THE GEOMETRIC MINIMAL MODELS OF ANALYTIC SPACES

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ABSTRACT. This paper shows that an analytic space X has a unique maximal model through which every proper surjective morphism from a non-singular analytic space to X factors. This is called the geometric minimal model of X and characterized by the contraction property of rational curves. Some other properties such as functoriality, direct product property, quotient property of the geometric minimal model are also studied here. The relation of the geometric minimal model with Mori's minimal model is shown.

1. INTRODUCTION

It is well known that a two dimensional singularity (X, x) has a good model called the minimal resolution $f : Y \to X$, that is :

(1) all resolutions of (X, x) factor through f and

(2) Y itself is non-singular.

For the higher dimensional case, such a resolution no longer exists. Then, seeking a good model of a higher dimensional singularity in the Minimal Model Conjecture, people modify the conditions (1) and (2) into a condition (3) which is equivalent to (1) and (2) in the two-dimensional case.

(3) Y has $\mathbb{Q}$-factorial terminal singularities and K_Y is f-nef.

As is known, such a model exists for a three-dimensional singularity ([11]), but the existence for the higher dimensional case is still conjecture.

On the other hand, the minimal resolution $f : Y \to X$ of a two-dimensional singularity is also characterized as follows:

(1) all resolutions of (X, x) factor through f and

($2'$) $f : Y \to X$ is maximal among those satisfying (1).

This paper shows that such a model also uniquely exists for the higher dimensional case.

We also prove that the following model $f : Y \to X$ uniquely exists for any dimension:

($1'$) all proper surjective morphisms $\tilde{X} \to X$ from non-singular $\tilde{X}$ factor through f and

(2″) $f : Y \to X$ is maximal among those satisfying (1′).

This model is called the geometric minimal model of X and is proved to coincide with the model satisfying (1) and (2′) if $\dim X \neq 2$.

The geometric minimal model is characterized by the contraction of rational curves: let $\tilde{X} \to X$ be a proper surjective morphism from non-singular $\tilde{X}$, then every rational curve on $\tilde{X}$ which is mapped to a point on X is mapped to a point on the geometric minimal model Y; on the contrary, the maximal model $Y \to X$ with the following property is the geometric minimal: there is a proper bimeromorphic morphism $Z \to Y$ such that every rational curve on Z which is mapped to a point on X is mapped to a point on Y. By this, we can see that the geometric minimal model of a toric variety is the identity and the geometric minimal model of a cone over an Abelian variety is the corresponding line bundle over the Abelian variety.

In §2, we prove the existence of a maximal model through which all resolutions factor. The proof is done under a more general setting, and the existence of the geometric minimal model is also proved.

In §3, we prove the characterization of the geometric minimal model by the contraction of rational curves. The coincidence of the geometric minimal model and the model with (1) and (2′) for $\dim \neq 2$ is also proved here.

In §4, we prove miscellaneous properties of the geometric minimal model: the functoriality, the liftability of group actions, the quotient property, the direct product property, and the relation with Mori's minimal model for 3-dimensional case.

Throughout this paper, an analytic space X is an analytic $\mathbb{C}$-space in Hironaka's sense [9], i.e.,

(i) for every point x of X, there exists an open neighbourhood such that the restriction of X onto it is $\mathbb{C}$-isomorphic to a local analytic $\mathbb{C}$-space,

(ii) the underlying topological space of X is countable at infinity, i.e., X is a union of countably many compact subsets, and

(iii) X is a Hausdorff space.

Analytic spaces are always assumed to be irreducible and reduced except for Lemma 2.4.

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2. The greatest common factor of a category of proper surjective morphisms

Definition 2.1. For an analytic space X, denote by $\mathcal{A}(X)$ the category whose objects are proper surjective morphisms $X' \to X$ from analytic spaces X' to X and whose morphisms are morphisms over X.

In this category, we introduce an order between the objects according to the existence of morphisms: i.e., $(X' \to X) \geq (X'' \to X)$, if there is a morphism $X' \to X''$ over X. In this case we say that $(X' \to X)$ is greater than $(X'' \to X)$.

Definition 2.2. For a subcategory $\mathcal{F}$ of $\mathcal{A}(X)$, an object $f : Y \to X$ of $\mathcal{A}(X)$ is called a *common factor* of $\mathcal{F}$, if every object of $\mathcal{F}$ is greater than f.

A maximal object among the common factors of $\mathcal{F}$ is called a *greatest common factor of* $\mathcal{F}$. We sometimes say that Y is a common factor or a greatest common factor instead that $Y \to X$ is.

For an object $\tilde{X} \to X$ and a common factor $Y \to X$ of $\mathcal{F}$, there is a morphism $\tilde{X} \to Y$ over X by definition. We call this morphism a *factorizing morphism*. If $Y \to X$ is bimeromorphic, then a factorizing morphism is unique.

Theorem 2.3. *Assume that in a subcategory $\mathcal{F}$ of $\mathcal{A}(X)$ there is an object $X' \to X$ which is bimeromorphic.*

Then, for every common factor of $\mathcal{F}$ there is a greatest common factor which is greater. Moreover, a greatest common factor is unique.

If for every object $X' \to X$ of $\mathcal{F}$, X' is normal, then the greatest common factor of $\mathcal{F}$ is normal.

To prove the theorem, we need the following lemma and definition.

Lemma 2.4. *Let $\cdots E_{i+1} \xrightarrow{\varphi_{i+1}} E_i \xrightarrow{\varphi_i} E_{i-1} \cdots \to E_0$ be a sequence of proper surjective morphisms of reduced (not necessarily irreducible) analytic spaces. For each $i \geq 0$, let $\psi_i : E \to E_i$ be a proper surjective morphism of reduced analytic spaces. Assume the commutativity $\psi_i = \varphi_{i+1} \circ \psi_{i+1}$ for every $i \geq 0$.*

Then, for every compact subset $K \subset E_0$, there exists a number $r \geq 0$ such that φ_i restricted onto the inverse image of K is of finite fibers for all $i \geq r$.

Proof. Since we restrict onto a compact subset, we may assume that the number of irreducible components of E is finite. Then, we may assume that the number of irreducible components of E_i's is constant, say s, because they are bounded by that of E. Let E_i^j $(j = 1, .., s)$ be the irreducible components of E_i and $\varphi_{i+1}(E_{i+1}^j) = E_i^j$. Note that for every j there is an irreducible component E^j of E which is mapped onto E_i^j. Then, for each j, we may assume that $\dim E_i^j$'s are constant for all i, because they are bounded by $\dim E$. Now let us prove the lemma by induction on $n = \dim E$.

Assume $n = 1$. Then $\dim E_i^j = 0$ for all i or $\dim E_i^j = 1$ for all i. Both cases are trivial.

Assume $n \geq 2$ and the lemma holds true for the dimension less than n. Since it is sufficient to prove the finiteness of φ_i on each irreducible component E_i^j', we may assume that E_i's and E are all irreducible. Let $\dim E_i = k$ and $\Phi_i = \varphi_1 \cdots \varphi_{i-1} \circ \varphi_i$. Define $D_i \subset E_i$ as $\{x \in E_i \mid \dim_x \Phi_i^{-1} \circ \Phi_i(x) > 0\}$, then it is a proper closed subset of E_i. Next define $D \subset E$ as $\{y \in E \mid \dim_y \psi_0^{-1} \circ \psi_0(y) > n - k\}$, then it is also a proper closed subset in E and satisfies $D_i \subset \psi_i(D)$. This inclusion is proved as follows:

For any point $x \in D_i$, take an irreducible component D_{i0} of $\Phi_i^{-1} \circ \Phi_i(x)$ containing x, then $\dim D_{i0} > 0$. Let $D' \subset E$ be an irreducible closed subset dominating D_{i0} and $y \in D'$ a point corresponding to x. Then, according to a general theory, $\dim D' \geq \dim D_{i0} + n - k$. By $D' \subset \psi_0^{-1} \circ \psi_0(y)$, we obtain that $\dim_y \psi_0^{-1} \circ \psi_0(y) > n - k$, which yields $x \in \psi_i(D)$.

Now we obtain a sequence $\cdots \psi_i(D) \to \psi_{i-1}(D) \to \cdots \to \psi_0(D)$ of proper surjective morphisms of reduced schemes which are dominated by D of dimension $< n$. By induction hypothesis, $\psi_i(D) \to \psi_{i-1}(D)$ is of finite fibers for $i >> 0$. On the other hand $\varphi_i : E_i \to E_{i-1}$ is of finite fibers outside of D_i, therefore φ_i is of finite fibers on the whole of E_i for $i >> 0$. □

Definition 2.5. Let K be a compact subset of X. We say that a common factor $Y \to X$ is *maximal on* K, if for every common factor $Z \to X$ greater than Y we have $Z|_K = Y|_K$, where $|_K$ means the inverse image of K.

Proof of Theorem 2.3. First note that all common factors of $\mathcal{F}$ are bimeromorphic to X by the assumption of the theorem. Let $Y \to X$ be a common factor of $\mathcal{F}$. We will prove that for a compact subset K, there exists a common factor $Y' \to X$ greater than Y and maximal on K. If Y is not maximal on K, there is a common factor $Y_1 \to X$ greater than $Y = Y_0$ and not equal to $Y_0 \to X$ on K. Next, if Y_1 is not maximal on K, there is $Y_2 \to X$ as above. Successively we obtain a sequence:

$$\cdots \to Y_i \xrightarrow{\varphi_i} Y_{i-1} \to \cdots \xrightarrow{\varphi_1} Y_0$$

of common factors of $\mathcal{F}$. Let $X' \to X$ be an object of $\mathcal{F}$ and $\psi_i : X' \to Y_i$ the factorizing morphism for every i. Since Y_i's are bimeromorphic to X, the commutativity $\varphi_i \circ \psi_i = \psi_{i-1}$ follows for every i. Take the normalizations X'^* and Y_i^* of X' and Y_i, respectively. Then by the lemma, there is an integer $r = r(K)$ such that $Y_{i+1}^* \to Y_i^*$ (every $i > r$) is a proper bimeromorphic morphism between normal spaces, and its restriction onto the inverse images of K is of finite fibers, which is therefore an isomorphism by the Zariski Main Theorem (for analytic version, see [6, 4.9]). Hence, on K, every common factor in

$$\cdots \to Y_i \xrightarrow{\varphi_i} Y_{i-1} \to \cdots \xrightarrow{\varphi_{r+1}} Y_r$$

is dominated by the common normalization $\pi_i : Y^* \to Y_i$. Therefore, the coherent $\mathcal{O}_{Y_r}$-module $\pi_{r*}\mathcal{O}_{Y^*}$ has an ascending chain of submodules:

$$\mathcal{O}_{Y_r} \subset \rho_{r+1*}\mathcal{O}_{Y_{r+1}} \subset \ldots \subset \rho_{i*}\mathcal{O}_{Y_i} \subset \ldots,$$

where $\rho_i : Y_i \to Y_r$ is the composite $\varphi_r \circ \cdots \circ \varphi_i$. By the coherency of $\pi_{r*}\mathcal{O}_{Y^*}$ this chain should terminate at a finite stage. Thus, we obtain a common factor $Y' \to X$ greater than Y and maximal on K.

Since X is countable at infinity, $X = \bigcup_{i=1}^{\infty} K_i$ with K_i compact and $K_i \subset K_{i+1}$. Take a sequence

$$\cdots \to Y_i' \xrightarrow{\varphi_i'} Y_{i-1}' \to \cdots \xrightarrow{\varphi_1'} Y_0 = Y$$

of common factors such that $Y_i' \to X$ is maximal on K_i. Then,

$$\cdots \to Y_i' \xrightarrow{\varphi_i'} Y_{i-1}' \to \cdots \xrightarrow{\varphi_{n+1}'} Y_n'$$

are all isomorphic on K_n.

Now, as in [10, 4.1.7], we can define an analytic space $Y_\infty = \varprojlim_r Y_r'$, which is again a common factor of $\mathcal{F}$ and greater than all Y_i''s. This Y_∞ is a greatest common factor of $\mathcal{F}$. Indeed, if $Z \to X$ is a common factor greater than Y_∞, then for every K_i, $Z|_{K_i} = Y_i'|_{K_i} = Y_\infty|_{K_i}$ by the maximality of Y_i' on K_i. Hence $Z = Y_\infty$.

Let $Y \to X$ and $Y' \to X$ be greatest common factors. Then the unique component Z of $Y \times_X Y'$ dominating Y and Y' is again a common factor of $\mathcal{F}$ and greater than Y and Y'. By the maximality of Y and Y', Z coincides with Y and Y', which yields the uniqueness of the greatest common factor.

If every object of $\mathcal{F}$ is normal, the normalization of a common factor is also a common factor. So the greatest common factor is normal. □

Remark 2.6. Instead of Lemma 2.4, we can prove the theorem differently: Consider a sequence of common factors as in the proof of Theorem 2.3. Each Y_i is dominated by a common object $\tilde{X} \xrightarrow{f_i} Y_i$. Let $\mathcal{J}_{i,a}$ be the defining ideal of $f_i^{-1}f_i(a)$ in $\tilde{X}$. On a compact subset K on Y_0, consider the Hilbert-Samuel function $H_{i,a}$ of $\mathcal{O}_{\tilde{X},a}/\mathcal{J}_{i,a}$

for $a \in f_0^{-1}(K)$. One can easily prove that there exists $n(K)$ such that for any $n \geq n(K)$ and $a \in f_0^{-1}(K)$, the equality $H_{n,a} = H_{n(K),a}$ holds by means of Zariski upper-semicontinuity ([2, lemmas 7.1, 7.2], [5, Lemma 3.10 and Definition 3.11]) and the stabilization property of Hilbert-Samuel function ([3, Theorem 5.2.1]). This equality yields the equality of the ideals $\mathcal{J}_{n,a} = \mathcal{J}_{n(K),a}$, and therefore $Y_n \to Y_{n(K)}$ is finite on K. The rest of the proof follows in the same way as above.

Remark 2.7. Consider the case that X is an algebraic variety. Theorem 2.3 holds also for the category $\mathcal{A}lg(X)$ whose objects are proper surjective morphisms of algebraic varieties to X. The proof is easier. In Lemma 2.4, there is a number r so that φ_i $(i \geq r)$ is of finite fibers for the global E_i's. Therefore one can prove Theorem 2.3 immediately by showing the ascending chain condition, without using Y_∞.

Example 2.8. There are many subcategories of $\mathcal{A}(X)$ which satisfy the condition of the theorem.

$\mathcal{R}es(X)$: the category of resolutions of the singularities of X.

$\mathcal{S}m(X)$: the category of proper surjective morphisms $X' \to X$, where X' is non-singular. Here, we note that the dimension of X' may be bigger than the dimension of X.

$\mathcal{C}(X)$ (resp. $\mathcal{L}t(X)$, $\mathcal{T}(X)$) : the category of proper bimeromorphic morphisms $X' \to X$ with canonical singularities (resp. log-terminal singularities, terminal singularities) on X'.

These are of course nonempty categories due to the results of [9], [1], [4], [5] and [13].

We can also think of the following categories for a toric variety X (cf., 2.7):

$\mathcal{R}es(X)_{tor}$: the category of toric resolutions of the singularities of X.

$\mathcal{S}m(X)_{tor}$: the category of toric proper surjective morphisms $X' \to X$, where X' is non-singular.

We may expect that the greatest common factors of $\mathcal{S}m(X)$ and of $\mathcal{S}m(X)_{tor}$ are different even for the same X.

Example 2.9. For a two-dimenstional analytic space X, the greatest common factor of $\mathcal{R}es(X)$ is the minimal resolution.

Definition 2.10. The greatest common factor of $\mathcal{S}m(X)$ is called the *geometric minimal model* of X.

Definition 2.11. For an algebraic variety X, the greatest common factor of $\mathcal{S}m(X)$ in the category $\mathcal{A}lg(X)$ is called the *algebraic geometric minimal model* which is not studied in the following sections (cf. 2.7).

3. The geometric minimal model and rational curves

In this section we study some properties of the geometric minimal model of an analytic space X.

First, we state a basic lemma, which is well known and used often in this paper.

Lemma 3.1. *Let $f : Z \to X$ be a proper surjective morpphism and $g : Z \to Y$ a morphism. Assume X is normal and g is constant on $f^{-1}(x)$ for every $x \in X$.*

Then, there is a morphism $\varphi : X \to Y$ such that $g = \varphi \circ f$.

Proof. Let $\Gamma \subset X \times Y$ be the reduced image of the proper morphism $(f, g) : Z \to X \times Y$ and $\Phi : \Gamma \to X$ the restriction of the projection $X \times Y \to X$ onto Γ. Then, $\Phi^{-1}(x)$ consists of one point. Now we obtain that $\Phi : \Gamma \to X$ is a bijective proper bimeromorphic morphism. Since X is normal, Φ is an isomorphism by Zariski's Main Theorem. □

The following is a basic property of the geometric minimal model.

Theorem 3.2. *Let X be an analytic space of dimension ≥ 1, $\varphi : Y \to X$ a common factor of $\mathcal{S}m(X)$ and $f : X' \to X$ an arbitrary object of $\mathcal{S}m(X)$.*

Then every rational curve $l \subset X'$ which is mapped to a point on X is mapped to a point on Y by the factorizing morphism $X' \to Y$.

Proof. It is sufficient to construct a commutative diagram with f, γ and $f \circ \alpha = \gamma \circ \beta$ being in $\mathcal{S}m(X)$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\beta} & X'' \\ \alpha \downarrow & & \gamma \downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

such that there is a curve $\tilde{l}$ on $\tilde{X}$ with $\alpha(\tilde{l}) = l$ and $\beta(\tilde{l}) = p$ a point. This is because γ and f factor through $\varphi : Y \to X$, and the image of l on Y coincides with the image of $\tilde{l}$ and therefore with the image of p.

Now we construct $\tilde{X}$ and $\tilde{l}$. First we may assume that $\dim X' = n \geq 3$. In fact, if $\dim X' = 1$, then there is no such rational curve. If $\dim X' = 2$, take the product $X' \times \mathbb{P}^1$. Then, the composite $X' \times \mathbb{P}^1 \xrightarrow{p_1} X' \to X$ is also an object of $\mathcal{S}m(X)$, where p_1 is the projection to the first factor. Take a section l' of p_1 over l and replace X', l with $X' \times \mathbb{P}^1$, l'. Now $\dim X' \geq 3$; then there is a proper bimeromorphic morphism $\alpha_1 : X_1 \to X'$ with a non-singular rational curve $l_1 \subset X_1$ corresponding onto l such that the normal sheaf $N_{l_1/X_1} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by [8]. The following construction is also found in [8]: take the blow-up $\alpha_2 : \tilde{X} \to X_1$ with the center l_1 and define $\alpha := \alpha_1 \circ \alpha_2$.

Then the exceptional divisor E of α_2 is $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ and its normal sheaf is $p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^{n-2}}(-1)$, where p_i is the projection to the i-th factor and n is the dimension of X'. Then, there is a proper bimeromorphic morphism $\beta : \tilde{X} \to X''$ whose restriction onto E is the projection $E \xrightarrow{p_2} \mathbb{P}^{n-2}$. By the normal bundle of E, we can see that X'' is non-singular. Since $f \circ \alpha$ is constant on $\beta^{-1}(x)$ for every $x \in X''$, we obtain a morphism $\gamma : X'' \to X$ with $\gamma \circ \beta = f \circ \alpha$ by Lemma 3.1. Now, take a section $\tilde{l} = \mathbb{P}^1 \times \{P\} \subset E$ over l_1, where P is a point of $\mathbb{P}^{n-2}$. Then $\beta(\tilde{l})$ is a point, which shows that $\tilde{l}$ is a curve we wanted. □

Theorem 3.3. *Let X be an analytic space of dimension ≥ 3, $\varphi' : Y' \to X$ a common factor of $\mathcal{R}es(X)$ and $f : X' \to X$ an arbitrary object of $\mathcal{R}es(X)$.*

Then every rational curve $l \subset X'$ which is mapped to a point on X is mapped to a point on Y' by the factorizing morphism $X' \to Y'$.

Proof. Since all objects in $\mathcal{R}es(X)$ are of dimension ≥ 3, we can construct a commutative diagram in $\mathcal{R}es(X)$ in the same way as in the proof of the previous theorem. □

Theorem 3.4. *Let $Y \to X$ be a bimeromorphic object of $\mathcal{A}(X)$ satisfying the following condition (R):*

(R) There exists a proper bimeromorphic morphism $\tilde{Y} \to Y$ such that every rational curve $l \subset \tilde{Y}$ which is mapped to a point in X is mapped to a point in Y.

Then $Y \to X$ is a common factor of $\mathcal{S}m(X)$.

Proof. Let $X' \to X$ be an arbitrary object of $\mathcal{S}m(X)$. Then, $X' \times_X \tilde{Y} \to X'$ is bimeromorphic on a component Z of $X' \times_X \tilde{Y}$. Then, by [9, Main Theorem II'] and [10, Corollary 1], for every point $x \in X'$ there are a relatively compact neighbourhood U_r, $(r \geq 1)$ of x, a non-singular analytic space U_1, bimeromorphic morphisms $f : U_1 \to U_r$ and $U_1 \to Z|_{U_r}$ such that f is the composite $f_{r-1} \circ f_{r-2} \circ \cdots \circ f_1$ of blow-ups $f_i : U_i \to U_{i+1}$ with non-singular centers and the following diagram is commutative:

$$\begin{array}{ccccccccc}
U_1 & \xrightarrow{f_1} & U_2 & \to & \dots & \xrightarrow{f_{r-1}} & U_r & \subset & X' \\
\downarrow & & & & & & & & \\
Z & & & & & & & & \downarrow \\
\downarrow & & & & & & & & \\
\tilde{Y} & \to & Y & & & \to & & & X.
\end{array}$$

Therefore, $f_1^{-1}(y) = \mathbb{P}^s$ $(s \geq 0)$ for every $y \in U_2$.

For a rational curve $l \subset f_1^{-1}(y) \subset U_1$, if the image of l on $\tilde{Y}$ is not a point then it is a rational curve. So l is mapped to a point in Y by the condition (R), which yields the existence of a morphism $U_2 \to Y$ by Lemma 3.1. In the same way we induce the morphisms $U_i \to Y$ for $i = 2, .., r$. Now we complete the proof of the statement that $X' \to X$ factors through $Y \to X$. □

Corollary 3.5. *Let X be an analytic space of dimension $\neq 2$.*

Then, the greatest common factor of $\mathcal{R}es(X)$ coincides with the geormetric minimal model.

Proof. Since $\mathcal{R}es(X) \subset \mathcal{S}m(X)$, it follows that the greatest common factor $Y' \to X$ of $\mathcal{R}es(X)$ is greater than the geometric minimal model $Y \to X$. For $\dim X = 1$, Y and Y' are normal and bimeromorphic to X, therefore they coincide.

For $\dim X \geq 3$, it is sufficient to prove that $Y' \to X$ is a common factor of $\mathcal{S}m(X)$. This follows from Theorem 3.3 and Theorem 3.4. □

Theorem 3.6. *The geometric minimal model of an analytic space X is the maximal bimeromorphic object among the objects satisfying (R) in $\mathcal{A}(X)$.*

Proof. This is clear by Theorem 3.4, because the geometric minimal model also satisfies (R) by Theorem 3.2. □

Corollary 3.7. *If $Y \to X$ is the geometric minimal model, then for every non-trivial bimeromorphic morphism $Z \to Y$, there is a rational curve on Z which is mapped to a point on Y.*

Proof. If $Y \to X$ is the geometric minimal model, then Z is not a common factor of $\mathcal{S}m(X)$ by the maximality of Y. Therefore, by Theorem 3.4, for a resolution $\tilde{Z} \to Z$ of singularities of Z there is a rational curve $l \subset \tilde{Z}$ which is mapped to a point on X and to a curve l' on Z. This curve l' is a required rational curve on Z. □

Now, by the above theorem, the following is obvious.

Corollary 3.8. *If X is an analytic space of dimension two, then the geometric minimal model of X is a surface obtained by contracting all rational exceptional curves of the minimal resolution.*

Example 3.9. Let X be an affine cone over a non-singular variety A, which means that there is a line bundle Y over A with the zero section of negative normal bundle and the zero section is contracted to a point of X in Y. We will denote the zero section of Y also by A.

(1) If A is an Abelian variety or the direct product of non-singular curves of positive genera. Then Y is a resolution of the singularities on

X and also the geometric minimal model of X, since A has no rational curve (see Theorem 3.4).

(2) If A is a K3-surface, then there is a rational curve with the positive self-intersection number. Then, the whole of A should be contracted to a point in the geometric minimal model; therefore the geometric minimal model of X is the identity.

(3) Let A be a ruled surface with $q \geq 1$ and $p : A \to C$ the ruling, then there is a contraction $Y \to Y'$ of the zero section A to a curve C in Y by [7, Theorem 1]. Then Y' is the geometric minimal model of X.

Example 3.10. Let $X \subset \mathbb{C}^4$ be a hypersurface defined by a homogeneous polynomial f such that the hyperplane in $\mathbb{P}^3$ defined by f is non-singular and hyperbolic. Then, the blow-up $\varphi : Y \to X$ of X at the origin is the geometric minimal model of X, because there is no rational curve in a fiber of φ and Y is non-singular.

Example 3.11. Let X be a toric variety, then the geometric minimal model is the identity. Indeed, for every toric resolution $f : X' \to X$, each fiber $f^{-1}(x)$ of f is a connected union of rational varieties, so it should be mapped to a point on the geometric minimal model.

4. Miscellaneous properties on the geometric minimal models

Proposition 4.1. *(Functoriality) Let X and X' be analytic spaces and $\varphi : Y_X \to X$ and $\varphi' : Y_{X'} \to X'$ the geometric minimal models of X and X', respectively.*

Then a proper surjective morphism $f : X \to X'$ is uniquely lifted to $\tilde{f} : Y_X \to Y_{X'}$ such that $\varphi' \circ \tilde{f} = f \circ \varphi$.

Proof. The composition with f gives an injection $\mathcal{S}m(X) \to \mathcal{S}m(X')$. Therefore the greatest common factor of the image of $\mathcal{S}m(X)$ is greater than that of $\mathcal{S}m(X')$. □

Example 4.2. Let $\pi : X \to S$ be a proper surjective morphism to a surface S. Let S' be a surface obtained by contracting all rational exceptional curves of the minimal resolution of S. If the geometric minimal model of X is the identity (for example, X is toric), then π factors through S'.

Corollary 4.3. *Let X be an analytic space and $\varphi : Y \to X$ the geometric minimal model of X.*

Then every automorphism $\sigma : X \to X$ is lifted to $\sigma' : Y \to Y$ such that $\varphi \circ \sigma' = \sigma \circ \varphi$.

Proof. By Proposition 4.1, σ is lifted to $\sigma' : Y \to Y$ and σ^{-1} is lifted to $\sigma'' : Y \to Y$. By the uniqueness of the lifting, σ' and σ'' are the inverse morphism of each other. □

By this corollary, we obtain that every action of a finite group on X is lifted to the geometric minimal model. What follows is a discussion of the quotient by a finite group.

Proposition 4.4. *Let X be an analytic space and G a finite group acting on X. Let $\varphi : Y_X \to X$ and $\varphi' : Y_{X/G} \to X/G$ be the geometric minimal models of X and X/G, respectively.*

Then there is a morphism $Y_X/G \to Y_{X/G}$ over X/G.

Proof. By the functoriality of the gemometric minimal models (cf. Proposition 4.1), there is a morphism $Y_X \to Y_{X/G}$. By the uniqueness of this morphism, it is a G-equivariant morphism with the trivial action on $Y_{X/G}$. Hence the morphism factors through Y_X/G. □

The following is obvious from the above proposition.

Corollary 4.5. *Let X be an analytic space and G a finite group acting on X. If the geometric minimal model of X is the identity, then the geometric minimal model of X/G is also the identity.*

Example 4.6. Let $X = \mathbb{C}^n/G$ by a finite group G, then the geometric minimal model of X is the identity.

Example 4.7. The equality $Y_X/G \simeq Y_{X/G}$ does not hold in general. Indeed, take a line bundle $Y' = \mathbf{V}(\mathcal{O}_{\mathbb{P}^1}(m))$ over $\mathbb{P}^1$, where m is a positive integer. Let $\pi : C \to \mathbb{P}^1$ be a double cover from an elliptic curve C and $G = \mathbb{Z}/(2)$ the Galois group. Then G acts on $Y = \mathbf{V}(\pi^*\mathcal{O}_{\mathbb{P}^1}(m))$ and also on X, which is a variety obtained by contracting the zero section of Y. The quotient X/G has a resolution Y', therefore it is the quotient of $\mathbb{C}^2$ by a finite group of order m. Hence the geometric minimal model of X/G is the identity, but the geometric minimal model of X is Y, and $Y/G \simeq Y'$.

In order to prove the direct product property of the geometric minimal model, we state the following lemma.

Lemma 4.8. *Let X and X' be analytic spaces, $\mathcal{F}$ a subcategory of $\mathcal{A}(X)$ satisfying the condition of Theorem 2.3. Let Y_X be the greatest common factor of $\mathcal{F}$ and $f : Z \to X'$ an object of $\mathcal{A}(X')$. Let $\mathcal{F} \times Z$ be the category of the morphisms $\tilde{X} \times Z \to X \times X'$ with $\tilde{X} \in \mathcal{F}$.*

Then $Y_X \times Z$ is the greatest common factor of $\mathcal{F} \times Z$.

Proof. It is clear that $Y_X \times Z$ is a common factor of $\mathcal{F} \times Z$. By Theorem 2.3, there is the greatest common factor Y' of $\mathcal{F} \times Z$. So for every object $\tilde{X} \to X$ of $\mathcal{F}$, there is a factorizing morphism $\varphi : \tilde{X} \times Z \to Y'$ of $\tilde{X} \times Z \to X \times X'$. On the other hand, there is a morphism $\psi : Y' \to Y_X \times Z$ by the maximality of Y'. Therefore for every point $z \in Z$, $\tilde{X} \times \{z\} \to X \times \{f(z)\}$ is factored as follows:

$$\tilde{X} \times \{z\} \xrightarrow{\varphi} \psi^{-1}(Y_X \times \{z\}) \xrightarrow{\psi} Y_X \times \{z\} \to X \times \{f(z)\}.$$

Then, by the maximality of Y_X, $\psi^{-1}(Y_X \times \{z\}) \xrightarrow{\psi} Y_X \times \{z\}$ is an isomorphism. Now we obtain that $\psi : Y' \to Y_X \times Z$ is bijective and $\psi^{-1}((y, z)) \simeq \{(y, z)\}$ for every point $(y, z) \in Y_X \times Z$, which yields that ψ is an isomorphism. □

Proposition 4.9. *(Direct product) Let X and X' be analytic spaces, Y_X, $Y_{X'}$ and $Y_{X \times X'}$ the geometric minimal models of X, X' and $X \times X'$, respectively.*

Then $Y_X \times Y_{X'} \simeq Y_{X \times X'}$ over $X \times X'$.

Proof. First for any objects $\tilde{X} \to X$ and $\tilde{X}' \to X'$ of $\mathcal{S}m(X)$ and $\mathcal{S}m(X')$, respectively, $\tilde{X} \times \tilde{X}' \to X \times X'$ factors through $Y_{X \times X'}$. For every $\tilde{X}'$ in $\mathcal{S}m(X')$, $Y_X \times \tilde{X}'$ is the greatest common factor of $\mathcal{S}m(X) \times \tilde{X}'$ by Lemma 4.8. So there is a morphism $Y_X \times \tilde{X}' \to Y_{X \times X'}$. Then by Lemma 4.8 again, $Y_X \times Y_{X'}$ is the maximal common factor of $Y_X \times \mathcal{S}m(X')$. So there is a morphism $Y_X \times Y_{X'} \to Y_{X \times X'}$.

In order to prove the existence of the inverse morphism $Y_{X \times X'} \to Y_X \times Y_{X'}$, it is sufficient to prove that $Y_X \times Y_{X'}$ is a common factor of $\mathcal{S}m(X \times X')$. Let $\tilde{X}$ and $\tilde{X}'$ be resolutions of Y_X and $Y_{X'}$, respectively, and l a rational curve on $\tilde{X} \times \tilde{X}'$ which is mapped to a point on $X \times X'$. By Theorem 3.4, it is sufficient to prove that l is mapped to a point on $Y_X \times Y_{X'}$. If the image of l by the projection $\tilde{X} \times \tilde{X}' \xrightarrow{p_1} \tilde{X}$ is not a point, then it is a rational curve, and therefore is mapped to a point on Y_X. In the same way, l is mapped to a point on $Y_{X'}$ through the projection $\tilde{X} \times \tilde{X}' \xrightarrow{p_2} \tilde{X}'$. Hence l is mapped to a point on $Y_X \times Y_{X'}$. □

Next we consider an action of a complex Lie group.

Proposition 4.10. *Let G be a complex Lie group acting on an analytic space X.*

Then the action is lifted to the geometric minimal model of X.

Proof. Let $\sigma : G \times X \to X$ be the morphism defining the action, $p_1 : G \times X \to G$ the projection to the first factor and Y the geometric minimal model of X. Since the geometric minimal model of $G \times X$ is $G \times Y$ by Proposition 4.9, the automorphism $(p_1, \sigma) : G \times X \to G \times X$ is lifted to $\Phi : G \times Y \tilde{\to} G \times Y$. The composite $p_2 \circ \Phi : G \times Y \to Y$ defines the lifted action of G on Y, where $p_2 : G \times Y \to Y$ is the projection to the second factor. □

4.11. It is natural to ask if the geometric minimal model is a local property, i.e., if $\varphi : Y \to X$ is the geometric minimal model of X and $U \subset X$ is open, then $\varphi^{-1}(U) \to U$ is the geometric minimal model of U. The following example shows that the geometric minimal model is not local.

Example 4.12. Here we construct $U \subset X$ such that the geometric minimal model of X is the identity but that of U is not.

Let S be a compact rational surface with only one simple elliptic singularity p on it. Let $V = S \times \mathbb{P}^1$ and $C \not\ni p$ be a non-singular very ample divisor on $S_0 = S \times \{0\}$. Take the blow-up $X' \to V$ with the center C, then the proper transform S_0' of S_0 on X' has the negative normal bundle on X'. Hence, there is a morphism $f : X' \to X$ which contracts S_0' to a point Q of X. Since X' has singularities (simple elliptic singularities)$\times\mathbb{P}^1$, there is a resolution $g : \tilde{X} \to X'$ whose exceptional divisor E is isomorphic to (elliptic curve)$\times\mathbb{P}^1$. Then, the composite $f \circ g : \tilde{X} \to X$ factors through the geometric minimal model $Y \to X$ of X. The proper transform S_0'' of S_0' on $\tilde{X}$ is contracted to a point on Y, because S_0'' is rational. As $S_0'' \cap E$ has the trivial normal bundle in E, the other elliptic fibers are all contracted to points on Y, therefore Y must be isomorphic to X.

On the other hand, for $U = X \setminus \{Q\}$, the restriction $(f \circ g)^{-1}(U) \to U$ is a resolution of singularities on U and has no rational curves in each fiber of $f \circ g$, therefore it is the geometric minimal model.

Proposition 4.13. *Let X be a three-dimensional analytic space with at worst canonical singularities.*

Then the geometric minimal model of X is the identity.

Proof. By [12], there is a resolution $f : \tilde{X} \to X$ of the singularities of X whose exceptional divisors are all ruled surfaces. Then, the geometric minimal model $\varphi : Y \to X$ of X is either an isomorphism or a contraction of the union C of finite curves. Assume that φ is not an isomorphism and let $g : \tilde{X} \to Y$ be the factorizing morphism. Then, all components of C are rational, otherwise $R^1\varphi_*\mathcal{O}_Y \neq 0$, which contradicts the edge sequence of the Leray spectral sequence:

$$E_2^{p,q} = R^p\varphi_* R^q g_* \mathcal{O}_{\tilde{X}} \Longrightarrow E^{p+q} = R^{p+q} f_* \mathcal{O}_{\tilde{X}}.$$

If the curve C is not contained in the singular locus, then we can construct a resolution $\tilde{X}' \to X$ with a rational curve l on $\tilde{X}'$ mapped to a point on X and mapped to a curve on Y, a contradiction. Thus, C is contained in the singular locus of Y. Note that Y is again of index one and has rational singularities. So general points of C are cDV singularities in Y. For an irreducible component C_i of C, take the blow-up $h : Y' \to Y$ with the center C_i.

Then, in case Y has compound A_1, D_n $(n \geq 4)$, E_6, E_7, E_8 singularities on general points of C_i, h has an irreducible exceptional divisor E and $h^{-1}(p)$ is an irreducible rational curve for a general $p \in C_i$. Then, E is rational and is not contained in the singular locus of Y'. Therefore, by Theorem 3.2, E must be mapped to a point on Y, a contradiction.

In case Y has compound A_n $(n \geq 2)$ singularities at general points of C_i, then $h^{-1}(p)$ is the union of two $\mathbb{P}^1$ with one intersection point. This implies that an exceptional divisor E of h has a generic section on C_i, therefore again E is rational and is not contained in the singular locus of Y', a contradiction in the same way as above. □

Corollary 4.14. *Let X be a sufficiently small neighbourhood of a log-terminal singularity of dimension three.*

Then the geometric minimal model of X is the identity.

Proof. We may assume that X is a cyclic quotient of an analytic space with at worst canonical singularities. Then, by Corollary 4.5 and Proposition 4.13 we obtain the statement. □

Corollary 4.15. *Let X be an analytic space of dimension three, $Y' \to X$ the relative canonical model in Mori's sense and $Y \to X$ the geometric minimal model.*

Then there is a morphism $\varphi : Y' \to Y$ over X. Therefore, there is a morphism $Y'' \to Y$ over X, where Y'' is a relative minimal model in Mori's sense.

Proof. By 4.1, the geometric minimal model of Y' is greater than Y over X. Here, by Proposition 4.13, the geometric minimal model of Y' is the identity.

□

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