ON COMPLEX ANALYTIC COMPACTIFICATIONS OF $\mathbb{C}^3$

by

Mikio Furushima

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

MPI/87-19

ON COMPLEX ANALYTIC COMPACTIFICATIONS OF $\mathbb{C}^3$

by

Mikio Furushima

Introduction. Let X be an n-dimensional connected compact complex manifold and A be an analytic subset of X. We say the pair (X,A) a complex analytic compactification of $\mathbb{C}^n$ if $X - A$ is biholomorphic to $\mathbb{C}^n$. If X admits a Kähler metric, we say the pair (X,A) a Kähler compactification of $\mathbb{C}^n$. Then, Hirzebruch (Problem 27 in [4]) proposed the following

Problem H. Determine all the compactifications of $\mathbb{C}^n$ with the second Betti number $b_2(X) = 1$.

For $n = 1$, it is easy to see that $(X,A) \cong (\mathbb{P}^1, \infty)$. For $n = 2$, Remmert-Van de Ven [10] proved that $(X,A) \cong (\mathbb{P}^2, \mathbb{P}^1)$, where $A = \mathbb{P}^1$ is a line on $X = \mathbb{P}^2$. For $n \geq 3$, Problem H is still open.

In the paper [2], the author considered the following special case of Problem H for $n = 3$.

Problem. Determine all the Kähler compactifications (X,A) of $\mathbb{C}^3$ such that A has at most isolated singular points.

Then we have the following

Theorem 1 ([2]). (X,A) be a Kähler compactification of $\mathbb{C}^3$ such that A has at most isolated singular points. Then, A is an irreducible normal divisor on X, the line bundle $[A]$ defined by A is positive on X, and the canonical divisor $K_X = -rA$ $(1 \leq r \leq 4)$. Especially, X is a Fano 3-fold of index r with $b_2(X) = 1$. The structure of (X,A) is determined by the index r

as follow:

(1) $r = 4 \Rightarrow (X,A) \cong (\mathbb{P}^3, \mathbb{P}^2)$, where $A = \mathbb{P}^2$ is a hyperplane on $X = \mathbb{P}^3$.

(2) $r = 3 \Rightarrow (X,A) \cong (\mathbb{Q}^3, \mathbb{Q}^2_0)$, where $\mathbb{Q}^3$ is a non-singular quadric hypersurface in $\mathbb{P}^4$ and $\mathbb{Q}^2_0$ is a quadric cone which is a hyperplane section.

(3) $r = 2 \Rightarrow (X,A) \cong (V_5, H_5)$, where V_5 is a Fano 3-fold of degree 5 in $\mathbb{P}^6$ and H_5 is a hyperplane section with a rational double point (A_4-singularity).

(4) $r = 1 \Rightarrow (X,A) \cong ?$. A is not a cone over a non-singular compact algebraic curve of genus $q \geq 0$.

Moreover, $X - A \cong \mathbb{C}^3$ in each case of $r \geq 2$.

In this paper, we shall consider the case of $r = 1$. Our main result is the following

<u>Theorem</u> 2. Let (X,A) be as in Theorem 1. Assume that the index $r = 1$. Then we have $(X,A) \cong (V_{22}, H_{22})$, where V_{22} is a Fano 3-fold of degree 22 in $\mathbb{P}^{13}$ and H_{22} is a hyperplane section of V_{22} which is a normal rational surface.

<u>Remark</u>. There exists such a pair (V_{22}, H_{22}), but the author does not know whether $V_{22} - H_{22} \cong \mathbb{C}^3$.

Question. Is there a hyperplane section H_{22} of V_{22} such that $V_{22} - H_{22} \cong \mathbb{C}^3$?

§ 1. General results.

Let (X,A) be a Kähler compactification of $\mathbb{C}^3$ such that A has at most isolated singular points. By Hartogs theorem, A is an analytic subset of pure codimension one. Since A has at most hypersurface singular points, A is a normal Gorenstein surface, that is, we can define the canonical divisor K_A on A. Since $\mathbb{C}^3$ is connected at infinity, A is connected, hence A is an irreducible normal Gorenstein surface. Then the general properties can be summarized as follow:

Proposition 1 ([2]). Let (X,A) be as above. Then

(1) $H^1(X;\mathbb{Z}) \cong H^1(A;\mathbb{Z}) \cong 0$

(2) $H^2(X;\mathbb{Z}) \cong H^2(A;\mathbb{Z}) \cong \mathbb{Z}$.
$H^2(X;\mathbb{Z})$ is generated by the first Chern class $c_1([A])$ of the line bundle $[A]$ defined by A and $H^2(A;\mathbb{Z})$ is generated by $c_1(N_A)$, where $N_A = [A]|A$ is the normal bundle.

(3) The Euler number $\chi(X) = 4 - b_3(A)$, where $b_3(A) = \dim H^3(A;\mathbb{R})$.

(4) The line bundle $[A]$ is positive on X and the canonical divisor $K_X = -rA$ $(1 \leq r \leq 4)$. Especially, X is a Fano 3-fold of index r with the second Betti number $b_2(X) = 1$.

(5) $H^i(X,\mathcal{O}_X) = 0$ for $1 \leq i \leq 3$.

(6) $H^i(A,\mathcal{O}_A) = 0$ for $1 \leq i \leq 2$ if $r \geq 2$
$H^1(A,\mathcal{O}_A) = 0$ and $H^2(A,\mathcal{O}_A) \cong \mathbb{C}$ if $r = 1$.

A projective algebraic normal Gorenstein surface A is called a (singular) Del Pezzo (resp. a singular $K-3$ surface) if $-K_A$ is positive on A (resp. $-K_A = 0$ and $H^1(A,\mathcal{O}_A) = 0$). Then we have

<u>Proposition 2.</u> If $r \geq 2$, then A is a (singular) Del Pezzo surface with $\mathrm{Pic}\, A \cong \mathbb{Z}\cdot N_A$. If $r = 1$, then A is a singular $K-3$ surface with $\mathrm{Pic}\, A \cong \mathbb{Z}\cdot N_A$.

§ 2. The structure of A in case of $r = 1$.

In this section, assume that $r = 1$. Then A is a singular K - 3 surface with $\mathrm{Pic}\, A \cong \mathbb{Z}\cdot N_A$ and the singular points of A are hypersurface singular points. Let Sing A be the set of the singular locus of A and S be the set of singular points of A which are not rational double points. Let $\pi : M \longrightarrow A$ be the minimal resolution of singular points of A and put $B = \pi^{-1}(\mathrm{Sing}\, A)$, $C = \pi^{-1}(S) = \bigcup_{i=1}^{s_0} C_i$, and $s = \dim H^2(B:\mathbb{R})$. Let us denote the canonical divisor on M by K_M.

Lemma 1. $S \neq \phi$.

Proof. Let us consider the following exact sequence of cohomology group (see [1]):

$$\longrightarrow H^1(A;\mathbb{R}) \longrightarrow H^1(M;\mathbb{R}) \longrightarrow H^1(B;\mathbb{R}) \longrightarrow H^2(A;\mathbb{R}) \longrightarrow H^2(M,\mathbb{R})$$
$$\longrightarrow H^2(B;\mathbb{R}) \longrightarrow 0.$$

Assume that $S = \phi$. Then we have $K_M = 0$ and $H^1(B;\mathbb{R}) = 0$. Since $H^1(A,\mathcal{O}_A) = 0$ implies $H^1(A;\mathbb{R}) = 0$, by the exact sequence above, $H^1(M;\mathbb{R}) = 0$. Thus M is a K - 3 surface. Since $b_2(A) = 1$ and A is algebraic, we have $b^+(A) = 1$. On the other hand, by Brenton [1], $b^+(A) = b^+(M)$. Thus we have $b^+(M) = 1$. This is a contradiction.

Q.E.D.

Corollary 1. M is a ruled surface over a non-singular compact algebraic curve R of genus $q = \dim H^1(M,\mathcal{O}_M)$.

Proof. Since $S \neq \phi$, we have $-K_M = \Sigma n_i C_i$ $(n_i > 0,\ n_i \in \mathbb{Z})$. Thus $P_m(M) = \dim H^0(M,\mathcal{O}(mK_M)) = 0$ for $m > 0$. By the classification of surfaces, we have the claim.

Q.E.D.

Lemma U ([11]).

(1) If $q \neq 1$, then S consists of one point with $p_g = \dim (R^1\pi_*\mathcal{O}_M)_S = q + 1$.

(2) If $q = 1$, then S consists of either one point with $p_g = 2$ or two points with $p_g = 1$, in second case of (2), both of the two points are simple elliptic.

Lemma 2. S consists of one point with $p_g = q + 1$ and $b_2(M) = s + 1$.

Proof. Assume that S consists of two points. By Lemma U, these two points are simple elliptic and $C = \pi^{-1}(S) = C_1 \cup C_2$, where C_1, C_2 are distinct sections of M. Since $b_2(A) = 1$, by the exact sequence in Lemma 1, we have $b_1(M) = b_1(B)$ and $b_2(M) = s + 1$. Since $2 = b_1(M) = b_1(B) \geq b_1(C) = b_1(C_1) + b_1(C_2) = 2 + 2 = 4$, this is a contradiction.

Q.E.D.

Let Z be the fundamental cycle of S with respect to the resolution $\pi : M \longrightarrow A$. Then,

Lemma 3 (see Proposition 2 in [3]).

(1) $q = 0 \Rightarrow M$ is a rational surface and $-K_M = Z$

(2) $q \neq 0 \Rightarrow$ there exists an irreducible component C_{i_1} of C such that C_{i_1} is a section of M and the rest

$\overline{C - C_{i_1}} = \bigcup_{i \neq i_1} C_i$ is either empty or contained in the singular fibres of M, and $-K_M = Z + C_{i_1}$.

Corollary 2 (see Corollary 1 in [3]). Assume that $q \neq 0$. Then

(1) $(C_{i_1} \cdot Z) = 2 - 2q$

(2) $(Z \cdot Z) \leq (C_{i_1} \cdot C_{i_1})$

Lemma 4

(1) $q \neq 0 \Rightarrow b_2(M) \leq 9 - 4q + \sqrt{9+8q}$

(2) $q = 0 \Rightarrow 11 \leq b_2(M) \leq 13.$

Proof. Assume that $q \neq 0$. By Noether formula,

$$10 - 8q = (K_M \cdot K_M) + b_2(M) \tag{2.1}$$

Since $-K_M = Z + C_{i_1}$, we have

$$(K_M \cdot K_M) = (Z \cdot Z) + 2(Z \cdot C_{i_1}) + (C_{i_1} \cdot C_{i_1}) \tag{2.2}$$

By Corollary 2 and (2.1), (2.3),

$$b_2(M) = 6 - 4q - (Z \cdot Z) - (C_{i_1} \cdot C_{i_2}) \tag{2.3}$$

$$\leq 6 - 4q - 2(Z \cdot Z) \tag{2.4}$$

since $S = \{\text{one point}\}$ is a hypersurface singular point of A, we have,

$$\begin{cases} (Z \cdot Z) \geq -n \quad \text{(Wagreich [12])} & (2.5) \\ p_g \geq \frac{1}{2}(n-1)(n-2) \quad \text{(Yau [13])} & (2.6) \end{cases}$$

Since $p_g = q+1$, by (2.5) and (2.6), we have

$$-(Z\cdot Z) \leq \frac{1}{2}(3+\sqrt{9+8q}) \tag{2.7}$$

By (2.4) and (2.7), we have finally the inequality

$$b_2(M) \leq 9 - 4q + \sqrt{9+8q} \ .$$

This proves (1). Next, assume that $q = 0$. By Noether formula,

$$b_2(M) = 10 - (K_M \cdot K_M) \tag{2.8}$$

since $p_g = 1$ and S is a hypersurface singularity, by Laufer [14], we have $Z^2 = -1, -2, -3$. By Lemma 3 - (1) and (2.8), we have the inequality $11 \leq b_2 \leq 13$.

Q.E.D.

<u>Corollary 3.</u> $0 \leq q \leq 3$.

<u>Proof.</u> Assume that $q \neq 0$. Then, by Lemma 4 - (1), we have

$$2 \leq b_2(M) \leq 9 - 4q + \sqrt{9+8q}$$

This implies $q \leq 3$.

Q.E.D.

Now, since the index $r = 1$, X is a Fano 3-fold of index 1 with $b_2(x) = 1$ and A is a hyperplane section of X. We put $g = \frac{1}{2}(A\cdot A\cdot A)_X + 1$. The number g is called the "genus" of the Fano 3-fold X. Then by Iskovskih, we have

<u>Lemma</u> 5 ([6],[7]). Let g be the genus of a Fano 3-fold X of index 1 and $b_2(X) = 1$. Then,

g	2	3	4	5	6	7	8	9	10	12
$\frac{1}{2}b_3(X)$	52	30	20	14	10	7	5	3	2	0

Table 1

Remark. The list of the classification of Fano 3-folds due to Iskovskih [6] is incomplete, in fact, S. Mukai-H.Umemura [8] gave an example of a Fano 3-fold of index 1 and the genus $g = 12$ which is overlooked by Iskovskih. But Table 1 is justified by S. Mukai, who has recently succeeded in classifying Fano 3-folds of index 1 with $b_2(X) = 1$ applying the theory of vector bundles on $K-3$ surfaces. According to his theory, such a X can be represented as a complete intersection of a homogeneous space (see [9]).

Lemma 6. $q = \frac{1}{2}b_3(X)$

Proof. $2q = b_1(M) = b_3(M) = b_3(A) = b_3(X)$.

By Corollary 3, Lemma 5 and Lemma 6, we have $(g,q) = (9,3)$, $(10,2)$ or $(12,0)$. If $q = 3$, then, by Proposition 2 - (1), $b_2(M) < 3$, that is $b_2(M) = 2$. Then, M is a $\mathbb{P}^1$-bundle over R of genus $q = 3$ (see Corollary 1), and thus A is a cone over R. This contradicts Theorem 1 - (4). Therefore $q \neq 3$. On the other hand, by Lemma 7 below, we must have $b_2(M) \geq 4$.

Lemma 7. Assume that $q \neq 0$. Then, there exists unique exceptional curve of the first kind in every singular fiber of M and then another irreducible components of the singular fiber are all contained in $B = \pi^{-1}(\mathrm{Sing}\ A)$.

Proof. See Proposition 7 in [3].

Therefore $(g,q) = (10,2)$ or $(12,0)$. In case of $q = 2$, by Proposition 2-(1), and the fact above, we have $4 \leq b_2(M) \leq 6$.

Next, we shall determine the type of singular fiber of M in case of $q = 2$.

Lemma 8. Assume that $q = 2$. Let Z, C_{i_1}, s, B be as above. We put $(C_{i_1} \cdot C_{i_1}) = e < 0$. Then we have $-4 \leq (Z \cdot Z) \leq -3$ and

(1) $(Z \cdot Z) = -4 \Rightarrow (e,s) = (-2,3), (-3,4), (-4,5)$

(2) $(Z \cdot Z) = -3 \Rightarrow (e,s) = (-3,3)$.

Proof. Since $q = 2$, $4 \leq b_2(M) \leq 6$. By (2.4), $4 \leq b_2(M) \leq 6 - 8 - 2(Z \cdot Z)$, namely, $3 \leq -(Z \cdot Z)$. By (2.7), $-(Z \cdot Z) \leq 4$. Therefore $-4 \leq (Z \cdot Z) \leq -3$. Since $(Z \cdot Z) = -3$ or -4 and $4 \leq b_2(M) \leq 6$, by Corollary 2-(2) and (2.3), we have that

(i) $b_2(M) = 6 \Rightarrow (e,s) = (-4,5)$ and $(Z \cdot Z) = -4$

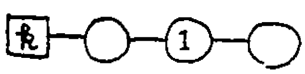
(ii) $b_2(M) = 5 \Rightarrow (e,s) = (-3,4)$ and $(Z \cdot Z) = -4$

(iii) $b_2(M) = 4 \Rightarrow (e,s) = (-3,3)$ and $(Z \cdot Z) = -3$, or
$(e,s) = (-2,3)$ and $(Z \cdot Z) = -4$

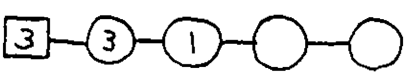
Q.E.D.

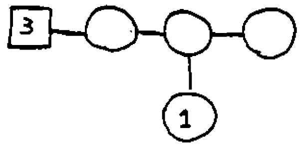
By Lemma 7, Lemma 8 and the fact that $\mathrm{Sing}\, A \setminus S$ consists of rational double points, we have the following

Proposition 2. Assume that $q = 2$. Then the structure of M as a ruled surface can be described as Table 2:

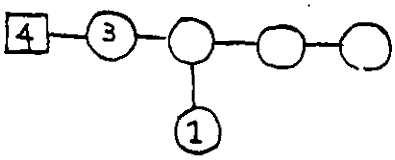
(i) 

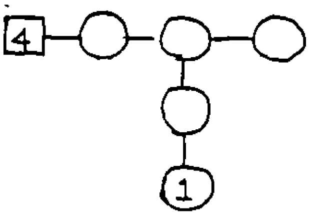
(k = 2 or 3)

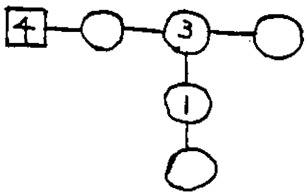
(ii) 

(iii) 

(iv) 

(v) 

(vi) 

(vii) 

(viii) 

Table 2

<u>Notation.</u> In proposition 2, the vertex $\boxed{k}$ represents a non-singular compact algebraic curve of genus 2 with the self-intersection number $-k$, $\textcircled{k}$ a non-sinulgar rational curve with the self-intersection number $-k$, and we denote $\textcircled{2}$ simply by $\bigcirc$.

§ 3. Proof of Theorem

Let (X,A), M, π and g be as before. Then,

Lemma 9. $\mathrm{Pic}\, A \cong \mathbb{Z}[D]$, where D is a canonical curve of genus g such that $\deg D = 2g - 2$ and $D \cap \mathrm{Sing}\, A = \phi$ (in our case $g = 10, 12$).

Proof. X is a Fano 3-fold of degree $2g - 2$ in $\mathbb{P}^{g+1}$ (see [7]) and A is a hyperplane section. For a sufficiently general hyperplane section H. $N_A \sim [H]|_A \sim [D]$ on A (linearly equivalent), where D is a non-singular canonical curve of genus g with $\deg D = (A\cdot A\cdot A) = 2g - 2$ and $D \cap \mathrm{Sing}\, A = \phi$.

Q.E.D.

We shall prove Theorem 2. We have only to show $q \neq 2$. Then $q = 0$, namely, X is a Fano 3-fold of degree 22 in $\mathbb{P}^{13}$ and A is a hyperplane section which is normal and rational. Assume that $q = 2$. Then M is a ruled surface over a non-singular compact algebraic curve of genus 2. We put $D^* = \pi^{-1}(D) \hookrightarrow M$. Since $D \cap \mathrm{Sing}\, A = \phi$, $D^* \cong D$, $(D^*\cdot D^*)_M = 2g - 2$ and $(D^*\cdot B_i)_M = 0$ for every exceptional curve $B_i \subset B = \pi^{-1}(\mathrm{Sing}\, A)$. Let e_i $(0 \leq i \leq k)$ be a basis of the cohomology group $H^2(M;\mathbb{Z})$, which is chosen as follow: e_0 is the class of the negative section of M and e_i $(1 \leq i \leq k)$ is that of a singular fibers of M. These can be chosen easily if we can see the type of singular fibers of M. Then we have

$$D^* = \sum_{i=0}^{k} \alpha_i e_i \quad (\alpha_i \in \mathbb{Z}) \tag{3.1}$$

In case of $(g,q) = (10,2)$, by Proposition 2, we determine the structure of M as a ruled surface case Table 2). Thus, we can easily choose a basis $\{e_i\}$ $(0 \leq i \leq k)$, and have that

(i) the intersection number $(e_i \cdot e_j)_M$ is determined by the graph in Table 2.

(ii) $(D^* \cdot e_i)_M = 0$ if e_i is the class of exceptional curve in $B = \pi^{-1}$ (Sing A),

(iii) $(D^* \cdot D^*)_M = 2g - 2 = 18$.

(iv) $d_{i_0} = (D^* \cdot e_{i_0}) \neq 0$, where e_{i_0} is the class of the exceptional curve of the first kind which is obtained by the blowing up.

By (3.1) and the assertions (i) - (iv) above, we have the equations concerning to α_i, d_{i_0} over $\mathbb{Z}$. Finally, we can show by easy calculations that these equations have no solution over $\mathbb{Z}$ (see Appendix).

Therefore $q = 0$ and $g = 12$. This completes the proof of Theorem 2.

§4 An example

Let M be a rational surface which is obtained from $\mathbb{P}^2$ by 12 times blowing-ups, and the configuration of exceptional curves on M be as Figure 1, where

(i) C_i's are all non-singular rational curves.

(ii) E_i $(i = 1,2)$, C_i $(i \neq 5,9)$ are the exceptional curves obtained by the blowing-ups.

(iii) $(C_1 \cdot C_1) = (C_5 \cdot C_5) = -3$, $(C_i \cdot C_i) = -2$ if $i \neq 1,5$
$(E_i \cdot E_i) = -1$ $(i = 1,2)$.

(iv) D^* is a non-singular compact algebraic curve of genus 12 with $\deg D^* = (D^* \cdot D^*) = 22$, which is the proper transform of a curve of degree 11 in $\mathbb{P}^2$ with a singular point.

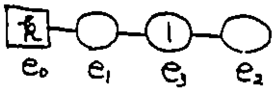
(v) $(D^* \cdot E_1) = 2$, $(D^* \cdot E_2) = 3$ and $(D^* \cdot C_i) = 0$ $(1 \leq i \leq 12)$.

We put $C = \bigcup_{i=1}^{12} C_i$. Then the intersection matrix $((C_i \cdot C_j))$ is negative definite, hence C is an exceptional curve on M. We put $A = M/C$. Then A is a singular $K-3$ surface with a hypersurface singular point (in fact, $P_g = 1$), and $\mathrm{Pic}\, A \cong \mathbb{Z}[D]$, where D is the image of D^* in A. We find that $D \cap \mathrm{Sing}\, A = \phi$ and $\deg D = (D^* . D^*) = 22$.

Assume that A is a hyperplane section of a Fano 3-fold X of degree 22 in $\mathbb{P}^{13}$. Then we can see that $X - A$ is an affine 3-fold with $b_i(X - A) = b_i(\mathbb{C}^3)$ for $i \geq 0$.

Question. Is $X - A$ a homology 3-cell ?

Appendix

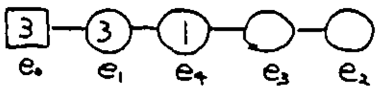
(i) 

(k = 2 or 3)

$D^* = \sum_{i=0}^{3} \alpha_i e_i \quad (\alpha_i \in \mathbb{Z})$. Then we have

$$\begin{cases} \alpha_1 - k\alpha_0 & = 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 & = 0 \\ \alpha_3 - 2\alpha_2 & = 0 \\ \alpha_2 - \alpha_3 + \alpha_1 & = d_3 \\ \alpha_3 \cdot d_3 & = 18 \end{cases}$$

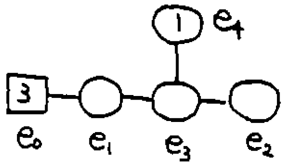
$\therefore \ \alpha_0^2 = \dfrac{36}{2k-1}$ (k = 2.3). Therefore $\alpha_0 \notin \mathbb{Z}$.

(ii) 

$D^* = \sum_{i=0}^{4} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$

$$\begin{cases} \alpha_1 - 3\alpha_0 & = 0 \\ \alpha_4 - 3\alpha_1 + \alpha_0 & = 0 \\ \alpha_3 - 2\alpha_2 & = 0 \\ \alpha_4 - 2\alpha_3 + \alpha_2 & = 0 \\ \alpha_1 - \alpha_4 + \alpha_3 & = d_4 \\ \alpha_4 \cdot d_5 & = 18 \end{cases}$$

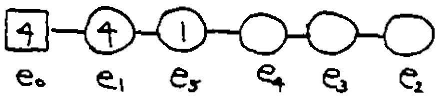
$\therefore \ \alpha_0^2 = \dfrac{27}{4}$. Therefore $\alpha_0 \notin \mathbb{Z}$.

(iii) 

$$D^* = \sum_{i=0}^{4} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$$

$$\begin{cases} \alpha_1 - 3\alpha_0 = 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 = 0 \\ \alpha_3 - 2\alpha_0 = 0 \\ \alpha_4 + \alpha_1 - 2\alpha_3 + \alpha_2 = 0 \\ \alpha_3 - \alpha_4 = d_4 \\ \alpha_4 \cdot d_4 = 18 \end{cases}$$

$\therefore \ \alpha_0^2 = 8$ Therefore $\alpha_0 \notin \mathbb{Z}$

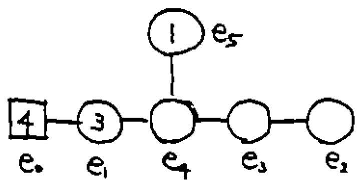
(iv) 

$$D^* = \sum_{i=0}^{5} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$$

$$\begin{cases} \alpha_1 - 4\alpha_0 = 0 \\ \alpha_5 - 4\alpha_1 + \alpha_0 = 0 \\ \alpha_3 - 2\alpha_2 = 0 \\ \alpha_4 - 2\alpha_3 + \alpha_2 = 0 \\ \alpha_5 - 2\alpha_5 + \alpha_3 = 0 \\ \alpha_1 - \alpha_5 + \alpha_4 = d_5 \\ \alpha_5 d_5 = 18 \end{cases}$$

$\therefore \ \alpha_0^2 = \frac{24}{5}$ Therefore $\alpha_0 \notin \mathbb{Z}$.

(v)

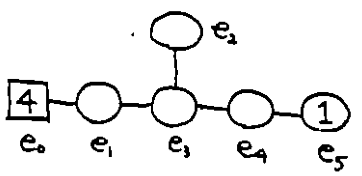


$$D^* = \sum_{i=0}^{5} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$$

$$\begin{cases} \alpha_1 - 4\alpha_0 = 0 \\ \alpha_4 - 3\alpha_1 + \alpha_0 = 0 \\ \alpha_3 - 2\alpha_2 = 0 \\ \alpha_4 - 2\alpha_3 + \alpha_2 = 0 \\ \alpha_5 + \alpha_1 - 2\alpha_4 + \alpha_3 = 0 \\ \alpha_4 - \alpha_5 = d_5 \\ d_5 \cdot \alpha_5 = 18 \end{cases}$$

$\therefore\ \alpha_0^2 = \frac{81}{16}$ Therefore $\alpha_0 \notin \mathbb{Z}$.

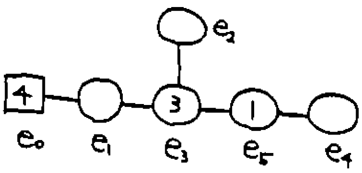
(vi)



$$D^* = \sum_{i=0}^{5} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$$

$$\begin{cases} \alpha_1 - 4\alpha_0 = 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 = 0 \\ \alpha_3 - 2\alpha_2 = 0 \\ \alpha_4 + \alpha_1 - 2\alpha_3 + \alpha_2 = 0 \\ \alpha_5 - 2\alpha_4 + \alpha_3 = 0 \\ \alpha_4 - \alpha_5 = d_5 \\ \alpha_5 d_5 = 18 \end{cases}$$

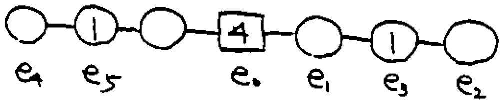
$\therefore\ \alpha_0^2 = 6$ Therefore $\alpha_0 \notin \mathbb{Z}$

(vii) 

$$D^* = \sum_{i=0}^{5} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$$

$$\begin{cases} \alpha_1 - 4\alpha_0 = 0 \\ \alpha_0 - 2\alpha_1 + \alpha_3 = 0 \\ \alpha_3 - 2\alpha_2 = 0 \\ \alpha_5 + \alpha_1 - 3\alpha_3 + \alpha_2 = 0 \\ \alpha_5 - 2\alpha_4 = 0 \\ \alpha_4 - \alpha_5 + \alpha_3 = d_5 \\ d_5 \cdot \alpha_5 = 18 \end{cases}$$

$\therefore \alpha_0^2 = \frac{16}{3}$ Therefore $\alpha_0 \notin \mathbb{Z}$.

(viii) 

$$D^* = \sum_{i=0}^{5} \alpha_i e_i \qquad (\alpha_i \in \mathbb{Z})$$

$$\begin{cases} \alpha_1 - 4\alpha_0 = 0 \\ \alpha_3 - 2\alpha_1 + \alpha_0 = 0 \\ \alpha_3 - 2\alpha_2 = 0 \\ \alpha_1 - \alpha_3 + \alpha_2 = d_3 \\ \alpha_5 - 2\alpha_4 = 0 \\ \alpha_4 - \alpha_5 = d_5 \\ \alpha_0 + \alpha_5 = 0 \\ \alpha_5 d_5 + \alpha_3 d_3 = 18 \end{cases}$$

$\therefore \alpha_0^2 = 6$ Therefore $\alpha_0 \notin \mathbb{Z}$.

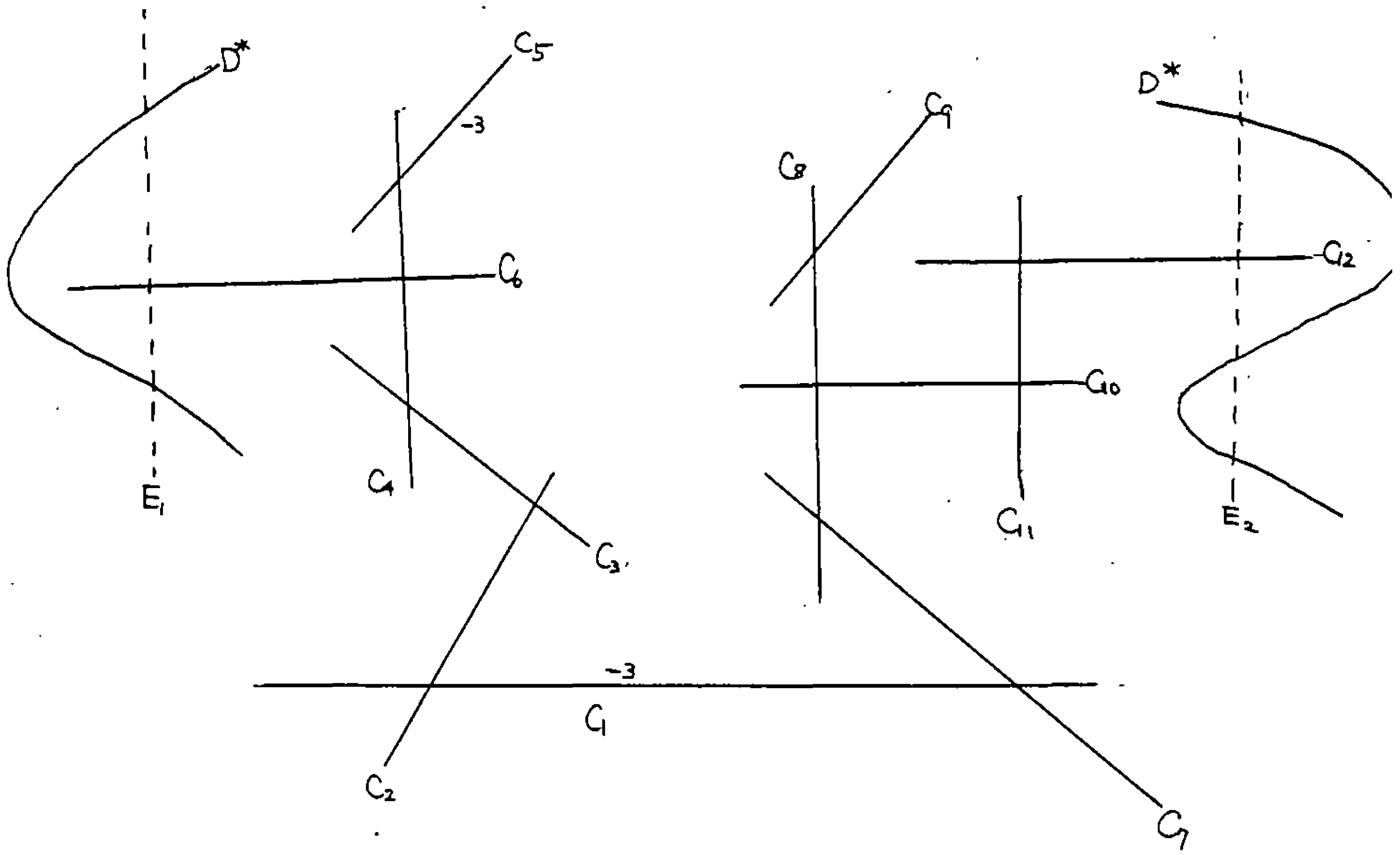


Figure 1

References

1. Brenton, L.: Some algebraicity criteria for singular surfaces. Invent. Math. 41, 129-147 (1977)

2. Furushima, M.: Singular Del Pezzo surfaces and complex analytic compactifications of the 3-dimensional complex affine space $\mathbb{C}^3$. Nagoya Math. J. 104, 1-28 (1986)

3. Furushima, M.: Singular K-3 surfaces with hypersurface singularities. Pacific J. Math. 125, 67-77 (1986)

4. Hirzebruch, F.: Some problems on differentiable and complex manifolds. Ann. Math. 60, 213-236 (1954).

5. Iskovskih, V.A.: Fano 3-folds I. Izv. Akad. Nauk. SSSR Ser. Mat. 41, 516-562 (1977)

6. Iskovskih, V.A.: Fano 3-folds II. Isv. Akad. Nauk. SSSR Ser. Mat. 42, 469-506 (1978)

7. Iskovskih, V.A.: Anticanonical models of three-dimensional algebraic varieties. J. Soviet Math. 13, 745-814 (1980)

8. Mukai, S., Umemura, H.: Minimal rational threefolds. Lect. Notes Math. 1016. Berlin, Heidelberg, New York: Springer 1983.

9. Mukai, S.: Curves, K-3 surfaces and Fano 3-folds of $g \leq 10$. (Preprint)

10. Remmert, R., Van de Ven, A.: Zwei Sätze über die komplex-projective Ebene. Niew Arch. Wisk. 8 (3), 147-157 (1960)

11. Umezu, Y.: On normal projective surfaces with trivial dualizing sheaf. Tokyo J. Math. 4, 343-354 (1981)

12. Wagreich, P: Elliptic singularities of surfaces. Amer. J. Math. 92, 419 - 454 (1970).

13. Yau, S.T.: On maximally elliptic singularities. Trans. A.M.S. 257, 269 - 329 (1980).

14. Laufer, H.: On minimally elliptic singularties. Amer. J. Math. 99, 1257 - 1295 (1977).