The maximum modulus of estimates for doubly nonlinear parabolic equations of the type of fast diffusion

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THE MAXIMUM MODULUS ESTIMATES FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS OF THE TYPE OF FAST DIFFUSION

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1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, $Q_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$, Γ_T is a parabolic boundary of Q_T, i.e., $\Gamma_T = S_T \cup [\bar{\Omega} \times (t=0)]$. Consider in Q_T equation

$$F[u] \doteqdot \partial u/\partial t - \operatorname{div} a(u, \nabla u) = f(x,t) \tag{1.1}$$

where $a = (a^1, \dots, a^n)$, $\nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$. Assume that $a^i(u,p)$, $i = 1, \dots, n$, are continuous on $\mathbb{R} \times \mathbb{R}^n$ and let for a.e. $(x,t) \in Q_T$ and any $u \in \mathbb{R}$, $p \in \mathbb{R}^n$

$$\begin{aligned}
&a(u,p) \cdot p \geq \nu_0 |u|^l |p|^m - \mu_0(|u|^\delta + 1),\ \nu_0 > 0,\ \mu_0 \geq 0,\\
&\delta \in (2, m+l) \quad \text{if} \quad m+l > 2,\ \delta = 2 \quad \text{if} \quad m+l \leq 2;\\
&|a(u,p)| \leq \mu_1 |u|^l |p|^{m-1} + \mu(|u|),\ \mu_1 \geq 0,\ \mu(s) \geq 0,\ \mu(s) \quad \text{is nondecreasing;}\\
&0 \leq f(x,t) \leq \mu_2,\ \mu_2 \geq 0;\ m > 1,\ l \geq 0.
\end{aligned} \tag{1.2}$$

Equations of the type (1.1), (1.2) are known as doubly nonlinear parabolic equations (DNPE) (see [1]-[5]). The aim of this paper is to obtain the maximum modulus estimates for generalized solutions of DNPE with the best possible condition

$$(m,l) \in D \backslash \omega,\ D \doteqdot \{m > 1, l \geq 0\},\ \omega \doteqdot \left\{(m,l) \in D : \frac{\sigma+1}{\sigma+2} \leq \frac{1}{m} - \frac{1}{n},\ \sigma = \frac{l}{m-1}\right\}. \tag{1.3}$$

Other results concerning L_∞ -estimates for DNPE were obtained in particular in [6], [7].

For the sake of breavity we limit ourselves by obtaining only the global estimates of the maximum modulus of generalized solutions. On the other hand from the point of view of the theory of existence of regular solutions for DNPE it is important to have such estimates for solutions of regularized Cauchy-Dirichlet problems. Taking into account this circumstance we obtain in this paper the global L_∞ -estimates for

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generalized solutions of the regularized Cauchy-Dirichlet problems of the type (see [6], [8])

$$(1.4)\quad F_{\epsilon,N}[u] \doteqdot \partial u/\partial t - \operatorname{div} a(\chi(u), \nabla u) = f(x,t) \quad \text{in} \quad Q_T,\ u = \psi + \epsilon \quad \text{on} \quad \Gamma_T$$

where

$$\chi(u) = \max(\epsilon, \min(u, N)),\ \epsilon > 0,\ N > \epsilon,$$

$$(1.5)\qquad \psi \in \overset{\circ}{W}{}^1_m(Q_T) \cap C^\infty(Q_T),\ \psi \geq 0 \quad \text{in} \quad Q_T,$$

and $a(u,p)$ and $f(x,t)$ are like as in condition (1.2).

Definition 1.1. Function u is a generalized solution of (1.4), (1.5) if $u \in C([0,T]; L_2(\Omega)) \cap W^{1,0}_m(Q_T)$, $u = \psi + \epsilon$ on Γ_T and for any $t \in (0,T]$ and $\phi \in \overset{\circ}{W}{}^1_m(Q_T)$

$$(1.6)\qquad \int\limits_\Omega u\phi dx|_0^t + \iint\limits_{Q_T} [-u\phi_t + a(\chi(u), \nabla u) \cdot \nabla\phi - f\phi] dxdt = 0.$$

It should be said at once that conditions (1.2) imply the following estimate

Lemma 1.1. *([8]) For any generalized solution of (1.4), (1.5) we have*

$$(1.7)\qquad inf(u, Q_T) \geq \epsilon.$$

We say that some constant c depends only on the data if c depends on $n, m, l, \sigma, \nu_0, \mu_0, \mu_1, \mu_2$, and $\sup(\psi, \bar{Q}_T)$. The main result of this paper is

Theorem 1.1. *Let conditions (1.2), (1.3) be fulfilled. Let u be a generalized solution of Cauchy-Dirichlet problem (1.4), (1.5) with any $\epsilon > 0$, $N > \epsilon$. Then*

$$(1.8)\qquad sup(\chi(u), Q_T) \leq c_1$$

where constant c_1 depends only on the data.

Remark 1.1. From (1.7), (1.8) it follows that $\chi(u) = u$ a.e. in Q_T if $N \geq c_1$. Hence we have

Corollary 1.1. *Let conditions (1.2), (1.3) be fulfilled and let u be a generalized solution of Cauchy-Dirichlet problem (1.4), (1.5) with any $\epsilon > 0$ and $N \geq c_1$ where constant c_1 is defined by Theorem 1.1. Then*

$$(1.9)\qquad sup(u, Q_T) \leq c_1.$$

Using Theorem 1.1 and our results on Hölder estimates for DNPE (see [9]-[11]) we can derive existence of Hölder continuous weak solutions of Cauchy-Dirichlet problem for appropriate class of equations (1.1)-(1.3).

Similar results can be obtained in the same way for more general DNPE of the type

$$\partial u/\partial t - \operatorname{div} a(x,t,u,\nabla u) = a_0(x,t,u,\nabla u) \tag{1.10}$$

satisfying conditions

$$\begin{aligned} &a(x,t,u,p)\cdot p \geq \nu_0|u|^l|p|^m - \mu_0(|u|^\delta+1),\\ &|a(x,t,u,p)| \leq \mu_1|u|^l|p|^{m-1} + \mu(|u|),\\ &|a_0(x,t,u,p)| \leq \mu_2[(|u|^l|p|^{m-1})^\theta + |u|^{\delta-1}+1],\ 0<\theta<1, \end{aligned} \tag{1.11}$$

with the same $m, l, \delta, \nu_0, \mu_0, \mu_1, \mu_2$ and $\mu(|u|)$ as in (1.2), (1.3).

At the end of this paper we give some counterexample which shows that in the case (see (1.3))

$$(m,l)\in\omega \tag{1.12}$$

generalized subsolutions of the model equation

$$\partial u/\partial t - \operatorname{div}[|u|^l|\nabla u|^{m-2}\nabla u] = 0 \quad \text{in} \quad B_1(0)\times[0,1] \tag{1.13}$$

can be unbounded as $x\to 0$. From here it follows that it is impossible to obtain at least local L_∞ -estimates for generalized solutions of (1.13) in the case (1.12). In this sense condition (1.3) is sharp.

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2. Some auxiliary propositions

Proposition 2.1. *Let function $g(u)$ satisfies a Lipschitz condition uniformly on $\mathbb{R}$ and its derivative $g'(u)$ be continuous everywhere on $\mathbb{R}$ with the possible exception of finitely many points at which $g'(u)$ has a discontinuity of the first order. Let function $u\in C([0,T];L_2(\Omega))\cup W_m^{1,0}(Q_T)$ satisfies for all $t\in(0,T]$ and any $\phi\in \overset{\circ}{W}{}^1_m(Q_T)$ the integral identity*

$$\int_\Omega u\phi dx\Big|_0^t + \iint_{Q_T}[-u\phi_t + f_i\phi_{x_i} + f_0\phi]dxdt = 0 \tag{2.1}$$

where $f_i \in L_{m'}(Q_T)$, $i = 0,1,\dots,n$, $1/m + 1/m' = 1$, $m > 1$. *Assume that* $u = \varphi$ *on* S_T *for some* $\varphi \in W^1_m(Q_T)$. *Then for any* $t \in (0,T]$ *we have*

$$\int_\Omega [G(u) - ug(\varphi)]dx|_0^t + \tag{2.2}$$
$$+ \iint_{Q_T} [ug'(\varphi)\varphi_t + f_i(g'(u)u_{x_i} - g'(\varphi)\varphi_{x_i}) + f_0(g(u) - g(\varphi))]dxdt = 0$$

where $G(u) = \int_0^u g(\xi)d\xi$.

Proposition 2.1 is well-known (see, for example, [6]). We shall use the following inequality (the close inequality was proved in [6]).

Proposition 2.2. *Let function* u *satisfy conditions*

$$u \in C([0,T]; L_q(\Omega)),\ \nabla(|u|^s u) \in L_m(Q_T),\ s = (q - 2 + \frac{\sigma}{\sigma+1})/m, \tag{2.3}$$
$$q \geq \frac{\sigma+2}{\sigma+1},\ \sigma = \frac{l}{m-1},\ m \geq 1,\ l \geq 0.$$

Assume that $u = 0$ *on* S_T. *Then*

$$\iint_{Q_T} |u|^\beta dxdt \leq c \iint_{Q_T} |\nabla(|u|^s u)|^m dxdt \left(\sup_{t\in[0,t]} \int_\Omega |u|^q dx\right)^{\frac{m}{n}} \tag{2.4}$$

where $\beta = (1 + \frac{m}{n})q + m - 2 + \frac{\sigma}{\sigma+1}$ *and constant* c *depends only on* n, m, *and* l *(in particular* c *independent of* q*).*

Proof. For any $v \in \overset{\circ}{W}{}^1_m(\Omega) \cap L_r(\Omega)$, $m \geq 1$, $r > 0$, we have (see, for example, [12])

$$\left(\int_\Omega |v|^{\hat\beta} dx\right)^{1/\hat\beta} \leq c^\lambda \left(\int_\Omega |\nabla v|^m dx\right)^{\lambda/m} \left(\int_\Omega |v|^r dx\right)^{(1-\lambda)/r} \tag{2.5}$$

where $\lambda = m/\hat\beta$, $\hat\beta = (n+r)m/n$, c depends only on n, m, and r. Set

$$\hat\beta = \beta/(s+1),\ r = q/(s+1),\ v = |u|^s u$$

where q and s are such like in (2.3) (in particular $r = q/(s+1)$ may be estimated from above by some constant depending only on n, m, and l). Then for a.e. $t \in [0,T]$ we derive from (2.5) inequality

$$\int_\Omega |u|^\beta dx \leq c^m \int_\Omega |\nabla(|u|^s u)|^m dx \left(\int_\Omega |u|^q dx\right)^{\frac{m}{n}}. \tag{2.6}$$

Integrating (2.6) with respect to t we obtain (2.4). Proposition 2.2 is proved.

3. The proof of Theorem 1.1

In the case $m+l \geq 2$ estimate (1.8) can be derived directly from [6]. The novelty of Theorem 1.1 is concerned with the case of equations of the type of fast diffusion, i.e., with the case $m+l<2$. Therefore we prove Theorem 1.1 here assuming that $m+l<2$. Our proof in this case is appropriate development of the Moser method of obtaining L_∞ -estimates.

Apply Proposition 2.1 in the case

$$g(\xi) = \tilde{\xi}^{(\sigma+1)(p-1)},\ \tilde{\xi} \doteqdot \sup(0, \chi(\xi)-\lambda),\ \epsilon \leq \lambda \leq N,\ p \geq 2,\ \varphi = \epsilon. \tag{3.1}$$

Then

$$\int_\Omega G(u)dx|_0^t + \iint_{Q_T} (\sigma+1)(p-1)\tilde{u}^{(\sigma+1)(p-1)-1}[\delta \nabla u \cdot \nabla \tilde{u} + a(\chi(u), u), \nabla u)\cdot \nabla \tilde{u}]dxdt$$

$$= \iint_{Q_T} f\tilde{u}^{(\sigma+1)(p-1)}dxdt \tag{3.2}$$

where $G(u) = \int_0^u \tilde{\xi}^{(\sigma+1)(p-1)}d\xi$. In particular we have taken into account that $\tilde{\varphi} = \sup(0, \epsilon-\lambda) = 0$, $g = g(\tilde{\varphi}) = 0$ for $\varphi = \epsilon$. Using that

$$\frac{d\tilde{u}}{du} = \operatorname{sign}[\lambda \leq u \leq N]$$

where sign A denotes the characteristic function of set A we have

$$\nabla \tilde{u} = \operatorname{sign}[\lambda \leq u \leq N]\nabla u,\ |\nabla u| \geq |\nabla \tilde{u}|,\ G(u) \geq \frac{\tilde{u}^{(\sigma+1)(p-1)+1}}{(\sigma+1)(p-1)+1}. \tag{3.3}$$

Denote $v = \tilde{u}^{\sigma+1}$. Then

$$|\nabla v|^m \leq \tilde{u}^{l+\sigma}|\nabla \tilde{u}|^m,\ v^{-\frac{\sigma}{\sigma+1}}|\nabla v|^m \leq \tilde{u}^l|\nabla \tilde{u}|^m. \tag{3.4}$$

Using also the trivial inequalities

$$\chi(u) \geq \tilde{u},\ (\chi(u))^2 \leq 2(\tilde{u}^2+\lambda^2) \tag{3.5}$$

we derive from (1.2) and (3.2)-(3.4) that

$$\int_\Omega v^{p-1+\frac{1}{\sigma+1}}dx|^t + \iint_{Q_t} v^{p-2}|\nabla v|^m dxdt \leq \int_\Omega G(u)dx|^{t=0}+$$

$$+cp\iint_{Q_t}(v^{p-1+\frac{1}{\sigma+1}} + v^{p-1-\frac{1}{\sigma+1}} + v^{p-1})dxdt. \tag{3.6}$$

Denote

$$(3.7) \qquad p = q + \frac{\sigma}{\sigma+1},\ q \geq \frac{\sigma+2}{\sigma+1}.$$

Then

$$p-1+\frac{1}{\sigma+1} = q,\ p-2 = q-2+\frac{\sigma}{\sigma+1},\ p-1-\frac{1}{\sigma+1} = q-\frac{2}{\sigma+1},\ p-1 = q-\frac{1}{\sigma+1}.$$

Obviously that for any $q \geq \frac{\sigma+2}{\sigma+1}$ we have

$$(3.8) \qquad v^{q-\frac{1}{\sigma+1}} \leq c\left(q^{\frac{q}{q-\frac{1}{\sigma+1}}} v^q + q^{-q}\right),\ v^{q-\frac{2}{\sigma+1}} \leq c\left(q^{\frac{q}{q-\frac{2}{\sigma+1}}} v^q + q^{-q}\right)$$

with some constant $c = c(\sigma) > 1$. Then choosing $\lambda = \epsilon + \sup(\psi, \Omega \times [t=0]) \leq N$ we derive from (3.6)-(3.8) that

$$(3.9) \qquad \sup_{t\in[0,T]} \int_\Omega v^q dx + \iint_{Q_T} v^{q-2+\frac{\sigma}{\sigma+1}} |\bigtriangledown v|^m dxdt \leq cq^c \left(\iint_{Q_T} v^q dxdt + q^{-q}\right)$$

where constant $c > 1$ is independent of q. From (3.9) it follows obviously that

$$\left(\sup_{t\in[0,T]} \int_\Omega v^q dx\right)^{\frac{m}{n}} \iint_{Q_T} v^{q-2+\frac{\sigma}{\sigma+1}} |\bigtriangledown v|^m dxdt \leq cq^c \left(\iint_{Q_T} v^q dxdt + q^{-q}\right)^b$$

where $b \doteqdot 1 + \frac{m}{n}$. Then using Proposition 2.2 we obtain

$$(3.10) \qquad \iint_{Q_T} v^{bq+\kappa} dxdt \leq cq^c \left(\iint_{Q_T} v^q dxdt + q^{-q}\right)^b,\ \kappa \doteqdot m-2+\frac{\sigma}{\sigma+1}.$$

Obviously that $\kappa < 0$. Really we have

$$\kappa = m+l-2+\left(\frac{l}{m+l-1} - l\right) = m+l-2+l\frac{2-m-l}{m+l-1} = \frac{(m+l-2)(m-1)}{m+l-1} < 0.$$

Remark that

$$(3.11) \qquad 0 < |\kappa| \leq 2-m-l < 1,\ 0 < \frac{|\kappa|}{bq+\kappa} < \frac{n|\kappa|}{mq} < \frac{c}{q},\ \forall q \geq \frac{\sigma+2}{\sigma+1}.$$

Finally it is easy to see that for all $q \geq \frac{\sigma+2}{\sigma+1}$ we have

$$(3.12)\quad \frac{bq+\kappa}{q} \geq b+\left(m-2+\frac{\sigma}{\sigma+1}\right)\frac{\sigma+1}{\sigma+2} = 1+[m\left(\frac{1}{n}+\frac{\sigma+1}{\sigma+2}\right)-1] \doteq k > 1$$

because the square braces in (3.12) are strictly positive in view of condition (1.3) which is equivalent to condition

$$(3.13)\qquad m > \frac{n(\sigma+2)}{n(\sigma+1)+\sigma+2},\ \sigma = \frac{l}{m-1},\ m > 1,\ l \geq 0.$$

In particular in view of (3.12), (3.13) we have

$$(3.14)\qquad bq+\kappa \geq kq,\ k > 1,\ q \geq \frac{\sigma+2}{\sigma+1}.$$

Denote

$$(3.15)\qquad \| v \|_p^p \doteq \iint_{Q_T} |v|^p dxdt/|Q_T|,\ p > 0.$$

Then from (3.10), (3.15) it follows that

$$(3.16)\qquad \left(\| v \|_{bq+\kappa}\right)^{\frac{bq+\kappa}{bq}} \leq c^{1/q} q^{c/q} \left(\| v \|_q + q^{-1}\right),\ q \geq \frac{\sigma+2}{\sigma+1},$$

and hence in view of (3.13)-(3.16) we have

$$(3.17)\qquad \| v \|_{kq} \leq \| v \|_{bq+\kappa} \left[c^{1/q} q^{c/q} (\| v \|_q + q^{-1})\right]^{1+\frac{|\kappa|}{bq+\kappa}}.$$

Denote

$$(3.18)\qquad q_n = \frac{\sigma+2}{\sigma+1} k^n,\ y_n \doteq \| v \|_{q_n},\ n = 0, 1, \ldots$$

Then from (3.17) it follows that

$$(3.19)\qquad y_{n+1} \leq \left[c^{1/k^n} c^{n/k^n} (y_n + 1/k^n)\right]^{1+\frac{|\kappa|}{bq_n+\kappa}},\ n = 0, 1, \ldots$$

where constant $c > 1$ is independent of ϵ, N and number n. Set

$$(3.20) \qquad z_n = \max(e, y_n),\ n = 0, 1, \dots$$

Then from (3.11), (3.19), (3.20) and taking into account that

$$e \leq \left[c^{1/k^n} c^{n/k^n} z_n\right]^{1+\frac{|\kappa|}{bq_n+\kappa}},\ n = 0, 1, \dots,$$

we derive inequalities

$$(3.21) \qquad z_{n+1} \leq c^{(n+1)/k^n} \left(z_n + c/k^n\right)^{1+c/k^n},\ n = 0, 1, \dots$$

Using the Lagrange formulae we can estimate for all $n = 0, 1, \dots$

$$z_n + c/k^n \leq z_n^{1+c/k^n} = z_n + z_n^{1+\theta c/k^n} (\ln z_n) c/k^n,\ 0 < \theta < 1,$$

because $\ln z_n \geq 1$, $z_n^{1+\theta c/k^n} \geq 1$. Then we can rewrite (3.21) as

$$(3.22) \qquad z_{n+1} \leq c^{(n+1)/k^n} z_n^{1+c/k^n},\ n = 0, 1, \dots$$

because $(1 + c/k^n)(1 + c/k^n) \leq 1 + c_1/k^n$ for appropriate constant c_1. We prove that

$$(3.23) \qquad \sup_{n\in\mathbf{N}} y_n \leq \sup_{n\in N} z_n \leq M \doteqdot c^{\sum_{\nu=0}^{\infty}((\nu+1)/k^\nu) \prod_{\nu=0}^{\infty}(1+c/k^\nu)} z_0^{\prod_{\nu=0}^{\infty}(1+c/k^\nu)}$$

where the infinite product $\prod_{\nu=0}^{\infty}(1+c/k^\nu)$ and series $\sum_{\nu=0}^{\infty}(\nu+1)/k^\nu$ are obviously convergent. To prove (3.23) it suffices to convince ourselves that for any $n = 1, 2, \dots$

$$(3.24) \qquad z_n \leq c^{\sum_{\nu=0}^{n-1}((\nu+1)/k^\nu) \prod_{\nu=0}^{n-1}(1+c/k^\nu)} z_0^{\prod_{\nu=0}^{n-1}(1+c/k^\nu)}.$$

We prove this assertion by mathematical induction. For $n = 1$ inequality (3.24) is obvious in view of (3.22) in the case $n = 0$. To complete the proof, it suffices to show that if (3.24) holds for some $n \geq 1$, then

$$(3.25) \qquad z_{n+1} \leq c^{\sum_{\nu=0}^{n}((\nu+1)/k^\nu) \prod_{\nu=0}^{n}(1+c/k^\nu)} z_0^{\prod_{\nu=0}^{n}(1+c/k^\nu)}.$$

Indeed, from (3.24) it follows that

$$(3.26) \qquad \ln z_n \leq \sum_{\nu=0}^{n-1}((\nu+1)/k^\nu) \left(\prod_{\nu=0}^{n-1}(1+c/k^n)\right) \ln c + \left(\prod_{\nu=0}^{n-1}(1+c/k^\nu)\right) \ln z_0$$

while (3.22) yields

$$\ln z_{n+1} \le ((n+1)/k^n)\ln c + (1+c/k^n)\ln z_n. \tag{3.27}$$

Substituting (3.26) in (3.27), we find that

$$\begin{aligned} \ln z_{n+1} &\le ((n+1)/k^n)\left(\prod_{\nu=0}^{n}(1+c/k^\nu)\right)\ln c + \sum_{\nu=0}^{n-1}((\nu+1)k^\nu)\left(\prod_{\nu=0}^{n}(1+c/k^\nu)\right)\ln c + \\ &+ \prod_{\nu=0}^{n}(1+c/k^\nu)\ln z_0 = \sum_{\nu=0}^{n}((\nu+1)/k^\nu)\left(\prod_{\nu=0}^{n}(1+c/k^\nu)\right)\ln c + \prod_{\nu=0}^{n}(1+c/k^\nu)\ln z_0 \end{aligned} \tag{3.28}$$

from which (3.25) follows. Thus (3.24) holds for all $n = 0, 1, \dots$, consequently, so does (3.23). Because

$$\sup(\hat{u}, Q_T) = \lim_{n\to\infty} y_n \le \sup_{n\in\mathbb{N}} y_n$$

we have in view of (3.23) and definition of $\hat{u}$

$$\sup(\tilde{u}, Q_T) \le M^{\frac{1}{\sigma+1}} \tag{3.29}$$

where M is defined by the right hand side of (3.23) with

$$z_0 = \max\left(e, \| v \|_{\frac{\sigma+2}{\sigma+1}}\right),\ \| v \|_{\frac{\sigma+2}{\sigma+1}} = \left(\iint_{Q_T} \tilde{u}^{\sigma+2}dxdt/|Q_T|\right)^{\frac{\sigma+1}{\sigma+2}}. \tag{3.30}$$

To complete the proof of estimate (1.8) we have to estimate now $\iint_{Q_T} \tilde{u}^{\sigma+2}dxdt$ where $\tilde{u} = \sup(0, \chi(u) - \lambda)$, $\lambda = \epsilon + \sup(\psi, \Omega \times [t=0])$. Consider now (3.1) in the case $p = 2$, $\lambda = \epsilon$. Then in view of (3.6)-(3.8) (in the case $q = \frac{\sigma+2}{\sigma+1}$, $\lambda = \epsilon$) and taking into account that function ψ is bounded on $\Omega \times [t=0]$ (see (1.5)) we obtain for any $t \in (0, T]$ inequalities

$$\int_\Omega \bar{u}^{\sigma+2}dx|^t + \iint_{Q_T} \bar{u}^{lm'} | \nabla \bar{u}|^m dxdt \le c\left(\iint_{Q_T} \bar{u}^{\sigma+2}dxdt + 1\right) \tag{3.31}$$

where $\bar{u} = \sup(0, \chi(u) - \epsilon)$. Using the Gronwall inequality we derive from (3.31) that

$$\iint_{Q_T} \bar{u}^{\sigma+2}dxdt \le \sup_{t\in[0,T]}\int_\Omega \bar{u}^{\sigma+2}dxdt + \iint_{Q_T} \bar{u}^{lm'} | \nabla \bar{u}|^m dxdt \le c \tag{3.32}$$

with some constant c independent of ϵ and N. Because $\tilde{u} \le \bar{u}$ we obatin estimate

$$(3.33) \qquad \iint\limits_{Q_T} \tilde{u}^{\sigma+2} dxdt \le c$$

and hence (see (3.29), (3.23), (3.30))

$$(3.34) \qquad \sup(\tilde{u}, Q_T) \le c$$

with some constant c independent of ϵ and N. Finally in view of (3.1) and (3.34) we have

$$(3.35) \qquad \sup(\chi(u), Q_T) \le \sqrt{2}\,[\sup(\tilde{u}, Q_T) + \lambda] \le c_1$$

with $c_1 = \sqrt{2}\,(c + 1 + \sup(\psi, \Omega \times [t = 0]))$ if $\epsilon \in (0, 1)$. Theorem 1.1 is proved.

In proving existence of Holder continuous weak solutions for DNPE of the type (1.1)-(1.3) useful the following

Theorem 1.2. *Let conditions (1.2), (1.3) are fulfilled and let u be a generalized solution of Cauchy-Dirichlet problem (1.4), (1.5) with any $\epsilon > 0$ and $N \ge c_1$ where constant c_1 is defined by Theorem 1.1. Then*

$$(3.36) \qquad \iint\limits_{Q_T} u^l |\nabla u|^m \le c_2$$

where constant c_2 depends on the data.

Proof. Apply Proposition 2.1 in the case

$$(3.37) \qquad g(\xi) = \sup(0, \chi(\xi) - \epsilon),\ \varphi = \epsilon.$$

Then using Remark 1.1 we have $g'(u) = 1$ a.e. in Q_T and hence

$$(3.38) \qquad \int\limits_{\Omega} G(u)dx|_0^T + \iint\limits_{Q_T} a(\chi(u), \nabla u) \cdot \nabla u \, dxdt = \iint\limits_{Q_T} fu \, dxdt$$

where (again in view of Remark 1.1) we have

$$(3.39) \quad G(u) \ge \frac{1}{2}(u - \epsilon)^2,\ a(\chi(u), \nabla u) \cdot \nabla u \ge \nu_0 u^l |\nabla u|^m - \mu_0(|u|^2 + 1) \text{ a.e. in } Q_T.$$

Using estimate (3.8) we derive from (3.38), (3.39) estimate (3.36) with some constant c_2 depending on c_1, ν_0^{-1}, μ_0 and μ_2. Theorem 1.2 is proved.

4. A counterexample

In this section we show that if condition (1.3) is violated then at least local boundedness of generalized solution cannot be proved. We use constructions of an counterexample by DiBenedetto ([13], p. 130-133) who showed similar necessity of condition (1.3) in the case $l = \sigma = 0$.

Let $r \geq \frac{\sigma+2}{\sigma+1}$ and $a \in (0,1)$ be given constants and consider the function

$$y = z^{\frac{1}{\sigma+1}},\ z = \frac{(a^2 - |x|^2)^2}{|x|^{\frac{n}{r}} ln^2 |x|^2}. \tag{4.1}$$

Let B_a denote the ball of radius a in $\mathbb{R}^n$ centered at the origin. Obviously $z \in L_r(B_a)$ and $z \notin L_{r+\epsilon}(B_a)$, $\forall \epsilon \in (0,1)$.

Introduce also the function

$$w = (1 - ht)_+ y \tag{4.2}$$

where $h > 1$ is to be chosen, and consider the Cauchy-Dirichlet problem

$$\left.\begin{array}{l} \partial u/\partial t - \operatorname{div}[|u|^l |\nabla u|^{m-2} \nabla u] = 0 \quad \text{in} \quad Q \doteqdot B_a \times (0,1],\ m > 1,\ l \geq 0, \\ u = 0 \quad \text{on} \quad S_T,\ u = y \quad \text{on} \quad B_a \times [t = 0]. \end{array}\right\} \tag{4.3}$$

Lemma 4.1. *Assume that*

$$\lambda_r \doteqdot n(m - \frac{\sigma+2}{\sigma+1}) + rm = 0,\ r \geq \frac{\sigma+2}{\sigma+1},\ \sigma = \frac{l}{m-1},\ m > 1, l \geq 0. \tag{4.4}$$

The constants $a \in (0,1)$ and $h > 1$ can be determined a priori so that function w defined by (4.2) is a non-negative generalized subsolution of (4.3) such that $w \in C([0,1]; L_{r(\sigma+1)}(B_a))$, $w^{\sigma+1} \in \overset{\circ}{W}{}^{1,0}_m(Q)$, but $w \notin L_\infty(Q_T)$.

Proof. Denote $a = e^{-k}$, $k > 1$. In [13] it is proved that

$$\left.\begin{array}{l} \operatorname{div}(|\nabla z|^{m-2} \nabla z) \geq 0 \quad \text{on} \quad \mathcal{E}_k^{(1)} \doteqdot [\frac{2}{3}e^{-2k} \leq |x|^2 < e^{-2k}], \\ \operatorname{div}(|\nabla z|^{m-2} \nabla z) \geq -\gamma \dfrac{z^{m-1}}{|x|^m} \quad \text{on} \quad \mathcal{E}_k^{(2)} \doteqdot [|x|^2 < \frac{2}{3}e^{-2k}] \end{array}\right\} \tag{4.5}$$

with some constant $\gamma = \gamma(m,n,r) > 0$. Then with w given by (4.2), (4.1) we compute in $[0 < |x| < a]$

$$\begin{aligned} L(w) &\doteqdot \frac{\partial w}{\partial t} - \operatorname{div}[w^l |\nabla w|^{m-2} \nabla w] = \frac{\partial w}{\partial t} - \operatorname{div}[|\nabla w^{\sigma+1}|^{m-2} \nabla w^{\sigma+1}] = \\ &= -hz^{\frac{1}{\sigma+1}} - (1-ht)_+^{(\sigma+1)(m-1)} \operatorname{div}[|\nabla z|^{m-2} \nabla z]. \end{aligned} \tag{4.6}$$

From (4.5), (4.6) it follows that $Lw \leq 0$ on $\mathcal{E}_k^{(1)}$ while on $\mathcal{E}_k^{(2)}$ we have

$$L(w) \leq z^{\frac{1}{\sigma+1}}\left(-h + \gamma\frac{z^{m-1-\frac{1}{\sigma+1}}}{|x|^m}\right). \tag{4.7}$$

By calculation on $\mathcal{E}_k^{(2)}$

$$\gamma\frac{z^{m-1-\frac{1}{\sigma+1}}}{|x|^m} \leq \gamma(a^2-|x|^2)^{2(m-1-\frac{1}{\sigma-1})}|x|^{-\frac{\lambda_r}{r}} \leq \gamma^*(k) \tag{4.8}$$

where we have used the fact that $\lambda_r = 0$. Therefore

$$L(w) \leq z(-h+\gamma^*(k)).$$

Choosing $h = \gamma^*(k)$ proves that

$$L(w) \leq 0 \quad \text{on} \quad [0<|x|<a]\times[0,1]. \tag{4.9}$$

Using (4.9) it is easy to prove (exactly in the same way as in [13]) that indeed w is a weak solution of (4.3) in the whole $B_a \times [0,1]$. Obviously that from (4.1), (4.2), and (4.4) it follows that $w \in C([0,1]; L_{r(\sigma+1)}(B_a))$, $w^{\sigma+1} \in \overset{\circ}{W}{}_m^{1,0}(Q_T)$ while $w \notin L_{r(\sigma+1)+\epsilon}(Q_T)$, $\forall\epsilon \in (0,1)$. In particular $w \in C([0,1]; L_{\sigma+2}(B_a))$ and $w \notin L_\infty(Q_T)$, $\forall\epsilon \in (0,1)$. Lemma 4.1 is proved.

Remark 4.1. Obviously that conditions (4.4) and (1.12) are equivalent. Really (4.4) is equivalent to inequalities

$$m\left(n+\frac{\sigma+2}{\sigma+1}\right) \leq n\frac{\sigma+2}{\sigma+1},\ m>1, l\leq 0 \tag{4.10}$$

which can be rewritten as (1.12) and on the contrary from (4.10) it follows, that (4.4) hold with some $r \geq \frac{\sigma+2}{\sigma+1}$. Lemma 4.1 shows that for generalized solutions of (1.13) (and hence of (1.1), (1.2)) the local L_∞ -estimate can not be established in the case (4.10).

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