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MPI 96-37

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Abstract

m-soliton solutions of $(1+2)$ -dimensional Davey-Stewartson equation are constructed explicitly by means of known general solution of two-dimensional matrix Toda chain [1]. These solitons are expressed in terms of n-linear different solutions of the independent pair of linear matrix Schrodinger equations (each in $(1+1)$ dimension). Their potentials are arbitrary hermitian matrix functions of variables (t, x) or (t, y) respectively.

1 Introduction

This paper is a continuation of our recent work [1] in which general solution of equations of matrix Toda chain with fixed ends was represented in explicit form. Here we apply this solution to construct multi-soliton solutions

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of $(1+2)$ dimensional matrix Davey-Stewartson equation. We have chosen specially this nontrivial example to demonstrate the powerful method of discrete transformation [3],[4] and its applicability in different cases such as multidimensional integrable systems as well as integrable systems with internal structures.

We define the matrix Davey-Stewartson equation as the system of two equations for two matrix functions u, v of dimension s:

$$
\begin{aligned}
&iu_t + au_{xx} + bu_{yy} - 2au\int dy(vu)_x - 2b\int dx(uv)_y u = 0 \\
&-iv_t + av_{xx} + bv_{yy} - 2a\int dy(vu)_x v - 2bv\int dx(uv)_y = 0
\end{aligned}
\tag{1.1}
$$

where a, b are arbitrary numerical parameters (we will choose them below as $a = b = 1$) and x, y are the coordinates of two-dimensional space. In the case $s = 1$, when the order of multipliers is not essential (1.1) is the usual Davey-Stewartson equation for $v = u*$ [2].

2 Discrete substitution

By direct but tedious computations one can become convinced that the system (1.1) is invariant with respect to the following change of the unknown (matrix) functions;

$$\tilde{u} = v^{-1}, \quad \tilde{v} = [vu - (v_x v^{-1})_y]v \equiv v[uv - (v^{-1}v_y)_x] \tag{2.1}$$

The substitution (2.1) is the discrete transformation [3],[4] with respect to which all the equations of the matrix Davey-Stewartson hierarchy are invariant. In the case of a one-dimensional matrix Schrodinger equation this substitution was mentioned in [5].

The substitution (2.1) is invertible and the "old" functions u, v may be represented in terms of the new ones as:

$$v = (\tilde{u})^{-1}, \quad u = [\tilde{u}\tilde{v} - (\tilde{u}_y\tilde{u}^{-1})_x]\tilde{u} \equiv \tilde{u}[\tilde{v}\tilde{u} - (\tilde{u}^{-1}\tilde{u}_x)_y] \tag{2.2}$$

The substitution (2.1) may be rewritten in the form of an infinite chain of equations

$$((v_n)_x v_n^{-1})_y = v_n v_{n-1}^{-1} - v_{n+1}v_n^{-1}, \quad (u_{n+1} = v_n^{-1}) \tag{2.3}$$

where by (v_{n-1}, u_{n-1}) is to be understood as the result of the n-times application of the substitution (2.1) to some given matrix-functions (v_0, u_0).

Generally the chain (2.3) is infinite in both directions, but it may be interrupted by appropriate boundary conditions. The case when $v_{-1}^{-1} = v_{N+1} = 0$ we shall call the matrix Toda chain with fixed ends.

In the scalar case $s = 1$ the general solution of the Toda chain with fixed ends was found in [6] for all series of semisimple algebras except for E_7, E_8. In [7] this result was reproduced in terms of invariant root techniques applicable to all semisimple series.

The general solution of the matrix Toda chain with fixed ends in explicit form was found in [1]. In the present paper we will use this result for construction the multi-soliton solution of matrix Davey-Stewartson equation (1.1).

3 General strategie

Now we formulate the problem; it is necessary to find solution of the system (1.1) under additional condition of reality $u = v*$ where $z*$ means the hermitian conjugation of matrix z.

System (1.1) obviously possesses solution for which $u_0 = 0$. In this case first equation is satisfied automatically and the second one for unknown function v_0 may be rewritten as

$$-i\dot{v}_0 + (v_0)_{xx} + (v_0)_{yy} + V_1(t,x)v_0 + v_0 V_2(t,y) = 0$$

where V_1, V_2 are arbitrary $s \times s$ matrix functions of their arguments (the arising of terms of such kind is connected with that circumstance that the integrals $\int dx(uv)y$, $\int dy(uv)_x$ in the equations of the system (1.1) are determined only up to arbitrary functions of (t, y) or (t, x) arguments correspondingly).

Of course the condition of reality is not satisfied for this solution. But after application to it successively times discrete transformation (2.1) it will be possible to come to solution for which condition of reality will be satisfied.

To clarify situation let us consider the solution u, v for which condition of reality is satisfied $u = v*$. Let us applicate direct (2.1) and inverse (2.2) discrete transformation to this solution and denote results as u_1, v_1 and u_{-1}, v_{-1} respectively. It is not difficult to check that $u_{-1} = v*_1$ and $v_{-1} = u*_1$.

Continuing this procedure we obtain $u_{-m} = v*_m$ and $v_{-m} = u*_m$, where index $\pm m$ means m-times application of direct and inverse transformation for the initial solution for which condition of reality is satisfied. So if we begin from the solution with $u_0 = 0, v_0$ and after 2m -times application of discrete transformation obtain the solution of the form $u_{2m} = v*_0, v_{2m} = 0$ then for solution in the "middle" of the chain u_{m+1}, v_{m+1} condition of reality will be satisfied automatically.

Explicit form of solution of the system (2.3) for which $u_0 = v_{-1}^{-1} = v_{2m} = 0$ is known from our recent paper [1] (we have called it there as the matrix Toda chain with fixed end points). This solution has the form

$$v^0 = \sum_{r=1}^{2m} X_r(x)Y_r(y) \tag{3.1}$$

where X, Y are arbitrary $s \times s$ -matrix functions of their arguments.

So for construction of the solution of the problem as it was formulated in the beginning of this section it is necessary to discharge the following steps-find explicit expression for u_{2m}, find such the dependence of matrix functions X, Y on time argument that v_0 would be solution of the equation of the beginning of this section and at last satisfy condition of reality $u_{2m} = v*_0$. After this u_{m+1}, v_{m+1} will give us some partial (m-soliton) solution of the problem.

4 Scalar case

To obtain some experience in calculations at first we will consider the scalar case s=1, for which much of necessary calculations steps are wellknown and much simpler then in the general matrix case.

Solution of equations of discrete transformation in this case is coincided with solution of Toda chain with fixed ends and has the form [4]

$$u_k = \frac{Det_{k-1}}{Det_k} \quad v_k = \frac{Det_{k+1}}{Det_k}, \quad Det_{-1} \equiv 0, \quad Det_0 \equiv 1 \tag{4.1}$$

where Det_k are the principle minors of the matrix

$$\begin{pmatrix} v^0 & v^0_x & v^0_{xx} & .... \\ v^0_y & v^0_{xy} & v^0_{xxy} & ..... \\ v^0_{yy} & v^0_{xyy} & v^0_{xxyy} & ..... \\ ... & .... & .... & ..... \\ ... & .... & .... & ..... \end{pmatrix}$$

and v^0 is determined by (3.1), where X_r, Y_r are arbitrary scalar functions of their arguments.

From the formulae above it is not difficult to conclude that if $v^0 = \sum_{r=1}^{2m} X_r(x)Y_r(y)$, then $v_{2m} = 0$ and u_{2m} may be represented in the form

$$u_{2m} = \sum_{r=1}^{2m} \tilde{Y}_r(x)\tilde{X}_r(y) \tag{4.2}$$

where

$$\tilde{X}_r(x) = \frac{W_{2m-1}(X_1, X_2, ..., X_{r-1}, X_{r+1}, ...X_{2m})}{W_{2m}(X_1, X_2, ...X_{2m})}$$

and by W_k we denote the determinant of Vrosnki constructed from the functions in the brackets. The same expressions take place for $\tilde{Y}_r$ with obvious change $X \to Y$.

By help of (4.2) the conditions of reality may be rewritten in the form

$$\tilde{X}_r = X^*_{r'}, \quad \tilde{Y}_r = Y^*_{r'} \tag{4.3}$$

where under r' it is necessary to understand some of the possible $(2m)!$ permutations of $2m$ indexes r.

The equation for v_0 of the last section may be rewritten in terms of X_r, Y_r functions in following terms: each of the functions X_r, Y_r satisfy one-dimensional Schrodinger type equations potentials of which $V_1(t,x), V_2(t,y)$ are arbitrary functions of corresponding arguments:

$$-i\dot{X}_r + (X_r)_{xx} + V_1(t,x)X_r = 0 \quad - i\dot{Y}_r + (Y_r)_{xx} + Y_rV_2(t,y) = 0 \tag{4.4}$$

To satisfy (4.3) and (4.4) in explicit form it will be suitable represent functions $X_r(t,x), Y_r(t,u)$ in Frobenious like form

$$X_1 = \phi_1, \quad X_r = \phi_1 \int dx\phi_2 ... \int dx\phi_r \quad (2 \le s \le 2m) \tag{4.5}$$

All functions X_r are the solutions of the same equation (4.4) which in terms of functions ϕ_k may be rewritten as a system of equations

$$\dot{\phi}_r = (\phi_r(\ln \phi_r \prod_{k=1}^{r-1} \phi_k^2)')' \tag{4.6}$$

We have included imaginary unity i into time variable , which will be considered as pure imiganary from this moment.

We will use permutation $(2m, 2m-1, ...2, 1)$ in the conditions of reality (4.3) (for this case we can fulfill all calculations up to the end). In terms of functions ϕ_k we have for this condition

$$\phi*_r = \phi_{2m-r+2} \quad (r = 2, 3, ...2m), \quad \phi_{m+1} = \phi*_{m+1} = \frac{1}{\prod_{k=1}^{m} \phi_k \phi*_k} \tag{4.7}$$

System (4.6) of course is invariant with respect to such conditions of reality, what can be easily checked independently.

In terms of the new unknown functions $y_r = \prod_{k=1}^{r} \phi_k$ system (4.6) takes the form

$$\dot{(\frac{y_r}{y_{r-1}})} = (\frac{y_r}{y_{r-1}}(\ln y_r y_{r-1})')' \tag{4.8}$$

Let us consider now the equation of this system with $r = m+1$. Keeping in mind the condition of reality $y*_m = \frac{1}{y_{m+1}}$ we obtain

$$\dot{(\frac{1}{y_m y*_m})} = (\frac{1}{y_m y*_m}(\ln \frac{y_m}{y*_m}')' \tag{4.9}$$

From the last relation we can conclude that function $\frac{1}{y_m}$ is solution of one-dimensional Schrodinger equation potential of which W is arbitrary real function both times and spaces coordinates.

$$\dot{(\frac{1}{y_m})} + (\frac{1}{y_m})'' = W\frac{1}{y_m}, \quad W = W^* \tag{4.10}$$

Now let us consider equation from (4.8) with $r = m$

$$\dot{(\frac{y_m}{y_{m-1}})} = (\frac{y_m}{y_{m-1}}(\ln y_m y_{m-1})')' \tag{4.11}$$

and partially resolve it

$$\frac{y_m}{y_{m-1}} = z', \quad (\ln y_m y_{m-1})' = \frac{\dot{z}}{z'} \tag{4.12}$$

Excluding from the last system function y_{m-1} after some simple calculations we come to conclusion that function $\frac{z}{y_m}$ satisfy equation (4.10) exactly the same equation which function $\frac{1}{y_n}$ satisfy. Denoting these solutions by u_1, u_2 we obtain

$$\frac{1}{y_m} = u_1, \quad z = \frac{u_1}{u_2}, \quad \frac{1}{y_{m-1}} = \frac{z'}{y_m} = \frac{u_1 u_2' - u_2 u_1'}{u_1} \tag{4.13}$$

To continue further calculations it will be important for us two relations of equivalence both of which can be checked by help of well known Jacobi identity for determinants. Let we have some semi-limited in up and left directions matrix T. Under T_n we will understand its principle minors of n-th order counted from its left upper angle (for the corresponding matrices of this determinants we conserve the same notations without any confusions). $\vec{T}^s$ means the (semi-limited) matrix T with crossing out its s-th column; $\downarrow T^s$ means the matrix T with striking out its s-th row; $\downarrow \vec{T}^s$ means the matrix T with crossing out both s-th column and row simultaneously.

In this notations the Yacobi identity takes the form

$$T_n \downarrow \vec{T}_n^n - \vec{T}_n^n \downarrow T_n^n = T_{n+1} T_{n-1} \tag{4.14}$$

We want now to concretisize matrix T : let its first line consists from different (linear independent) solutions of equation (4.10) $u_1, u_2, ...$, the second one from its derivatives of first order $u_1', u_2', ...$, the third one from the derivatives of the second order $u_1'', u_2'', ...$ and so on (in other words this is construction of matrix of Vronski determinant). For so arisen matrix we will use notation U. Then as a direct corollary of this definition and (4.14) we obtain two identities

$$\frac{U_{n+1} U_{n-1}}{U_n^2} = (\frac{\vec{U}_n}{U_n})', \quad (\ln \frac{U_{n+1}}{U_{n-1}})' = \frac{U_n (\dot{\vec{U}_n}) - \vec{U}_n \dot{U}_n}{U_{n+1} U_{n-1}} \tag{4.15}$$

By help of the last identity general solution of the systems (4.8) and (4.6) may be represented in the form

$$y_{n-s}^{-1} = \frac{U_{s+1}}{U_s} \quad , \phi_{n+1} = u_1 u*_1$$

$$\phi_{n-s-1} = \frac{U_{s+1}U_{s-1}}{U_s^2}(0 \leq s \leq n-3), \quad \phi_1 = \frac{U_{n-1}}{U_n} \tag{4.16}$$

5 Matrix case

The aim of this section to generalize the above results on the matrix case. We emphasize that absolutely all formulae of last section admit such generalization but some times the known for us proofs are not sufficiently simple and we will omit them keeping in mind that reader will be able to do this better us or check the corresponding expressions by the methods of computer mathematics.

Let us introduce the following notations

$$R_n \equiv v_n^{-1}\dot{v}_n, \quad S_n^q \equiv \sum_{k=0}^{n-1}(\dot{S}_k^{q-1} + R_k S_k^{q-1})$$

with abbreviations of derivatives with respect to space coordinates $F_y \equiv \dot{F}, F_x \equiv F'$. As a direct corollary of (2.1) it arisen the recurrent relations for introduced above values S_n^q:

$$S_n^q = [(S_{n-1}^1)']^{-1}(S_{n-1}^{q+1})' \tag{5.1}$$

with boundary conditions

$$S_1^q = v_0^{-1}(v_0)_{yyy..y}$$

In this notations the explicit expression for v_{n+1} takes the form

$$v_{n+1} = v_n(S_{n+1}^1)' = v_0(S_1^1)'(S_2^1)'....(S_{n+1}^1)' \tag{5.2}$$

and may be calculated by help of above (5.1) recurrent relations.

In matrix case formula (4.2) conserve its form but connection of the "finally" matrices $\tilde{X}_r$ with initial ones X_r has some more complicate structure

$$(\tilde{X}_1)^{-1} = X_1(T_1^1)'(T_2^1)'....(T_{2m-1}^1)' \tag{5.3}$$

The functions T_s^1 introduced in (5.3) are partial case of the set of functions T_n^q which satisfy the following set of recurrent relations (compare with (5.1)):

$$T_n^q = [(T_{n-1}^1)']^{-1}(T_{n-1}^{q+1})' \tag{5.4}$$

with boundary conditions

$$T_1^q = X_1^{-1} X_{q+1}$$

All other functions $\tilde{X}_r$ have to be obtained from (5.3) by help of of one of $2m$ circle permutations of the lower indexes of the initial functions X_s.

Explicit expressions for $\tilde{Y}_r$ may be obtained from the corresponding expressions for matrices X_r by the operation of formal transposition and changing the index of differentiation. For instance (5.3) takes the form

$$(\tilde{Y}_1)^{-1} = (\dot{Q}_{2m-1}^1)....(\dot{Q}_1^1)Y_1 \tag{5.5}$$

In whole analogue to (5.4) it take place the recurrent relations for functions Q_n^s

$$Q_n^s = [(Q_{n-1}^1)^{\cdot}]^{-1}(Q_{n-1}^{s+1})^{\cdot}$$

with corresponding boundary conditions

$$Q_1^s = Y_{s+1} Y_1^{-1}$$

Representation of the initial functions X_r in Frobenious like form (with taking into account of the order of multipliers)

$$X_1 = \phi_1, \quad X_r = \phi_1 \int dx \phi_2 ... \int dx \phi_r, \quad (2 \leq s \leq 2m)$$

allows us by help of (5.3) find explicit expressions for finally functions

$$\tilde{X}_r = \int dx \phi_{2m-r+1} ... \int dx \phi_2 (\phi_1 \phi_2,,,, \phi_{2m})^{-1}, (2 \leq s \leq 2m),$$

$$(X_{2m})^{-1} = (\phi_1 \phi_2 .... \phi_{2m})$$

The fact that all functions X_r are the solution of the same equation (4.4) in terms of matrix ϕ_s functions takes the form of system of equations

$$-(\phi_s)_t + (2(\phi_1 \phi_2 ... \phi_{s-1})^{-1}(\phi_1 \phi_2 ... \phi_{s-1})' \phi_s + \phi_s')' = 0 \tag{5.6}$$

Condition of reality (also for the case of permutation of the previous section), compatible with the last system contains their form (4.7)

$$\phi_r^* = \phi_{2m-r+2} \quad (r = 2, 3, ...2m),$$

$$\phi_{m+1}^{-1} = (\phi_{m+1}^*)^{-1} = (\phi_1\phi_2...\phi_m) * (\phi_1\phi_2...\phi_)$$

From $(m+1)$-th equation of system (5.6) together with reality conditions we can conclude that matrix function $y_m^{-1} = (\phi_1\phi_2...\phi_m)$ is the solution of linear Schrodinger equation

$$\dot{(\frac{1}{y_m})} + (\frac{1}{y_m})'' = W\frac{1}{y_m}, \quad W = W^* \tag{5.8}$$

with Hermitian matrix potential W.

As in the previous section in terms of the new functions $y_m = \phi_1\phi_2....\phi_m$ system (5.6) may be written in the form

$$-(y_{r-1}^{-1}y_r)_t = (y_{r-1}^{-1}y_r' - (y_{r-1}^{-1})'y_r)' \tag{5.9}$$

Partially resolving m-th equation of the system (5.9) by the obvious substitution

$$y_{m-1}^{-1}y_m = z', \quad y_{m-1}^{-1}y_m' - (y_{m-1}^{-1})'y_m = z_t$$

and excluding the function y_{m-1} from the last equality we come to conclusion that function zy_m^{-1} satisfy the same equation (5.8) as the function y_m^{-1}. Denoting two linear independent solution of this equation by u_1, u_2 we obtain finally

$$y_m^{-1} = u_1, \quad y_{m-1} = (u_2u_1^{-1})'u_1$$

We represent now general solution of the system (5.9) . For proving of below formulae it is necessary to use the same procedure (more exactly its generalization on the matrix case) as in the scalar case

$$y_{m-r}^{-1} = (U_r^1)'....(U_1^1)'u_1' \tag{5.10}$$

where matrix functions U_n^q are determined by recurrent relations

$$U_n^q = [(U_{n-1}^1)']^{-1}(U_{n-1}^{q+1})'$$

with the boundary conditions

$$U_1^r = u_{r+1}u_1^{-1}$$

and functions u_r satisfy Schrodinger like equation (5.8).

To come to the finally result it is necessary to repeat all calculations with respect to Y_r functions, express them via solution of one-dimensional matrix Schrodinger equation potential of which is arbitrary Hermitian matrix of arguments (t, y). Then by help of formulae (5.2) reconstruct v_{m+1} by known X_r, Y_r. This will be solution of Davey-Stewartson equation which satisfy reality condition $u_{m+1} = v_{m+1}^*$.

6 The simplest example of one soliton solution

In this case $m = 1$ and corresponding formulae of the last section takes the form

$$v_0 = X_1Y_1 + X_2Y_2, \quad X_1 = \phi_1, X_2 = \phi_1(\int dx\phi_2) \quad , Y_1 = \psi_1, Y_2 = (\int dx\psi_2)\psi_1$$

For explicit expression for $u_1 = v*_1$ by which defined one-soliton solution we have

$$u_1 = (v_0)^{-1} = \psi_1^{-1}(1 + \int dx\phi_1 \int dy\psi_1)^{-1}\phi_1^{-1} \tag{6.1}$$

Matrix functions $\phi(t,x), \psi(t,y)$ are connected with solutions of one-dimensional Schrodinger equations by relations

$$\phi_1 = U, \quad \phi_2^{-1} = U^*U, \quad \psi_1 = V, \quad \psi_2^{-1} = VV^*$$

$$U_t + U_{xx} + W_1(t,x)U = 0, \quad V_t + V_{yy} + VW_2(t,y) = 0, \quad W_{1,2} = W_{1,2}^*$$

7 Conclusion

The main result of the paper is explicit expressions for m-soliton solutions for (1 + 2) dimensional matrix Davey-Stewartson equation. By means of the corresponding formulae of sections 4-5 it is possible to represent them via m linear independent solutions of the pair of one-dimensional (1 + 1) Schrodinger equations with Hermitian matrix potentials.

On the group theoretical level this means that we have found the realization of the finite-dimensional representation of the group of integrable mappings [4]. This viewpoint is beyond our concrete calculations.

We want to emphasize that the restriction to the finite-dimensional matrix case is absolutely nonessential. We have never used this fact and moreover, the dimension doesn't contribute in any expression. For instance, in one soliton solution, U, V can be considered as the second quantisized wave functions of the one dimensional Schrodinger equation. Most likely it is the bridge to the problem of quantum groups in two dimensions.

A.L. is grateful to the Max-Planck Institute for Mathematic in Bonn for the hospitality, where this work was prepared partly.

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