H-torsor. Suppose H acts on Z on the left. Define a free left action of H on $X \times Z$ by

$$h \cdot (x, z) \mapsto (x \cdot h^{-1}, h \cdot z) \tag{B.6}$$

We denote by $X \times_H Z$ the quotient of $X \times Z$ by H. Let $q : X \times Z \to X \times_H Z$ the canonical quotient map. We define twisted external product of sheaves as follows:

- Let $\mathcal{F}$ and $\mathcal{G}$ be H-equivariant perverse sheaves on X and Z. Then $\mathcal{F} \boxtimes \mathcal{G}$ is a perverse sheaf on $X \times Z$ equivariant with respect to the action (B.6). Thus, we obtain a canonical perverse sheaf on $X \times_H Z$.
- Suppose $\mathcal{G}$ is as above, but $\mathcal{F}$ is now a perverse sheaf on Y. Then $p^*\mathcal{F}$ is an H-equivariant perverse sheaf on X. The construction of the previous paragraph applies to give us a perverse sheaf $\mathcal{F}\tilde{\boxtimes}\mathcal{G}$ on $X \times_H Z$. Roughly speaking, $\mathcal{F}\tilde{\boxtimes}\mathcal{G}$ is $\mathcal{F}$ "along the base" and $\mathcal{G}$ "along the fiber".
- By definition, $\tilde{\boxtimes}$ is a functor

$$\tilde{\boxtimes} : \mathscr{P}_H(X) \times \mathscr{P}_H(Z) \to \mathscr{P}(X \times_H Z), \quad \text{satisfying } q^*\mathcal{F} \cong (p^*\mathcal{F}) \boxtimes \mathcal{G}. \tag{B.7}$$

- The functor $\tilde{\boxtimes}$, with the property expressed in (B.7), makes sense in the following more general situation: Y is a "strict ind-scheme" of ind-finite type over k, Z is a strict ind-scheme of ind-finite type over k equipped with a "nice action" of a pro-algebraic group H over k. (For the notions "strict ind-scheme" and "nice action" see [Gai01].) In this case, although X need not be of ind-finite type, $X \times_H Z$ remains of ind-finite type, since it is a fibration over Z with fibers isomorphic to Y. The fact that it is nonetheless legitimate to work with H-equivariant perverse sheaves on X is explained in [Nad05, §2.2].
- More generally, suppose that $H' < H$ is a proalgebraic subgroup such that $A := H/H'$ is an algebraic group. Let $\mathcal{M}_0$ be a multiplicative local system on A. Let $\mathcal{M}$ be the pullback of $\mathcal{M}_0$ to H. Suppose $\mathcal{F}$ and $\mathcal{G}$ are $(H, \mathcal{M}^{-1})$- and $(H, \mathcal{M})$-equivariant perverse sheaves on X and Z, respectively; more precisely (to deal with the case that X may not be ind-finite), we let $\mathcal{F}$ be the pullback of an $(A, \mathcal{M}_0^{-1})$-equivariant local system on Y (cf. Remark 69). Then $\mathcal{F}\tilde{\boxtimes}_{H'}\mathcal{G}$ is (untwisted) equivariant with respect to the action of A which descends from (B.6). Hence, it descends to a canonical perverse sheaf $\mathcal{F}\tilde{\boxtimes}\mathcal{G}$ on $X \times_H Z$. Thus, $\tilde{\boxtimes}$ also defines a functor

$$\mathscr{P}_{(H,\mathcal{M}^{-1})}(X) \times \mathscr{P}_{(H,\mathcal{M})}(Z) \to \mathscr{P}(X \times_H Z).$$

B.6.1. *Twisted external product of local systems.* Let Y' (resp. Z') denote a locally closed subscheme of Y (resp. Z) of dimension d (resp. d'). Let $X' \subseteq X$ denote the restriction of the G-torsor X to Y'. Let $d'' := \dim(X' \times_H Z')$. Suppose $\mathcal{L}$ is a local system on Y' and $\mathcal{L}'$ is an H-equivariant local system on Z'. The proof of the following lemmas are left to the reader.

Lemma 72. *$\mathcal{L}[d]\tilde{\boxtimes}\mathcal{L}'[d'] \cong \mathcal{L}''[d'']$, where $\mathcal{L}''$ is a local system on $X' \times_H Z'$.*

Lemma 73. *Let*

$$j : X' \hookrightarrow \overline{X'}, \qquad j' : Z' \hookrightarrow \overline{Z'}, \qquad j'' : X' \times_H Z' \hookrightarrow \overline{X' \times_H Z'}.$$

Assume that $j_!(\mathcal{L}[d])$*,* $j'_!(\mathcal{L}'[d'])$*, and* $j''_!(\mathcal{L}''[d''])$ *are perverse.*[11] *Then*

$$j_!(\mathcal{L}[d]) \widetilde{\boxtimes} j'_!(\mathcal{L}'[d']) \cong j''_!(\mathcal{L}''[d'']).$$

B.7. **Trace of Frobenius.** Let $\mathbb{F}_q$ be a field with q elements. Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. The *Frobenius substitution* $\varphi \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the automorphism $x \mapsto x^q$ of $\overline{\mathbb{F}}_q$. The *geometric Frobenius* Fr_q, or simply the Frobenius, is the inverse of φ.

Let X be a separated scheme of finite type over $\mathbb{F}_q$. Let $x : \mathrm{Spec}(\mathbb{F}_q) \to X$ be an $\mathbb{F}_q$-point of X, and let $\bar{x}$ be a geometric point lying above x. If $\mathcal{G} \in \mathscr{D}(X)$, then the fiber $\mathcal{G}_{\bar{x}}$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space on which $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acts [Del80, §1.1.7]. We denote by $\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{G})(x) \in \overline{\mathbb{Q}}_\ell$ the trace of Frobenius acting on this vector space. Thus, we obtain the trace function of $\mathcal{G}$

$$\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{G}) : X(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell.$$

Similarly, we have trace functions $\mathrm{Tr}(\mathrm{Fr}_{q^n}, \mathcal{G}) : X(\mathbb{F}_{q^n}) \to \overline{\mathbb{Q}}_\ell$ for all $n \geq 1$; see [Lau87, §0.9 and §1.1.1]. Note that with our conventions

$$\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{G}(n)) = q^{-n}\,\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{G}), \tag{B.8}$$

where $\mathcal{G}(n)$ denotes the n^{th} Tate twist of $\mathcal{G}$ [BD06, §E.1].

B.7.1. *Character sheaves on connected commutative algebraic groups.* Suppose $\mathcal{L}$ is a one-dimensional character sheaf on a connected algebraic group G over $\mathbb{F}_q$ (Definition 66). The property $m^*\mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$ ensures that the trace function is a one-dimensional character $\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{L}) : G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$. For a general noncommutative algebraic group, there may exist one-dimensional characters of $G(\mathbb{F}_q)$ which do not arise in this manner; see, e.g., [Boy07, §1.5.5].

If G is *commutative*, then every one-dimensional character of $G(\mathbb{F}_q)$ can be obtained as the trace of Frobenius function of a one-dimensional character sheaf on G. To see this, let

$$0 \to G(\mathbb{F}_q) \to G \xrightarrow{\mathrm{Fr}_q - \mathrm{id}} G \to 0$$

denote the Lang central extension. Let $\eta : G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell^\times$ be a character. Pushing forward the above central extension by η^{-1}, we obtain a one-dimensional local system $\mathcal{N}$ on G. One can check that $\mathrm{Tr}(\mathrm{Fr}_q, \mathcal{N}) = \eta$; see [Del77, Sommes Trig], [Lau87, Example 1.1.3], and [BD06, §1.8].

Remark 74. If G is commutative, then for every integer n, we have a "norm map" $N_n : G(\mathbb{F}_{q^n}) \to G(\mathbb{F}_q)$. Namely, $N_n(x) = \prod_{i=0}^{n-1} \mathrm{Fr}_q^i(x)$, with the product taken in $G(\mathbb{F}_{q^n})$ (cf. [Del77, Sommes Trig, §1.6]). Let $\eta_n := \eta \circ N_n$. Then, one can show that $\mathrm{Tr}(\mathrm{Fr}_{q^n}, \mathcal{N} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}) = \eta_n$; see *op. cit.* or [Lau87, §1.1.3.3].

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