ENUMERATIVE COMBINATORICS OF INVARIANTS OF CERTAIN COMPLEX THREEFOLDS WITH TRIVIAL CANONICAL BUNDLE

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ABSTRACT. Minimal Calabi-Yau models can be roughly classified by studying the behaviour of the linear form being induced by their second Chern class on their nef cone. Strict positivity of $[c_2]_X$ on $\overline{\mathrm{Amp}(X)}$ leads to CY models X of *general type*. We consider a wide class of such models, namely 3-dimensional well-formed quasismooth complete intersections of hypersurfaces $X = X_{\mathbf{d}} = X_{(d_1,\dots,d_k)}$ in a weighted projective space $\mathbb{P}^{m-1}(\mathbf{w})$ with vanishing amplitude. We give *explicit* formulae for various *invariants* depending on two types of functions in the variables $\mathbf{w}$ and $\mathbf{d}$. Functions defined by the residua of some symmetric polynomial expressions of $\mathbf{w}$ and $\mathbf{d}$ on the one hand, and enumerating functions of weighted partitions on the other. If X admits non-pathological stratifications, these formulae enable us to determine the delta genus $\Delta(X, \mathbf{L}_X)$ arising from the natural polarization with respect to $\mathbf{L}_X$. We give a partial generalization of results of Oguiso whenever $\Delta(X, \mathbf{L}_X) \leq 2$, and present, for $k = 1$, the "geographical chart" of the pairs $(\mathbf{L}_X^3, [c_2]_X(\mathbf{L}_X))$ even in the case in which $\Delta(X, \mathbf{L}_X) \geq 3$.

Moreover, we describe the construction and some basic properties of the *toroidal crepant desingularizations* of X's and compute their invariants by using certain "local-global principles" concerning the combinatorially controllable contributions of the exceptional divisors to the corresponding invariants of the starting point models. Finally, $[c_2]$ -forms, "triple couplings" and "testing bilinear forms" pave the way for the development of *a formal algorithm*, by means of which one can mostly decide if two distinct toroidal crepant desingularizations have definitely different diffeomorphism (resp. homotopy) types or not.

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Introduction

There are at least two reasons which have made the study of *Calabi-Yau manifolds* so attractive during the last decade. The former is that they represent the high dimensional analogues of $K3$-surfaces and are naturally expected to inhabit in some very interesting "moduli space landscapes", both from algebraic geometrical and from differential geometrical point of view. The latter is their pivotal role in the framework of the development of certain *conformal field theories*, like those corresponding to the so called *one-loop semiclassical non-linear sigma models*, (CY manifolds are used as the best candidates for being fibers of the "target spaces" of these sigma models. For an introduction to these themes we refer to the book of Hübsch [65].)

In algebraic geometry, *threefolds with trivial canonical bundle* occupy a very special place within the "3-dimensional cosmography" and one hopes that the methods which will be required for the solution of a number of important open problems regarding period maps, diffeomorphism types, possible bounds of Betti or Hodge numbers, existence and possible "enumeration" of rational curves etc., will considerably promote the whole classification programme of higher dimensional algebraic varieties. For a wonderful survey article written under this perspective, see Friedman [42].

In theoretical physics, on the other hand, where certain *concrete* constructions are needed, a string propagation in a Calabi-Yau background can be expressed geometrically in a convenient way, only in connection with predeterminating Landau-Ginzburg effective Langrangians. By Witten's generalized "LG/CY-correspondence" [129, § 5], one concludes that the most "favourable" CY manifolds have to be either *hypersurfaces* or *complete intersections* embedded in projective spaces, in weighted projective spaces or products thereof, in general toric varities or even in Grassmannians.

The case of complete intersections in a product of *usual* projective spaces is discussed in great detail in the above mentioned book of Hübsch [65]. The next CY threefolds coming into question, namely *quasismooth hypersurfaces* in a 4-dimensional weighted projective space (or, more general, in a 4-dimensional Fano toric variety), together with their crepant desingularizations, have been the focal point of many researches during the past few years. Experimental observation at the beginning [15], showing a remarkable "dualism" between the non-trivial Hodge numbers $h^{1,1}$ and $h^{1,2}$ of the desingularized models, turned out later not to be an irony of fate, but the revelation of an exciting *symmetry* with a deep geometrical interpretation and inestimable, up to now, futuristic consequences. For a first mathematical approach to this symmetry the reader is referred to the articles of Roan [102], Morrison [91] and Batyrev [7] and to the collected papers in [131].

However, very little is known for the corresponding *complete intersection case.* In our work we attempt to enlighten that part of *enumerative combinatorics* which is necessary for the description of the invariants and of other important numbers characterizing the desingularized complete intersections in a weighted projective space. Although our results could be valid (with minor modifications) in a more general setting, we prefer to restrict ourselves to weighted projective spaces, as these lead directly to problems on *linear diophantine equations* or, if you wish, on *linear programming* depending on systems of certain "weights".

More precisely, the organization of the paper is as follows. After reviewing some basic facts from Wilson's classification theory of *minimal CY models* in § 1, we explain all the essential details of our construction in § 2, give the formulae for the corresponding invariants, and show how the *delta genera distribution* depends on the *denumerants of weighted partitions*. (We should notice that most of the results of § 2 are actually *independent* of the dimension and of the amplitude of X's, although in the end we focus attention on the CY threefold/model case. For certain interesting new aspects of applications of higher dimensional complete intersections in weighted projective spaces with *negative* amplitude for the realization of some useful, suitably modified (i.e. "non-classical") Landau-Ginzburg theories, see Schimmrigk [105].)

Using toric geometry in § 3, we describe the distinctive features of the T_{N_G}-equivariant crepant resolutions of 2- and 3-dimensional *Gorenstein cyclic quotient singularities* $Z(N_G, \Sigma_0)$, such as the nature of the occuring exceptional prime divisors, their enumeration by their types, their intersection numbers, and the elementary transformation mechanism. Globalizing this resolution process along the components of the singular loci of our X's in § 4, we define "toroidal" crepant desingularizations and compute their non-trivial Hodge numbers in two different ways. *Intrinsically*, with the step by step recognition of the singularity types, and, when possible, *explicitly* (in terms of $\mathbf{w}$ and $\mathbf{d}$) by making use of relative Milnor fibrations, eventually after a rearrangement of the defining polynomials. ($h^{1,1}$ equals obviously the *Picard number* and $h^{1,2}$ "counts" *the moduli number of complex structures.*)

Section 5 deals with $[c_2]$ *-forms* and *intersection trilinear forms* (or, in other words, "topological triple couplings") of the desingularized models Y. Their evaluations at the members of the canonical $\mathbb{Q}$-bases of the Picard group of Y are encoded partially in the local informations coming from the data of the "toric triangles" lying over the dissident points of X, and partially in the global geometry of the exceptional divisors and of the pull-back divisor $\mathbf{L}_Y$ on Y. These evaluations lend to the various desingularization spaces Y's a significant topological characterization, which, in connection with classification results of Wall, Jupp, Žubr and Sullivan, allows us to *distinguish* (in most of the cases) *diffeomorphism* (resp. *homotopy) types*. This method is indicated in § 6, where an arithmetical example is examined thoroughly. On the other hand, the formulae of our main theorems of § 5 seem to have direct applications to physically important CY threefolds, as they describe the "unquantized" part of the $(1,1)$ *-level Yukawa couplings*, and they have been already used for computations of some special examples in [64].

Finally, the appendix of § 7 is entirely devoted to the *pure combinatorial ingredients* of our formulae, namely to the *pt*-functions which date back to the monumental work of Euler on the "Partitio Numerorum". Apart from some historical remarks, we manifest here their immediate interpretation as *Ehrhart quasipolynomials* of a dilated special rational convex polyhedron. In addition, in the case where this (or a closely related to it) polyhedron is *integral*, we give formulae which express the *pt*-functions by means of the volumes of appropriate polyhedral faces.

Basic notations and definitions

(i) We denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}_{\geq 0}, \mathbb{R}$ and $\mathbb{C}$ the set of natural, non-negative integer, integer, rational, non-negative real, real and complex numbers respectively.
(ii) "gcd" and "lcm" are abbreviations for *greater common divisor* and *lower common multiple*.
For $l \in \mathbb{N}$ and $m \in \mathbb{Z}$, we denote by $[m]_l$ the integer which satisfies $0 \leq [m]_l < l$ and $m \equiv [m]_l(\text{mod}l)$. Furthermore, for $n, k \in \mathbb{N}_0$, $n \geq k$, we set:

$$\binom{n}{k} := \frac{n^{[k]}}{k!}, \quad \text{where} \quad n^{[k]} := n(n-1)\cdots(n-k+1).$$

(iii) $\mathbb{Z}/(n\mathbb{Z})$, $n \in \mathbb{N}$, will denote the cyclic group of order n and $\zeta_n := \exp(\frac{2\pi\sqrt{-1}}{n})$ the "first" $n-th$ primitive root of unity.
(iv) $|S|$ or $\sharp(S)$ are used to express the number of elements of a finite set S. For $S \subset \mathbb{N}_0$, δ_S denotes *the characteristic function of* S, i.e.

$$\delta_S(s) = \begin{cases} 1, & \text{for} \quad s \in S \\ 0, & \text{otherwise} \end{cases}$$

On the other hand, $\delta_{p,q}$ denotes the usual *Kronecker symbol.*
(v) Let A be a local ring with maximal ideal $\mathfrak{M}$. A is called *regular* (resp. *normal*) if $\dim(A) = \dim(\mathfrak{M}/\mathfrak{M}^2)$ (resp. if its localizations are integrally closed domains). A sequence $\{a_1, \ldots .a_s\}$ of elements of A is called *regular sequence* if $A \neq (a_1, \ldots, a_s)A$ and if for all $i \in \{0, \ldots, s-1\}$, a_{i+1} is not a zero divisor in $A/(a_1, \ldots, a_i)A$. The *depth* of A is defined to be the maximum of the lengths of regular sequences $\{a_1, \ldots, a_s\}$ with $a_i \in \mathfrak{M}$, $\forall i$, $1 \leq i \leq s$. A is called *Cohen-Macaulay* if $\dim(A) = \text{depth}(A)$. If A is Cohen-Macaulay, then A is called *Gorenstein* whenever $\text{Ext}_A^{\dim(A)}(A/\mathfrak{M}, A) \cong A/\mathfrak{M}$.
(vi) In section 2 we shall consider certain graded commutative rings $A = \bigoplus_{d \geq 0} A_d$ with $A_0 = \mathbb{C}$ the field of complex numbers and A finitely generated as $\mathbb{C}$-algebra. We denote by $\mathfrak{M} := A_+ := \bigoplus_{d>0} A_d$ the unique maximal ideal of such a ring A. A *graded A-module* is an A-module M, together with a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ such that $A_d \cdot M_e \subset M_{d+e}$. For any graded A-module M, and for any $n \in \mathbb{Z}$, we define the *twisted module* $M(n)$ by shifting n places to the left, i.e. $M(n)_d = M_{d+n}$.
(vii) For $q \in \mathbb{N}_0$, let Ext_A^q denote the derived functors of Hom_A within the category $\mathcal{GM}(A)$ of graded A-modules. $H_{\mathfrak{M}}^q : \mathcal{GM}(A) \to \mathcal{GM}(A)$ is defined to be the functor which sends a graded A-module M to the $q-th$ *algebraic local cohomology group*

$$H_{\mathfrak{M}}^q(M) := \varinjlim_{l} \text{Ext}_A^q(A/\mathfrak{M}^l, M)$$

of M supported at $\mathfrak{M}$ (cf. [55], [56]).
(viii) Let A be a graded ring and M a graded A-module as in (vi).

$$X = \text{Proj}(A) := \{\mathfrak{p} \in \text{Spec}(A) | \mathfrak{p} \quad \text{homogeneous and} \quad \mathfrak{p} \not\supseteq A_+\}$$

will denote, as usual, the projective scheme associated to A, $\tilde{M}$ the $\mathcal{O}_X$-module sheaf associated to M on X, $\mathcal{O}_X(n) := A(n)^\sim$, $n \in \mathbb{Z}$, $M(n)^\sim$ the twisted sheaf

associated to $M(n)$ and $\tilde{M}(n) := \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ (cf. [53, Ch. II, § 2.5 - § 2.6.] or [61, Ch. II, § 5]). Note that, if A is generated as a $\mathbb{C}$-algebra by A_1, then $M(n)^{\sim} \cong \tilde{M}(n)$, but in general this is not true.
(ix) The *Poincaré-series* of a $\mathbb{Z}$-graded vector space $A = \bigoplus_{\nu \in \mathbb{Z}} A_\nu$ with finite-dimensional homogeneous components is defined as the formal Laurent series

$$\mathcal{P}(A;x) := \sum_{\nu \in \mathbb{Z}} (\dim A_\nu) x^\nu$$

Correspondingly, the *Poincaré series* of a projective scheme X is the formal series

$$\mathcal{P}(X,x) := \sum_{n \in \mathbb{Z}} (\dim(H^0(X, \mathcal{O}_X(n))) x^n$$

(x) Let A be again a graded $\mathbb{C}$-algebra. $\Omega_A^1 := \Omega_{A/\mathbb{C}}^1$ denotes the A-module of *Kähler $\mathbb{C}$-differentials* of A and $\Omega_A^p := \wedge^p \Omega_A^1$, $\forall p$, $p \in \mathbb{N}_0$. Furthermore, if for a homogeneous $h \in A_+$, $A_{(h)}$ is the subring of elements of degree 0 in the localized ring A_h, then $\{\mathrm{Spec}(A_{(h)}) | h$ homogeneous element of $A_+\}$ is a basis of $X = \mathrm{Proj}(A)$ and the $\mathcal{O}_X$-module sheaf of germs of *p- forms* Ω_X^p can be defined by globalization, so that

$$\Omega_X^p|_{\mathrm{Spec}(A_{(h)})} \cong \Omega^p_{\mathrm{Spec}(A_{(h)})} \cong (\Omega^p_{A(h)})^{\sim}.$$

(By $(\Omega_X^p)^\vee := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^p, \mathcal{O}_X)$ and $(\Omega_X^p)^{\vee\vee}$ we shall denote the *dual* and the *bidual* of the sheaf of p-forms on X respectively.)
(xi) By a *complex variety* we mean an integral, separated algebraic scheme over $\mathbb{C}$. A complex variety is *complete* if its structural morphism to $\mathrm{Spec}(\mathbb{C})$ is proper. If X is a complex variety, then a point $x \in X$ (resp. the whole space X) will be called *regular, normal, Cohen-Macaulay* or *Gorenstein* if the local ring $\mathcal{O}_{X,x}$ (resp. all local rings $\mathcal{O}_{X,x}$, $\forall x \in X$) is (resp. are) of this type. In particular, we set

$$\mathrm{Reg}(X) := \{x \in X : \mathcal{O}_{X,x} \quad \text{regular}\} \quad \text{and} \quad \mathrm{Sing}(X) := X \setminus \mathrm{Reg}(X)$$

for the *regular* and the *singular locus* of X respectively. A (closed) *subvariety* Y of X is a closed integral subscheme of X. A subvariety Y of X with $\mathrm{codim}_X(Y) = 1$ is especially called *a prime divisor* of X. A *Weil divisor* is an element of the free abelian group which is generated by the prime divisors of X.
(xii) Let X be a *normal* r-dimensional complex variety. If D is a Weil divisor of X, let $\mathcal{O}_X(D)$ denote the corresponding divisorial sheaf (cf. [98, App. to § 1]). D is called *Cartier divisor* if $\mathcal{O}_X(D)$ is invertible. For r Cartier divisors $D_1, \ldots, D_r$, for which $W := \bigcap_{i=1}^r \mathrm{supp}(D_i)$ is *complete*, one defines their *intersection number* as $(D_1 \cdot D_2 \cdots D_r) := \deg_W(D_1 \cdots D_r) \in \mathbb{Z}$ (see e.g. [46, Ch. 2]). Moreover, if $j : \mathrm{Reg}(X) \hookrightarrow X$ is the natural inclusion of the regular locus of X into X, we define $\tilde{\Omega}_X^\bullet := j_*(\Omega^\bullet_{\mathrm{Reg}(X)}) = j_*(j^*\Omega_X^\bullet)$. $\omega_X := \tilde{\Omega}_X^r$ is called the *canonical* or *dualizing sheaf* of X. Note that: X is Gorenstein $\Longleftrightarrow$ X is Cohen-Macaulay and ω_X is invertible. On the other hand, we define $\bar{\Omega}_X^\bullet := \pi_*\Omega_Y^\bullet$, where $\pi : Y \to X$ is an arbitrary desingularization of X. We have an inclusion $\bar{\Omega}_X^p \hookrightarrow \tilde{\Omega}_X^p$, and X has at

most *rational singularities* $\Longleftrightarrow X$ is Cohen-Macaulay and $\bar{\Omega}^\bullet_X \cong \tilde{\Omega}^\bullet_X$.
(xiii) A complex variety is called *V-variety* (or rather *Q-variety*) if it has at most *quotient singularities.* If X is a projective V-variety, then

$$\bar{\Omega}^\bullet_X \cong \tilde{\Omega}^\bullet_X \cong (\Omega^\bullet_X)^{\vee\vee}$$

because any quotient singularity is rational and X is normal.
(xiv) For a complex variety X of dimension r, we denote by $b_i(X) := \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$, $0 \leq i \leq 2r$, $e(X) := \sum_{i=0}^{2r}(-1)^i b_i(X)$, $h^i(X, \mathcal{F}) := \dim_{\mathbb{C}} H^i(X, \mathcal{F})$, $\chi(X, \mathcal{F}) := \sum_{i=0}^{\infty}(-1)h^i(X, \mathcal{F})$, the $i-th$ Betti number of X, the topological Euler-Poincaré characteristic of X, the dimension of the $i-th$ cohomology group of a coherent sheaf $\mathcal{F}$ over X and the corresponding Euler-Poincaré characteristic of $\mathcal{F}$ over X respectively. Pic(X) will denote the *Picard group* of X, i.e. the group of isomorphism classes of invertible sheaves (or line bundles) over X. (Line bundles will be identified with linear equivalence classes of Cartier divisors.)
(xv) A pair $(X, \mathbf{L})$ consisting of a normal complete complex variety and an ample (resp. nef and big) bundle $\mathbf{L}$ over X is called a *polarized* (resp. *quasi-polarized*) *variety.* Fujita's *delta genus* of an r-dimensional polarized variety $(X, \mathbf{L})$ is defined by

$$\Delta(X, \mathbf{L}) := r + \mathbf{L}^r - h^0(X, \mathbf{L})$$

and turns out to be a very powerful invariant of $(X, \mathbf{L})$ as it leads to a partial (or, sometimes, complete) classification of such pairs, when it takes values which are small enough. (For an introduction to the corresponding classification theories and adjunction techniques we refer to Fujita's monograph [45].)

§ 1. Calabi-Yau models

This section is introductory and serves as a reminder of certain fundamental properties of CY threefolds and of their singular analogues.

Definition 1.1. By a *Calabi-Yau threefold (CY threefold)* we mean 3-dimensional complete, projective, smooth complex variety Y with trivial canonical class and $h^1(Y, \mathcal{O}_Y) = 0$. (Note that $h^2(Y, \mathcal{O}_Y) = h^{2,0}(Y) = 0$ by Serre and Hodge duality).

Thanks to Yau's verification of Calabi's conjecture [130], the representative of any such threefold Y in the analytic category admits a Ricci-flat metric. The topological Euler-Poincaré characteristic of Y is given by

$$e(Y) = 2(h^{1,1}(Y) - h^{1,2}(Y)) = 2(b_2(Y) + 1) - b_3(Y) \tag{1.1}$$

From the exponential cohomology sequence we get $\mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z})$ and $\rho(Y) = h^{1,1}(Y) = b_2(Y)$, where $\rho(Y)$ denotes the *Picard number* of Y. On the other hand, the second non-trivial Hodge number $h^{1,2}(Y)$ of Y express the number of parameters for the complex structure on Y in the following sense:

Theorem 1.2. *(Bogomolov [12], Tian [118], Todorov [119]) The first order deformations of a CY threefold Y are unobstructed, and the corresponding local moduli space of Y is smooth and has dimension $h^{1,2}(Y) = h^1(Y, \Theta_Y)$.*

Moreover, general structure theorems, due to Beauville, Bogomolov, Kobayashi and Michelson, inform us that a CY threefold Y has finite fundamental group unless some finite unramified covering of it is either an abelian threefold or is decomposable into a product of a K-3 surface with an elliptic curve (see [8], [9]). Up to these two cases, in which $e(Y) = 0, Y$ has the whole $SU(3)$ as holonomy group.

Definition 1.3. Let Y be a complete, smooth (but not necessarily projective) complex threefold with $h^1(Y, \mathcal{O}_Y) = h^2(Y, \mathcal{O}_Y) = 0$ and trivial canonical class. We define:

$$\begin{aligned}
[c_2]_Y : \mathrm{Pic}(Y) \ni \mathcal{O}_Y(D) &\longmapsto (c_2(Y) \smile c_1(\mathcal{O}_Y(D)))[Y] \in \mathbb{Z}, \\
q_Y : (\mathrm{Pic}(Y))^3 \ni (\mathcal{O}_Y(D_1), \mathcal{O}_Y(D_2), \mathcal{O}_Y(D_3)) &\longmapsto (D_1 \cdot D_2 \cdot D_3) = \\
&(c_1(\mathcal{O}_Y(D_1)) \smile c_1(\mathcal{O}_Y(D_2)) \smile c_1(\mathcal{O}_Y(D_3)))[Y] \in \mathbb{Z}
\end{aligned}$$

the linear form on $\mathrm{Pic}(Y)$ induced by the second Chern class of Y and the trilinear symmetric form induced by intersection numbers respectively. In the physics literature, in the case in which Y is a CY threefold, the latter is usually called *the unquantized topological Yukawa coupling form.* (Remark: We shall use the notations $[c_2]_Y^{\mathbb{Q}}$ and $q_Y^{\mathbb{Q}}$ (resp. $[c_2]_Y^{\mathbb{R}}$ and $q_Y^{\mathbb{R}}$) if we work with $\mathrm{Pic}_{\mathbb{Q}}(Y) := \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\mathrm{Pic}_{\mathbb{R}}(Y)$) instead of $\mathrm{Pic}(Y)$.)

As it is known from the classification theory of simply connected, compact, oriented, 6-dimensional $\mathcal{C}^\infty$-differentiable manifolds with vanishing second Stiefel-Whitney class, developed by Wall [122], Žubr [132] and Sullivan [114], the diffeomorphism type of simply connected complex threefolds Y satisfying the properties of 1.3. is determined, up to finite possibilities, by means of the quadruple $(H^2(Y,\mathbb{Z}), b_3(Y), -2[c_2]_Y, q_Y)$. In particular, if $H^3(Y,\mathbb{Z})$ is assumed to be torsion-free, this quadruple classifies Y (up to a diffeomorphism) uniquely. (For analogous classification theorems up to an orientation-preserving homotopy equivalence or up to a homeomorphism, see Jupp [68] and Žubr [132], [133].)

Unfortunately, these theorems cannot be applied directly in concrete examples, because
(i) there is no satisfactory way to check whether two symmetric trilinear forms are equivalent up to change of basis or not,
(ii) it is often very difficult to find explicit integer bases of $H^2(Y,\mathbb{Z})$, and
(iii) there is not always adequate information available about the torsion part of $H^3(Y,\mathbb{Z})$.

In practice, one tries to develop methods to distinguish, if possible, diffeomorphism (resp. homotopy) types, just by keeping necessary conditions of the above theorems and by introducing further controllable numerical invariants, which could hopefully be different for the regarded threefolds. Motivated by similar considerations of Green and Hübsch ([49, p. 314], [65, p. 174]), we give the following definition:

Definition 1.4. Let Y be a complex threefold as in 1.3. We define

$$\varphi_Y : (\mathrm{Pic}(Y))^4 \ni (\mathcal{O}_Y(D_1), \mathcal{O}_Y(D_2), \mathcal{O}_Y(D_3), \mathcal{O}_Y(D_4)) \longmapsto$$
$$(q_Y(\mathcal{O}_Y(D_1), \mathcal{O}_Y(D_2), \mathcal{O}_Y(D_3)) \cdot [c_2]_Y(D_4) + \text{cyclic permutations}) \in \mathbb{Z}$$

φ_Y is a symmetric quadrilinear form, which induces a bilinear form:

$$\beta_Y : (\mathrm{Sym}^2(\mathrm{Pic}(Y)))^2 \to \mathbb{Z} \tag{1.2}$$

(We just define the image of a pair of decomposable elements of $\mathrm{Sym}^2(\mathrm{Pic}(Y))$ under β_Y to be the evaluation of φ_Y at its members and we extend linearly.) β_Y will be called *the testing bilinear form* of Y.

The negation direction of the statement of the next lemma will be very useful.

Lemma 1.5. *Let Y_1, Y_2 be two simply connected complex threefolds satisfying the properties mentioned in 1.3. A necessary condition, under which Y_1 and Y_2 have the same diffeomorphism (resp. homotopy) type, is the identification of their Betti numbers and the existence of an isomorphism $f : H^2(Y_1,\mathbb{Z}) \stackrel{\cong}{\to} H^2(Y_2,\mathbb{Z})$, such that $[c_2]_{Y_1}(\cdot) = [c_2]_{Y_2}(f(\cdot))$ and $q_{Y_1}(\cdot,\cdot,\cdot) = q_{Y_2}(f(\cdot), f(\cdot), f(\cdot))$.*
In particular, in this case, β_{Y_i} (resp. $\beta^{\mathbb{Q}}_{Y_i}, \beta^{\mathbb{R}}_{Y_i}$), $i = 1,2$, will be equivalent as $\mathbb{Z}$- (resp. $\mathbb{Q}-, \mathbb{R}-$) bilinear forms.

Proof. It follows from the vanishing of the second Stiefel-Whitney class. □

We shall come back to it with an example in section 6.

Let us now turn our attention to the singular models.

Definition 1.6. ([126], [127]) A complete, projective, normal 3-dimensional complex variety X with trivial dualizing sheaf and $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$ is called *CY model* if it allows at most rational Gorenstein singularities (i.e. canonical singularities of index 1 in Reid's terminology [98], [99]) and if there is a (necessarily crepant) desingularization $\pi : Y \to X$ of X with Y a CY threefold. A *CY contraction* of a CY model X_1 is defined to be a birational morphism $f : X_1 \to X_2$ to another CY model X_2, such that $\rho(X_2) < \rho(X_1)$. A CY model is called *minimal* if it does not admit any CY contraction.

Definition 1.7. Let X be a CY model and $\pi : Y \to X$ a crepant desingularization of X. We define the linear form $[c_2]_X : \mathrm{Pic}(X) \to \mathbb{Z}$ by $[c_2]_X(\mathcal{O}_X(D)) := [c_2]_Y(\pi^* \mathcal{O}_X(D))$, for all Cartier divisors D on X. (Note that $[c_2]_X$ is essentially independent of the concrete choice of π.)

If we now denote by $\mathrm{Amp}(X)$ *the ample cone* of X in $\mathrm{Pic}_{\mathbb{R}}(X)$, generated by the real classes of ample Cartier divisors, its closure $\overline{\mathrm{Amp}(X)}$ parametrizes the real classes of nef Cartier divisors and is dual to *Mori's cone* $\overline{\mathrm{NE}}(X)$ consisting of the real classes of effective 1-cycles. We call $\overline{\mathrm{Amp}(X)}$ *the nef cone* of X. By a result of Miyaoka [87, thm. 6.6., p. 468] we deduce:

Theorem 1.8. *The linear form $[c_2]_X^{\mathbb{R}}$, which is associated to a CY model X, takes non-negative values on the nef cone $\overline{\mathrm{Amp}(X)} \subset \mathrm{Pic}_{\mathbb{R}}(X)$ of X.*

Various properties of the nef cone $\overline{\mathrm{Amp}(X)}$ of CY models X have been studied extensively by Wilson [126], [127], [128], who proposed to use $[c_2]_X^{\mathbb{R}}$ in a role parallel to the one played by the canonical divisor in the classification theory of compact complex surfaces, in order to achieve a first type separation for X's. Wilson's rough classification of *minimal* CY models is outlined in the following table:

	Behaviour of $[c_2]_X^{\mathbb{R}}$ on $\overline{\mathrm{Amp}(X)}$	Type of X
(a)	$[c_2]_X^{\mathbb{R}}$ is trivial on $\overline{\mathrm{Amp}(X)}$	abelian quotient type
(b)	non-trivial but not strictly positive	fibering type (?)
(c)	strictly positive on $\overline{\mathrm{Amp}(X)}$	general type

Shepherd-Barron and Wilson [107] proved that the threefolds (a) can always be represented as quotients of abelian threefolds by (not necessarily freely acting) finite groups. Wilson [128] investigated certain models belonging to case (b) and conjectured the existence of fiber space structure for any such X. In fact, cases (a) and (b) include minimal CY models of "special type" and there should be a complete "fine" classification for them, whereas (c) constitutes the "general case" in which, analogously to the surfaces of general type, there are still a lot of open questions arising from "geographical problems". For instance, a minimal CY model of general type X is equipped with a canonical polarization coming from c_2 in a natural way. If $\mathbf{L}$ is an ample line bundle on X and $\Delta(X, \mathbf{L})$ the corresponding delta genus, what kind of lattice regions should be expected to be covered by its values? In which regions do $\mathbf{L}^3$ and $[c_2]_X(\mathbf{L})$ reside? Finally, if $\Delta(X, \mathbf{L}) \geq 3$, what kind of relationships are there between them (and eventually the topological invariants of X) besides the standard RR-inequality $[c_2]_X(\mathbf{L}) \leq 10\ \mathbf{L}^3$? (Is it possible to get any absolute or relative new bounds?)

In the present paper we construct minimal CY models of general type by considering certain quasismooth complete intersections X in a weighted projective space with vanishing amplitude. Especially, we emphasize the combinatorial complexity of the above mentioned numbers, and we study the forms q_Y and $[c_2]_Y$ of some natural crepant desingularizations Y of X in detail.

§ 2. Complete intersections in weighted projective spaces

In this section we recall briefly some basic facts from the theory of complete intersections in a weighted projective space, we prove a Lefschetz-type theorem for dimensions ≥ 3 and we give combinatorial formulae which enable the determination of all the interesting invariants and of the delta genera. For an introduction to the theory of weighted projective spaces, the reader is referred to the expository articles of Delorme [28], Dolgachev [32] and Beltrametti-Robbiano [10].

Definition 2.1. For $m \in \mathbb{N}$, let $\mathbb{P}^{m-1} = \mathbb{P}^{m-1}(\mathbf{1})$ denote the usual complex $(m-1)$ -dimensional projective space. If $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{N}^m$, we define $S(\mathbf{w})$ to be the polynomial algebra $\mathbb{C}[z_1, \ldots, z_m]$ over $\mathbb{C}$, graded by the condition $\deg(z_i) = w_i, \forall i, 1 \leq i \leq m$. The $(m-1)$ -dimensional *weighted projective space (w.p.s.)* $\mathbb{P}^{m-1}(\mathbf{w})$ is defined as the irreducible normal projective variety

$$\mathbb{P}^{m-1}(\mathbf{w}) := \mathrm{Proj}(S(\mathbf{w})).$$

$\mathbb{P}^{m-1}(\mathbf{w})$ is isomorphic to $\mathbb{P}^{m-1}/\prod_{i=1}^{m}(\mathbb{Z}/w_i\mathbb{Z})$, and the canonical projection

$$p(\mathbf{w}) : \mathbb{P}^{m-1} \longrightarrow \mathbb{P}^{m-1}(\mathbf{w})$$

corresponds to the canonical ramified covering:

$$[x_1, \ldots, x_m] \mapsto [z_1, \ldots, z_m], \quad z_i := x_i^{w_i}, \quad \forall i, \quad 1 \leq i \leq m,$$

with Galois group $\prod_{i=1}^{m}(\mathbb{Z}/w_i\mathbb{Z})$. Equivalently, one defines $\mathbb{P}^{m-1}(\mathbf{w})$ as the geometric quotient $(\mathbb{C}^m \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by:

$$\mathbb{C}^* \times (\mathbb{C}^m \setminus \{0\}) \ni (t, (z_1, \ldots, z_m)) \longmapsto (t^{w_1} z_1, \ldots, t^{w_m} z_m) \in (\mathbb{C}^m \setminus \{0\}).$$

Its associated projection map will be denoted by

$$\pi(\mathbf{w}) : (\mathbb{C}^m \setminus \{0\}) \longrightarrow \mathbb{P}^{m-1}(\mathbf{w}).$$

We shall say that $\mathbf{w}$ is *reduced* (resp. *normalized*) if $\gcd(w_1, \ldots, w_m) = 1$ (resp. if $\gcd(w_1, \ldots, \hat{w}_i, \ldots, w_m) = 1, \forall i, 1 \leq i \leq m$). $\mathbb{P}^{m-1}(\mathbf{w})$ is called *well-formed* if $\mathbf{w}$ is normalized.

Definition 2.2. For $m \in \mathbb{N}$ and $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{N}^m$ an arbitrary m-tuple of weights, we define

$$\overline{w}_i := \frac{w_i}{\gcd(w_1, \ldots, w_m)},$$

$$\rho_i(\mathbf{w}) := \gcd(\overline{w}_1, \ldots, \widehat{\overline{w}_i}, \ldots, \overline{w}_m),$$

and

$$w_i' := \frac{\overline{w}_i}{\operatorname{lcm}(\rho_1(\mathbf{w}),\ldots,\widehat{\rho_i(\mathbf{w})},\ldots,\rho_m(\mathbf{w}))} \quad \text{for all} \quad i, \quad 1 \leq i \leq m.$$

$\bar{\mathbf{w}} := (\overline{w}_1, \ldots, \overline{w}_m)$ (resp. $\mathbf{w}' := (w_1', \ldots, w_m')$) will be called the *reduction* (resp. *the normalization) of* $\mathbf{w}$.

Proposition 2.3. *([10,3.A.3, 3.C.5]) There exist natural isomorphisms:*

$$\mathbb{P}^{m-1}(\mathbf{w}) \cong \mathbb{P}^{m-1}(\bar{\mathbf{w}}) \cong \mathbb{P}^{m-1}(\mathbf{w}')$$

.

Remark 2.4. In contrast to the case of a usual projective space, the twisted sheaves $\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n) = S(\mathbf{w})(n)^{\sim}$ on a weighted projective space $\mathbb{P}^{m-1}(\mathbf{w})$ are, in general, not so "well-behaved". For instance:

(a) It may happen that $\mathcal{O}_{\mathbb{P}^{m-1}_{(\mathbf{w})}}(n_1) \cong \mathcal{O}_{\mathbb{P}^{m-1}_{(\mathbf{w})}}(n_2)$ with $n_1 \neq n_2$.

(b) Even if $\mathbf{w} = \mathbf{w}'$, $\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n)$ is not always invertible.

(c) A sheaf $\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n), n > 0$, could be invertible but not ample if $\mathbf{w} \neq \mathbf{w}'$.

(d) The canonical homomorphism

$$\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n_1) \otimes \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n_2) \to \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n_1 + n_2)$$

induced by the natural multiplication

$$S(\mathbf{w})(n_1) \otimes S(\mathbf{w})(n_2) \to S(\mathbf{w})(n_1 + n_2)$$

may be not an isomorphism.

For counterexamples and further discussion see [32, § 1.5] and [10, 3D]. The pathologies of this kind are mainly due to the number theoretical relations between the weights and to the existence of singularities on $\mathbb{P}^{m-1}(\mathbf{w})$. Mori [90] studied the largest open subset, for which most of the nice properties of the twisted sheaves, which are valid for unweighted spaces, can be preserved unchanged. Finally, Dimca and Dimiev [31] proved that this open set is nothing but the regular locus $\operatorname{Reg}(\mathbb{P}^{m-1}(\mathbf{w}))$ of $\mathbb{P}^{m-1}(\mathbf{w})$.

Theorem 2.5. *([31])* $\mathbb{P}^{m-1}(\mathbf{w})$ *is a V-variety with only cyclic quotient singularities, and its singular locus can be written as a union*

$$\operatorname{Sing}(\mathbb{P}^{m-1}(\mathbf{w})) = \bigcup_{I \subset \{1,2,\ldots,m\}} \{\mathbb{P}_I(\mathbf{w}) \mid c(\mathbf{w}, I) > 1\},$$

where

$$\mathbb{P}_I(\mathbf{w}) := \mathbb{P}_I := \mathbb{P}^{m-1}(\mathbf{w}) \cap \{z_i = 0, \forall i \in I\}$$

and

$$c(\mathbf{w}, I) := c_I := \gcd(w_j \mid j \in \{1, \ldots, m\} \setminus I).$$

Definition 2.6. Let $\mathbf{w} \in \mathbb{N}^m$ be an m-tuple of weights, $\bar{\mathbf{w}}$ its reduction, $\mathbf{w}'$ its normalization and $\rho_i(\mathbf{w}), 1 \leq i \leq m$, defined as in 2.2. Since $\gcd(\bar{w}_i, \rho_i(\mathbf{w})) = 1$, there exist two unique integers $\gamma_i(n; \mathbf{w})$ and $\varepsilon_i(n; \mathbf{w})$ with

$$n = \gamma_i(n; \mathbf{w})\bar{w}_i + \varepsilon_i(n; \mathbf{w})\rho_i(\mathbf{w}),\ 0 \leq \gamma_i(n; \mathbf{w}) < \rho_i(\mathbf{w}) \quad \text{for all} \quad i, \quad 1 \leq i \leq m,$$

and for all $n \in \mathbb{Z}$. We define

$$\theta(n; \mathbf{w}) := \frac{n - \sum_{i=1}^{m} \overline{w}_i \gamma_i(n; \mathbf{w})}{\operatorname{lcm}(\rho_1(\mathbf{w}), \ldots, \rho_m(\mathbf{w}))} \tag{2.1}$$

It is easy to see that $\theta(n; \mathbf{w}) \in \mathbb{Z}$, for all $n \in \mathbb{Z}$.

Proposition 2.7. ([10,3.C.1, 3.C.7.]) For all $n \in \mathbb{Z}$, we have:

$$\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(\gcd(w_1, \ldots, w_m) \cdot n) \cong \mathcal{O}_{\mathbb{P}^{m-1}(\bar{\mathbf{w}})}(n) \cong \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w}')}(\theta(n; \mathbf{w}))$$

Proposition 2.8. ([10, 4.B.7], [104, Th.2.7]) Let $m \in \mathbb{N}, \mathbf{w} \in \mathbb{N}^m$ and $\mathbb{P}^{m-1}(\mathbf{w})$ be the corresponding w.p.s. Then:

(i) $\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n)$ is coherent and Cohen-Macaulay, for all $n \in \mathbb{Z}$.

(ii) The sheaf $\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(\operatorname{lcm}(w_1, \ldots, w_m))$ is ample.

(iii) In the case, in which $\mathbf{w} = \mathbf{w}', \operatorname{Pic}(\mathbb{P}^{m-1}(\mathbf{w}))$ is generated by the class $[\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(\operatorname{lcm}(w_1, \ldots, w_m))]$.

For general ampleness criteria of twisted sheaves, see [28, § 2] or [10, § 4 B].

Proposition 2.9. *([32, § 1.4]) A w.p.s. $\mathbb{P}^{m-1}(\mathbf{w})$ has the following properties:*

(i) The Serre homomorphism $S(\mathbf{w}) \to \bigoplus_{n \geq 0} H^0(\mathbb{P}^{m-1}(\mathbf{w}), \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n))$ is a graded isomorphism.

(ii) $H^s(\mathbb{P}^{m-1}(\mathbf{w}), \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n)) = 0$, for $1 \leq s \leq m-2$ and for all $n \in \mathbb{Z}$.

(iii) For $n \in \mathbb{N}_0$, the natural map

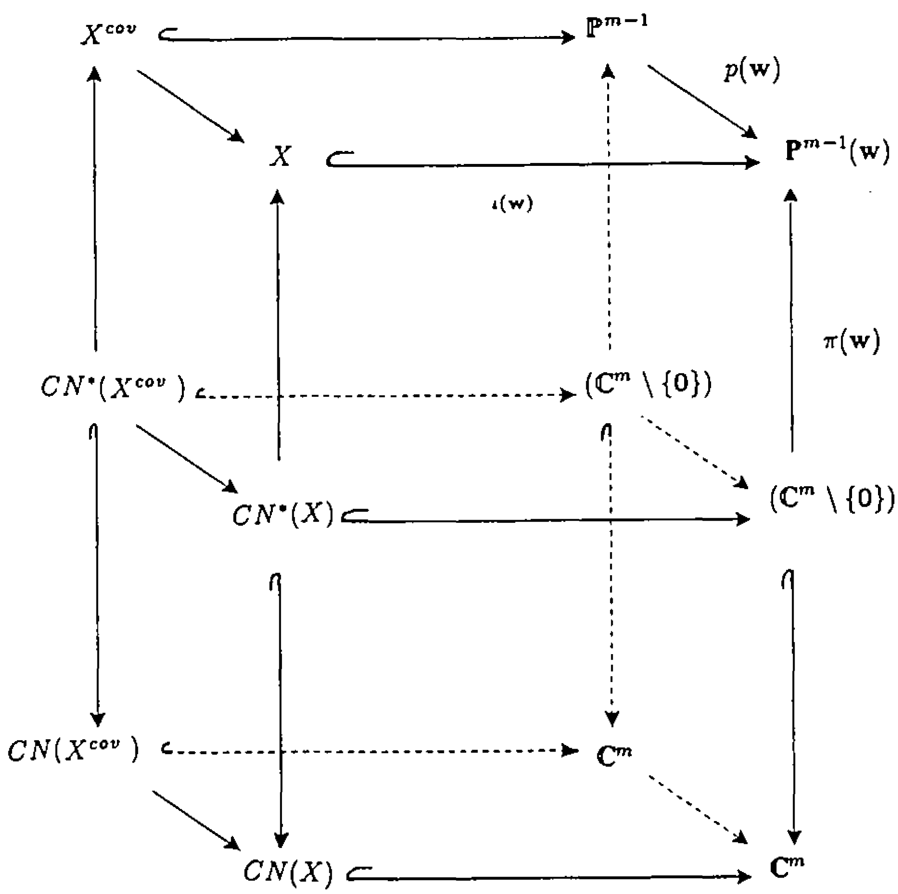
$$H^0(\mathbb{P}^{m-1}(\mathbf{w}), \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n)) \times H^{m-1}(\mathbb{P}^{m-1}(\mathbf{w}), \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(-n - \sum_{i=1}^{m} w_i)) \to$$
$$H^{m-1}(\mathbb{P}^{m-1}(\mathbf{w}), \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(-\sum_{i=1}^{m} w_i)) \cong \mathbb{C}$$

is a perfect pairing.

Definition 2.10. Let $X \overset{\iota(\mathbf{w})}{\hookrightarrow} \mathbb{P}^{m-1}(\mathbf{w})$ be a closed subvariety of $\mathbb{P}^{m-1}(\mathbf{w})$ and

$$p(\mathbf{w}) : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}(\mathbf{w}), \quad \pi(\mathbf{w}) : (\mathbb{C}^m \setminus \{0\}) \to \mathbb{P}^{m-1}(\mathbf{w})$$

the maps introduced in 2.1. $X^{cov} := p(\mathbf{w})^{-1}(X)$ is defined to be the variety which sits over X via the covering map $p(\mathbf{w})$. $CN^*(X) := \pi(\mathbf{w})^{-1}(X)$ is called *the punctured affine quasicone over* X. The *affine quasicone* $CN(X)$ *over* X is the scheme-theoretic closure of $CN^*(X)$ in $\mathbb{C}^m$. The interrelation of these objects to each other can be described by means of two cubes built of commutative diagrams:



X is said to be *quasismooth (q.s.)* if $CN^*(X)$ is overall smooth.

Remark 2.11. We should note here that X can be identified with the geometric quotient $CN^*(X)/\mathbb{C}^*$ with respect to the action, which was introduced in 2.1. Furthermore, the quasismoothness of X does not, in general, offer any guarantee for the smoothness of X^{cov}. Of course, wide classes of quasismooth subvarieties X of $\mathbb{P}^{m-1}(\mathbf{w})$, as for example the class of BP-like complete intersections (see 2.16.) being defined by means of sufficiently general polynomials, have always smooth X^{cov}'s.

Proposition 2.12. *(cf. [32, 3.1.6]) All quasismooth closed subvarieties X of $\mathbb{P}^{m-1}(\mathbf{w})$ are V-varieties.*

Proof. Let X be the zero locus of the $\mathbf{w}$-homogeneous polynomials $f_1, \ldots, f_k$,

$$U_i := \{[z_1, \ldots, z_m] \in \mathbb{P}^{m-1}(\mathbf{w}) \mid z_i \neq 0\}$$

the standard cover of $\mathbb{P}^{m-1}(\mathbf{w})$ and

$$V_i := \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid z_i = 1\} \cap CN(X), \forall i, 1 \leq i \leq m.$$

Then

$$V_i = \{(z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_m) \in \mathbb{C}^m \mid$$
$$f_j(z_1, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_m) = 0, \forall j, 1 \le j \le k\}.$$

If we would assume that there were a singular point

$$\mathbf{z}^0_{(i)} = (z^0_1, \dots, z^0_{i-1}, 1, z^0_{i+1}, \dots, z^0_m)$$

on V_i, then the Euler formula

$$\frac{\partial f_j}{\partial z_i}(\mathbf{z}^0_{(i)}) = -\frac{1}{w_i} \sum_{\substack{l=1 \\ l \neq i}}^{m} w_l z^0_l \frac{\partial f_j}{\partial z_l}(\mathbf{z}^0_{(i)})$$

would imply

$$\operatorname{rank}\left(\left. \frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_m)} \right|_{\mathbf{z}^0_{(i)}} \right) \le \min(m-1, k) - 1 < \min(m, k),$$

contradicting to the quasismoothness of X. Thus, V_i is smooth and the chart $X \cap U_i$ of X can be represented via $\pi(\mathbf{w})|_{V_i} : V_i \to X \cap U_i$ as the quotient of V_i by the finite group $(\mathbb{Z}/w_i\mathbb{Z}) \subset \mathbb{C}^*$. □

Definitions 2.13. (i) A closed subvariety of codimension k in $\mathbb{P}^{m-1}(\mathbf{w})$ is called *well-formed* (or *in general position with respect to* $\operatorname{Sing}(\mathbb{P}^{m-1}(\mathbf{w}))$) if $\mathbb{P}^{m-1}(\mathbf{w})$ is well-formed and X contains no codimension $k+1$ singular stratum of $\mathbb{P}^{m-1}(\mathbf{w})$, i.e. $\operatorname{codim}_X(X \cap \operatorname{Sing}(\mathbb{P}^{m-1}(\mathbf{w}))) \ge 2$.
(ii) If $\mathcal{I}$ is a homogeneous ideal of the graded ring $S(\mathbf{w})$,

$$X = \operatorname{Proj}(S(\mathbf{w})/\mathcal{I}) \subset \mathbb{P}^{m-1}(\mathbf{w})$$

and $\mathcal{I}$ is generated by a regular sequence $\{f_j | 1 \le j \le k\}$ of homogeneous elements of $S(\mathbf{w})$, then X is called a *weighted projective* $(m-k-1)$*-dimensional (strict) complete intersection of the hypersurfaces* $\{f_j = 0\}$ (*c.i.*, for short) with *multidegree* $\mathbf{d} := (d_1, \dots, d_k)$, where $\deg(f_j) = d_j, \forall j, 1 \le j \le k$. We shall denote $S(\mathbf{w})/\mathcal{I}$ by $A(X)$. $A(X) = \bigoplus_{n \ge 0} A(X)_n$ is a graded $\mathbb{C}$-algebra with

$$A(X)_n := S(\mathbf{w})_n/(S(\mathbf{w})_n \cap \mathcal{I}) \quad \text{and} \quad \operatorname{Spec}(A(X)) = CN(X).$$

Moreover we shall make use of the notation

$$X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$$

to express a sufficiently general element of the family of all w.c.i. of multidegree $\mathbf{d}$.

Proposition 2.14. *(Bertini-type quasismoothness criterion).*
Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(w_1, \dots, w_m)$ be a c.i. with defining polynomials $f_1, \dots, f_k$. Then X is quasismooth for $f_1, \dots, f_k$ general enough (i.e. for polynomials, the coefficient systems of which are parametrized by appropriate Zariski-dense open subsets of certain $\mathbb{C}^{\kappa}$'s) if and only if for all possible non-empty index-sets $I_r := \{i_1, \dots, i_r\} \subset M := \{1, 2, \dots, m\}$, there exists an integer $s = s(I_r)$, $0 \leq s \leq k$, and a splitting of the index-set $\{1, \dots, k\}$ into:

$$J_{\mathfrak{a}} = \begin{cases} \varnothing, & \text{if } \ s = 0 \\ \{j_1, \dots, j_s\}, & \text{if } \ 1 \leq s \leq k \end{cases} \quad \text{and} \quad J_{\mathfrak{b}} = \begin{cases} \{j_{s+1}, \dots, j_k\}, & \text{if } \ 0 \leq s \leq k-1 \\ \varnothing, & \text{if } \ s = k \end{cases}$$

satisfying the following property: In the monomial decomposition of $f_1, \dots, f_k$
(i) If $J_{\mathfrak{a}} \neq \varnothing$, there are at least s monomials of type

$$z_{i_1}^{a_{i_1}^{(j_\alpha)}} \cdots z_{i_r}^{a_{i_r}^{(j_\alpha)}}$$

with degree d_{j_α} $((a_{i_1}^{(j_\alpha)}, \dots, a_{i_r}^{(j_\alpha)}) \in (\mathbb{N}_0)^r \setminus \{(0, \dots, 0)\})$, $\forall \alpha$, $1 \leq \alpha \leq s$.
(ii) If $J_{\mathfrak{b}} \neq \varnothing$, and if for all β, $s+1 \leq \beta \leq k$, we set

$$N_r^{(j_\beta)} := \{n \in M \setminus I_r | \exists (b_{i_1}^{(j_\beta),n}, \dots, b_{i_r}^{(j_\beta),n}) \in (\mathbb{N}_0)^r : \\ b_{i_1}^{(j_\beta),n} w_{i_1} + \dots + b_{i_r}^{(j_\beta),n} w_{i_r} + w_n = d_{j_\beta}\},$$

$\nu(j_\beta) := \sharp(N_r^{(j_\beta)})$ and $N_r^{(j_\beta)} = \{y_1^{(j_\beta)}, \dots, y_{\nu(j_\beta)}^{(j_\beta)}\}$ is an enumeration of $N_r^{(j_\beta)}$, then there exist $\nu(j_\beta)$ monomials of degree d_{j_β}
(namely that of type $z_{i_1}^{b_{i_1}^{(j_\beta),y_\lambda^{(j_\beta)}}} \cdots z_{i_r}^{b_{i_r}^{(j_\beta),y_\lambda^{(j_\beta)}}} \cdot z_{y_\lambda^{(j_\beta)}}$, $1 \leq \lambda \leq \nu(j_\beta)$) with

$$|\{\bigcup_{\sigma=1}^{\tau} \bigcup_{\lambda-1}^{\nu(j_{t_\sigma})} y_\lambda^{(j_{t_\sigma})}\}| \geq r - s + \tau - 1, \quad \text{for all subsets} \quad \{t_1, \dots, t_\tau\} \subset \{s+1, \dots, k\}$$

consisting of τ elements, $1 \leq \tau \leq k - s$.

Sketch of proof. We generalize similar results of Fletcher [41, § I.5.] being valid for $k = 1$ and $k = 2$. In fact, one has to show that for a "generic choice" of the defining polynomials $f_1, \dots, f_k$, the rank of the Jacobian matrix $\left(\frac{\partial(f_1, \dots, f_k)}{\partial(z_1, \dots, z_m)} |_{\mathbf{z}=\mathbf{z}^0}\right)$ evaluated at a point $\mathbf{z}^0 \in CN(X)$ cannot be $< k$, except possibly for $\mathbf{z}^0 = \mathbf{0}$. Assume that (i) and (ii) hold for each $I_r \neq \varnothing$. By Bertini's theorem, the singularities of $CN(X)$ can occur only within $CN(X) \cap P_r$, where $P_r := \{(z_1, \dots, z_m) \in \mathbb{C}^m | z_{i_{r+1}} = \dots = z_{i_m} = 0\}$ and $\{i_{r+1}, \dots, i_m\} = \{1, \dots, m\} \setminus \{i_1, \dots, i_r\}$.
Let $\bar{P}_r$ be the stratum $\bar{P}_r := \{(z_1, \dots, z_m) \in P_r | z_{i_1} \neq 0, \dots, z_{i_r} \neq 0\}$. We expand our polynomials in terms of the variables $z_{i_{r+1}}, \dots, z_{i_m}$:

$$f_\alpha(\mathbf{z}) = g_\alpha(z_{i_1}, \dots, z_{i_r}) + \sum_{l=r+1}^{m} z_{i_l} h_\alpha^{i_l}(z_{i_1}, \dots, z_{i_r}) + \left\{ \begin{array}{l} \text{higher order terms} \\ \text{in} \quad z_{i_{r+1}}, \dots, z_{i_m} \end{array} \right\}$$

$$f_\beta(\mathbf{z}) = \sum_{l=r+1}^{m} z_{i_l} h_\beta^{i_l}(z_{i_1}, \dots, z_{i_r}) + \left\{ \begin{array}{l} \text{higher order terms} \\ \text{in} \quad z_{i_{r+1}}, \dots, z_{i_m} \end{array} \right\}$$

for $1 \leq \alpha \leq r$, $s+1 \leq \beta \leq k$, with g_α, $h_\alpha^{i_l}$, $h_\beta^{i_l}$ suitable polynomials in the variables $z_{i_1}, \dots, z_{i_r}$.
(a) Suppose first $s = k$. $\bar{P}_r$ is not a part of the base loci $\mathrm{Bs}(\mathbf{L}_j)$ of the linear systems

$$\mathbf{L}_j = \left\{ F_j(z_1, \dots, z_m; \lambda_{(a_1,\dots,a_m)}) = \sum \lambda_{(a_1,\dots,a_m)} z_1^{a_1} \dots z_m^{a_m}, \ \lambda_{(a_1,\dots,a_m)} \in \mathbb{C}^{\mathcal{A}_j} \right\},$$

$\mathcal{A}_j := \{(a_1, \dots, a_m) \in (\mathbb{N}_0)^m \,|\, \sum_{i=1}^m a_i w_i = d_j\}$, $1 \leq j \leq k$, parametrizing all quasihomogeneous polynomials of degree d_j w.r.t. the weights $(w_1, \dots, w_m)$. Thus $(f_j = 0)$ is non-singular along $\bar{P}_r$, $\forall j$, $1 \leq j \leq k$. Since $(g_\alpha = 0)$, $1 \leq \alpha \leq r$, determine free linear systems on $\bar{P}_r$, $\{d g_\alpha(\mathbf{z}^0) \,|\, 1 \leq \alpha \leq r\}$ are linearly independent for $\mathbf{z}^0 \in \bar{P}_r \cap CN^*(X)$. Hence, the transversality condition is fulfilled and $(\bigcap_{\alpha=1}^r (g_\alpha = 0) \cap \bar{P}_r) \setminus \{\mathbf{0}\} = \bar{P}_r \cap CN^*(X)$ is non-singular.

(b) Suppose now that $s \neq k$. By Bertini's theorem, $(f_\alpha = 0)$, $1 \leq \alpha \leq r$, are non-singular along $\bar{P}_r$. This means that
$\mathrm{Sing}(CN(X)) = (\bigcap_{\alpha=1}^r (g_\alpha = 0)) \cap (\bigcap_{\substack{j+1 \leq \beta \leq k \\ r+1 \leq l \leq m}} (h_\beta^{i_l} = 0))$.
It is an exercise of linear algebra to verify (from the above decompositions of f_α and f_β) that (i) and (ii) are equivalent to $\dim_{\mathbb{C}}(\mathrm{Sing}(CN(X))) = 0$, i.e. that the locus of $CN(X)$ consisting of that points, at which the Jacobian matrix has rank $\leq k-1$, is zero-dimensional. As $CN^*(X)$ is $\mathbb{C}^*$ -invariant (cf. 2.11.), we get $\mathrm{Sing}(CN(X)) \subset \{\mathbf{0}\}$. The converse can be proven similarly. □

Proposition 2.15. *([32, pr.2], [41, I.3.12, I.3.13]) A c.i.* $X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ *is well formed if and only if* $X_{\mathbf{d}}$ *satisfies one of the following equivalent conditions:*
(i)

- (a) $\mathbb{P}^{m-1}(\mathbf{w})$ *is well-formed and*
- (b) *for all* $p = 1, \dots, k$, *the gcd of any* $(m-k-2+p)$ *of the* w_i*'s divides at least* p *of the* d_j*'s.*

(ii) $m - k - \sharp\{i \in \{1, \dots, m\} : q | w_i\} + \sharp\{j \in \{1, \dots, k\} : q | d_j\} \geq 2$, *for all integers* $q \geq 2$. *(In particular, if* $X_{\mathbf{d}}$ *is quasismooth, then* $X_{\mathbf{d}}$ *is well-formed if and only if the above inequality is true for all prime numbers* $q \geq 2$.*)*

Definition 2.16. A c.i. of the form

$$(2.2) \quad X_{\mathbf{d}} = X_{(d_1,\dots,d_k)} = \{[z_1, \dots, z_m] \in \mathbb{P}^{m-1}(\mathbf{w}) | \sum_{i=1}^{m} \lambda_{ij} z_i^{\alpha_{ij}} = 0, \forall j, 1 \leq j \leq k\}$$

is called c.i. of *Brieskorn-Pham type (BP c.i.*, for short). Especially, if either $k=1$ or $d_1 = \ldots = d_k, k \geq 2, X_{\mathbf{d}}$ is called c.i. of *Fermat type*. A *BP-like c.i.* is defined to be a c.i. $X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$, for which $\mathrm{lcm}(w_1, \ldots, w_m) | d_j, \forall j, 1 \leq j \leq k$.

Using Prop. 2.15, we can easily verify, that any q.s. BP-like c.i., embedded in a well-formed w.p.s., is itself well-formed. The property of well-formedness of a q.s.c.i. turns out to be very important, but it was eluded by the authors of [32] and [10]. In fact, the following theorem, due to Dimca and Fletcher, reduces the examination of the validity of this property to dimension ≤ 2.

Theorem 2.17. *Let $X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a q.s.c.i. of dimension ≥ 3. Then either $X_{\mathbf{d}}$ is well-formed, or $X_{\mathbf{d}}$ is the intersection of a linear cone with other hypersurfaces (i.e. $d_j = w_i$ for some j and i). In the second case, $X_{\mathbf{d}}$ is isomorphic either to a q.s.c.i. of lower codimension or to a w.p.s.*

Proposition 2.18. *([30, prop. 8]) The singular locus of a well-formed q.s.c.i. $X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ is given by the intersection $\mathrm{Sing}(X_{\mathbf{d}}) = X_{\mathbf{d}} \cap \mathrm{Sing}(\mathbb{P}^{m-1}(\mathbf{w}))$, i.e. $\mathrm{Sing}(X_{\mathbf{d}}) = \bigcup_{I \subset \{1,\ldots,m\}} \{X_{\mathbf{d}}(I) | c_I > 1\}$, where $X_{\mathbf{d}}(I) := X_{\mathbf{d}} \cap \mathbb{P}_I$ in the notation of 2.5.*

Definition 2.19. Let $X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a q.s.c.i. The number

$$\mathrm{am}(X_{\mathbf{d}}) := \mathrm{am}(\mathbf{w}; \mathbf{d}) := \sum_{j=1}^{k} d_j - \sum_{i=1}^{m} w_i$$

will be called *the amplitude* of $X_{\mathbf{d}}$.

Proposition 2.20. *(Generalized adjunction formula, [10, 6.B9]) Let $X_{\mathbf{d}}$ be a well-formed q.s.c.i. Then there exists an isomorphism between its dualizing sheaf and its structure sheaf twisted $\mathrm{am}(\mathbf{w}; \mathbf{d})$ times, i.e.*

$$\omega_{X_{\mathbf{d}}} \cong \mathcal{O}_{X_{\mathbf{d}}}(\mathrm{am}(\mathbf{w}; \mathbf{d})) \tag{2.3}$$

(Examples in [41, I.3.15] show that we cannot drop the assumption of well-formedness of $X_{\mathbf{d}}$!)

If $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ is a q.s.c.i. of dimension $r := m-k-1$, the degeneration of the spectral sequence $\mathbf{E}_1^{p,q}(X) = H^q(X, \tilde{\Omega}_X) \Longrightarrow \mathbb{H}^{p+q}(X, \tilde{\Omega}_X^{\bullet}) = H^{p+q}(X, \mathbb{C})$ of hypercohomology (with respect to the complex $\tilde{\Omega}_X^{\bullet}$) at the term $\mathbf{E}_1$ gives rise to a filtration on the spaces $H^{p+q}(X, \mathbb{C})$, which coincides with that one of the usual Hodge structure. Hence, X admits a *pure Hodge structure*,

Hodge decomposition $\quad H^{s}(X, \mathbb{C}) \cong \bigoplus_{p+q=s} H^q(X, \tilde{\Omega}_X^p)$,

Hodge numbers $\quad h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X, \overline{\Omega}_X^p)$ and Serre duality isomorphisms:

$$H^q(X, \overline{\Omega}_X^p) \cong H^{r-q}(X, \overline{\Omega}_X^{r-q}), \forall p, q, \; 0 \leq p, q \leq r$$

(cf. [112, § 1]). On the other hand, according to the hard Lefschetz theorem for V-varieties, the maps

$$\mathbf{u}(q;p) : H^p(X,\mathbb{C}) \ni \xi \longmapsto c_1^q(\mathbf{L}) \smile \xi \in H^{p+2q}(X,\mathbb{C}),$$

induced by the class of an ample line bundle $\mathbf{L}$ over X, are isomorphisms, such that:

$$H^s(X,\mathbb{C}) \cong \bigoplus_{q\geq 0} \mathbf{u}(q; s-2q)(H^{s-2q}_{\mathrm{prim}}(X,\mathbb{C})), \forall s, 0 \leq s \leq 2r-2,$$

where

$$H^p_{\mathrm{prim}}(X,\mathbb{C}) := \mathrm{Ker}(\mathbf{u}(r-p+1;p) : H^p(X,\mathbb{C}) \to H^{2r-p+2}(X,\mathbb{C})),\ \forall p,\ 0 \leq p \leq r,$$

denote the so called *primitive cohomology groups of* X. As a consequence of the compatibility of the Hodge and Lefschetz decompositions we get:

$$H^s_{\mathrm{prim}}(X,\mathbb{C}) \cong \bigoplus_{p+q=s} H^{p,q}_{\mathrm{prim}}(X,\mathbb{C}),$$

where

$$H^{p,q}_{\mathrm{prim}}(X,\mathbb{C}) := H^q(X,\overline{\Omega}^p_X) \cap \mathrm{Ker}(\mathbf{u}(r-s+1;s)).$$

Moreover, if we set $h^{p,q}_{\mathrm{prim}}(X) := \dim_{\mathbb{C}} H^{p,q}_{\mathrm{prim}}(X,\mathbb{C})$, we obtain:

$$h^{p,q}(X) = \sum_{i=0}^{p} h^{p-i,q-i}_{\mathrm{prim}}(X) \tag{2.4}$$

and the application of the Lefschetz theorem for hyperplane sections gives the following:

Proposition 2.21. *For a q.s.c.i.* $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ *of dimension* $r = m-k-1$, *we have:*

$$h^{p,q}(X) = \delta_{p,q}, \quad \textit{for} \quad p+q \neq r \tag{2.5}$$

$$h^{p,q}_{\mathrm{prim}}(X) = h^{p,q}(X) - \delta_{p,q}, \quad \textit{for} \quad (p,q) \neq (0,0) \tag{2.6}$$

The above invariants (2.5) of X are called the *trivial* ones. The remaining, non trivial and most interesting, invariants of X, and their combinatorial expressions by means of the weights $w_1, \ldots, w_m$ and of the degrees of the defining w-homogeneous polynomials of X, have been studied by Hamm [59], [60] and Aleksandrov [2]. For the presentation of their formulae we need to introduce some special notations. Let

$$(2.7) \qquad W_0 := 1, \quad W_\lambda(y_1, \dots, y_m) := \sum_{1 \le i_1 < \dots < i_\lambda \le m} y_{i_1} \cdots y_{i_\lambda}$$

denote the elementary symmetric polynomials in the variables $y_1, \dots, y_m$ with weight $\lambda \in \mathbb{N}_0$,

$$(2.8) \qquad D_\lambda(y_1, \dots, y_k) := \sum_{\substack{j_1 + \dots + j_k = \lambda \\ j_1, \dots, j_k \ge 0}} y_1^{j_1} \cdots y_k^{j_k}$$

the symmetric polynomials in $y_1, \dots, y_k$ of degree λ, and

$$D_{\lambda,k}(y_1, \dots, y_k) := (-1)^\lambda y_1 \dots y_k \, D_\lambda(y_1, \dots, y_k)$$

with $D_{0,0} := 1$ and $D_{\lambda,0} := 0$, $\forall \lambda$, $\lambda \in \mathbb{N}$.

Theorem 2.22. *(Aleksandrov [1], [2, p.447])*
Let $X = X_{(d_1, \dots, d_k)} \subset \mathbb{P}^{m-1}(w_1, \dots, w_m)$ be a quasismooth c.i., $A(X)$ its graded coordinate ring and $\mathfrak{M}$ the maximal ideal of $A(X)$ corresponding to the zero point $0 \in CN(X)$. Suppose that the indices of the degrees of its defining polynomials are enumerated in order of size

$$d_1 = \dots = d_{k_1} < d_{k_1+1} = \dots = d_{k_{\tau-1}} < d_{k_{\tau-1}+1} = \dots = d_{k_\tau},$$

so that $k_\tau = k$, $k_0 = 0$, and set $g_\sigma := k_\sigma - k_{\sigma-1} - 1$, $\forall \sigma$, $1 \le \sigma \le \tau$.

Then the Poincaré-series of the graded $A(X)$-module $H_{\mathfrak{M}}^{m-k-q}(\Omega^q_{A(X)})$ is given, for $q = 1, \dots, m-k$, by the following formula:

$$(2.9) \qquad \mathcal{P}(H_{\mathfrak{M}}^{m-k-q}(\Omega^q_{A(X)}); x) = (-1)^{q+1} \sum_{\sigma=1}^{\tau} x^{(q+1)d_{k_\sigma}} \times$$

$$\times \left\{ \sum_{\substack{\lambda_1+\lambda_2+\lambda_3+\lambda_4 = g_\sigma \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0}} \binom{m-k-q+\lambda_1-1}{m-k-q-1} \binom{q+\lambda_2}{q} x^{-(m-k+\lambda_1+1)d_{k_\sigma}} \times \right.$$

$$\left. \times W_{m-\lambda_3}(\mu_\sigma(x; \mathbf{w})) D_{\lambda_4, k-g_\sigma-1}\left(\nu_\sigma(x; \mathbf{d})\right) \right\}$$

where $\mu_\sigma(x;\mathbf{w}), \nu_\sigma(x;\mathbf{d})$ abbreviate the rational function vectors:

$$\mu_\sigma(x;\mathbf{w}) := \left(\tfrac{x^{d_{k_\sigma}} - x^{w_1}}{x^{w_1}-1}, \dots , \tfrac{x^{d_{k_\sigma}} - x^{w_m}}{x^{w_m}-1} \right) \quad \textit{and}$$

$$\nu_\sigma(x;\mathbf{w}) := \left(\tfrac{x^{d_1}-1}{x^{d_{k_\sigma}} - x^{d_1}}, \dots , \tfrac{x^{d_{k_\sigma-1}}-1}{x^{d_{k_\sigma}} - x^{d_{k_\sigma-1}}}, \tfrac{x^{d_{k_\sigma+1}}-1}{x^{d_{k_\sigma}} - x^{d_{k_\sigma+1}}}, \dots , \tfrac{x^{d_k}-1}{x^{d_{k_\sigma}} - x^{d_k}} \right)$$

respectively.

Sketch of proof. Let $f_1, \dots, f_k$ be the defining polynomials of X,

$$X^{(0)} := \mathbb{P}^{m-1}(w), \quad X^{(p)} := X_{(d_1,\dots,d_p)},$$

$$\bar{f}_p := f_p|_{CN(X^{(p-1)})} \quad \text{and} \quad \partial f_p := \sum_{i=1}^{m} \frac{\partial f_p}{\partial z_i}$$

(w.r.t. a local coordinate system $\{z_1, \dots, z_m\}$ of $\mathbf{0}$), $\forall p, 1 \le p \le k$. Furthermore, let

$$\{\Omega^j_{\mathbb{C}^m} \xrightarrow{\partial f_p \wedge \cdot} \Omega^{j+1}_{\mathbb{C}^m}, j \in \mathbb{Z}\}$$

denote the Koszul-cocomplex defined by means of the left exterior multiplication by ∂f_p,

$$\Omega^j_{CN(X^{(p)})} := \Omega^j_{\mathrm{Spec}(A(X^{(p)}))} \cong \Omega^j_{\mathbb{C}^m} / \left(\sum_{l=1}^{p} f_l \Omega^j_{\mathbb{C}^m} + \partial f_l \wedge \Omega^{j-1}_{\mathbb{C}^m} \right) |_{CN(X^{(p)})}$$

and $\Omega^j_{f_p} := \Omega^j_{CN(X^{(p-1)})} / \partial \bar{f}_p \wedge \Omega^{j-1}_{CN(X^{(p-1)})}, \forall p, 1 \le p \le k, \forall j, j \in \mathbb{Z}$. By general de Rham-type lemma (see [50, Prop. 1.7. and Prop. 1.11.(i), pp. 241-242]) the sequences

$$0 \to \Omega^j_{f_p} \xrightarrow{\partial f_p \wedge \cdot} \Omega^{j+1}_{CN(X^{(p-1)})} \to \Omega^{j+1}_{f_p} \to 0 \tag{2.10}$$

$$0 \to \Omega^j_{f_p} \xrightarrow{\cdot f_p} \Omega^j_{f_p} \to \Omega^j_{CN(X^{(p)})} \to 0 \tag{2.11}$$

are exact, $\forall j, 0 \le j \le m-p$. The application of the functor $H^q_{\{0\}}(-)$ to (2.10) and (2.11), combined with Greuel's vanishing theorems ([51, pp.165-166]):

$$H^q_{\{0\}}(\Omega^j_{CN(X^{(p)})}) = 0,$$

$$\text{for} \quad \begin{cases} j + q \notin \{m-k, m-k+1\}, \\ (j,q) \notin \{(i, m-k) | 0 \le i \le m-k\} \cup \{(i,0) | m-k \le i \le m\} \end{cases}$$

and

$$H^q_{\{0\}}(\Omega^j_{f_p}) = 0,$$

$$\text{for} \quad \begin{cases} j+q \neq m-k-1, \\ (j,q) \notin \{(i,m-k-1) | 0 \leq i \leq m-k\} \cup \{(i,0) | m-k+1 \leq i \leq m\} \end{cases}$$

respectively, leads to four-term exact sequences of cohomology groups supported at $\{0\}$. The corresponding Poincaré-series must therefore satisfy the following recursive equations:

$$(2.12) \quad \mathcal{P}(H^q_{\{0\}}(\Omega^{m-p-q+1}_{f_p}); x) =$$
$$x^{-d_p}\mathcal{P}(H^{q-1}_{\{0\}}(\Omega^{m-p-q+2}_{f_p}); x) + \mathcal{P}(H^q_{\{0\}}(\Omega^{m-k-p}_{CN(X^{(k-p+1)})}; x)) - \mathcal{P}(H^{q-1}_{\{0\}}(\Omega^{m-k-q+1}_{CN(X^{(k-p+1)})}; x))$$
$$\text{for} \quad 1 \leq p \leq k, 1 \leq q \leq m-p$$

$$(2.13) \quad \mathcal{P}(H^q_{\{0\}}(\Omega^{m-p-q}_{CN(X^{(k-p)})}); x) =$$
$$\mathcal{P}(H^{q-1}_{\{0\}}(\Omega^{m-p-q+1}_{CN(X^{(k-p)})}); x) + (1 - x^{d_p})\mathcal{P}(H^q_{\{0\}}(\Omega^{m-p-q+1}_{f_p}); x)$$
$$\text{for} \quad 1 \leq p \leq k, 1 \leq q \leq m-p-1$$

The system of (2.12) and (2.13) has as solution:

$$(2.14) \quad \mathcal{P}(H^q_{\{0\}}(\Omega^{m-p-q}_{CN(X^{(k-p)})}); x) =$$
$$(-1)^{q-1}\sum_{j=p}^{k-1}\mathcal{P}(H^0_{\{0\}}(\Omega^{m-j-q}_{CN(X^{(k-j)})}); x)\mathrm{Res}_{t=0}\left[t^{-q}\frac{x^{-d_{k-j}}}{1+tx^{-d_{k-j}}}\prod_{i=k-j+1}^{k-p}\frac{x^{-d_i}-1}{1+tx^{-d_i}}\right]$$

for $0 \leq p \leq k, 1 \leq q \leq m-p-1$. The theorem is completed by using Grothendieck's local duality theorem [55, thm. 6.3.], which gives a perfect pairing

$$H^{m-k-q}_{\{0\}}(\Omega^q_{CN(X^{(k)})}) \times \mathrm{Ext}^q_{\mathcal{O}_{CN(X^{(k)})}}(\Omega^q_{CN(X^{(k)})}, \omega_{CN(X^{(k)})}) \to \mathbb{C},$$

and enables the computation of the Poincaré-series of the desired local cohomology groups, Greuel's and Hamm's computation of $\mathcal{P}(H^0_{\{0\}}(\Omega^{m-j-q}_{CN(X^{(k-j)})}); x)$ in (2.14) (see [52, Satz 3.1.]), simple duality for the highest dimension $q = m-p$, and "residue-acrobatics" with the symmetric polynomials (2.7), (2.8) (see [52, 3.9]), combined with

$$\mathcal{P}(H^{m-k-q}_{\mathfrak{M}}(\Omega^q_{A(X)}); x) = \mathcal{P}(H^{m-k-q}_{\{0\}}(\Omega^q_{CN(X)}); x).$$

□

Lemma 2.23. *Let $m \in \mathbb{N}, \mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{N}^m$, and $\mathbb{P}^{m-1}(\mathbf{w}) = \mathrm{Proj}(S(\mathbf{w}))$ be the corresponding w.p.s. with weights $\mathbf{w}$. Then:*

$$\mathcal{P}(S(\mathbf{w}); x) = \frac{1}{\prod_{i=1}^{m}(1 - x^{w_i})} \tag{2.15}$$

Proof. It follows directly from Prop. 2.9. □

Lemma 2.24. *For a q.s.c.i $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ we have:*

$$\mathcal{P}(A(X); x) = \frac{\prod_{j=1}^{k}(1 - x^{d_j})}{\prod_{i=1}^{m}(1 - x^{w_i})} \tag{2.16}$$

Proof. If we set $S^0 := S(\mathbf{w}), S^j := S(\mathbf{w})/(f_1, \ldots, f_j), \forall j, 1 \leq j \leq k$, where $f_1, \ldots, f_k$ are the defining polynomials of X, then

$$0 \to S^{j-1}(-d_j) \xrightarrow{\cdot f_j} S^{j-1} \to S^j \to 0$$

is exact as coming from an $S(\mathbf{w})$ -regular sequence. Thus,

$$\mathcal{P}(S^{j-1}; x) = \mathcal{P}(S^j; x) + x^{d_j}\mathcal{P}(S^{j-1}; x),$$

i.e.

$$\mathcal{P}(S^j; x) = (1 - x^{d_j})\mathcal{P}(S^{j-1}; x).$$

Substituting (2.15) for $\mathcal{P}(S^0; x)$, we get (2.16). □

As it has turned out, using either the Hodge filtration [60] or a Gysin-type exact sequence between local cohomology groups [2] and further vanishing theorems, the primitive parts of the non-trivial Hodge numbers of such an X are given by means of residue calculus on the rational functions (2.9) and (2.16):

Theorem 2.25. *(Formulae of Hamm and Aleksandrov)*
Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a q.s.c.i. of dimension $r = m - k - 1$. Then its non-trivial primitive Hodge numbers are computed by the following formulae, depending only on $\mathbf{w}$ and $\mathbf{d}$:

(2.17)
$$h^{p,q}_{\mathrm{prim}}(X) = h^{q,p}_{\mathrm{prim}}(X) = \mathrm{Res}_{x=0}\frac{1}{x}\mathcal{P}(H^{m-k-q}_{\mathfrak{M}}(\Omega^q_{A(X)}); x), \quad \text{for} \quad 1 \leq q < r,\ p+q = r$$

$$h^{r,0}_{\mathrm{prim}}(X) = h^{0,r}_{\mathrm{prim}}(X) = \mathrm{Res}_{x=0}\frac{1}{x^{\mathrm{am}(\mathbf{w},\mathbf{d})+1}}\mathcal{P}(A(X); x) \tag{2.18}$$

The forthcoming numbers, which are of fundamental importance and characterize q.s.c. intersections X, are the dimensions of their cohomology groups with coefficients taken from the twisted sheaves $\mathcal{O}_X(n)$.

Definition 2.26. Let $m \in \mathbb{N}$ and $\mathbf{w} = (w_1, \dots, w_m)$ an m-tuple of positive integers. We denote by $pt(n; \mathbf{w}), n \in \mathbb{N}_0$, the generating function determined by

$$\frac{1}{\prod_{i=1}^{m}(1 - x^{w_i})} = \sum_{n=0}^{\infty} pt(n; \mathbf{w}) x^n \tag{2.19}$$

(For reasons of convention we extend it to the whole $\mathbb{Z}$ by setting

$$pt(n; \mathbf{w}) = 0, \forall n, n \in \mathbb{Z} \setminus \mathbb{N}_0).$$

Obviously, $pt(n; \mathbf{w}) = \dim_{\mathbb{C}}(S(\mathbf{w})_n) = h^0(\mathbb{P}^{m-1}(\mathbf{w}), \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n))$.

Theorem 2.27. *Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a well-formed q.s.c.i. of dimension $r = m - k - 1$, whose ideal $\mathcal{I}$ is generated by a regular $S(\mathbf{w})$ -sequence $\{f_1, \dots, f_k\}$ with $\deg(f_j) = d_j, \mathbf{d} = (d_1, \dots, d_k)$, and $A(X) = S(\mathbf{w})/\mathcal{I}$. Then its cohomology groups with coefficients taken from the twisted sheaves $\mathcal{O}_X(n)$ are related to the graded parts of $A(X)$ by the isomorphisms*

$$H^i(X, \mathcal{O}_X(n)) \cong \begin{cases} A(X)_n, & \text{for } i = 0 \\ 0, & \text{for } 1 \leq i \leq r-1 \quad \text{or} \quad i \neq 0, r \\ A(X)_{\mathrm{am}(\mathbf{w},\mathbf{d})-n}, & \text{for } i = r \end{cases}$$

Moreover, the dimensions of the non-vanishing of them are computed by the following formulae, depending only on $\mathbf{w}$ and $\mathbf{d}$:

$$h^0(X, \mathcal{O}_X(n)) = pt(n; \mathbf{w}) + \sum_{j=1}^{k} (-1)^j \sum_{1 \leq \nu_1 < \nu_2 < \dots < \nu_j \leq k} pt(n - \sum_{\lambda=1}^{j} d_{\nu_\lambda}; \mathbf{w}) \tag{2.20}$$

$$h^r(X, \mathcal{O}_X(n)) = h^0(X, \mathcal{O}_X(\mathrm{am}(\mathbf{w}, \mathbf{d}) - n)), \forall n, n \in \mathbb{Z}. \tag{2.21}$$

Proof. For the proof of the first assertion we follow Dolgachev [32, § 3.2.]. We consider at first the long exact sequence

$$\begin{aligned} \dots &\to H^i_{\{0\}}(CN(X), \mathcal{O}_{CN(X)}) \to H^i(CN(X), \mathcal{O}_{CN(X)}) \\ &\to H^i(CN^*(X), \mathcal{O}_{CN(X)|CN^*(X)}) \cong H^i(CN^*(X), \mathcal{O}_{CN^*(X)}) \\ &\to H^{i+1}_{\{0\}}(CN(X), \mathcal{O}_{CN(X)}) \to H^{i+1}(CN(X), \mathcal{O}_{CN(X)}) \to \dots \end{aligned}$$

which is associated to the cohomology groups of $CN(X)$ with support $\{0\} = CN(X) \setminus CN^*(X)$ (see [55, Cor. 1.9., p.9], [56, Exp. II, Cor. 2.9., p.16]). Since $CN(X)$ is an affine variety, we have: $H^i(CN(X); \mathcal{O}_{CN(X)}) = 0, \forall i, i > 0$ (see [61, III.3.5]). Thus, for all $i > 0$: $H^i(CN^*(X), \mathcal{O}_{CN^*(X)}) \cong H^{i+1}_{\{0\}}(CN(X), \mathcal{O}_{CN(X)})$.

This group vanishes whenever $H^{i+1}_{\mathfrak{M}}(\mathcal{O}_{CN(X)}) \cong H^{i+1}_{\{0\}}(CN(X), \tilde{\mathcal{O}}_{CN(X)})$ is 0. As $CN(X)$ is an affine c.i., its structure sheaf is Cohen-Macaulay and $H^{i+1}_{\mathfrak{M}}(\mathcal{O}_{CN(X)}) = 0$, for $i+1 < \dim CN(X) = r+1$ (cf. [56, Exp. VII, prop. 1.2., cor. 1.4, pp. 78-80]). Furthermore,

$$\begin{aligned} H^i(CN^*(X), \mathcal{O}_{CN^*(X)}) &\cong H^i(CN^*(X), \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}} \mathcal{O}_{(\mathbb{C}^m \setminus \{0\})}) = \\ &= H^i(X, \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}} \pi(\mathbf{w})_* \mathcal{O}_{(\mathbb{C}^m \setminus \{0\})}) = \\ &= \bigoplus_{n \in \mathbf{Z}} H^i(X, \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}} \mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(n)) \cong \bigoplus_{n \in \mathbf{Z}} H^i(X, \mathcal{O}_X(n)). \end{aligned}$$

This means that $H^i(X, \mathcal{O}_X(n)) = 0$, $\forall i$, $1 \le i \le r-1$. Now since $A(X)$ is integrally closed, the Serre homomorphism $A(X) \to \bigoplus_{n \in \mathbf{Z}} H^0(X, \mathcal{O}_X(n))$, as in the unweighted case ([61, p. 188]), is a graded isomorphism, and therefore by (2.16)

$$\mathcal{P}(X; x) = \mathcal{P}(A(X); x) = \left(\prod_{j=1}^{k} (1 - x^{d_j}) \right) \left(\prod_{i=1}^{m} (1 - x^{w_i}) \right)^{-1}.$$

We write

$$\begin{aligned} \prod_{j=1}^{k} (1 - x^{d_j}) &= \prod_{j=1}^{k} \left(\sum_{n=0}^{\infty} (\delta_{\{0\}}(n) - \delta_{\{d_j\}}(n)) x^n \right) = \\ &= \sum_{n=0}^{\infty} [(\delta_{\{0\}}(n) - \delta_{\{d_1\}}(n)) \star \ldots \star (\delta_{\{0\}}(n) - \delta_{\{d_k\}}(n))] x^n \end{aligned}$$

where $\star$ denotes here the usual Cauchy multiplication. One checks directly that

$$(\delta_{\{0\}}(n) - \delta_{\{d_1\}}(n)) \star \ldots \star (\delta_{\{0\}}(n) - \delta_{\{d_k\}}(n)) =$$

$$= \begin{cases} 1, & \text{for } n = 0 \\ -1, & \text{for } n = d_{\nu_1} + \ldots + d_{\nu_j}, 1 \le \nu_1 < \ldots < \nu_j \le k,\ j \text{ odd} \\ 1, & \text{for } n = d_{\nu_1} + \ldots + d_{\nu_j}, 1 \le \nu_1 < \ldots < \nu_j \le k,\ j \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

After multiplication by $\prod_{i=1}^{m} (1 - x^{w_i})^{-1}$ we get (2.20).
Finally, the last isomorphism and (2.21) follow from Serre duality. □

Corollary 2.28. *For a well-formed q.s.c.i.* $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ *of dimension* $r = m - k - 1$ *we get:*

$$h^i(X, w_X(n)) = \begin{cases} h^0(X, \mathcal{O}_X(\mathrm{am}(\mathbf{w}, \mathbf{d}) + n)), \textit{for } i = 0 \\ 0, \textit{for } 1 \le i \le r-1 \\ h^0(X, \mathcal{O}_X(-n)), \textit{for } i = r \end{cases} \tag{2.22}$$

Proof. Obvious by the generalized adjunction formula (2.3) of prop. 2.20. ☐

Definition 2.29. Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a q.s.c.i and $\bar{\mathbf{w}}$ be the reduction of the weights $\mathbf{w}$ (as in 2.2). If $f_1, \ldots, f_k$ are the defining $\mathbf{w}$-homogeneous polynomials of X, we shall say that $\bar{X} = \bar{X}_{\bar{\mathbf{d}}} \subset \mathbb{P}^{m-1}(\bar{\mathbf{w}})$, defined by the $\bar{\mathbf{w}}$-homogeneous polynomials $f_1, \ldots, f_k$ with degrees $\deg(\bar{f}_j) = \bar{d}_j := \frac{d_j}{\gcd(w_1, \ldots, w_m)}$, is *the reduction of* X. Furthermore, if we suppose that $\bar{X}$ is not contained in any hyperplane $\{z_i = 0\}$, for $1 \le i \le m$, then we can determine a third q.s.c.i $X' = X'_{\mathbf{d}'} \subset \mathbb{P}^{m-1}(\mathbf{w}')$ coming from the *normalization* $\mathbf{w}'$ of $\mathbf{w}$ (in the notation of 2.2.) with defining $\mathbf{w}'$-homogeneous polynomials $f'_1, \ldots, f'_k$ of degrees

$$\deg(f'_j) = d'_j := \frac{\bar{d}_j}{\operatorname{lcm}(\rho_1(\mathbf{w}), \ldots, \rho_m(\mathbf{w}))}, \ \forall j, \ 1 \le j \le k.$$

(cf. [30, p. 186]).

Proposition 2.30. *([30, pp. 186-187]). Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a q.s.c.i and $\bar{X}$ its reduction. Assume that $\bar{X}$ is not contained in any hyperplane $\{z_i = 0\}$, for $1 \le i \le m$, and let X' denote the q.s.c.i coming from the normalization $\mathbf{w}'$ of $\mathbf{w}$. Then $X, \bar{X}$ and X' are isomorphic to each other.*

Remark 2.31. The above mentioned proposition informs us that under these relatively weak assumptions, we can consider the weights of q.s.c. intersections being normalized. Of course, this does not mean that the corresponding germs $(CN(X), 0), (CN(\bar{X}), 0), (CN(X'), 0)$ will have to be necessarily isomorphic.

Definition 2.32. A q.s.c.i $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ will be called *nondegenerate* if it has the following properties:
(a) its reduction $\bar{X}$ is not contained in any hyperplane $\{z_i = 0\}$, for $1 \le i \le m$, and
(b) X' fulfills, in addition, the condition (i) (b) of Prop. 2.15., i.e. X' is well-formed.

Proposition 2.33. *The degree of the twisted sheaf $\mathcal{O}_X(n)$, which is defined over a nondegenerate one dimensional q.s.c.i $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$, is given by the formula:*

$$\deg(\mathcal{O}_X(n)) = \tag{2.23}$$

$$\frac{1}{\gcd(w_1, \ldots, w_m)}\{h^0(X', \mathcal{O}_{X'}(\theta(n; \mathbf{w})) - h^0(X', \mathcal{O}_{X'}(\operatorname{am}(\mathbf{d}', \mathbf{w}') - \theta(n; \mathbf{w}))) + g(X') - 1\}$$

where X' denotes the space coming from the normalization $\mathbf{w}'$ of $\mathbf{w}, \theta(n; \mathbf{w})$ the function (2.1) introduced in 2.6., and $g(X') := h^1(X', \mathcal{O}_{X'}) = h^0(X', \mathcal{O}_{X'}(\operatorname{am}(\mathbf{d}', \mathbf{w}')))$ its genus.

Proof. By prop. 2.7.,

$$\deg(\mathcal{O}_X(n)) = \frac{1}{\gcd(w_1, \ldots, w_m)}\deg(\mathcal{O}_{\bar{X}}(n)) = \frac{1}{\gcd(w_1, \ldots, w_m)}\deg(\mathcal{O}_{X'}(\theta(n; \mathbf{w}))).$$

Since X' is well-formed and smooth, the usual Riemann-Roch formula for curves gives:

$$\deg(\mathcal{O}_{X'}(\theta(n;\mathbf{w}))) = h^0(X', \mathcal{O}_{X'}(\theta(n;\mathbf{w}))) - h^1(X', \mathcal{O}_{X'}(\theta(n;\mathbf{w}))) + (g(X') - 1).$$

The proof is completed by using the Serre duality equation

$$h^1(X', \mathcal{O}_{X'}(\theta(n;\mathbf{w}))) = h^0(X', \mathcal{O}_{X'}(\mathrm{am}(\mathbf{d}', \mathbf{w}') - \theta(n;\mathbf{w}))).$$

□

Let us now go into the description of the nature of the Picard groups of q.s.c. intersections. Since we are mainly interested in threefolds, we omit the consideration of the surface case referring the reader to Steenbrink [113], Cox [22] and Jong-Steenbrink [67] instead.

Theorem 2.34. *(Mori's weighted version of the classical Noether-Lefschetz theorem, [90, Th. 3.7.]) The Picard group* $\mathrm{Pic}(X)$ *of a smooth, well-formed c.i.* $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ *of dimension* ≥ 3 *is isomorphic to* $\mathbb{Z}$ *and is generated by the class* $[\mathcal{O}_X(1)]$.

Theorem 2.35. *(Dolgachev's generalization [32, 3.2.4 (i), 3.2.5]) The Picard group* $\mathrm{Pic}(X)$ *of a quasismooth, well-formed c.i.* $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ *of dimension* ≥ 3 *is isomorphic to* $\mathbb{Z}$ *and is generated by the class of an* $\mathbf{L}_X := \mathcal{O}_X(\eta_X)$*, for some* $\eta_X \in \mathbb{N}$.

Definition 2.36. Let $X = X_{\mathbf{d}}$ be a q.s.c.i. in $\mathbb{P}^{m-1}(\mathbf{w})$, $I \subset \{1, \ldots, m\}$ a (non-empty) index set and $X(I)$ the corresponding stratum on X (as in 2.18). We define:

$$\mathcal{V}(I) := \sharp\{\text{polynomials which vanish identically on } X(I)\}$$

X will be called *well-stratified* if
(a) $X(I)$ is a $(m-1-|I|)-(k-\mathcal{V}(I))$-dimensional q.s.c.i. for all I with

$$|I| \leq (m-1) - (k - \mathcal{V}(I)).$$

(b) for all I with $(m-1-|I|) - k - \mathcal{V}(I)) = 0$, $X(I)$ consists of finitely many points.
(c) $X(I) = \varnothing$ for all I with $|I| > (m-1) - (k - \mathcal{V}(I))$.

(Note that condition (c) is not superfluous! For instance, the intersection of two zero-dimensional hypersurfaces in a w.p.s. need not be empty.)

Next theorem strengthens Dolgachev's result in the well-stratifiedness case, gives a partial answer to a question of Beltrametti and Robbiano for dimension ≥ 3 [10, p. 155], and generalizes prop. 2.8. (iii).

Theorem 2.37. *Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a well-formed, well-stratified, q.s.c.i of dimension ≥ 3. Then $\mathrm{Pic}(X)$ is generated by the class of the ample bundle $\mathbf{L}_X := \mathcal{O}_X(\eta_X)$ with*

(2.24)

$$\eta_X = \mathrm{lcm}(\{\gcd(w_i | i \in I) | I \subset \{1, \dots, m\}, |I| \geq k + 1 - \mathcal{V}(\{1, \dots, m\} \setminus I)\}) =$$
$$= \mathrm{lcm}(\{\gcd(w_i | i \in I) | I \subset \{1, \dots, m\}, |I| = k + 1 - \mathcal{V}(\{1, \dots, m\} \setminus I)\})$$

Proof. By prop. 2.8. (iii), $\mathrm{Pic}(\mathbb{P}^{m-1}(\mathbf{w}))$ is generated by $[\mathcal{O}_{\mathbb{P}^{m-1}(\mathbf{w})}(\mathrm{lcm}(w_1, \dots, w_m))]$. Now, although X^{cov} is a not necessarily smooth c.i. (see 2.11.), $\mathrm{Pic}(X^{cov})$ is generated by $[\mathcal{O}_{X^{cov}}(1)], p(\mathbf{w})^*, (p(\mathbf{w})|_X)^*, \iota(\mathbf{w})^*$ are injective and $\iota(1)^*$ is an isomorphism by Grothendieck's version of Noether-Lefschetz theorem ([56, Exp. XII, Cor. 3.6. and 3.7., p. 153]).

$$\begin{array}{ccc}
\mathbb{Z} = \mathrm{Pic}(\mathbb{P}^{m-1}) & \xleftarrow{p(\mathbf{w})^*} & \mathrm{Pic}(\mathbb{P}^{m-1}(\mathbf{w})) = \mathrm{lcm}(w_1, \dots, w_m)\mathbb{Z} \\
\downarrow{\scriptstyle \iota(1)^*} & & \downarrow{\scriptstyle \iota(\mathbf{w})^*} \\
\mathbb{Z} = \mathrm{Pic}(X^{cov}) & \xleftarrow{(p(\mathbf{w})|_X)^*} & \mathrm{Pic}(X) = \eta_X \mathbb{Z}
\end{array}$$

Thus, $\mathrm{Pic}(X)$ is generated by the class of the ample line bundle $\mathbf{L}_X = \mathcal{O}_X(\eta_X)$, where η_X denotes the minimal positive integer which divides $\mathrm{lcm}(w_1, \dots, w_m)$ and for which $(p(\mathbf{w})|_X)^*(\mathcal{O}_X(\eta_X)) = \mathcal{O}_{X^{cov}}(\eta_X)$. In other words, η_X is the minimal divisor of $\mathrm{lcm}(w_1, \dots, w_m)$ for which

$$H^0(\mathrm{Spec}(A(X)_{(h)}), \mathcal{O}_X(\eta_X)) = A(X)(\eta_X)_{(h)} = \{\frac{g}{h^\nu} | g \in A(X)_{\nu s + \eta_X}\}$$

forms a free $A(X)_{(h)}$-module of rank 1, for all $h \in A(X)_s$ and for all $s \in \mathbb{N}$. For the determination of η_X we identify X with $CN^*(X)/\mathbb{C}^*$. Note that $\mathrm{Pic}(CN(X)) \cong \mathrm{Pic}(CN^*(X))$ is trivial ([56, Exp. XI, Cor. 3.10., p.130], [32, p.52]). The projection map $\pi(\mathbf{w})|_X$ induces a monomorphism

$$q_X : \mathrm{Pic}(CN^*(X)/\mathbb{C}^*) \hookrightarrow \mathrm{Pic}_{\mathbb{C}^*}(CN^*(X))$$

to the group of isomorphism classes of $\mathbb{C}^*$-line bundles over $CN^*(X)$. By the exact sequences

$$\begin{array}{ccccc}
 & & & \{1\} & \\
 & & & \downarrow & \\
 & & & \mathrm{Pic}(X) = \mathrm{Pic}(CN^*(X)/\mathbb{C}^*) & \\
 & & & {\scriptstyle q_X} \downarrow & \searrow {\scriptstyle \text{zero- map}} \\
\{1\} \to H^1_{alg}(\mathbb{C}^*, \mathrm{Aut}(\mathcal{O}_{CN^*(X)})) & \cong H^1_{alg}(\mathbb{C}^*, \mathbb{C}^*) & \cong \mathbb{Z} \to & \mathrm{Pic}_{\mathbb{C}^*}(CN^*(X)) & \to \mathrm{Pic}(CN^*(X)) = \{1\}
\end{array}$$

we deduce that the image of q_X consists of those $\mathbb{C}^*$-line bundles $\mathcal{L}$ over $CN^*(X)$, for which the isotropy groups $\{(\mathbb{C}^*)_{\mathbf{z}}, \mathbf{z} \in CN^*(X)\}$ act trivially on the fiber $\mathcal{L}_{\mathbf{z}}$ (cf. [75, §4, §5]). As these $\mathbb{C}^*$ -linearizations $\mathcal{L}_n$ of the trivial line bundle

$$CN^*(X) \times \mathbb{C} \to CN^*(X)$$

are parametrized by $n \in \mathbb{Z}$:

$$\mathcal{L}_n | \mathbb{C}^* \times (CN^*(X) \times \mathbb{C}) \ni (t, (\mathbf{z}, \lambda)) \longmapsto (t \cdot \mathbf{z}, t^n \lambda) \in CN^*(X) \times \mathbb{C},$$

for $\mathbf{z} = (z_1, \ldots, z_m) \in CN^*(X)$, $t \cdot \mathbf{z} = (t^{w_1} z_1, \ldots, t^{w_m} z_m)$,
we have $(\mathbb{C}^*)_{\mathbf{z}} \cong \mathbb{Z}/\gcd(w_i | i \in I_{\mathbf{z}})\mathbb{Z}$, where
$I_{\mathbf{z}} := \{i \in \{1, \ldots, m\} | z_i \neq 0\}$ with $|I_{\mathbf{z}}| \geq k + 1 - \mathcal{V}(\{1, \ldots, m\} \setminus I_{\mathbf{z}}) \geq 1$.
(The latter inequality comes from the well-stratifiedness of X.) Hence, if we set

$$Q_X(\mathbf{z}) := \{t \in \mathbb{C}^* : t^{w_i} = 1, \ \forall i, \ i \in I_{\mathbf{z}}\},$$

we get:

$$\eta_X = \min\{n \in \mathbb{N} | t^n = 1, \quad \text{for all} \quad t \in \bigcap_{\mathbf{z} \in CN^*(X)} Q_X(\mathbf{z})\}.$$

□

Corollary 2.38. *If $X = X_d$ is a well-formed q.s. hypersurface in $\mathbb{P}^{m-1}(\mathbf{w})$ of dimension ≥ 3 and its defining polynomial is general enough, then:*

(2.25)
$$\eta_X = \operatorname{lcm}(\{\gcd(w_{i_1}, w_{i_2}) | 1 \leq i_1 < i_2 \leq m\} \cup \{w_i, 1 \leq i \leq m, \quad \text{with} \quad w_i \nmid d\}).$$

Proof. Let $X = (f = 0)$. If the coefficients of f are sufficiently general w.r.t. each stratum (cf. proof of prop. 2.14), then X is well-stratified. For an index set $I \subset \{1, \ldots, m\}$ with $|I| = m - 1$ and $\{1, \ldots, m\} \setminus I = \{i\}$, $\mathbb{P}_I$ consists only of the point $[0, \ldots, 0, 1, 0, \ldots, 0]$ with the 1 in the $i-th$ position. X contains this point, i.e. $X(I) \neq \varnothing$ and $\mathcal{V}(I) = 1$, if and only if in the monomial decomposition of its defining polynomial there is no monomial involving only z_i. But this is equivalent to d not being a multiple of w_i. □

Corollary 2.39. *If X is a BP q.s.c.i. (2.2) of dimension ≥ 3, with $\alpha_{ij} \geq 2$, $\forall i$, $1 \leq i \leq m$, $\forall j$, $1 \leq j \leq k$, and all $(p \times p)$-subdeterminants of the matrix $(\lambda_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}$ are non-zero, $\forall p$, $1 \leq p \leq k$, then:*

$$\eta_X = \operatorname{lcm}(\{\gcd(w_{i_1}, \ldots, w_{i_{k+1}}) | 1 \leq i_1 < i_2 < \ldots < i_{k+1} \leq m\}) \tag{2.26}$$

Lemma 2.40. *Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a well-formed, well-stratified, q.s.c.i. of dimension $r \geq 3$ and $\mathbf{L}_X = \mathcal{O}_X(\eta_X)$ the generator of its Picard group. Then we have:*

$$\mathbf{L}_X^r = \frac{(\prod_{j=1}^k d_j)\eta_X^r}{\prod_{i=1}^m w_i} \tag{2.27}$$

Proof. It follows directly from the fact, that $(\mathcal{O}_{X^{\mathrm{cov}}}(\eta_X))^r = \eta_X^r(\mathcal{O}_{X^{\mathrm{cov}}}(1))^r = \eta_X^r(\prod_{j=1}^k d_j)$ and $(\mathcal{O}_{X^{\mathrm{cov}}}(\eta_X))^r = \deg(p(\mathbf{w})|_X)\mathbf{L}_X^r$, because

$$\mathcal{O}_{X^{\mathrm{cov}}}(\eta_X) = (p(\mathbf{w})|_X)^*(\mathbf{L}_X) \quad \text{and} \quad \deg(p(\mathbf{w})|_X) = \prod_{i=1}^m w_i.$$

□

Theorem 2.41. *The Δ-genus of a well-formed, well-stratified, q.s.c.i. $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ of dimension $r \geq 3$ with respect to $\mathbf{L}_X$ is given by the formula:*

$$\Delta(X, \mathbf{L}_X) = \tag{2.28}$$

$$r + \frac{(\prod_{j=1}^k d_j)\eta_X^r}{\prod_{i=1}^m w_i} - pt(\eta_X; \mathbf{w}) - \sum_{j=1}^k (-1)^j \sum_{1 \leq \nu_1 < \nu_2 < \ldots < \nu_j \leq k} pt(\eta_X - \sum_{\lambda=1}^j d_{\nu_\lambda}; \mathbf{w})$$

Proof. Obvious by the formulae (2.20) and (2.27). □

Remarks 2.42. (i) All well-formed q.s.c. intersections $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ with $r = m - k - 1 = 3$ and $\mathrm{am}(X) = 0$ are *minimal CY models of general type*, because they have always (full) crepant desingularizations (see § 4), their Picard number equals 1, and $([c_2]_X|_{\overline{\mathrm{Amp}(X)}}) > 0$. These models arose first in the physics literature in connection with the so called "Landau-Ginzburg potentials" (see [15], [74], [120], [129]). It should be mentioned, that the conditions of quasismoothness, well-formedness (cf. prop. 2.14 and 2.15) and of the vanishing amplitude are in fact very *restrictive*. This is the reason for which the expected degrees $\mathbf{d} = (d_1, \ldots, d_k)$ and weights $\mathbf{w} = (w_1, \ldots, w_m)$ for these X's have to move within *bounded* arithmetical regions and to be, in particular, *finitely many*. For example, there is no CY model of the regarded type with codimension $k \geq 5$, while only the intersection of four quadrics in the usual 7-dimensional projective space appears in codimension 4. For $1 \leq k \leq 3$, however, there are several thousands of allowable combinations $(\mathbf{d}; \mathbf{w})$, the number of which decreases as long as we increase k.
In the case where $k = 1$, Klemm and Schimmrigk [74] and, independently, Kreuzer and Skarke [79], gave a computer aided classification of all possible combinations for $(d; w_1, \ldots, w_5)$. They found 7555 combinations, the table of which covers a lot of pages (see preprint version of [74]). Recently Klemm [73] showed that there exist over 4200 (resp. 300) combinations corresponding to such models of codimension 2

(resp. 3) with $d_1, d_2 \leq 100$ (resp. $d_1, d_2, d_3 \leq 30$).
(ii) In [94, § 5] Oguiso studied polarized CY threefolds by means of their delta genera and came to the remarkable result, that *all* the polarized CY threefolds with $\Delta \leq 2$ have to be complete intersections of codimension ≤ 2 in a w.p.s. In the first step of his method, he makes use of the following Bertini-type theorem due to Fujita:

Theorem 2.43. *(Fujita [44], [45]) Let $(X, \mathbf{L})$ be a polarized smooth complex variety of dimension $r \geq 3$. Suppose that $\Delta(X, \mathbf{L}) \leq 2$ and $\mathbf{L}^r \geq 2$. Then* $\dim \mathrm{Bs}(|\mathbf{L}|) \leq 1$ *and all general members of $|\mathbf{L}|$ are smooth.*

If $(X, \mathbf{L})$ is a polarized CY threefold with $\Delta(X, \mathbf{L}) \leq 2$ and $\mathbf{L}^3 = 1$, then obviously $h^0(X, \mathbf{L}) \in \{2, 3\}$. If $\mathbf{L}^3 \geq 2$, then by 2.43. any general member S of $|\mathbf{L}|$ is smooth with ample canonical divisor $K_S = \mathbf{L}|_S$. This means that S is a minimal surface of general type with geometric genus $p_g(S) = h^0(X, \mathbf{L}) - 1$ and $K_S^2 = \mathbf{L}^3 \geq 2p_g(S) - 4 = 2h^0(X, \mathbf{L}) - 6$ (cf. [5, ch. VII, thm. 3.1., p. 210]). Thus, for $\Delta(X, \mathbf{L}) = 1$, it is necessarily $(h^0(X, \mathbf{L}), \mathbf{L}^3) \in \{(3, 1), (4, 2)\}$, and for $\Delta(X, \mathbf{L}) = 2$, $(h^0(X, \mathbf{L}), \mathbf{L}^3) \in \{(2, 1), (3, 2), (4, 3), (5, 4)\}$.

Oguiso's analysis on the corresponding graded rings $\oplus_{n \geq 0} H^0(X, n\mathbf{L})$ for the above 6 possible values of $(h^0(X, \mathbf{L}), \mathbf{L}^3)$ lead to the following:

Theorem 2.44. *(Oguiso's ($\Delta \leq 2$)-classification [94, thm. 5.1.]) Let $(X, \mathbf{L})$ be a polarized CY threefold with delta genus ≤ 2. Then X is a complete intersection of codimension ≤ 2 in w.p.s. and $\mathbf{L} = \mathbf{L}_X = \mathcal{O}_X(1)$. More precisely, for $\Delta(X, \mathbf{L}) = 1$, X is isomorphic either to an $X_8 \subset \mathbb{P}^4(1,1,1,1,4)$ or to an $X_{10} \subset \mathbb{P}^4(1,1,1,2,5)$. For $\Delta(X, \mathbf{L}) = 2$, X can be one of the following: $X_6 \subset \mathbb{P}^4(1,1,1,1,2)$, $X_{(2,6)} \subset \mathbb{P}^5(1,1,1,1,1,3)$, $X_{(3,6)} \subset \mathbb{P}^5(1,1,1,1,2,3)$, $X_{(4,6)} \subset \mathbb{P}^5(1,1,1,2,2,3)$ or $X_{(6,6)} \subset \mathbb{P}^5(1,1,2,2,3,3)$.*

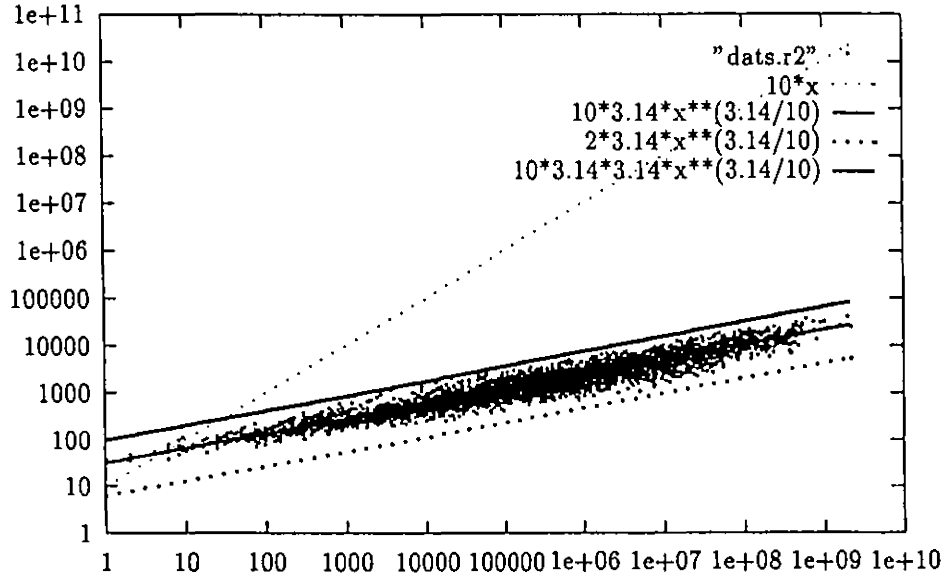
Theorems 2.43 and 2.44 are not true if one drops the assumption of the smoothness of X. Nevertheless, having formula (2.28) in hand, we can give the corresponding tables of CY models expressing well-formed, well-stratified q.s.c. intersections of codimension ≤ 2 in a w.p.s. with $\Delta(X, \mathbf{L}_X) \leq 2$, by using the "big classification tables" which were mentioned in 2.42. (i). Moreover, in the hypersurface case ($k = 1$), where the table of $(d; \mathbf{w})$'s is complete, we can win the whole picture of the "geographical placing" of the pairs $(\mathbf{L}_X^3, [c_2]_X(\mathbf{L}_X))$ which is in fact due to the numerical behaviour of η_X and of the *pt* -summands of $h^0(X, \mathcal{O}(\eta_X))$ in (2.20) (cf. comments at the end of § 1). The author is indebted to R. Schimmrigk for various computer checkings and for valuable remarks on the arising diagrams.

Proposition 2.45. *(i) The number $[c_2]_X(\mathbf{L}_X)$ within the class of minimal CY models $X = X_d \subset \mathbb{P}^4(\mathbf{w})$, being realized as hypersurfaces (with sufficiently general defining polynomials) in a four-dimensional w.p.s. and $(d; \mathbf{w})$'s running through the list of the above mentioned 7555 combinations, grows like:*

$$(2.29) \qquad [c_2]_X(\mathbf{L}_X) \sim 10\,\pi(\mathbf{L}_X^3)^{\frac{\pi}{10}} \quad \textit{and is bounded by}$$

(2.30) $$2\pi(\mathrm{L}_X^3)^{\frac{\pi}{10}} \le [c_2]_X(\mathrm{L}_X) \le 10\,\pi^2(\mathrm{L}_X^3)^{\frac{\pi}{10}}.$$

In logarithmic scales, the pairs $(\mathrm{L}_X^3, [c_2]_X(\mathrm{L}_X))$ *are given by the following diagram. (The line, which is indicated by faint dots, is the limiting* $(\Delta = 3)$ *-line.)*



(ii) The above class contains exactly 11 CY models with delta genus $\Delta(X, \mathrm{L}_X) \le 2$*. They are given by the following table:*

Nr.	*Model* $X = X_d$	L_X^3	$[c_2]_X(\mathrm{L}_X)$	$\Delta(X, \mathrm{L}_X)$	$e(X)$
(1)	$X_6 \subset \mathbb{P}^4(1,1,1,1,2)$	3	42	2	− 204
(2)	$X_8 \subset \mathbb{P}^4(1,1,1,1,4)$	2	44	1	− 296
(3)	$X_{10} \subset \mathbb{P}^4(1,1,1,2,5)$	1	34	1	− 288
(4)	$X_{12} \subset \mathbb{P}^4(1,2,2,3,4)$	2	32	2	− 138
(5)	$X_{12} \subset \mathbb{P}^4(1,1,2,2,6)$	4	52	2	− 250
(6)	$X_{14} \subset \mathbb{P}^4(1,2,2,2,7)$	2	44	1	− 212
(7)	$X_{15} \subset \mathbb{P}^4(1,3,3,3,5)$	3	42	2	− 124
(8)	$X_{18} \subset \mathbb{P}^4(1,1,1,6,9)$	9	102	2	− 542
(9)	$X_{18} \subset \mathbb{P}^4(1,2,3,3,9)$	3	42	2	− 188
(10)	$X_{24} \subset \mathbb{P}^4(1,1,2,8,12)$	8	92	2	− 482
(11)	$X_{36} \subset \mathbb{P}^4(1,2,3,12,18)$	6	72	2	− 362

Note that each of them can be defined by a Fermat polynomial.

(iii) Only 5 models lie on the $(\Delta = 3)$ *-line:* $[c_2]_X(\mathbf{L}_X) = 10\,\mathbf{L}_X^3$, *namely* $X_5 \subset \mathbb{P}^4$, $X_{16} \subset \mathbb{P}^4(1,1,3,3,8)$, $X_{20} \subset \mathbb{P}^4(1,4,5,5,5)$, $X_{26} \subset \mathbb{P}^4(2,2,3,6,13)$ *and* $X_{30} \subset \mathbb{P}^4(1,2,6,6,15)$.
(iv) Except for the cases, when X *is isomorphic to either* $X_{10} \subset \mathbb{P}^4(1,1,1,2,5)$ *or* $X_{12} \subset \mathbb{P}^4(1,2,2,3,4)$ *and* $\mathbf{L}_X^3 = \Delta(X, \mathbf{L}_X)$, *we have:*

$$\mathbf{L}_X^3 > \frac{6}{5}\Delta(X, \mathbf{L}_X). \tag{2.31}$$

Proof. Apply the formulae (2.25), (2.27), $[c_2]_X(\mathbf{L}_X) = 12\,h^0(X, \mathbf{L}_X) - 2\mathbf{L}_X^3$ (cf. (5.3)) and (2.28) to the table of the 7555 combinations of degrees and weights given in [74]. □

Similar results can be achieved for codimension $k = 2$.

Proposition 2.46. *There exist exactly 6 well-formed, well-stratified q.s.c. intersections* $X = X_{(d_1,d_2)} \subset \mathbb{P}^5(w_1, \ldots, w_6)$ *with* $\mathrm{am}(X) = 0$ *and* $\Delta(X, \mathbf{L}_X) \leq 2$:

Nr.	*Model* $X = X_{(d_1,d_2)}$	$\mathbf{L}_X^3$	$[c_2]_X(\mathbf{L}_X)$	$\Delta(X, \mathbf{L}_X)$
(1)	$X_{(2,6)} \subset \mathbb{P}^5(1,1,1,1,1,3)$	4	52	2
(2)	$X_{(3,6)} \subset \mathbb{P}^5(1,1,1,1,2,3)$	3	42	2
(3)	$X_{(4,6)} \subset \mathbb{P}^5(1,1,1,2,2,3)$	2	32	2
(4)	$X_{(6,6)} \subset \mathbb{P}^5(1,1,2,2,3,3)$	1	22	2
(5)	$X_{(6,10)} \subset \mathbb{P}^5(2,2,2,2,3,5)$	2	44	1
(6)	$X_{(6,12)} \subset \mathbb{P}^5(2,3,3,3,3,4)$	3	42	2

Moreover, 5 models of this kind lie on the $(\Delta = 3)$ *-line, namely* $X_{(4,4)} \subset \mathbb{P}^5(1,1,1,1,2,2)$, $X_{(6,8)} \subset \mathbb{P}^5(1,2,2,2,3,4)$, $X_{(10,12)} \subset \mathbb{P}^5(1,3,3,4,5,6)$, $X_{(10,15)} \subset \mathbb{P}^5(2,3,5,5,5,5)$ *and* $X_{(14,18)} \subset \mathbb{P}^5(2,2,6,6,7,9)$.

Proof. If X is a 2-dimensional well-formed, well-stratified q.s.c.i. in $\mathbb{P}^5(\mathbf{w})$, then $\mathrm{Pic}(X)$ is generated by the class of $\mathbf{L}_X = \mathcal{O}_X(\eta_X)$, where η_X, similarly to the hypersurface case of cor. 2.38., is given by $\eta_X = \mathrm{lcm}\{\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3\}$, with

$$\begin{aligned}
\mathcal{W}_1 &:= \{\gcd(w_{i_1}, w_{i_2}, w_{i_3}) \,|\, 1 \leq i_1 < i_2 < i_3 \leq 6\} \\
\mathcal{W}_2 &:= \{\gcd(w_{i_1}, w_{i_2}) \,|\, 1 \leq i_1 < i_2 \leq 6 \quad \text{with either} \quad \gcd(w_{i_1}, w_{i_2}) \nmid d_1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{or} \quad \gcd(w_{i_1}, w_{i_2}) \nmid d_2\} \\
\mathcal{W}_3 &:= \{w_i | 1 \leq i \leq 6 \quad \text{such that} \quad w_i \nmid d_1 \quad \text{and} \quad w_i \nmid d_2\}.
\end{aligned}$$

Applying the formula (2.28) for this η_X to the combinations of $(\mathbf{d};\mathbf{w})$'s which were found by [73], and taking into account that Δ grows rapidly after the first steps of a search procedure, we get only the above cases fulfilling the requirement $\Delta \leq 3$.□

Remark 2.47. Using elementary number theory, one can verify that, up to permutations of weights and degrees and up to different coefficients of the defining polynomials, there exist exactly 171 BP q.s.c. intersections of dimension 3 and of vanishing amplitude satisfying the assumptions of cor. 2.39. Namely 147 with codimension $k = 1$, 19 with $k = 2$, 4 with $k = 3$ and one with $k = 4$. In particular, by (2.26) and (2.28), we deduce that the minimal delta genus for such an X with $k \in \{2,3,4\}$ occurs when $X = X_{(6,6)} \subset \mathbb{P}^5(1,2,2,2,2,3)$ and $\Delta = 4$.

§ 3. Toric crepant resolutions of 2- and 3-dimensional Gorenstein cyclic quotient singularities

As it was mentioned in prop. 2.5. and 2.18., well-formed q.s.c. intersections have singular loci consisting of *cyclic quotient singularities (c.q.s.)* In particular, when the amplitude vanishes, the occuring c.q.s. are Gorenstein. To resolve them locally by crepant morphisms, we shall make use of the language of toric geometry as it is presented by Danilov [25], Oda [93] and Fulton [47].

Let us first review some preliminary definitions and facts and fix certain useful notations.

(i) For a lattice N of rank r, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denotes its dual lattice and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ their scalar extensions to the field of real numbers. A subset σ of $N_{\mathbb{R}}$ is said to be a *strongly convex rational polyedral cone* (SCRPC, for short) if $\sigma \cap (-\sigma) = \{0\}$ and if there exist $n_1, \ldots, n_\mu \in N$, s.t. $\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_\mu$. Its *dimension* $\dim(\sigma)$ is that of the smallest $\mathbb{R}$-subspace $\sigma + (-\sigma) = \mathbb{R}\sigma$ of $N_{\mathbb{R}}$ containing σ and its *relative interior* $\mathrm{int}(\sigma)$ (resp. its *relative boundary* $\partial\sigma$) is defined to be the usual interior (resp. the usual boundary) of σ regarded as a subset of the $\mathbb{R}$-vector space $\mathbb{R}\sigma$. Such a σ is called *simplicial* if $n_1, \ldots, n_\mu$ are linearly independent over $\mathbb{R}$. The *dual cone* $\check{\sigma}$ of σ is defined by $\check{\sigma} := \{x \in M_{\mathbb{R}} | < x, y > \geq 0, \forall y \in \sigma\}$ and turns out to be an r-dimensional SCRPC in $M_{\mathbb{R}}$. (Here $< , >: M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ denotes the natural $\mathbb{R}$-bilinear pairing). A subset τ of a SCRPC σ is called a *face* of σ (notation: $\tau \prec \sigma$) if it can be expressed as $\tau = \sigma \cap \{m_0\}^{\perp} := \{y \in \sigma | < m_0, y >= 0\}$ for some $m_0 \in \check{\sigma}$.

(ii) For a μ-tuple $(n_1, \ldots, n_\mu) \in N^\mu$ consisting of $\mathbb{R}$-linearly independent vectors, we define
$s(n_1, \ldots, n_\mu) := \{y \in N_{\mathbb{R}} | y = \sum_{i=1}^{\mu} \lambda_i n_i \text{ with } \sum_{i=1}^{\mu} \lambda_i = 1 \text{ and } \lambda_1, \ldots, \lambda_\mu \in \mathbb{R}_{\geq 0}\}$
to be the usual closed, affine simplex with vertices $n_1, \ldots, n_\mu$, and, for a given $s = s(n_1, \ldots, n_\mu)$, we set $\sigma(s) := \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_\mu$ to indicate the simplicial SCRPC arising from it after omitting of the affinity condition for its defining linear combinations.

(iii) If $\sigma \subset N_{\mathbb{R}}$ is a SCRPC, then the intersection $M \cap \check{\sigma}$ generates M as a group, is saturated, and is a finitely generated additive subsemigroup of M containing 0, i.e. there exist $m_1, \ldots, m_k \in M$, s.t. $M \cap \check{\sigma} = \mathbb{N}_0 m_1 + \ldots + \mathbb{N}_0 m_k$. If $\tau \prec \sigma$ with $\tau = \sigma \cap \{m_0\}^{\perp}$, then $M \cap \check{\tau} = M \cap \check{\sigma} + \mathbb{N}_0(-m_0)$.

(iv) Let now $T_N \cong (\mathbb{C}^*)^r$ be the r-dimensional algebraic torus defined by $T_N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Every $m \in M$ (resp. $n \in N$) assigns a character $\mathbf{e}(m) : T_N \ni t \longmapsto t(m) \in \mathbb{C}^*$ (resp. a 1-parameter subgroup $\gamma_n : \mathbb{C}^* \ni \lambda \longmapsto \gamma_n(\lambda) \in T_N$, $\gamma_n(\lambda)(m) = \lambda^{<m,n>}$, $\forall m \in M$) of T_N. This means that, after having chosen a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and its dual basis $\{m_1, \ldots, m_r\}$, we shall always get an isomorphism $T_N \ni t \longmapsto (u_1(t), \ldots, u_r(t)) \in (\mathbb{C}^*)^r$ for $u_j := \mathbf{e}(m_j)$, $1 \leq j \leq r$, and that $\{u_1, \ldots, u_r\}$ can therefore be considered as a coordinate system of T_N. On the other hand, for a SCRPC σ with $M \cap \check{\sigma} = \mathbb{N}_0 m_1 + \ldots + \mathbb{N}_0 m_k$, we associate to the finitely generated, normal $\mathbb{C}$-algebra $\mathbb{C}[M \cap \check{\sigma}]$ an affine variety $U_\sigma := \mathrm{Spec}(\mathbb{C}[M \cap \check{\sigma}])$, which can be written

$$U_\sigma = \{u : M \cap \check{\sigma} \to \mathbb{C} | u(0) = 1, u(m + m') = u(m)u(m'), \forall m, m' \in M \cap \check{\sigma}\}$$

with $\mathrm{e}(m)(u) := u(m), \forall m \in M \cap \check{\sigma}$ and $\forall u \in U_\sigma$. In the analytic category, U_σ, identified with its image under $(\mathrm{e}(m_1), \dots, \mathrm{e}(m_k)) : U_\sigma \to \mathbb{C}^k$, can be regarded as an analytic set determined by a system of equations of the form: (monomial) = (monomial). This complex analytic structure induced on U_σ is independent of the semigroup generators $\{m_1, \dots, m_k\}$ and each polynomial function $\mathrm{e}(m)$ on U_σ is holomorphic w.r.t. it. In particular, for $\tau \prec \sigma$, U_τ is an open subset of U_σ.

(v) A *fan* in $N \cong \mathbb{Z}^r$ is a collection Σ of SCRPCs in $N_\mathbb{R}$, s.t.(a) any face τ of $\sigma \in \Sigma$ belongs to Σ and (b) for $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. The union $|\Sigma| := \cup\{\sigma | \sigma \in \Sigma\}$ is called *the support of* Σ. Furthermore, we define $\Sigma(i) := \{\sigma \in \Sigma | \dim(\sigma) = i\}$, $0 \le i \le r$.

If $\rho \in \Sigma(1)$, then there exists a unique primitive vector $n(\rho) \in \mathbb{N} \cap \rho$ with $\rho = \mathbb{R}_{\ge 0} n(\rho)$ and each cone $\sigma \in \Sigma$ can be therefore written as

$$\sigma = \sum_{\rho \in \Sigma(1), \rho \prec \sigma} \mathbb{R}_{\ge 0} n(\rho).$$

The set $Sk^1(\sigma) := \{n(\rho) | \rho \in \Sigma(1), \rho \prec \sigma\}$ is called *the first skeleton* of σ. *The toric variety associated to a fan* (N, Σ) is the identification space
$Z(N, \Sigma) := ((\coprod_{\sigma \in \Sigma} U_\sigma)/ \sim)$ with $U_{\sigma_1} \ni u_1 \sim u_2 \in U_{\sigma_2} :\Longleftrightarrow$
$(\exists \tau \prec \sigma_1 \cap \sigma_2 : u_i \in U_\tau \subset U_{\sigma_i}$, for $i = 1, 2$, and $u_1 = u_2$ within $U_\tau)$.
$Z(N, \Sigma)$ admits a canonical T_N-action, which extends the group multiplication of $T_N = U_{\{0\}} : T_N \times Z(N, \Sigma) \ni (t, u) \longmapsto t \cdot u \in Z(N, \Sigma)$,
where, for $u \in U_\sigma$, $(t \cdot u)(m) := t(m) \cdot u(m)$, $\forall m \in M \cap \check{\sigma}$.

(vi) If we denote by $\mathrm{orb}(\sigma)$ (resp. $V(\sigma) := \overline{\mathrm{orb}(\sigma)}$) the orbit (resp. the closure of the orbit) of $\sigma \in \Sigma$ under this action, then

$$\Sigma \ni \sigma \longmapsto \mathrm{orb}(\sigma) = \{u : M \cap \sigma^\perp \to \mathbb{C}^* | u \quad \text{group homomorphism}\} \in \{T_N - \text{orbits in} \quad Z(N, \Sigma)\}$$

establishes an 1-1 correspondence. The T_N-orbits have the following properties:

(a) $\mathrm{orb}(\{0\}) = U_{\{0\}} = T_N$ and $\dim(\mathrm{orb}(\sigma)) = r - \dim(\sigma)$, $\forall \sigma \in \Sigma$.
(b) $\tau \prec \sigma \Longleftrightarrow \mathrm{orb}(\sigma) \subset V(\tau)$.
(c) For $\sigma \in \Sigma$, ,$\mathrm{orb}(\sigma)$ is the unique closed T_N-orbit in U_σ and $U_\sigma = \coprod\{\mathrm{orb}(\tau) | \prec \sigma\}$.
(d) For $\tau \in \Sigma$, we have $V(\tau) = \coprod\{\mathrm{orb}(\sigma) | \sigma \in \Sigma, \tau \prec \sigma\}$.
(e) For $\tau \in \Sigma$, $V(\tau) = Z(N(\tau), \mathrm{Star}(\tau))$ is itself a toric variety w.r.t. $N(\tau) := N/\mathbb{Z}(\tau \cap N), \mathrm{Star}(\tau) := \{\bar{\sigma} | \sigma \in \Sigma, \tau \prec \sigma\}$, where $\bar{\sigma} := (\sigma + \mathbb{R}\tau)/\mathbb{R}\tau$ denotes the image of σ in $N(\tau)_\mathbb{R} = N_\mathbb{R}/\mathbb{R}\tau$.

(vii) Let $Z(N, \Sigma)$ be the toric variety associated to a fan Σ and $N \cong \mathbb{Z}^r$. Then:
(a) for $\sigma \in \Sigma, U_\sigma$ is nonsingular $\Longleftrightarrow$ ($\exists$ $\mathbb{Z}$ $-$ basis$\{n_1, \dots, n_r\}$ of N and $k \le r$ with $\sigma = \sum_{i=1}^k \mathbb{R}_{\ge 0} n_i$) and $Z(N, \Sigma)$ is nonsingular $\Longleftrightarrow$ (U_σ is nonsingular, $\forall \sigma \in \Sigma$).
(b) $Z(N, \Sigma)$ is compact $\Longleftrightarrow$ Σ is finite and $|\Sigma| = N_\mathbb{R}$.

(viii) A *map of fans* $\varphi : (N', \Sigma') \to (N, \Sigma)$ is a $\mathbb{Z}$-linear homomorphism $\varphi : N' \to N$ whose scalar extension $\varphi : N'_\mathbb{R} \to N_\mathbb{R}$ satisfies the property: $(\forall \sigma' \in \Sigma'$, $\exists \sigma \in \Sigma : \varphi(\sigma') \subset \sigma)$. Such a φ induces an holomorphic map $\varphi_* : Z(N', \Sigma') \to Z(N, \Sigma)$ which is equivariant w.r.t. the action of $T_{N'}$ and T_N. Moreover, φ_* is proper

$\Longleftrightarrow$ ($\forall \sigma \in \Sigma$, $\Sigma'_\sigma := \{\sigma' \in \Sigma' | \varphi(\sigma') \subset \sigma\}$ is finite and $\varphi^{-1}(\sigma) = |\Sigma'_\sigma|$). In particular, if $N' = N$, $\varphi = \mathrm{id}$ and Σ' is a locally finite nonsingular subdivision of Σ, then $\mathrm{id}_\star$ is proper and birational and gives an equivariant desingularization of $Z(N, \Sigma)$.

(ix) For the group $T_N\mathrm{Div}(Z)$ of Weil divisors on a toric variety $Z = Z(N, \Sigma)$ we have $T_N\mathrm{Div}(Z) \cong \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}V(\rho)$. (For another approach via support functions see [93, § 2.1].)

Let us now come back to our c.q.s. and see how are they describable in terms of the above given toric glossary (i)-(ix). Suppose that $r \geq 2$ and $G \subset GL(r, \mathbb{C})$ is a finite cyclic group of order l containing no pseudoreflexions and being generated by $\mathrm{diag}(\zeta_l^{\alpha_1}, \ldots, \zeta_l^{\alpha_r})$, for suitable integers $0 \leq \alpha_1, \ldots, \alpha_r < l$. If $(\mathbb{C}^r/G, [0])$ is the germ of the corresponding quotient singularity, its underlying space $\mathbb{C}^r/G$ can be identified with a toric variety $Z(N_G, \Sigma_0)$ *of type* $(l; \alpha_1, \ldots, \alpha_r)$ as follows: $T_{N_G} := (\mathbb{C}^\star)^r/G$ is an r-dimensional algebraic torus with 1-parameter group N_G and with group of characters $M_G = \mathrm{Hom}_{\mathbb{Z}}(N_G, \mathbb{Z})$. Let $\{e_1 = (1, \ldots, 0), \ldots, e_r = (0, \ldots, 1)\}$ be the standard basis of $\mathbb{Z}^r$. Then

$$N_G = N_0 + \sum_{j=1}^{l-1} \mathbb{Z}\left(\frac{[j\alpha_1]_l}{l}, \ldots, \frac{[j\alpha_r]_l}{l}\right) \quad \text{with} \quad N_0 := \sum_{i=1}^{r} \mathbb{Z}e_i \quad \text{and}$$

$$M_G = \bigcap_{j=1}^{l-1} M_j \quad \text{with} \quad M_j := \{(m_1, \ldots, m_r) \in \mathbb{Z}^r | \sum_{i=1}^{r} m_i[j\alpha_i]_l \equiv 0 (\mathrm{mod}\ l)\}$$
$$\forall j,\ 1 \leq j \leq l-1.$$

Defining $\sigma_0 := \sum_{i=1}^r \mathbb{R}_{\geq 0} e_i$, $\Sigma_0 := \{\tau | \tau \prec \sigma_0\}$, and using the exact sequence

$$0 \to G \cong N_G/N_0 \to T_{N_0} \to T_{N_G} \to 0$$

we get as projection map: $\mathbb{C}^r = Z(N_0, \Sigma_0) \to Z(N_G, \Sigma_0) = \mathbb{C}^r/G$.

Proposition 3.1. *For $Z(N_G, \Sigma_0)$ the following conditions are equivalent:*
(i) $Z(N_G, \Sigma_0)$ is Gorenstein.
(ii) $\omega_{Z(N_G, \Sigma_0)}$ is trivial.
(iii) $\exists\,!\ m_0 \in M : \langle m_0, e_i \rangle = 1,\ \forall i,\ 1 \leq i \leq r$.

Proof. If follows from [98, footnote of p. 294] and Ishida's criteria [93, p. 126]. □

If $\varphi = \mathrm{id}_\star : Z(N_G, \Sigma'_0) \to Z(N_G, \Sigma_0) = \mathbb{C}^r/G$ is a T_{N_G}-equivariant desingularization of $Z(N_G, \Sigma_0)$, then a cone $\mathbb{R}_{\geq 0} n(\rho')$, $\rho' \in \Sigma'_0(1)$, determines a prime divisor $D_{n(\rho')} := V(\mathbb{R}_{\geq 0} n(\rho')) = Z(\mathrm{Star}(\mathbb{R}_{\geq 0} n(\rho')))$ on $Z(N_G, \Sigma'_0)$. So we have an 1-1 correspondence:

$$\{\text{exceptional prime divisors w.r.t. } \pi\} \longleftrightarrow \cup\{Sk^1(\sigma') | \sigma' \in \Sigma'_0\} \setminus \{e_1, \ldots, e_r\}.$$

D_{e_i} corresponds to *the strict transform* of $\{(z_1, \ldots, z_r) \in \mathbb{C}^r | z_i = 0\}/G$ w.r.t. π, $\forall i$, $1 \leq i \leq r$.

Proposition 3.2. *A T_{N_G}-equivariant resolution $\pi : Z(N_G, \Sigma_0') \to Z(N_G, \Sigma_0) = \mathbb{C}^r/G$ of a Gorenstein c.q.s. $Z(N_G, \Sigma_0)$ is crepant if and only if*

$$\cup\{Sk^1(\sigma')|\sigma' \in \Sigma_0'\} \subset \mathcal{H} := \{(x_1, \ldots, x_r) \in \mathbb{R}^r | \sum_{i=1}^{r} x_i = 1\}.$$

In this case we get $e(Z(N_G, \Sigma_0')) = |G|$.

Proof. Let $\sigma' \in \Sigma_0'$ and $\varphi \in H^0(U'_{\sigma'}, \omega_{Z(N_G,\Sigma_0')}|_{U'_{\sigma'}})$ be $\varphi = f\, du_1' \wedge \ldots \wedge du_r'$, w.r.t. local coordinates $u_1', \ldots, u_r'$ of $U'_{\sigma'}$. Then the zero order of f along any exceptional prime divisor $D_{n(\rho')}$, $\rho' \in \Sigma_0'(1)$, $\rho' \prec \sigma'$, equals (trace $(n(\rho'))) - 1$. So π is crepant if and only if the total union $\cup\{Sk^1(\sigma')|\sigma' \in \Sigma_0'(r)\}$ lies in the hyperplane $\mathcal{H}$. Moreover, $e(Z(N_G, \Sigma_0')) = \sharp(\Sigma_0'(r))$, which is equal to the multiplicity $[N_G : N_0] = |G|$ of σ_0, because $\sigma_0 = \cup\{\sigma'|\sigma' \in \Sigma_0'(r)\}$. □

For $r \geq 4$ it is not always possible to construct such crepant resolutions. Nevertheless, in dimension 2 and 3, relatively simple principles of the corresponding lattice geometry lead to the desired constructions.

Proposition 3.3. *For $r = 2$ and $Z(N_G, \Sigma_0)$ a Gorenstein c.q.s. of type $(l; \alpha_1, \alpha_2)$, there is a unique crepant desingularization $\pi : Z(N_G, \Sigma_0') \to Z(N_G, \Sigma_0)$ with $\Sigma_0' = \{\{\mathbb{R}_{\geq 0}(\frac{[j\alpha_1]_l}{l}, \frac{[j\alpha_2]_l}{l})|1 \leq j \leq l-1\}, \mathbb{R}_{\geq 0}e_1, \mathbb{R}_{\geq 0}e_2$ and their faces$\}$, being provided with $l - 1$ exceptional prime divisors $\cong \mathbb{P}^1$, which compose a Hirzebruch-Jung string.*

Let now $r = 3$ and $Z(N_G, \Sigma_0) = \mathbb{C}^3/G$ be of type $(l; \alpha_1, \alpha_2, \alpha_3)$. We define $s_0 := s(e_1, e_2, e_3)$ and

$$\Phi_G := \left\{ \left(\frac{[j\alpha_1]_l}{l}, \frac{[j\alpha_2]_l}{l}, \frac{[j\alpha_3]_l}{l} \right) \middle| \sum_{i=1}^{3} [j\alpha_i]_l = 1,\ 1 \leq j \leq l-1 \right\}$$

(If $G \subset SL(3, \mathbb{C})$, we can always assume, up to a generator change, that $\alpha_1 + \alpha_2 + \alpha_3 = l$.)

Proposition 3.4. *All toric crepant resolutions of a Gorenstein c.q.s. $Z(N_G, \Sigma_0) = \mathbb{C}^3/G$ are of the form $\pi : Z(N_G, \Sigma_0'(\mathcal{S})) \to Z(N_G, \Sigma_0)$, where $\mathcal{S}$ denotes a triangulation of $\Sigma_0 \cap \mathcal{H} = \{\tau \cap \mathcal{H} | \tau \prec \sigma_0\}$ with $s_0 \cap N_G = \Phi_G \coprod \{e_1, e_2, e_3\}$ as the sets of its vertices and $\Sigma_0'(\mathcal{S}) = \{\{0\}, \{\sigma(s)|s \in \mathcal{S}\}\}$. Moreover, they fulfill the following properties:*
(i)

$$\sharp(\mathrm{int}(s_0) \cap N_G) = \frac{1}{2}(l - \sum_{i=1}^{3} \gcd(\alpha_i, l)) + 1 \tag{3.1}$$

(ii) Let $D_n := V(\sigma(\{n\}))$ denote the prime divisor corresponding to an $n \in s_0 \cap N_G$.

(a) We have $\{$exc. pr. divisors w.r.t. $\pi\} = \cup\{D_n | n \in \Phi_G\}$.

(b) If $n \in \text{int}(s_0) \cap N_G$, then D_n is a rational surface coming from finitely many $T_{N_G(\sigma(\{n\}))}$-equivariant blow-ups either of $\mathbb{P}^2$ or of a Hirzebruch surface $\mathbb{F}_a := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$, $a \geq 0$.
(c) If $\partial s_0 \cap \Phi_G$ is non-empty and n one of its members lying on $s(e_{i_1}, e_{i_2})$, where $1 \leq i_1, i_2 \leq 3$, $i_1 \neq i_2$ and $\{i_3\} = \{1,2,3\} \setminus \{i_1, i_2\}$, then D_n represents a ruled fibration over the i_3-axis. Its fibers over the "punctured" i_3-axis are isomorphic to $\mathbb{P}^1$.

Proof. That the T_{N_G}-equivariant desingularizations of $Z(N_G, \Sigma_0)$ are parametrized by the above triangulations is obvious from prop. 3.1. and 3.2.
(i) It is easy to verify that $\sharp\Phi_G = \frac{1}{2}(l + \sum_{i=1}^{3} \gcd(\alpha_i, l)) - 2$ and $\sharp(\partial s_0 \cap \Phi_G) = \sum_{i=1}^{3} \gcd(\alpha_i, l) - 3$.
(ii) (a) and (c) are clear from the construction. (b) follows from [93, thm. 1.28]. □

Remarks 3.5.
(i) To each 1-simplex $s(n_1, n_2)$ of an $\mathcal{S}$ corresponds a curve $\mathcal{C}(n_1, n_2) := V(\sigma(s(n_1, n_2)))$. $\mathcal{C}(n_1, n_2)$ is compact $\Longleftrightarrow$ $\text{int}(s(n_1, n_2)) \subset \text{int}(s_0)$. In this case $\mathcal{C}(n_1, n_2) \cong \mathbb{P}^1$.
(ii) If n is as in 3.5. (ii) (c), $\text{Spl}^1(\mathcal{S})(n)$ denotes all 1-simplices of $\mathcal{S}$ which are connected with n having their second vertex in $\Phi_G \setminus \{n\}$ and $b(n) := \sharp(\text{Spl}^1(\mathcal{S})(n))$, then the fiber of $D_n \to \{i_3\text{-axis}\}$ over the zero point consists of a tree-configuration of $b(n)$ rational curves $\{\mathcal{C}(n, n_1), \ldots, \mathcal{C}(n, n_{b(n)})\}$ with

$$\mathcal{C}(n, n_{t_1}) \cap \mathcal{C}(n, n_{t_2}) = \begin{cases} \{\text{a point}\}, & \text{for } |t_1 - t_2| = 1,\ 1 \leq t_1, t_2 \leq b(n) \\ \varnothing, & \text{otherwise} \end{cases}$$

where $\text{Spl}^1(\mathcal{S})(n) = \{s(n, n_t) | 1 \leq t \leq b(n)\}$.

Proposition 3.6. *Let $Z(N_G, \Sigma_0) = \mathbb{C}^3/G$ be a Gorenstein c.q.s. and $Z(N_G, \Sigma_0'(\mathcal{S})) \to Z(N_G, \Sigma_0)$ be a crepant resolution w.r.t. $\mathcal{S}$.*
(i) For three distinct vertices of n_1, n_2, n_3 of $\mathcal{S}$ we have $(D_{n_1} \cdot D_{n_2} \cdot D_{n_3}) \neq 0 \Longleftrightarrow s(n_1, n_2, n_3)$ is a 2-simplex of $\mathcal{S}$. In this case $(D_{n_1} \cdot D_{n_2} \cdot D_{n_3}) = 1$.
(ii) If $n_1, n_2 \in s_0$, $s(n_1, n_2)$ is an 1-simplex of $\mathcal{S}$, but no both n_1 and n_2 belong to the same face of ∂s_0, then there exist exactly two vertices n_3, n_4 of $\mathcal{S}$, such that $s(n_1, n_2, n_3)$ and $s(n_1, n_2, n_4)$ are 2-simplices of $\mathcal{S}$, and the corresponding intersection numbers are related by

$$(D_{n_1}^2 \cdot D_{n_2}) n_1 + (D_{n_1} \cdot D_{n_2}^2) n_2 + n_3 + n_4 = 0 \tag{3.2}$$

(iii) If $n \in \text{int}(s_0)$ is a vertex of $\mathcal{S}$, then:

$$D_n^3 = 12 - \sharp(\text{Star}(\mathbb{R}_{\geq 0} n)(2)) \tag{3.3}$$

Proof. For (i) and (ii) apply the general techniques of [47, § 5].
(iii) Noether's formula gives $\chi(D_n, \mathcal{O}_{D_n}) = 1 \Longrightarrow \omega_{D_n}^2 = 12 - e(D_n)$. Since $\omega_{Z(N_G, \Sigma_0'(\mathcal{S}))} \cong \mathcal{O}_{Z(N_G, \Sigma_0'(\mathcal{S}))}$, we have $\omega_{D_n}^2 = D_n^3$ by adjunction. On the other hand

the topological Euler-Poincaré characteristic is nothing but $e(D_n) = e(V(\sigma(\{n\}))) = e(V(\mathbb{R}_{\geq 0}n)) = e(\mathrm{Star}(\mathbb{R}_{\geq 0}n)) = \sharp(\mathrm{Star}(\mathbb{R}_{\geq 0}n)(2))$ (cf. [47, p. 59]). □

Definition 3.7. Let $Z(N,\Sigma)$ be a 3-dimensional nonsingular toric variety, associated to a fan Σ (w.r.t. $N \cong \mathbb{Z}^3$), $\{n_1, n_2, n_3\}$, $\{n_1, n_2, n_4\}$ two $\mathbb{Z}$-bases of N and $\sigma_1, \sigma_2 \in \Sigma(3)$ two cones $\sigma_1 = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n_3$, $\sigma_2 = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n_4$ adjacent along $\tau_{1,2} = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_2$. If n_1, n_2, n_3 are coplanar and $n_1 + n_2 = n_3 + n_4$, then $\sigma_3 = \mathbb{R}_{\geq 0}n_1 + \mathbb{R}_{\geq 0}n_3 + \mathbb{R}_{\geq 0}n_4$, $\sigma_4 = \mathbb{R}_{\geq 0}n_2 + \mathbb{R}_{\geq 0}n_3 + \mathbb{R}_{\geq 0}n_4$ are adjacent along the 2-dimensional cone $\tau_{3,4} = \mathbb{R}_{\geq 0}n_3 + \mathbb{R}_{\geq 0}n_4$ and $Z(N, \widetilde{\Sigma})$, with $\widetilde{\Sigma} := (\Sigma \setminus \{\sigma_1, \sigma_2, \tau_{1,2}\}) \cup \{\sigma_3, \sigma_4, \tau_{3,4}\}$, is a nonsingular toric variety. In this case, $\widetilde{\Sigma}$ is called *elementary transformation* of Σ w.r.t. σ_1, σ_2 and $\tau_{1,2}$. (See Fig. 1.)

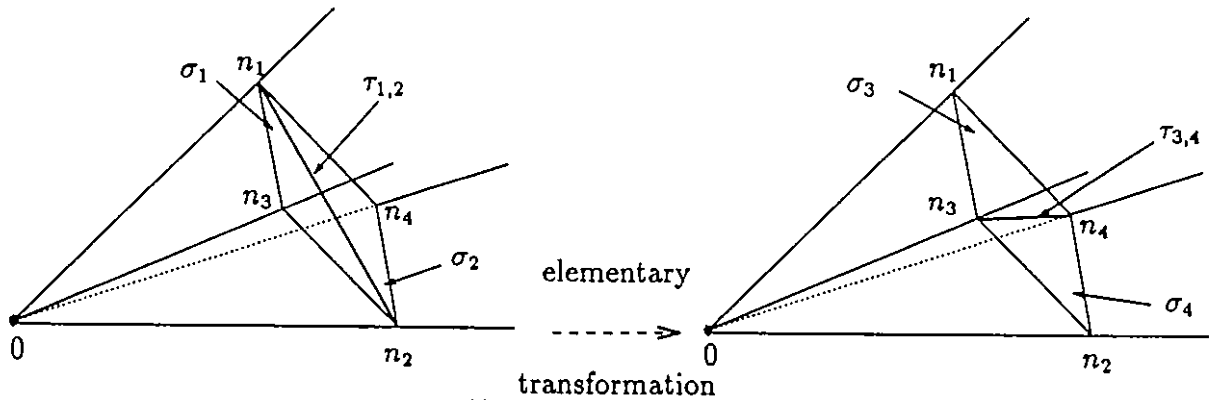


Fig. 1

Proposition 3.8. *Let Σ_1, Σ_2 be two nonsingular fans in $N \cong \mathbb{Z}^3$ with $|\Sigma_1| = |\Sigma_2|$. If we assume the existence of an m_0 in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, for which $\langle m_0, n(\rho_1)\rangle = \langle m_0, n(\rho_2)\rangle = 1$ for all $\rho_1 \in \Sigma_1(1)$ and all $\rho_2 \in \Sigma_2(1)$, then Σ_2 is obtained from Σ_1 by a finite succession of elementary transformations.*

Proof. See Danilov [26, prop. 2] or Oda [93, prop. 1.30. (ii)]. □

Corollary 3.9. *The fans $\Sigma_0'(S_i)$, $i = 1, 2$, which corresponds to two crepant resolutions $\pi_i : Z(N_G, \Sigma_0'(S_i)) \to Z(N_G, \Sigma_0)$, $i = 1, 2$, of a Gorenstein 3-dimensional cyclic quotient singularity, differ from each other by finitely many elementary transformations.*

Proof. Obvious by propositions 3.2. and 3.8. □

§ 4. Global toroidal crepant desingularizations

Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a system of weights, $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k$, and

$$X = X_{\mathbf{d}} = \{[z_1, \ldots, z_m] \in \mathbb{P}^{m-1}(\mathbf{w}) | f_1(z_1, \ldots, z_m) = \ldots = f_k(z_1, \ldots, z_m) = 0\}$$

a well-formed q.s.c.i. of dimension 3 (i.e. $m - k = 4$) with $\mathrm{am}(X) = 0$. Using the notations of § 2, we define:

$$\Gamma_0 := \{I \subset \{1, \ldots, m\} : |I| = 3 + \mathcal{V}(I), c_I > 1 \quad \text{and} \quad c_{I'} = 1, \forall I', I' \subsetneqq I\},$$

$$\hat{\Gamma}_0 := \{I \subset \{1, \ldots, m\} : |I| = 3 + \mathcal{V}(I) \quad \text{with} \quad c_I > 1, \quad c_{I \setminus \{i\}} > 1 \quad \text{for at least one } i \in I \quad \text{and} \quad c_I > c_{I \setminus \{i\}} \quad \text{for all that i's satisfying this property}\},$$

and $\Gamma_1 := \{I \subset \{1, \ldots, m\} : |I| = 2 + \mathcal{V}(I) \quad \text{and} \quad c_I > 1\}$.

Since X is normal and well-formed, we have $\mathrm{codim}_X(\mathrm{Sing}(X)) \geq 2$, and $\mathrm{Sing}(X)$ can therefore have at most 1-dimensional components. We write $\mathrm{Sing}(X)$ as the union of 0- and 1-dimensional singular strata

$$\mathrm{Sing}(X) = \mathrm{SSt}^0(X) \coprod \mathrm{SSt}^1(X) \tag{4.1}$$

where $\mathrm{SSt}^\rho(X) := \cup\{X(I) | I \in \Gamma_\rho\}$, $\rho = 0, 1$ (cf. prop. 2.18). Furthermore, we define $\mathrm{INP}(X) := \cup\{X(I) | I \in \hat{\Gamma}_0\}$. Without loss of generality, we shall treat here only the case in which none of the above strata is empty. We first fix an enumeration $\Gamma_1 = \{I_1, \ldots, I_\kappa\}$ of the index sets of Γ_1. Each $C_j := X(I_j)$ is an irreducible smooth curve and

$$\mathrm{SSt}^1(X) = \{C_j | 1 \leq j \leq \kappa\} \tag{4.2}$$

In addition, we introduce the enumerations

$$\{t_1^j, \ldots, t_{m-|I_j|}^j\} := \{1, \ldots, m\} \setminus I_j \tag{4.3}$$

$$\{s_1^j, \ldots, s_{k-\mathcal{V}(I_j)}^j\} := \{1, \ldots, k\} \setminus \{\rho \in \{1, \ldots, k\} : f_\rho|_{I_j} \equiv 0\} \tag{4.4}$$

to be used in the following:

Lemma 4.1. *For the curve C_j and any point $Q \in C_j \setminus \mathrm{INP}(X)$ there exist integers l_{C_j} and $\alpha_1^{(C_j)}, \alpha_2^{(C_j)} \geq 1$ with $\alpha_1^{(C_j)} + \alpha_2^{(C_j)} = l_{C_j}$, depending on the weights with indices in I_j and on the defining polynomials of X, such that:*
(i) the germ (X, Q) of X at Q is isomorphic to

$$(X, Q) \cong (\mathbb{C}^2/G_{C_j} \times \mathbb{C}, [0] \times \{0\}) \tag{4.5}$$

where $(\mathbb{C}^2/G_{C_j}, [0])$ is a c.q.s. of type $(l_{C_j}; \alpha_1^{(C_j)}, \alpha_2^{(C_j)})$, and
(ii) X near C_j looks like:

$$(X, C_j) \cong ((\mathcal{O}_{C_j}(\alpha_1^{(C_j)}) \oplus \mathcal{O}_{C_j}(\alpha_2^{(C_j)}))/G_{C_j}, C_j) \tag{4.6}$$

Proof. (i) Since X is well-stratified and $Q \notin \mathrm{INP}(X)$ (i.e. it is not possible for Q to contain more than $2 + \mathcal{V}(I_j)$ coordinates equal to zero), we have:

$$\mathrm{rank}\left(\frac{\partial(f_{s_1^j}|_{C_j}, \ldots, f_{s^j_{k-\mathcal{V}(I_j)}}|_{C_j})}{\partial(z_{t_1^j}, \ldots, z_{t^j_{m-|I_j|}})}\,|_Q\right) = k - \mathcal{V}(I_j) \quad \text{and}$$

$$\mathrm{rank}\left(\frac{\partial(f_1, \ldots, f_k)}{\partial(z_1, \ldots, z_m)}\,|_Q\right) = k.$$

Thus, by implicit function theorem, a local chart of the V-variety X centered at Q will have as coordinates $z_{q_1^j}$, $z_{q_2^j}$, where $\{q_1^j, q_2^j\} \subset I_j$, together with a third one $z_{q_3^j}$ expressing the restriction on C_j ($q_3^j \in \{t_1^j, \ldots, t^j_{m-|I_j|}\}$). Note that the complex plane determined by $z_{q_1^j}, z_{q_2^j}$ is equipped with the action

$$(z_{q_1^j}, z_{q_2^j}) \longmapsto (\zeta_{c_{I_j}}^{[w_{q_1^j}]_{c_{I_j}}} z_{q_1^j}, \zeta_{c_{I_j}}^{[w_{q_2^j}]_{c_{I_j}}} z_{q_2^j}).$$

We end the proof just by setting $l_{C_j} := c_{I_j} = c(\mathbf{w}, I_j) = \gcd(w_{t_1^j}, \ldots, w_{t^j_{m-|I_j|}})$, $\alpha_1^{(C_j)} := [w_{q_1^j}]_{c_{I_j}}$ $\alpha_2^{(C_j)} := [w_{q_2^j}]_{c_{I_j}}$, and taking into account that $\mathrm{am}(X) = 0$.

(ii) We can use the above description or, alternatively, apply the tubular neighbourhood theorem to the affine quasicones over C_j and X. For the punctured quasicones we have $(CN^*(X), CN^*(C_j)) \cong (CN^*(C_j) \times \mathbb{C}^2/G_{C_j},\ CN^*(C_j))$. Letting $\mathbb{C}^*$ act on them in the usual way (cf. 2.11.) we get (4.6). □

Let us now give enumerations to the points of the singular locus of X. We set:

$$\mathrm{SSt}^0(X) = \{P_i | 1 \leq i \leq \lambda\} \quad \text{and} \tag{4.7}$$

$$\mathrm{INP}(X) = \{Q_\iota | 1 \leq \iota \leq \mu\} \quad \text{as well as} \tag{4.8}$$

$$\mathrm{INP}(X) \cap C_j = \{Q_{\nu_j}^{(j)} | 1 \leq \nu_j \leq \xi_j\},\ 1 \leq j \leq \kappa \tag{4.9}$$

to indicate an enumeration of the points of $\mathrm{INP}(X)$ sitting on C_j. It should be mentioned here that $\max\{\xi_j | 1 \le j \le \kappa\} \le \mu \le \sum_{j=1}^{\kappa} \xi_j$, because (for $\kappa \ge 2$) it may happen that there are indices $1 \le j,\ j' \le \kappa,\ j \ne j'$, such that $Q_\iota = Q_{\nu_j}^{(j)} = Q_{\nu_{j'}}^{(j')}$, for certain $1 \le \iota \le \mu$ and $1 \le \nu_j \le \xi_j$, $1 \le \nu_{j'} \le \xi_{j'}$. (This is exactly the case, in which C_j and $C_{j'}$ have at least one intersection point.)

Lemma 4.2. *For any $P_i \in X(I)$, $I \in \Gamma_0$, there exist integers l_{P_i} and $\alpha_1^{(P_i)}, \alpha_2^{(P_i)}, \alpha_3^{(P_i)} \ge 1$ with $\alpha_1^{(P_i)} + \alpha_2^{(P_i)} + \alpha_3^{(P_i)} = l_{P_i}$, such that*

$$(X, P_i) \cong (\mathbb{C}^3 / G_{P_i}, [0]) \tag{4.10}$$

i.e. the germ of X at P_i is isomorphic to that of a c.q.s. of type $(l_{P_i};\ \alpha_1^{(P_i)}, \alpha_2^{(P_i)}, \alpha_3^{(P_i)})$. (The same holds true if we consider a Q_ι instead of P_i).

Proof. Exactly as in the proof of 4.1. one finds indices $\{q_1, q_2, q_3\} \subset I$, such that $z_{q_1}, z_{q_2}, z_{q_3}$ represent local coordinates of X centered at P_i with respect to the action:

$$(z_{q_1}, z_{q_2}, z_{q_3}) \longmapsto (\zeta_{c_I}^{[w_{q_1}]_{c_I}} z_{q_1},\ \zeta_{c_I}^{[w_{q_2}]_{c_I}} z_{q_2},\ \zeta_{c_I}^{[w_{q_3}]_{c_I}} z_{q_3}).$$

Since the acting cyclic group lies in $SL(3, \mathbb{C})$, we set $l_{P_i} := c_I$ and make use of the "normalization" of the exponents

$$(\alpha_1^{(P_i)}, \alpha_2^{(P_i)}, \alpha_3^{(P_i)}) := \begin{cases} ([w_{q_1}]_{c_I}, [w_{q_2}]_{c_I}, [w_{q_3}]_{c_I}), \text{ if } \sum_{\rho=1}^{3} [w_{q_\rho}]_{c_I} = c_I \\ (c_I - [w_{q_1}]_{c_I}, c_I - [w_{q_2}]_{c_I}, c_I - [w_{q_3}]_{c_I}), \text{ if } \sum_{\rho=1}^{3} [w_{q_\rho}]_{c_I} = 2c_I \end{cases}$$

□

The points of $\mathrm{SSt}^0(X)$ are *the isolated points* of $\mathrm{Sing}(X)$. The points of $\mathrm{INP}(X)$ will be called *individual*. The justification of the choice of this name comes from the fact, that for a $Q_{\nu_j}^{(j)} \in \mathrm{INP}(X) \cap C_j$, the group $G_{Q_{\nu_j}^{(j)}}$ has order strictly bigger than that one of the group G_{C_j}. The union $\mathrm{SSt}^0(X) \coprod \mathrm{INP}(X)$ of isolated and individual points of $\mathrm{Sing}(X)$ constitutes the set of the so called *dissident points* in Reid's terminology (see [98, Cor. 1.14., p. 281]), i.e. the set of points of the threefold X which are not of *compound Du Val type*. In other words, the *compound Du Val locus*, in our case, is composed of the points of $\mathrm{SSt}^1(X) \setminus \mathrm{INP}(X)$, and each point $Q \in C_j \setminus \mathrm{INP}(X)$ is by lemma 4.1. of type $cA_{l_{C_j}-1}$.

Let us now consider appropriate open neighbourhoods $\mathcal{U}_{P_i}$, resp. $\mathcal{U}_{Q_\iota}$ of P_i, resp. of Q_ι, $1 \le i \le \lambda$, $1 \le \iota \le \mu$, such that $\mathcal{U}_{P_i} \cong Z(N_{G_{P_i}}, \Sigma_0) = \mathbb{C}^3 / G_{P_i}$ and $\mathcal{U}_{Q_\iota} \cong Z(N_{G_{Q_\iota}}, \Sigma_0) = \mathbb{C}^3 / G_{Q_\iota}$ respectively. Since $Q_\iota = Q_{\nu_j}^{(j)}$ for some $1 \le \nu_j \le \xi_j$, $1 \le j \le \kappa$, we can take a tubular neighbourhood $\mathcal{U}_{C_j}$ of C_j, such that

$$\mathcal{U}_{C_j} \cap \mathcal{U}_{Q_{\nu_j}^{(j)}} \cong V(\tau^{(j)}) = Z(N_{G_{Q_{\nu_j}^{(j)}}}(\tau^{(j)}), \mathrm{Star}(\tau^{(j)})) \tag{4.11}$$

with $\tau^{(j)} = \mathbb{R}_{\geq 0}\, e_{v_1^{(j)}} + \mathbb{R}_{\geq 0}\, e_{v_2^{(j)}}$, $\{v_1^{(j)}, v_2^{(j)}\} \in \{\{1,2\},\{2,3\},\{3,1\}\}$.

Fig. 2 shows these neighbourhoods of two curves of $\mathrm{SSt}^1(X)$ having an individual intersection point.

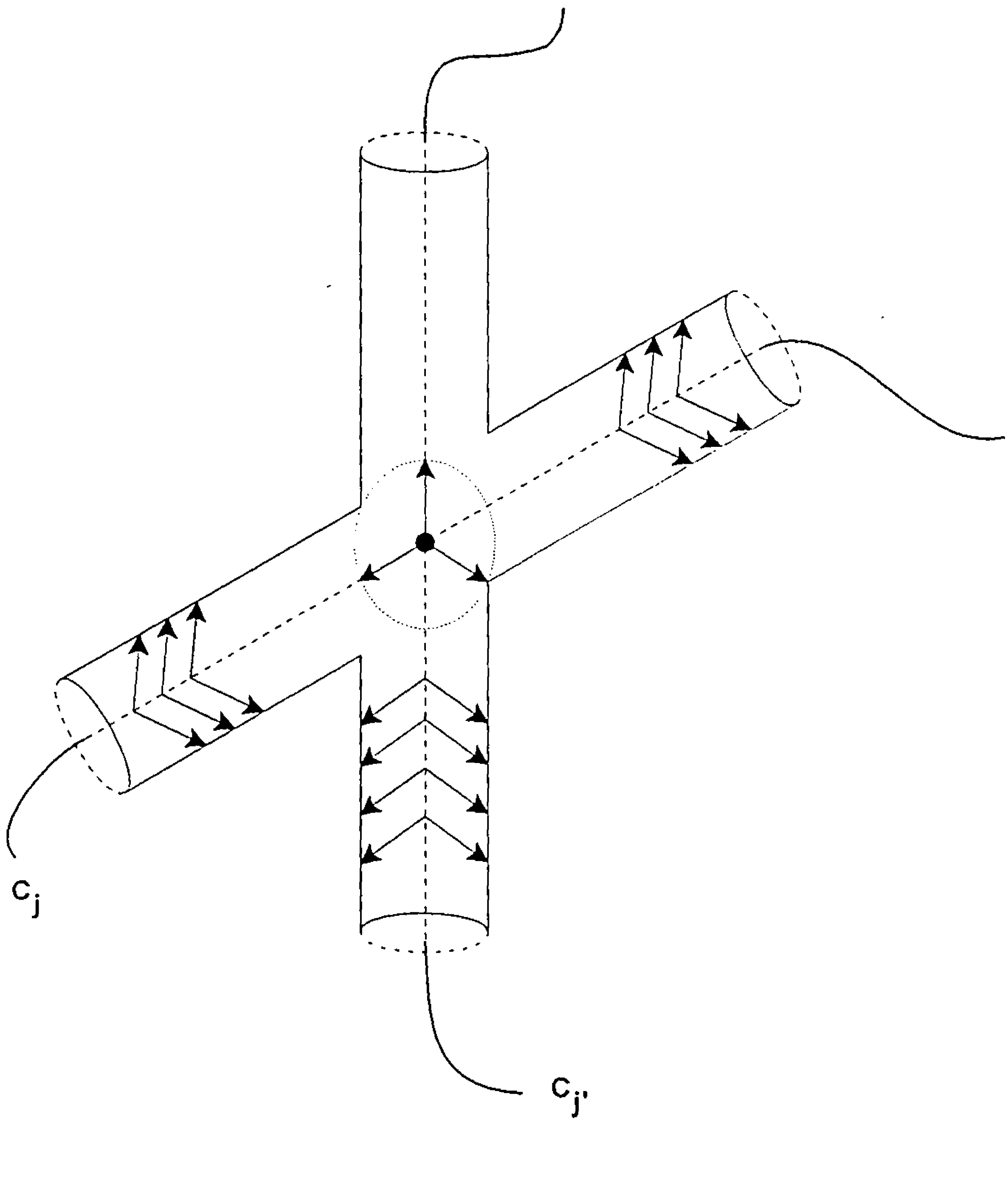


Fig. 2

Using prop. 3.4 we construct toric crepant resolutions

$$(4.12)\qquad \pi_{P_i}(\mathcal{S}_{P_i}) : (Z(N_{G_{P_i}}, \Sigma_0'(\mathcal{S}_{P_i})), \mathcal{E}_{P_i}) \to (Z(N_{G_{P_i}}, \Sigma_0), P_i)$$

$$(4.13)\qquad \pi_{Q_\iota}(\mathcal{S}_{Q_\iota}) : (Z(N_{G_{Q_\iota}}, \Sigma_0'(\mathcal{S}_{Q_\iota})), \mathcal{E}_{Q_\iota}) \to (Z(N_{G_{Q_\iota}}, \Sigma_0), Q_\iota)$$

with $\mathcal{E}_{P_i} := \{D_n^{(P_i)} | n \in \text{int}(s_0) \cap N_{G_{P_i}}\}$, $\mathcal{E}_{Q_\iota} := \{D_n^{(Q_\iota)} | n \in \text{int}(s_0) \cap N_{G_{Q_\iota}}\}$. (All the above divisors are assumed to be endowed with the *reduced* space structure.) Analogously, we construct uniquely determined crepant resolutions

$$(4.14)\qquad \pi_{C_j} : Z(N_{G_{C_j}}, \Sigma_0') \times C_j \to \mathcal{U}_{C_j} \cong Z(N_{G_{C_j}}, \Sigma_0) \times C_j$$

by applying prop. 3.3. along the normal sheaf $\mathcal{N}_{C_j/X}$ of C_j in X. (4.11) give rise to the compatibility conditions

$$(4.15)\qquad \pi_{C_j}|_{\mathcal{U}_{C_j} \cap \mathcal{U}_{Q_{\nu_j}^{(j)}}} = \pi_{Q_{\nu_j}^{(j)}}(\mathcal{S}_{Q_{\nu_j}^{(j)}})|_{\mathcal{U}_{C_j} \cap \mathcal{U}_{Q_{\nu_j}^{(j)}}}$$

and enable us to glue (4.12), (4.13) and (4.14) together in order to define a total number of

$$(\prod_{i=1}^{\lambda} \sharp\{\text{triangulations} \quad \mathcal{S}_{P_i}\})(\prod_{\iota=1}^{\mu} \sharp\{\text{triangulations} \quad \mathcal{S}_{Q_\iota}\})$$

global resolutions

$$(4.16)\qquad \pi : (Y = Y(\mathcal{S}_1, \ldots, \mathcal{S}_{P_\lambda}, \mathcal{S}_{Q_1}, \ldots, \mathcal{S}_{Q_\mu}), \mathcal{E}(X)) \to (X, \text{Sing}(X))$$

of the singularities of X, with Y's obtained from X by replacing $\text{Sing}(X)$ by $\mathcal{E}(X)$. Their *exceptional loci* can be written as

$$\mathcal{E}(X) := \left(\coprod_{i=1}^{\lambda} \mathcal{E}_{P_i}\right) \coprod \left(\left(\bigcup_{j=1}^{\kappa} \mathcal{E}_{C_j}\right) \cup \left(\coprod_{\iota=1}^{\mu} \mathcal{E}_{Q_\iota}\right)\right),$$

where $\mathcal{E}_{C_j} = \{D_{r_j}^{(C_j)} | 1 \le r_j \le l_{C_j} - 1\}$ express the union of the $l_{C_j} - 1$ prime divisors lying over C_j. All $\pi|_{D_{r_j}^{(C_j)}} : D_{r_j}^{(C_j)} \to C_j$ are smooth *ruled fibrations* provided with two sections $\cong C_j$ and allowing, in general, exceptional fibers. Indeed, if $Q_{\nu_j}^{(j)}$ is an individual point of C_j, it is $\text{supp}(D_{r_j}^{(C_j)}) \cap \mathcal{U}_{Q_{\nu_j}^{(j)}} = \text{supp}(D_{n^{(\nu_j, r_j)}})$, i.e. the support of a non-compact divisor, realized in the corresponding triangle $s_0 = s(e_1, e_2, e_3)$ by the vertex $n^{(\nu_j, r_j)}$ of $\mathcal{S}_{Q_{\nu_j}^{(j)}}$, and if (in the notation of 3.5. (ii))

$$(4.17)\qquad \text{Spl}^1(\mathcal{S}_{Q_{\nu_j}^{(j)}})(n^{(\nu_j, r_j)}) = \{\mathcal{C}(n^{(\nu_j, r_j)}, n_{t^{(\nu_j, r_j)}}^{(\nu_j, r_j)}) | 1 \le t^{(\nu_j, r_j)} \le b(n^{(\nu_j, r_j)})\}$$

then

$$\pi^{-1}(Q_{\nu_j}^{(j)}) \cap \mathcal{E}_{C_j} = \bigcup_{r_j=1}^{l_{C_j}-1} \left(\bigcup_{t^{(\nu_j, r_j)}=1}^{b(n^{(\nu_j, r_j)})} \mathcal{C}(n^{(\nu_j, r_j)}, n_{t^{(\nu_j, r_j)}}^{(\nu_j, r_j)})\right)$$

(i.e. exceptional fibers occur whenever $b(n^{(\nu_j, r_j)}) > 1$, see fig. 3).

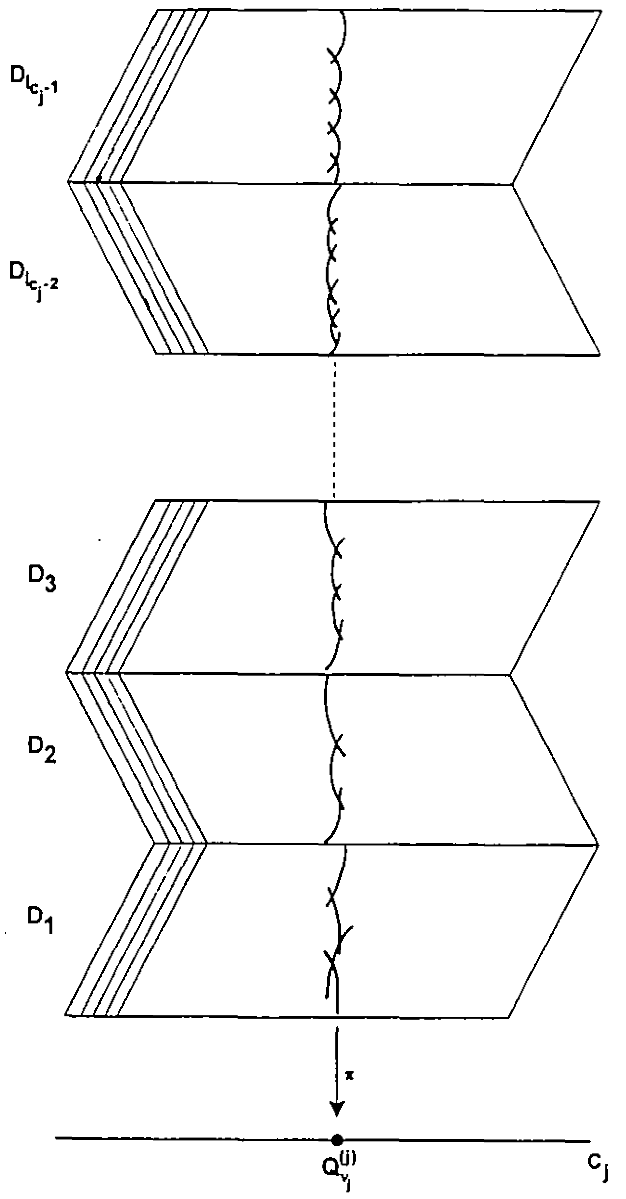


Fig. 3

Proposition 4.3. *The global desingularizations (4.16) of X are all crepant, and each of them differs from another one by finitely many simple flops, being realized by elementary transformations which take place within the fans* $\{\Sigma_0'(\mathcal{S}_{P_i}),\ \Sigma_0'(\mathcal{S}_{Q_\iota}) | 1 \leq i \leq \lambda,\ 1 \leq \iota \leq \mu\}$.

Proof. The verification of the first assertion follows from the construction and [99, thm. 1.14., p. 142]. Now every rational curve $C = C(n_1, n_2)$ representing an 1-simplex in one of the above triangulations $\mathcal{S}_{P_i}, \mathcal{S}_{Q_\iota}$, which is a diagonal of a convex quadrilateral (determined by lattice points), has normal bundle $\mathcal{N}_{C/Y} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ and can therefore be flopped (see [78, 2.3.2.1]). Simple flopping, in our case, means the replacement of the one diagonal of a regarded convex lattice quadrilateral by the other, i.e. the application of an elementary transformation. Hence, the second assertion follows from cor. 3.9. □

Definition 4.4. Although X and Y's are not themselves toric varieties, we shall call (4.16) *the toroidal crepant desingularizations of* X, since they can be completely described by means of their local toric data.

Remarks 4.5. (i) Most of Y's are projective, but, in general, there is no guarantee that *all* of them will be projective. This is why we shall deal here with both possibilities. (Of course, all projective ones are CY threefolds.) For the construction of pathological triangulations for crepant toric resolutions of abelian quotient singularities which lead, after gluing together, to nonprojective threefolds with trivial canonical bundle, as well as for a combinatorial method of how one discribes the projective ones, we refer to [24].
(ii) In both cases, Y's admit Hodge decomposition according to a theorem of Deligne (see [27, prop. 5.3., p. 121]). Moreover, all Y's have *the same* Hodge numbers, because the surgery of "flopping type" does not have any influence upon them (cf. [77, § 4] or [78, § 5]).
(iii) X can have crepant desingularizations other than the toroidal ones. Nevertheless, a general theorem of Kawamata-Matsuki [70] and Kollár [77, Cor. 5.6.] informs us that the number of all projective crepant desingularizations has to be always finite. The toroidal crepant desingularizations are, so to say, the ones which can be characterized, from the combinatorial point of view, in the best possible manner, because the corresponding "flopping loci" are easily controllable.

Before proceeding to the determination of the Hodge numbers of Y's, we have to introduce some more useful notations. We set $a(P_i) := \sharp(\mathcal{E}_{P_i})$, $a(Q_\iota) := \sharp(\mathcal{E}_{Q_\iota})$ and we fix the enumerations

$$\mathcal{E}_{P_i} = \{D^{(P_i)}_{n_{\mathfrak{p}_i}} | 1 \leq \mathfrak{p}_i \leq a(P_i)\},\ \mathcal{E}_{Q_\iota} = \{D^{(Q_\iota)}_{n_{\mathfrak{q}_\iota}} | 1 \leq \mathfrak{q}_\iota \leq a(Q_\iota)\} \tag{4.18}$$

By Prop. 3.4. (i) we get

$$a(P_i) = \frac{1}{2}(l_{P_i} - \sum_{\rho=1}^{3} \gcd(\alpha^{(P_i)}_\rho, l_{P_i})) + 1 \tag{4.19}$$

$$\text{and} \quad a(Q_\iota) = \frac{1}{2}(l_{Q_\iota} - \sum_{\rho=1}^{3} \gcd(\alpha^{(Q_\iota)}_\rho, l_{Q_\iota})) + 1 \tag{4.20}$$

Theorem 4.6. *The Hodge numbers of the toroidal crepant desingularization spaces* Y *of* X *are the following:*

$$h^{0,0}(Y) = h^{0,3}(Y) = h^{3,0}(Y) = h^{3,3}(Y) = 1, h^{p,q}(Y) = 0, \forall p, q, p \neq q, p + q \neq 3,$$

$$h^{1,1}(Y) = \sum_{i=1}^{\lambda} a(P_i) + \sum_{\iota=1}^{\mu} a(Q_\iota) + \sum_{j=1}^{\kappa} l_{C_j} - (\kappa - 1) \tag{4.21}$$

$$h^{1,2}(Y) = h^{1,2}(X) + \sum_{j=1}^{\kappa} (l_{C_j} - 1) h^0(C_j, \mathcal{O}_{C_j}(\mathrm{am}(C_j))) \tag{4.22}$$

($h^{1,2}(X) = h^{1,2}_{\text{prim}}(X)$ *and* $h^0(C_j, \mathcal{O}_{C_j}(\text{am}(C_j)))$ *are known from the formulae (2.17) and (2.20).)*

Proof. By (2.5) we know that $h^{p,q}(X) = 1, \forall p, 0 \le p \le 3$ and $h^{p,q}(X) = 0$, for $p + q \ne 3, p \ne q$. Since $\text{am}(X) = 0$, formulae (2.6), (2.16) and (2.18) give $h^{3,0}(X) = h^{0,3}(X) = 1$. The desingularization process alters only the remaining non-trivial Hodge numbers $h^{1,1} = h^{2,2}$ and $h^{1,2} = h^{2,1}$. Making use of the Mayer-Vietoris homology (or cohomology) sequences (see [24], [101]), we deduce the following additive splitting for the Betti numbers coming into question:

$$b_2(Y) = b_2(X) + \sharp\,(\text{exceptional prime divisors w.r.t. } \pi) \tag{4.23}$$

$$b_3(Y) = b_3(X) + \sum_{j=1}^{\kappa} (l_{C_j} - 1) b_1(C_j) \tag{4.24}$$

The number of the exceptional prime divisors being located only over P_i (resp. Q_ι) is $a(P_i)$ (resp. $a(Q_\iota)$), while over C_j lie exactly $l_{C_j} - 1$ ruled fibrations. Summing them up and setting $b_2(Y) = h^{1,1}(Y)$, $b_2(X) = 1$, we get (4.21) by (4.23). Finally, (4.22) follows from (4.24), because $h^{1,2}(Y) = \frac{1}{2} b_3(Y) - 1$, $b_3(X) = 2(1 + h^{1,2}(X))$ and $b_1(C_j) = 2h^1(C_j, \mathcal{O}_{C_j}) = 2h^0(C_j, \mathcal{O}_{C_j}(\text{am}(C_j)))$. □

Remarks 4.7. (i) By [32, 3.2.4. (ii)$'$] X is simply connected. Thus, Y's are also simply connected, because the fibers of $\pi : Y \to X$ are simply connected.
(ii) The above formulae depend, of course, on the defining polynomials of X. However, they turn out to be very efficient if one examines polynomials with special prescribed monomial decompositions, without demanding the satisfaction of any other extra conditions.

On the other hand, in certain cases, if each stratum $X(I)$ is assumed to be defined by polynomials, which are general enough, and if $f_1, \ldots, f_k$ can be rearranged in such a way that $(f_1 = \ldots = f_\rho = 0)$ is quasismooth, $\forall \rho, 1 \le \rho \le k$, then one can use a "gluing technique" of the so called *relative Milnor fibers*, in order to get *explicit* formulae for $h^{1,1}$ and $h^{1,2}$, depending *only* on $\mathbf{w}$ and $\mathbf{d}$. This computational method will be applied below. Previous results concerning the hypersurface case (i.e. when $k = 1$) are due to Vafa [120] and Roan [101].

Another computation method for the case of hypersurfaces, which are embedded in a Gorenstein toric Fano variety, was recently presented by Batyrev [7]. Batyrev's approach is mainly based on the study of the ambient space and its associated polyhedron and it is also applicable to partial crepant resolutions of arbitrary dimension. The regarded Hodge numbers are expressible by the numbers of the integral points of certain polyhedron faces. Moreover, for some families of hypersurfaces in spaces corresponding to reflexive polyhedra, the classical involutive duality of convex sets leads to a first precise mathematical interpretation of the so called *mirror phenomena*, which had been initially observed by the physicists within the framework of investigations of special conformal field theories.

Definition 4.8. For ρ, $1 \le \rho \le k$, let X^ρ be a w.c.i.

(4.25)
$$\{[z_1, \ldots, z_m] \in \mathbb{P}^{m-1}(w_1, \ldots, w_m) | f_1(z_1, \ldots, z_m) = \ldots = f_\rho(z_1, \ldots, z_m) = 0\}$$

with $\deg(f_\rho) = d_\rho$ and $X := X^k := X_{(d_1,\dots,d_k)}$, $X^0 := \mathbb{P}^{m-1}(\mathbf{w})$. X will be called *overall well-stratified* (w.r.t. the above enumeration of its defining polynomials) if X^ρ is well-stratified (see 2.36) for all ρ's.

Definition 4.9. Let $X = X_{(d_1,\dots,d_k)}$ be an overall well-stratified w.c.i. with a fixed enumeration (4.25) of its defining polynomials. For all ρ, $1 \le \rho \le k$, $(CN(X^\rho), 0)$ is the zero locus of the holomorphic function germ $f_\rho : (CN(X^{\rho-1}), 0) \to (\mathbb{C}, 0)$. For a sufficiently small $\varepsilon > 0$ we consider the open ball

$$B_\varepsilon(0) := \{\mathbf{z} \in \mathbb{C}^m : \| \mathbf{z} \| < \varepsilon\}.$$

$CN(X^\rho) \cap \partial\overline{B_\varepsilon(0)}$ is a $(m-\rho-2)$ -connected $(2(m-\rho)-1)$ -dimensional orientable C^∞ -differentiable manifold and $CN(X^\rho) \cap \overline{B_\varepsilon(0)}$ is homeomorphic to the cone $\{t\mathbf{z} | 0 \le t \le 1, \mathbf{z} \in CN(X^\rho) \cap \partial\overline{B_\varepsilon(0)}\}$ over $\mathrm{Lk}(\rho) := CN(X^\rho) \cap \partial\overline{B_\varepsilon(0)}$. $\mathrm{Lk}(\rho)$ is called *the relative link* of the origin in $CN(X^\rho)$ w.r.t. $CN(X^{\rho-1})$. By assumption, the set of critical points of $f_\rho|_{CN(X^{\rho-1})}$ consists only of the zero point. We can therefore choose $\varepsilon >> \varepsilon'$ small enough in order to construct two fibrations

$$\hat{f}_\rho : (f_\rho|_{CN(X^{\rho-1})})^{-1}(\overline{B_{\varepsilon'}(0)}) \cap B_\varepsilon(0) \setminus \{0\} \to \overline{B_{\varepsilon'}(0)} \setminus \{0\}$$

and

$$\varphi_\rho := \frac{\hat{f}_\rho}{\| \hat{f}_\rho \|} : \mathrm{Lk}(\rho - 1) \setminus \mathrm{Lk}(\rho) \to S^1.$$

The inclusion $\partial\overline{B_{\frac{\varepsilon'}{2}}(0)} \hookrightarrow (B_{\varepsilon'}(0) \setminus \{0\})$ is an homotopy equivalence and consequently all topological properties of the first fibration are preserved (up to homotopy) by its restriction over the circle $\partial\overline{B_{\frac{\varepsilon'}{2}}(0)}$. By identifying $\partial\overline{B_{\frac{\varepsilon'}{2}}(0)}$ with the unit circle S^1 via the map $\tau \longmapsto \frac{2\tau}{\varepsilon'}$, we regard this new restricted fibration over S^1, which we shall denote again by $\hat{f}_\rho$. $\hat{f}_\rho$ and φ_ρ are (in this sense) fiberwise diffeomorphic in the category of the locally trivial fibrations over S^1 and define *the relative Milnor fibration* of $CN(X^\rho)$ w.r.t. $CN(X^{\rho-1})$. The fibers F^ρ are homotopic to a bouquet $S^{m-\rho} \vee \ldots \vee S^{m-\rho}$ of $(m-\rho)$ -spheres (see [58]). The number of these spheres is called the *relative Milnor number* $\mathrm{mil}(F^\rho)$ of $CN(X^\rho)$ w.r.t. $(X^{\rho-1})$. It is easy to see that the topological Euler-Poincaré characteristic of F^ρ is given by $e(F^\rho) = 1 + (-1)^{m-\rho}\mathrm{mil}(F^\rho)$ and $\mathrm{mil}(F^\rho) = \mathrm{rk}(\tilde{H}_{m-\rho}(F^\rho, \mathbb{Z}))$. F^ρ is furthermore diffeomorphic to $\{(z_1, \dots, z_m) \in \mathbb{C}^m | f_1(z_1, \dots, z_m) = \ldots = f_{\rho-1}(z_1, \dots, z_m) = 0,\ f_\rho(z_1, \dots, z_m) = 1\}$ and its associated *characteristic automorphism* $\mathfrak{h}_\rho : F^\rho \to F^\rho$, coming from the fiber transport via the standard generator of $\pi_1(S^1, \{1\})$, is given by $\mathfrak{h}_\rho(z_1, \dots, z_m) := \zeta_{d_\rho} \cdot (z_1, \dots, z_m) = (\zeta_{d_\rho}^{w_1}, \dots, \zeta_{d_\rho}^{w_m} z_m)$.

For an X as above and indices $0 \le r_\rho \le d_\rho - 1$, $0 \le s_\rho \le d_\rho - 1$, $1 \le \rho \le k$, we introduce the following index-sets and enumerate them increasingly:

$$N_\rho(r_\rho) := \left\{ i | 1 \le i \le m : \frac{r_\rho w_i}{d_\rho} \in \mathbb{Z} \right\} = \left\{ \nu_1^{(r_\rho)}, \dots, \nu_{\alpha_\rho(r_\rho)}^{(r_\rho)} \right\},$$

$$\Xi_\rho(r_\rho) := \left\{ e | 1 \le e \le \rho : \frac{r_\rho d_e}{d_\rho} \in \mathbb{Z} \right\} = \left\{ \xi_1^{(r_\rho)}, \dots, \xi_{\beta_\rho(r_\rho)}^{(r_\rho)} \right\},$$

$$T(r_\rho, s_\rho) := \left\{ t | 1 \le t \le \alpha(r_\rho) : \frac{s_\rho w_{\nu_t(r_\rho)}}{d_\rho} \in \mathbb{Z} \right\} = \left\{ \tau_1^{(r_\rho, s_\rho)}, \dots, \tau_{\gamma_\rho(r_\rho, s_\rho)}^{(r_\rho, s_\rho)} \right\},$$

$$V(r_\rho, s_\rho) := \left\{ l | 1 \le l \le \beta(r_\rho) : \frac{s_\rho d_{\xi_l(r_\rho)}}{d_\rho} \in \mathbb{Z} \right\} = \left\{ \upsilon_1^{(r_\rho, s_\rho)}, \dots, \upsilon_{\varepsilon_\rho(r_\rho, s_\rho)}^{(r_\rho, s_\rho)} \right\},$$

$$\mathcal{L}^{(j)} := \begin{cases} \{r_k | 1 \le r_k \le d_k - 1 \quad \text{with} \quad \alpha_k(r_k) - \beta_k(r_k) = 2\}, & \text{if} \quad j = k \\ \{r_j | 1 \le r_j \le d_j - 1 \quad \text{with} \quad \alpha_k(r_j) - \beta_k(r_j) = 2 \quad \text{and} \\ N_j(r_j) \ne N_l(r_l), \forall l, j+1 \le l \le k \quad \text{and} \\ \forall r_l, 0 \le r_l \le d_l - 1\}, & \text{if} \quad j \ne k \end{cases}$$

where the j in the last expression is bounded by $1 \le j \le k$ and $\{j : 1 \le j \le k | \mathcal{L}^{(j)} \ne \varnothing\} = \{j_1, \dots, j_q\}$. Furthermore we need the following abbreviations:

$$\mathfrak{Z}(r_\rho, s_\rho) := \left(\prod_{l=1}^{\varepsilon_\rho(r_\rho, s_\rho)} d_{\xi^{(r_\rho)}_{\upsilon_l(r_\rho, s_\rho)}} \right) \left(\prod_{t=1}^{\gamma_\rho(r_\rho, s_\rho)} w_{\nu^{(r_\rho)}_{\tau_t^{(r_\rho, s_\rho)}}} \right)^{-1},$$

$$\mathfrak{Y}_\rho(r_\rho, s_\rho) :=$$

$$= \sum_{i=0}^{\mathfrak{e}(\rho)} (-1)^i D_i \left(d_{\xi^{(r_\rho)}_{\upsilon_1(r_\rho, s_\rho)}}, \dots, d_{\xi^{(r_\rho)}_{\upsilon^{(r_\rho, s_\rho)}_{\varepsilon_\rho(r_\rho, s_\rho)}}} \right) W_{\mathfrak{e}(\rho) - i} \left(w_{\nu^{(r_\rho)}_{\tau_1^{(r_\rho, s_\rho)}}}, \dots, w_{\nu^{(r_\rho)}_{\tau^{(r_\rho, s_\rho)}_{\gamma_\rho(r_\rho, s_\rho)}}} \right),$$

defined by means of the summetric polynomials (2.7), (2.8), with $\mathfrak{e}(\rho) := \gamma_\rho(r_\rho, s_\rho) - \varepsilon_\rho(r_\rho, s_\rho)$,

$$e(r_\rho, s_\rho) := \begin{cases} 0, & \text{if} \quad \mathfrak{e}(\rho) < 0 \\ \mathfrak{Z}(r_\rho, s_\rho) \cdot \mathfrak{Y}(r_\rho, s_\rho), & \text{if} \quad \mathfrak{e}(\rho) \ge 0 \end{cases}$$

$$g_{r_\rho}^{(\rho)} := \frac{1}{d_\rho} \sum_{s_\rho = 0}^{d_\rho - 1} (1 - e(r_\rho, s_\rho))$$

and

$$\mathfrak{G}_{r_{j_\lambda}}^{(j_\lambda)} := \begin{cases} 0, & \text{if} \quad j_\lambda = 1 \\ \sum_{\sigma=1}^{j_\lambda - 1} \tilde{g}_{r_\sigma}^{(\sigma)}, & \text{if} \quad j_\lambda \ge 2 \end{cases}$$

with

$$\tilde{g}^{(\sigma)}_{r_\sigma} := \begin{cases} 0, & \text{if} \quad \nexists r_\sigma,\, 0 \le r_\sigma \le d_\sigma - 1, \quad \text{with} \quad N_\sigma(r_\sigma) = N_{i_\lambda}(r_{j_\lambda}) \\ g^{(\sigma)}_{\hat{r}_\sigma}, & \text{if} \quad \exists \hat{r}_\sigma = r_\sigma \quad \text{with} \quad 0 \le \hat{r}_\sigma \le d_\sigma - 1 \quad \text{and} \\ N_\sigma(\hat{r}_\sigma) = N_{j_\lambda}(r_{j_\lambda}),\, \forall \sigma,\, 1 \le \sigma \le j_\lambda - 1 \end{cases}$$

for $1 \le \sigma \le j_\lambda - 1$ and $1 \le \lambda \le q$.

Lemma 4.10. *Let $F^\rho(r_\rho) := (F^\rho)^{\mathfrak{h}_\rho^{r_\rho}}$ denote the fixed point set of $\mathfrak{h}_\rho^{r_\rho}$. Suppose $F^\rho(r_\rho) \neq \varnothing$. Then*

$$F^\rho(r_\rho) = \{(z_1, \dots, z_m) \in \mathbb{C}^m \,|\, f_j(z_1, \dots, z_m) = 0,\, \forall j,\, 1 \le j \le \rho - 1,\, j \in \Xi_\rho(r_\rho), \\ f_\rho(z_1, \dots, z_m) = 1 \quad \text{and} \quad z_i = 0,\, \forall i,\, 1 \le i \le m,\, i \notin N_\rho(r_\rho)\}$$

is an affine complete intersection of dimension $\alpha_\rho(r_\rho) - \beta_\rho(r_\rho)$ w.r.t. the weights $\{w_{\nu_1^{(r_\rho)}}, \dots, w_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}\}$ in the $\alpha_\rho(r_\rho)$ variables $z_{\nu_1^{(r_\rho)}}, \dots, z_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}$ with no singular points other than the origin. $F^\rho(r_\rho)$ is also diffeomorphic to the relative Milnor fiber w.r.t.

$$f_\rho : (\{(z_{\nu_1^{(r_\rho)}}, \dots, z_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}) \in \mathbb{C}^{\alpha_\rho(r_\rho)} | f_j(z_{\nu_1^{(r_\rho)}}, \dots, z_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}) = 0, \\ \forall j,\, 1 \le j \le \rho - 1,\, j \in \Xi_\rho(r_\rho)\}, 0) \to (\mathbb{C}, 0).$$

Proof. Let $t := \zeta_{d_\rho}^{r_\rho}$ and consider the maps $\psi_\rho := (f_1, \dots, f_\rho) : \mathbb{C}^m \to \mathbb{C}^\rho$, resp. $\mathbb{C}^m \ni \mathbf{z} \longmapsto {}^t\psi_\rho(\mathbf{z}) := \psi_\rho(t \cdot \mathbf{z}) \in \mathbb{C}^\rho$, where $t \cdot \mathbf{z} := (t^{w_1} z_1, \dots, t^{w_m} z_m)$. the chain rule gives

$$\mathbf{D}({}^t\psi_\rho(\mathbf{z})) = (\mathbf{D}\psi_\rho(t \cdot \mathbf{z})) \cdot \operatorname{diag}(t^{w_1}, \dots, t^{w_m}) = \operatorname{diag}(t^{d_1}, \dots, t^{d_\rho}) \cdot (\mathbf{D}\psi_\rho(z)).$$

For $\mathbf{z} \in F^\rho(r_\rho)$ we have $t \cdot \mathbf{z} = \mathbf{z}$ and therefore

$$t^{w_i} \frac{\partial f_j}{\partial z_i} = t^{d_j} \frac{\partial f_j}{\partial z_i},\, \forall i,\, 1 \le i \le m,\, \forall j,\, 1 \le j \le \rho.$$

This means that

$$\frac{\partial f_j}{\partial z_i} = 0 \quad \text{if} \quad \begin{cases} \text{either} & i \notin N_\rho(r_\rho) \quad \text{and} \quad j \in \Xi_\rho(r_\rho) \\ \text{or} & i \in N_\rho(r_\rho) \quad \text{and} \quad j \notin \Xi_\rho(r_\rho). \end{cases}$$

On the other hand the functions $\{f_j | j \notin \Xi_\rho(r_\rho)\}$ are constant on

$$\{\mathbf{z} \in \mathbb{C}^m | t \cdot \mathbf{z} = \mathbf{z}\} = \{\mathbf{z} \in \mathbb{C}^m | z_i = 0,\, \forall i,\, 1 \le i \le m,\, i \notin N_\rho(r_\rho)\}$$

and therefore vanish. Thus, $F^\rho(r_\rho)$ has the above form. Moreover, for every point $\mathbf{z}^0 \in (F^\rho(r_\rho) \setminus \{\mathbf{0}\})$, we get for the Jacobian block matrix:

$$\mathrm{rk}\left(\frac{\partial f_j}{\partial z_i}\Big|_{\mathbf{z}=\mathbf{z}^0}\right)_{\substack{i\in N_\rho(r_\rho)\\ j\in\Xi_\rho(r_\rho)}} = \beta_\rho(r_\rho)$$

and $F^\rho(r_\rho)$ cannot have $\mathbf{z}^0$ as a singular point. □

Now $\mathfrak{h}_\rho$ induces the automorphism:

$$\mathfrak{h}_\rho(r_\rho) : F^\rho(r_\rho) \ni (z_{\nu_{1(r_\rho)}}, \dots, z_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}) \longmapsto (\zeta_{d_\rho}^{w^{(r_\rho)}_{\nu_1}} z_{\nu_1^{(r_\rho)}}, \dots, \zeta_{d_\rho}^{w_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}} z_{\nu^{(r_\rho)}_{\alpha_\rho(r_\rho)}}) \in F^\rho(r_\rho)$$

and the cyclic group $\langle\mathfrak{h}_\rho(r_\rho)\rangle$ generated by $\mathfrak{h}_\rho(r_\rho)$ has the order

$$d_{r_\rho} := \frac{d_\rho}{\gcd\{w_i | i \in N_\rho(r_\rho)\}}$$

For $0 \le r_\rho \le d_\rho - 1$, $0 \le s_\rho \le d_\rho - 1$, we get $F^\rho(r_\rho)^{\mathfrak{h}_\rho(r_\rho)^{s_\rho}} = F^\rho(r_\rho) \cap F^\rho(s_\rho) =:$ $F^\rho(r_\rho, s_\rho)$ and $F^\rho(r_\rho)^{\mathfrak{h}_\rho(r_\rho)^{s_\rho}} = F^\rho(r_\rho)^{\mathfrak{h}_\rho(r_\rho)^{s'_\rho}}$ if $0 \le s'_\rho \le d_\rho - 1$ with $s_\rho \equiv s'_\rho(\mathrm{mod}(d_{r_\rho}))$. As in the previous lemma, we can conclude that $F^\rho(r_\rho, s_\rho)$ is an affine c.i. of dimension $\mathfrak{e}(\rho)$ with $\mathbf{0}$ as the only singular point (for $\mathfrak{e}(\rho) \ge 0$). From the Lefschetz fixed point formula we deduce:

$$1 - e(F^\rho(r_\rho)/\langle\mathfrak{h}_\rho\rangle) = 1 - \frac{1}{d_{r_\rho}} \sum_{s_\rho=0}^{d_\rho-1} e(F^\rho(r_\rho)^{\mathfrak{h}_\rho(r_\rho)^{s_\rho}}) =$$

$$(4.26) \qquad = 1 - \frac{1}{d_\rho} \sum_{s_\rho=0}^{d_\rho-1} e(F^\rho(r_\rho, s_\rho)) = \frac{1}{d_\rho} \sum_{s_\rho=0}^{d_\rho-1} (1 - e(F^\rho(r_\rho, s_\rho))).$$

Theorem 4.11. *Let $X = X_{(d_1,\dots,d_k)} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a well-formed, overall well-stratified c.i. with $m - k = 4$, $\mathrm{am}(X) = 0$. Then the non-trivial Hodge numbers of the toroidal crepant desingularizations $\pi : Y \to X$ of X are given by:*

$$(4.27) \qquad h^{1,1}(Y) = \frac{1}{2}e(Y) + h^{1,2}(Y), \quad \text{where}$$

$$(4.28) \qquad e(Y) = \sum_{\rho=1}^{k} \sum_{r_\rho=0}^{d_\rho-1} g^{(\rho)}_{r_\rho} \quad \text{and}$$

$$(4.29) \qquad h^{1,2}(Y) = h^{1,2}(X) - \frac{1}{2}\left(\sum_{\lambda=1}^{q} \sum_{r_{j_\lambda} \in \mathcal{L}^{(j_\lambda)}} (\mathfrak{G}^{(j_\lambda)}_{r_{j_\lambda}} + g^{(j_\lambda)}_{r_{j_\lambda}})\right)$$

Proof. (4.27) is obvious. By prop. 3.2. we can see that for all strata X_I of X and $I \subsetneq M \subset \{1, \dots, m\}$,

$$\pi : (\pi^{-1}(X_I \setminus \cup\{X_M : I \subsetneq M\})) \to X_I \setminus \cup\{X_M : I \subsetneq M\}$$

is a fibration with e(fiber) $= c_I$ (cf. 2.5., 2.18.). The stratification of Y gives

$$e(Y) = \sum_I \left(\sum_{I \subset J} (-1)^{|I|+|J|} e(X_J) \right) c_I. \tag{4.30}$$

By assumption, $X_I^\rho := X^\rho \cap \mathbb{P}_I$ (in the notation of 2.5.) is a q.s.c.i. Since X_I^ρ appears as the support of a *subcomplex* in a *topological triangulation* of $X_I^{\rho-1}$ (see [83]), the exact cohomology sequence concerning the complements $U_I^\rho := X_I^{\rho-1} \setminus X_I^\rho$ ([13, p. 52]):

$$\ldots \to H^i(U_I^\rho, \mathbb{Q}) \to H^i(X_I^{\rho-1}, \mathbb{Q}) \to H^i(X_I^\rho, \mathbb{Q}) \to H^{i+1}(U_I^\rho, \mathbb{Q}) \to \ldots$$

yields: $e(U_I^\rho) = e(X_I^{\rho-1}) - e(X_I^\rho)$, $\forall \rho$, $1 \le \rho \le k$. Summing these k equations together, we get for $X_I := X_I^k$:

$$e(X_I) = e(\mathbb{P}_I) - \sum_{\rho=1}^{k} e(U_I^\rho) \tag{4.31}$$

If we set $\mathcal{I}_\rho := \{I \subset \{1, \ldots, m\} | U_I^\rho \neq \varnothing\}$, then $e(Y)$, according to (4.30) and (4.31), can be written as:

$$e(Y) = \sum_I [e(\mathbb{P}_I) - e(\cup\{\mathbb{P}_J | I \subsetneqq J\})] c_I - \sum_{\rho=1}^{k} \sum_{I \in \mathcal{I}_\rho} [e(U_I^\rho) - e(\cup\{U_J^\rho | I \subsetneqq J\})] c_I \tag{4.32}$$

Combining this with $e(\mathbb{P}_I) - e(\cup\{\mathbb{P}_J | I \subsetneqq J\}) = \begin{cases} 1, & \text{if } |I| = m-1 \\ 0, & \text{otherwise} \end{cases}$

we have:

$$e(Y) = \sum_{i=1}^{m} w_i - \sum_{\rho=1}^{k} \sum_{I \in \mathcal{I}_\rho} [e(U_I^\rho) - e(\cup\{U_J^\rho | I \subsetneqq J\})] \tag{4.33}$$

Defining $U_I'^\rho := U_I^\rho \setminus \cup\{U_J^\rho | I \subsetneqq J\}$, we show similarly

$$\begin{aligned} &e(U_I'^\rho) = e(U_I^\rho) - e(\cup\{U_J^\rho | I \subsetneqq J\}) \quad \text{and} \\ &e(U_I^\rho) = \sum_{I \subset J} e(U_J'^\rho), \quad \forall I \in \mathcal{I}_\rho. \end{aligned} \tag{4.34}$$

The transition from the local to the global data can be achieved via the relative Milnor fibers F^ρ, by identifying $F^\rho / \langle \mathfrak{h}_\rho \rangle$ and $U^\rho = X^{\rho-1} \setminus X^\rho$ by means of the projection

$$\mathfrak{q}_\rho : F^\rho \ni (z_1, \dots, z_m) \longmapsto [z_1, \dots, z_m] \in U^\rho, \quad \text{for all} \quad \rho, \quad 1 \leq \rho \leq k.$$

Moreover, for $I \in \mathcal{I}_\rho$, $\mathfrak{q}_\rho^{-1}(U_I^\rho)$ is $\mathfrak{h}_\rho$ -invariant and

(4.35)

$$\sharp\{r_\rho | 0 \leq r_\rho \leq d_\rho - 1 : F^\rho(r_\rho)/\langle \mathfrak{h}_\rho \rangle \supset U_I^\rho\} =$$
$$\sharp\{r_\rho | 0 \leq r_\rho \leq d_\rho - 1 : \mathfrak{h}_\rho^{r_\rho} = \ \text{id} \ \text{on} \ \mathfrak{q}_\rho^{-1}(U_I^\rho)\} =$$
$$= \sharp\{r_\rho | 0 \leq r_\rho \leq d_\rho - 1 : \frac{d_\rho}{c_I} | r_\rho\} = c_I.$$

Hence (4.33), (4.34) give:

$$e(Y) = \sum_{\rho=1}^{k} \left[d_\rho - \sum_{I \in \mathcal{I}_\rho} e(U_I'^\rho) c_I \right] \quad (\text{since} \ \sum_{i-1}^{m} w_i = \sum_{\rho=1}^{k} d_\rho)$$
$$\overset{(4.35)}{=} \sum_{\rho=1}^{k} \left[d_\rho - \sum_{r_\rho=0}^{d_\rho-1} (\sum \{ e(U_J'^\rho) | F^\rho(r_\rho)/\langle \mathfrak{h}_\rho \rangle \supset U_J^\rho \}) \right]$$
$$\overset{(4.34)}{=} \sum_{\rho=1}^{k} \left[d_\rho - \sum_{r_\rho=0}^{d_\rho-1} e(F^\rho(r_\rho)/\langle \mathfrak{h}_\rho \rangle) \right]$$
$$= \sum_{\rho=1}^{k} \sum_{r_\rho=0}^{d_\rho-1} [1 - e(F^\rho(r_\rho)/\langle \mathfrak{h}_\rho \rangle)]$$
$$\overset{(4.26)}{=} \sum_{\rho=1}^{k} \sum_{r_\rho=0}^{d_\rho-1} g_{r_\rho}^{(\rho)},$$

because $e(F^\rho(r_\rho, s_\rho)) = 1 + (-1)^{\epsilon(\rho)} \text{mil}(F^\rho(r_\rho, s_\rho)) = e(r_\rho, s_\rho)$ by the formulae of Greuel and Hamm in [52, Cor. 3.8. (b), p. 76]. So (4.28) is proven. Now if we define $J_{r_{j_\lambda}} := \{1, \dots, m\} \setminus N_{j_\lambda}(r_{j_\lambda})$, $\forall \lambda$, $1 \leq \lambda \leq q$, we have

$$\text{SSt}^1(X) = \{X_{J_{j_\lambda}} | 1 \leq \lambda \leq q\}.$$

(Warning (*) ! $X_{J_{j_\lambda}}$ represents the *same curve* for all r_{j_λ}'s of the form $r_{j_\lambda} = t_{j_\lambda} \cdot \frac{d_{j_\lambda}}{c_{J_{r_{j_\lambda}}}}$, $1 \leq t_{j_\lambda} \leq c_{J_{r_{j_\lambda}}}$.)

On the other hand, for all λ, $1 \leq \lambda \leq q$, $1 \leq r_{j_\lambda} \leq d_{j_\lambda} - 1$, we get:

$$b_1(X_{J_{r_{j_\lambda}}}) = 2 - e(X_{J_{r_{j_\lambda}}}) = 2 - e(\mathbb{P}_{J_{r_{j_\lambda}}}) + \sum_{\rho=1}^{j_\lambda} e(U_{J_{r_{j_\lambda}}}^\rho) =$$
$$= 2 - \{(m-1) - [(m-2) - \beta_{j_\lambda}(r_{j_\lambda})] + 1\} + \sum_{\rho=1}^{j_\lambda} e(U_{J_{r_{j_\lambda}}}^\rho) =$$

(Note that exactly $j_\lambda - \beta_{j_\lambda}(r_{j_\lambda})$ of the $U^{\rho}_{J_{r_{j_\lambda}}}$'s are empty !)

$$= - \sum_{u=1}^{\beta_{j_\lambda}(r_{j_\lambda})} (1 - e(U^{\xi_u^{(r_{j_\lambda})}}_{J_{r_{\xi_u^{(r_{j_\lambda})}}}})) = \quad (\text{here is} \quad \xi^{(r_{j_\lambda})}_{\beta_{j_\lambda}(r_{j_\lambda})} = j_\lambda)$$

$$= - \sum_{u=1}^{\beta_{j_\lambda}(r_{j_\lambda})} g^{(\xi_u^{(r_{j_\lambda})})}_{r_{\xi_u^{(r_{j_\lambda})}}} = - \sum_{p=1}^{\beta_{j_\lambda}(r_{j_\lambda})-1} g^{(\xi_u^{(r_{j_\lambda})})}_{r_{\xi_p^{(r_{j_\lambda})}}} - g^{(j_\lambda)}_{r_{j_\lambda}}$$

by the identification of $U^{(\xi_u^{(r_{j_\lambda})})}_{J_{r_{\xi_u^{(r_{j_\lambda})}}}}$ with $F^{\xi_u^{(r_{j_\lambda})}}(r_{\xi_u^{(r_{j_\lambda})}})/\langle \mathfrak{h}_{\xi_u^{(r_{j_\lambda})}} \rangle$ and for all $r_{\xi_u^{(r_{j_\lambda})}}$'s for which $N_{\xi_u^{(r_{j_\lambda})}}(r_{j_\lambda}) = N_{j_\lambda}(r_{j_\lambda})$ holds.

If C is an irreducible curve in $\mathrm{SSt}^1(X)$ with $C = X_{J_{r_{j_\lambda}}}$ for some $\lambda \in \{1, \dots, q\}$, we have exactly $c_{J_{r_{j_\lambda}}} - 1$ exceptional prime divisors of Y lying over C. To express $\Sigma\{(l_C - 1)b_1(C) | C \in \mathrm{SSt}^1(X)\}$ in terms of $\mathbf{d}$ and $\mathbf{w}$, it is sufficient (by (*)) to sum our $b_1(X_{J_{r_{j_\lambda}}})$'s over all λ's and r_{j_λ}'s included in $\mathcal{L}^{(j_\lambda)}$. The representatives of the required additional summands, which occur w.r.t. each stratum, can be obviously abbreviated by $\mathfrak{G}^{(j_\lambda)}_{r_{j_\lambda}}$. Hence, the above sum equals

$$\sum_{\lambda=1}^{q} \sum_{r_{j_\lambda} \in \mathcal{L}^{(j_\lambda)}} (\mathfrak{G}^{(j_\lambda)}_{r_{j_\lambda}} + g^{(j_\lambda)}_{r_{j_\lambda}})$$

and (4.29) follows directly from (4.24). $\square$

§ 5. Intersection forms and $[c_2]$-linear forms

Let X be a well-formed, well-stratified q.s.c.i.
$X = X_{\mathbf{d}} = \{[z_1, \ldots, z_m] \in \mathbb{P}^{m-1}(\mathbf{w}) | f_1(z_1, \ldots, z_m) = \ldots = f_k(z_1, \ldots, z_m) = 0\}$ of dimension 3 with $\mathrm{am}(X) = 0$, such that all curves C_j in $\mathrm{SSt}^1(X)$ are *nondegenerate* in the sense of 2.32. (We shall keep here the notations, which were introduced at the beginning of § 4, in order to avoid lenghty repetitions.) By theorem 2.37., $\mathrm{Pic}(X)$ is generated by the class of $\mathbf{L}_X = \mathcal{O}_X(\eta_X)$ (with η_X given by (2.24)), and $(X, \mathbf{L}_X)$ is polarized. In this section, we shall study the forms $q_Y^{\mathbb{Q}}$ and $[c_2]_Y^{\mathbb{Q}}$, which are associated to the crepant toroidal desingularizations $\pi : Y \to X$ of X (cf. 1.3.), by using their evaluations at the members of the natural $\mathbb{Q}$-basis

$$\mathcal{B}_Y := \{c_1(\mathbf{L}_Y), \{c_1(\mathcal{O}_Y(D)) | D \,\text{exc. prime divisor} \in \mathcal{E}(X)\}\}$$

of $\mathrm{Pic}_{\mathbb{Q}}(Y)$, where $\mathbf{L}_Y := \pi^* \mathbf{L}_X$.

Theorem 5.1. *Let $\pi : Y \to X$ be a crepant toroidal desingularization of X. Then:*
(i)

$$q_Y(\mathbf{L}_Y, \mathbf{L}_Y, \mathbf{L}_Y) = \mathbf{L}_Y^3 = \mathbf{L}_X^3 \tag{5.1}$$

(ii)

$$(Y, \mathbf{L}_Y) \quad \textit{is a quasi-polarized threefold}$$

(iii)

$$h^i(Y, \mathbf{L}_Y) = \begin{cases} 0, & \textit{for } i > 0 \\ h^0(X, \mathbf{L}_X), & \textit{for } i = 0 \end{cases} \tag{5.2}$$

(iv)

$$[c_2]_Y(\mathbf{L}_Y) =: (c_2(Y) \cdot \mathbf{L}_Y) = (c_2(X) \cdot \mathbf{L}_X) = 12h^0(X, \mathbf{L}_X) - 2\mathbf{L}_X^3 \tag{5.3}$$

($\mathbf{L}_X^3$ and $h^0(X, \mathbf{L}_X) = h^0(X, \mathcal{O}_X(\eta_X))$ are given by (2.27) and (2.20).)

Proof. (i) Obvious by $\mathbf{L}_Y^3 = \deg(\pi)\mathbf{L}_X^3$ and $\deg(\pi) = 1$.
(ii) For an arbitrary curve C on Y the projection formula gives: $(\mathbf{L}_Y \cdot C) = (\mathbf{L}_X \cdot (\pi_* C)) \geq 0$ and therefore $\mathbf{L}_Y$ inherits its numerical effectiveness from that of $\mathbf{L}_X$. $\mathbf{L}_Y$ is also big, since $\mathbf{L}_Y^3 = \mathbf{L}_X^3 > 0$ by (2.27).
(iii) *1st case:* Let us first consider the case where Y is projective. Applying Kawamata-Viehweg vanishing theorem to $\mathbf{L}_Y$ (cf. [76, 1.8.]) we get
$h^i(Y, \omega_Y \otimes \mathbf{L}_Y) = h^i(Y, \mathbf{L}_Y) = 0$, $\forall i$, $i > 0$.
2nd case: Let now Y be arbitrary. We shall show that, even in this case, the above equality remains valid. Since X contains only quotient singularities, i.e. special *rational* singularities, we have

$$R^i\pi_*\mathcal{O}_Y \cong \begin{cases} \mathcal{O}_X, & \text{for } i = 0 \\ 0, & \text{for } i > 0 \end{cases}$$

which means that, by the projection formula for direct image sheaves (see [47, 12.2.3.2, p. 402]), one obtains:

$$\begin{aligned} R^i\pi_*\mathbf{L}_Y = R^i\pi_*(\pi^*\mathbf{L}_X) &\cong R^i(\pi_*(\mathcal{O}_Y \otimes \pi^*\mathbf{L}_X)) \\ &\cong (R^i\pi_*\mathcal{O}_Y) \otimes \mathbf{L}_X \cong \begin{cases} \mathbf{L}_X, & \text{for } i = 0 \\ 0, & \text{for } i > 0 \end{cases} \end{aligned}$$

On the other hand, the Leray spectral sequence $\{\mathbf{E}_r = \bigoplus_{i,j\geq 0} \mathbf{E}_r^{i,j}\}$, which is associated to $\pi : Y \to X$, converges to the term $\mathbf{E}_2 = \ldots = \mathbf{E}_\infty$ with $\mathbf{E}_2^{i,j} = H^i(X, R^j\pi_*\mathbf{L}_Y)$ abuting to $H^{i+j}(Y, \mathbf{L}_Y)$. Thus, for all $i > 0$: $H^i(X, \mathbf{L}_X) \cong H^i(X, R^0\pi_*\mathbf{L}_Y) \cong H^i(Y, \mathbf{L}_Y)$ (comp. [61], Exc. III 8.1., p. 252), and it is sufficient to prove that $H^i(X, \mathbf{L}_X) = 0, \forall i, i > 0$. This can be done by applying a suitable version of Grauert-Riemenschneider vanishing theorem (see [108, thm. 7.80. (f), pp. 157-158]) and taking into account the ampleness of $\mathbf{L}_X$ and the triviality of ω_X.

Finally, from the connectedness of the fibers of π and the projection formula, we obtain $H^0(Y, \mathbf{L}_Y) \cong H^0(X, \mathbf{L}_X)$. q.e.d.

(iv) Since $\omega_Y \cong \mathcal{O}_Y$, Atiyah-Singer-Hirzebruch version of Riemann-Roch theorem (cf. [5, p. 20], [63, p. 155 and p. 187]) express the Euler-Poincaré characteristic

$$\chi(Y, \mathbf{L}_Y) = \sum_{i=0}^{3} (-1)^i h^i(Y, \mathbf{L}_Y)$$

of the sheaf of holomorphic sections of $\mathbf{L}_Y$ as follows:

$$\chi(Y, \mathbf{L}_Y) = \frac{\mathbf{L}_Y^3}{6} + \frac{(c_2(Y) \cdot \mathbf{L}_Y)}{12} \tag{5.4}$$

Hence, (5.4), combined with (5.1) and (5.2), implies

$$(c_2(Y) \cdot \mathbf{L}_Y) = 12\chi(Y, \mathbf{L}_Y) - 2\mathbf{L}_Y^3 = 12h^0(X, \mathbf{L}_X) - 2\mathbf{L}_X^3.$$

□

Lemma 5.2. *For a prime divisor D of a smooth, compact complex threefold Y with trivial canonical bundle we have:*

$$K_D^2 = D^3 \tag{5.5}$$

$$(c_2(Y) \cdot \mathcal{O}_Y(D)) = 12\chi(D, \mathcal{O}_D) - 2D^3 \tag{5.6}$$

Proof. (a) The adjunction formula and $\omega_Y \cong \mathcal{O}_Y$ give

$$\omega_D \cong \omega_Y \otimes \mathcal{O}_Y(D)|_D \cong \mathcal{O}_D(D) \cong \mathcal{N}_{D/Y} \Rightarrow K_D^2 = c_1^2(\omega_D)([D]) = c_1^2(\mathcal{N}_{D/Y})([D]) = D^3.$$

(b) If $\mathcal{T}_D$, resp. $\mathcal{T}_Y$ is the tangent bundle of D, resp. of Y, then the normal bundle sequence $0 \to \mathcal{T}_D \to \mathcal{T}_{Y|D} \to \mathcal{N}_{D/Y} \to 0$, combined with Noether's formula and (5.5), implies

$$\begin{aligned} c(\mathcal{T}_{Y|D}) = c(\mathcal{T}_D)c(\mathcal{N}_{D/Y}) = c(D)c(\omega_D) \Rightarrow \\ (1 + (c_2(Y) \cdot \mathcal{O}_Y(D))) = (1 + c_1(D) + c_2(D))(1 - c_1(D)) \Rightarrow \\ (c_2(Y) \cdot \mathcal{O}_Y(D)) = -K_D^2 + e(D) = 12\chi(D, \mathcal{O}_D) - 2D^3. \end{aligned}$$

□

We start the computation of the intersection numbers containing exceptional divisors from $\mathcal{E}(X)$ by considering firstly the case of a D sitting over a dissident point of X. (The computation will be done for a fixed Y, i.e. for a fixed choice of triangulations $\mathcal{S}_{P_1}, \ldots, \mathcal{S}_{P_\mu}, \mathcal{S}_{Q_1}, \ldots, \mathcal{S}_{Q_\lambda}$.)

Theorem 5.3. *The intersection numbers of an exceptional prime divisor D within $(\coprod_{i=1}^{\lambda} \mathcal{E}_{P_i}) \coprod (\coprod_{\iota=1}^{\mu} \mathcal{E}_{Q_\iota})$ with elements of $\mathcal{E}(X) \cup \{\mathbf{L}_Y\}$ are given by the following formulae:*
(i) If $D = D_{n_{\mathfrak{p}_i}}^{(P_i)} \in \mathcal{E}_{P_i}$ (resp. $D = D_{n_{\mathfrak{q}_\iota}}^{(Q_\iota)} \in \mathcal{E}_{Q_\iota}$) with $n_{\mathfrak{p}_i}$ (resp. $n_{\mathfrak{q}_\iota}$) denoting a vertex of $\Phi_{G_{P_i}}$ (resp. of $\Phi_{G_{Q_\iota}}$), then:

$$(5.7) \qquad D^3 = 12 - \sharp(\mathrm{Star}(\mathbb{R}_{\geq 0} n_{\mathfrak{p}_i})(2)) \quad (\text{resp.}\, D^3 = 12 - \sharp(\mathrm{Star}(\mathbb{R}_{\geq 0} n_{\mathfrak{q}_\iota})(2)))$$

(5.8)
$$(c_2(Y) \cdot D) = 2\sharp(\mathrm{Star}(\mathbb{R}_{\geq 0} n_{\mathfrak{p}_i})(2)) - 12 \quad (\text{resp.}\, (c_2(Y) \cdot D) = 2\sharp(\mathrm{Star}(\mathbb{R}_{\geq 0} n_{\mathfrak{q}_\iota})(2)) - 12)$$

(ii) For $D = D_{n_{\mathfrak{p}_i}}^{(P_i)} \in \mathcal{E}_{P_i}$ and $D' = D_{n_{\mathfrak{p}'_i}}^{(P_i)} \in \mathcal{E}_{P_i}$ with $1 \leq \mathfrak{p}_i, \mathfrak{p}'_i \leq a(P_i)$, $\mathfrak{p}_i \neq \mathfrak{p}'_i$, one computes $(D^2 \cdot D')$ and $(D \cdot D'^2)$ by means of a single vectorial $\mathbb{Z}$-linear dependence equation:

$$(5.9) \qquad (D^2 \cdot D') n_{\mathfrak{p}_i} + (D \cdot D'^2) n_{\mathfrak{p}'_i} + n(\mathfrak{p}_i, \mathfrak{p}'_i) + n(\mathfrak{p}'_i, \mathfrak{p}_i) = 0$$

where $n(\mathfrak{p}_i, \mathfrak{p}'_i)$, $n(\mathfrak{p}'_i, \mathfrak{p}_i)$ denote the unique vertices of $\Phi_{G_{P_i}} \coprod \{e_1, e_2, e_3\}$ for which $s(n_{\mathfrak{p}_i}, n_{\mathfrak{p}'_i}, n(\mathfrak{p}_i, \mathfrak{p}'_i))$ and $s(n_{\mathfrak{p}_i}, n_{\mathfrak{p}'_i}, n(\mathfrak{p}'_i, \mathfrak{p}_i))$ form two distinct 2-simplices of $\mathcal{S}_{P_i}$. (The intersection numbers $(D^2 \cdot D')$, $(D \cdot D'^2)$ can be computed by the same method, if D and D' correspond to vertices of $\Phi_{G_{Q_\iota}} \coprod \{e_1, e_2, e_3\}$ and Q_ι is an individual point. We do not exclude the case in which both D and D' correspond to points of

$\partial s_0 \setminus \{e_1, e_2, e_3\}$, *i.e.* $(D, D') \in (\mathcal{E}_{C_j})^2$, *where* $Q_\iota = Q_{\nu_j}^{(j)} \in C_j$ *for some* j.)
(iii) The intersection numbers involving $\mathbf{L}_Y$ *vanish, i.e.*

$$(5.10) \qquad (\mathbf{L}_Y^2 \cdot D) = (\mathbf{L}_Y \cdot D^2) = 0$$

(iv) All the other possible triples lead to vanishing intersection numbers.

Proof. (i) The equality (5.7) is a reformulation of (3.3) applied to the c.q.s. $Z(N_{G_{P_i}}, \Sigma_0)$ (resp. $Z(N_{G_{Q_\iota}}, \Sigma_0)$). Since $\chi(D, \mathcal{O}_D) = 1$, (5.6) and (5.7) give (5.8).
(ii) is an immediate consequence of prop. 3.6. (ii).
(iii) As $\mathbf{L}_X$ is ample, $n\mathbf{L}_X$ will be very ample for some $n >> 0$. If $D \in \mathcal{E}_{P_i}$ (resp. $D \in \mathcal{E}_{Q_\iota}$) and if we choose a general member $\mathbf{M}$ of the linear system $|n\mathbf{L}_X|$, such that $P_i \notin \mathbf{M}$ (resp. $Q_\iota \notin \mathbf{M}$), then $\mathrm{supp}(\pi^*\mathbf{M}) \cap \mathrm{supp}(D) \neq \varnothing$, $\pi^*\mathbf{M}|_D \sim 0 \Rightarrow n^2(\mathbf{L}_Y^2 \cdot D) = ((\pi^*(n\mathbf{L}_X))^2 \cdot D) = ((\pi^*\mathbf{M})^2 \cdot D) = (\pi^*\mathbf{M}|_D)^2 = 0$ and $n(\mathbf{L}_Y \cdot D^2) = ((\pi^*(n\mathbf{L}_X) \cdot D^2) = ((\pi^*\mathbf{M}) \cdot D^2) = ((\pi^*\mathbf{M}|_D) \cdot (D|_D)) = 0$, i.e.

$$(\mathbf{L}_Y^2 \cdot D) = (\mathbf{L}_Y \cdot D^2) = 0.$$

(iv) is obvious. □

Our next step will be the description of the intersection numbers which contain a divisor D located over a curve $C_j \in \mathrm{SSt}^1(X)$, $1 \leq j \leq \kappa$. At first we shall need some technical lemmata.

Lemma 5.4. *For each* $1 \leq j \leq \kappa$, $1 \leq r_j \leq l_{C_j} - 1$, *one contracts all* (-1)*-curves of* $D_{r_j}^{(C_j)}$ *by a birational morphism* $\varphi_{r_j}^{(j)} : D_{r_j}^{(C_j)} \to \bar{D}_{r_j}^{(C_j)}$, *which factors into a composite of* $\sum_{\nu_j=1}^{\xi_j}(b(n^{(\nu_j, r_j)}) - 1)$ *blow-downs. Each* $\bar{D}_{r_j}^{(C_j)}$ *is endowed with the structure of a minimal (i.e. geometrically) ruled surface by* $\bar{\pi}_{r_j}^{(j)} : \bar{D}_{r_j}^{(C_j)} \to C_j$, *so that* $\bar{\pi}_{r_j}^{(j)} \circ \varphi_{r_j}^{(j)} = \pi|_{D_{r_j}^{(C_j)}}$. *Moreover,*

$$(5.11) \qquad K^2_{D_{r_j}^{(C_j)}} = K^2_{\bar{D}_{r_j}^{(C_j)}} - \sum_{\nu_j=1}^{\xi_j} (b(n^{(\nu_j, r_j)}) - 1)$$

Proof. We consider the pull-back

$$(\pi|_{D_{r_j}^{(C_j)}})^*([Q_{\nu_j}^{(j)}]) = \sum_{t^{(\nu_j, r_j)}=1}^{b(n^{(\nu_j, r_j)})} \mathfrak{l}_{t^{(\nu_j, r_j)}} C(n^{(\nu_j, r_j)}, n_{t^{(\nu_j, r_j)}}^{(\nu_j, r_j)})$$

of the divisor $[Q_{\nu_j}^{(j)}]$, $1 \leq \nu_j \leq \xi_j$, under $\pi|_{D_{r_j}^{(C_j)}} : D_{r_j}^{(C_j)} \to C_j$ (cf. (4.17)), where $\mathfrak{l}_{t^{(\nu_j, r_j)}} \in \mathbb{N}$.
It should be first mentioned that $\gcd(\mathfrak{l}_{t^{(\nu_j, r_j)}} | 1 \leq t^{(\nu_j, r_j)} \leq b(n^{(\nu_j, r_j)})) = 1$. (Indeed, if the corresponding fiber were *multiple*, then by [5, Ch. III, Lemma 8.3., p. 91] one could conclude that $\mathcal{O}$ (fiber) would be a torsion bundle and therefore

that $H^1(\text{fiber}, \mathbb{Z}) \neq 0$. But this would be not true, because the fiber is simply connected.) Now since

$$-2 = (K_{D^{(C_j)}_{r_j}} \cdot (\text{generic fiber})) = (K_{D^{(C_j)}_{r_j}} \cdot ((\pi|_{D^{(C_j)}_{r_j}})^*([Q^{(j)}_{\nu_j}])))$$

there exists at least one index $\check{t}^{(\nu_j,r_j)} \in \{1, \ldots, b(n^{(\nu_j,r_j)})\}$, such that $(K_{D^{(C_j)}_{r_j}} \cdot \mathcal{C}(n^{(\nu_j,r_j)}, n^{(\nu_j,r_j)}_{\check{t}(\nu_j,r_j)})) < 0$. Making use of Zariski's fibration lemma [5, III. 8.2. (9), (10), p. 90], we deduce that $(\mathcal{C}(n^{(\nu_j,r_j)}, n^{(\nu_j,r_j)}_{\check{t}(\nu_j,r_j)}))^2 < 0$. Thus $(\mathcal{C}(n^{(\nu_j,r_j)}, n^{(\nu_j,r_j)}_{\check{t}(\nu_j,r_j)}))^2 = -1$ (see [5, III.2.2., p. 72]). We blow it smoothly down by Castelnuovo-Enriques contractibility criterion [5, III. 4.1., p. 78], play again the same game for the new fibration, and proceed succesively. After $\sum_{\nu_j=1}^{\xi_j}(b(n^{(\nu_j,r_j)}) - 1)$ steps we obtain $\bar{\pi}^{(j)}_{r_j} : \bar{D}^{(C_j)}_{r_j} \to C_j$. (5.11) is obvious. □

Lemma 5.5. *For a curve $C_j = X(I_j) \in \mathrm{SSt}^1(X)$ $1 \leq j \leq k$, and an $n \in \mathbb{Z}$, we have:*

(5.12)

$$\begin{aligned}\deg(\mathcal{O}_{C_j}(n)) = \frac{1}{l_{C_j}} \Big\{ & h^0(C'_j, \mathcal{O}_{C'_j}(\theta(n; w_{t^j_1}, \ldots, w_{t^j_{m-|I_j|}}))) - \\ & - h^0(C'_j, \mathcal{O}_{C'_j}(\mathrm{am}(C'_j) - \theta(n; w_{t^j_1}, \ldots, w_{t^j_{m-|I_j|}}))) + \\ & + h^0(C'_j, \mathcal{O}_{C'_j}(\mathrm{am}(C'_j))) - 1 \Big\}\end{aligned}$$

where C'_j is the q.s.c.i. coming from the normalization $w'_{t^j_1}, \ldots, w'_{t^j_{m-|I_j|}}$ of the weights $w_{t^j_1}, \ldots, w_{t^j_{m-|I_j|}}$ (cf. 2.29).

Proof. Apply (2.23) with $l_{C_j} = \gcd(w_{t^j_1}, \ldots, w_{t^j_{m-|I_j|}})$. □

Lemma 5.6. *For the union $\mathcal{E}_{C_j}$ of the exceptional prime divisors lying over a curve $C_j = X(I_j) \in \mathrm{SSt}^1(X)$, $1 \leq j \leq k$ and indices $2 \leq r_j \leq l_{C_j} - 1$, we have:*

(5.13)

$$\mathcal{O}_{D^{(C_j)}_{r_j-1} \cap D^{(C_j)}_{r_j}}(D^{(C_j)}_{r_j-1}) \cong \mathcal{O}_{C'_j}(\theta(r_j - \alpha_2^{(C_j)}; w_{t^j_1}, \ldots, w_{t^j_{m-|I_j|}})) \quad \text{and}$$

(5.14)

$$\mathcal{O}_{D^{(C_j)}_{r_j-1} \cap D^{(C_j)}_{r_j}}(D^{(C_j)}_{r_j}) \cong \mathcal{O}_{C'_j}(\theta(1 - r_j + \alpha_2^{(C_j)}; w_{t^j_1}, \ldots, w_{t^j_{m-|I_j|}}))$$

respectively. (The numbers $\alpha_1^{(C_j)}$ and $\alpha_2^{(C_j)}$ are determined by lemma 4.1.)

Proof. The application of resolutions (4.14) to the singularities occuring along $\mathcal{N}_{C_j/X}$ and (4.6) give rise to the following relations (cf. [103, thm. 1]):

$$\mathcal{O}_{D^{(C_j)}_{r_j-1}\cap D^{(C_j)}_{r_j}}(D^{(C_j)}_{r_j-1}) \cong \mathcal{O}_{C_j}(r_j \cdot \alpha_1^{(C_j)}) \otimes \mathcal{O}_{C_j}((r_j - l_{C_j}) \cdot \alpha_2^{(C_j)})$$

$$\mathcal{O}_{D^{(C_j)}_{r_j-1}\cap D^{(C_j)}_{r_j}}(D^{(C_j)}_{r_j}) \cong \mathcal{O}_{C_j}((1-r_j) \cdot \alpha_1^{(C_j)}) \otimes \mathcal{O}_{C_j}((l_{C_j} - r_j + 1) \cdot \alpha_2^{(C_j)})$$

To get (5.13) and (5.14), we make use of $r_j\alpha_1^{(C_j)} + (r_j - l_{C_j})\alpha_2^{(C_j)} = l_{C_j}(r_j - \alpha_2^{(C_j)})$, $(1-r_j)\alpha_1^{(C_j)} + (l_{C_j} - r_j + 1)\alpha_2^{(C_j)} = l_{C_j}(1 - r_j + \alpha_2^{(C_j)})$ and of the isomorphisms of prop. 2.7. □

Theorem 5.7. *The intersection numbers of an exceptional prime divisor within $\mathcal{E}_{C_j}$ with elements of $\mathcal{E}_{C_j} \cup \{\mathbf{L}_Y\}$ (resp. of $\mathcal{E}(X) \cup \{\mathbf{L}_Y\}$) are given by the following formulae:*
(i) For $1 \le r_j \le l_{C_j} - 1$ and $g(C_j) = h^1(C_j, \mathcal{O}_{C_j})$ the genus of the curve C_j, we have:

(5.15)
$$(D^{(C_j)}_{r_j})^3 = 8(1 - g(C_j)) - \sum_{\nu_j=1}^{\xi_j} (b(n^{(\nu_j, r_j)}) - 1)$$

(5.16)
$$(c_2(Y) \cdot \mathcal{O}_Y(D^{(C_j)}_{r_j})) = -4(1 - g(C_j)) + 2(\sum_{\nu_j=1}^{\xi_j} (b(n^{(\nu_j, r_j)}) - 1))$$

(5.17)
$$(\mathbf{L}_Y^2 \cdot D^{(C_j)}_{r_j}) = 0$$

(5.18)
$$((D^{(C_j)}_{r_j})^2 \cdot \mathbf{L}_Y) = -2\deg(\mathcal{O}_{C_j}(\eta_X))$$

(The latter is computable by (5.12)).

(ii) For group orders $|G_{C_j}| = l_{C_j} \ge 3$ and indices $2 \le r_j \le l_{C_j} - 1$, we get:

(5.19)
$$((D^{(C_j)}_{r_j-1})^2 \cdot D^{(C_j)}_{r_j}) = \deg(\mathcal{O}_{C'_j}(\theta(r_j - \alpha_2^{(C_j)}; w_{t_1^j}, \ldots, w_{t^j_{m-|I_j|}})))$$

(5.20)
$$((D^{(C_j)}_{r_j})^2 \cdot D^{(C_j)}_{r_j-1}) = \deg(\mathcal{O}_{C'_j}(\theta(1 - r_j + \alpha_2^{(C_j)}; w_{t_1^j}, \ldots, w_{t^j_{m-|I_j|}})))$$

(5.21)
$$(D^{(C_j)}_{r_j-1} \cdot D^{(C_j)}_{r_j} \cdot \mathbf{L}_Y) = \deg(\mathcal{O}_{C_j}(\eta_X))$$

(iii) The intersection numbers involving $D^{(C_j)}_{r_j}$ and other divisors of $(\coprod_{i=1}^{\lambda} \mathcal{E}_{P_i}) \coprod (\coprod_{i=1}^{\mu} \mathcal{E}_{Q_i})$ are already given by thm. 5.3. (ii). Moreover, if

$\kappa \geq 2$, $1 \leq j, j' \leq \kappa$, $j \neq j'$, $C_j \cap C_{j'} \neq \varnothing$, $C_j \cap C_{j'} = \{Q_1^{(j,j')}, \ldots, Q_{\mathfrak{e}_{(j,j')}}^{(j,j')}\}$ denotes an enumeration of their common individual points and $D_{r_j}^{(C_j)}$ (resp. $D_{r_{j'}}^{(C_{j'})}$) is realized in the triangulation $\mathcal{S}_{Q_\rho^{(j,j')}}$ by $D_{n(r_j)}^{(\rho)}$ (resp. by $D_{n(r_{j'})}^{(\rho)}$), $\forall \rho$, $1 \leq \rho \leq \mathfrak{e}_{(j,j')}$, then

$$((D_{r_j}^{(C_j)})^2 \cdot D_{r_{j'}}^{(C_{j'})}) = \sum_{\rho=1}^{\mathfrak{e}_{(j,j')}} ((D_{n(r_j)}^{(\rho)})^2 \cdot D_{n(r_{j'})}^{(\rho)})$$

$$(D_{r_j}^{(C_j)} \cdot (D_{r_{j'}}^{(C_{j'})})^2) = \sum_{\rho=1}^{\mathfrak{e}_{(j,j')}} (D_{n(r_j)}^{(\rho)} \cdot (D_{n(r_{j'})}^{(\rho)})^2)$$

(which are again known from thm. 5.3. (ii)).

(iv) All the other intersection numbers are zero.

Proof. We shall examine each case separately.
(i) (a) (5.15) follows from (5.5), (5.11), and from the fact, that the self-intersection number of the geometrically ruled surface $\bar{D}_{r_j}^{(C_j)}$, which was defined by lemma 5.4, is given by $K^2_{\bar{D}_{r_j}^{(C_j)}} = 8(1 - g(C_j))$ (see [61, ch. V, cor. 2.11., p. 374]).
(b) Since $12\chi(D_{r_j}^{(C_j)}, \mathcal{O}_{D_{r_j}^{(C_j)}}) - 2(D_{r_j}^{(C_j)})^3 = 12(1-g(C_j)) - 2K^2_{D_{r_j}^{(C_j)}}$, (5.6) combined with (5.11) gives (5.16).
(c) As $\mathbf{L}_X$ is ample, $n\mathbf{L}_X$ will be very ample for some $n >> 0$. If we consider two general members $\mathbf{M}_1, \mathbf{M}_2$ of linear system $|n\mathbf{L}_X|$, such that

$$\mathrm{supp}(\mathbf{M}_1) \cap \mathrm{supp}(\mathbf{M}_2) \cap C_j = \varnothing, \quad \text{then}$$
$$\mathrm{supp}(\pi^*\mathbf{M}_1) \cap \mathrm{supp}(\pi^*\mathbf{M}_2) \cap \mathrm{supp}(D_{r_j}^{(C_j)}) = \varnothing$$

for all $1 \leq r_j \leq l_{C_j} - 1$ and

$$n^2(\mathbf{L}_Y^2 \cdot D_{r_j}^{(C_j)}) = ((\pi^*(n\mathbf{L}_X)) \cdot (\pi^*(n\mathbf{L}_X)) \cdot D_{r_j}^{(C_j)}) = $$
$$= ((\pi^*\mathbf{M}_1) \cdot (\pi^*\mathbf{M}_2) \cdot D_{r_j}^{(C_j)}) = 0,$$

$$\text{i.e.} \quad (\mathbf{L}_Y^2 \cdot D_{r_j}^{(C_j)}) = 0.$$

(d) $((D_{r_j}^{(C_j)})^2 \cdot \mathbf{L}_Y)$ cannot, in general, vanish, because it contains many informations coming from the underlying curve C_j, as we get:

(5.22)
$$((D_{r_j}^{(C_j)})^2 \cdot \mathbf{L}_Y) = (\omega_{D_{r_j}^{(C_j)}} \cdot \mathbf{L}_Y|_{D_{r_j}^{(C_j)}}) = (\mathcal{O}_{D_{r_j}^{(C_j)}}(K_{D_{r_j}^{(C_j)}})(\pi|_{D_{r_j}^{(C_j)}})^*(\mathbf{L}_X|_{C_j})) =$$
$$= (\mathcal{O}_{D_{r_j}^{(C_j)}}(K_{D_{r_j}^{(C_j)}}) \cdot (\pi|_{D_{r_j}^{(C_j)}})^*(\mathcal{O}_{C_j}(G))) = (K_{D_{r_j}^{(C_j)}} \cdot ((\pi|_{D_{r_j}^{(C_j)}})^*(G))),$$

where $G = \Sigma_{i=1}^{\tau}\mathfrak{d}_i G_i$, $\mathfrak{d}_i \in \mathbb{Z}, 1 \leq i \leq \tau$, denotes the divisor of C_j which is associated to the line bundle $\mathbf{L}_X|_{C_j} = \mathcal{O}_{C_j}(\eta_X)$. Suppose that $b(n^{(\nu_j,r_j)}) > 1$ for all $1 \leq \nu_j \leq \xi_j$.
(The case in which $b(n^{(\nu_j,r_j)}) = 1$, for some ν_j's, can be treated similarly.) If we assume, without loss of generality, that the set of curves, which are contracted by $\varphi_{r_j}^{(j)}$, is $\{\mathcal{C}(n^{(\nu_j,r_j)}, n^{(\nu_j,r_j)}_{t^{(\nu_j,r_j)}}) | 2 \leq t^{(\nu_j,r_j)} \leq b(n^{(\nu_j,r_j)})\}$, then we can describe the relationship between the canonical divisors of $D_{r_j}^{(C_j)}$ and $\tilde{D}_{r_j}^{(C_j)}$ as follows:

$$(5.23) \qquad K_{D_{r_j}^{(C_j)}} \sim (\varphi_{r_j}^{(j)})^*(K_{\tilde{D}_{r_j}^{(C_j)}}) + \sum_{\nu_j=1}^{\xi_j}\left(\sum_{t^{(\nu_j,r_j)}=2}^{b(n^{(\nu_j,r_j)})} \mathcal{C}(n^{(\nu_j,r_j)}, n^{(\nu_j,r_j)}_{t^{(\nu_j,r_j)}})\right)$$

Note that

$$(5.24) \qquad K_{\tilde{D}_{r_j}^{(C_j)}} \sim (-2)\,(\text{a section of } \;\; \bar{\pi}_{r_j}^{(j)}) + (\bar{\pi}_{r_j}^{(j)})^*(K_{C_j} + E_{r_j}^{(j)})|_{F_{r_j}^{(j)}}$$

where $E_{r_j}^{(j)}$ is a divisor of C_j with $\deg(E_{r_j}^{(j)}) = C_j^2$ and $F_{r_j}^{(j)}$ a fiber of

$$\bar{\pi}_{r_j}^{(j)} : \tilde{D}_{r_j}^{(C_j)} \to C_j \quad \text{(cf. [61, ch. V, lemma 2.10., p. 373])}.$$

$\mathbf{L}_X|_{C_j}$ is ample. So there is again an $n >> 0$ for which $n(\mathbf{L}_X|_{C_j})$ is very ample. If $\mathbf{N}_j$ is a general member of $|n(\mathbf{L}_X|_{C_j})|$, such that

$$\text{supp}(\mathbf{N}_j) \cap \{Q_1^{(j)}, \ldots, Q_{\xi_j}^{(j)}\} = \varnothing, \quad \text{then}$$
$$\text{supp}((\pi|_{D_{r_j}^{(C_j)}})^*(\mathbf{N}_j)) \cap \left(\bigcup_{\nu_j=1}^{\xi_j} \bigcup_{t^{(\nu_j,r_j)}=2}^{b(n^{(\nu_j,r_j)})} \mathcal{C}(n^{(\nu_j,r_j)}, n^{(\nu_j,r_j)}_{t^{(\nu_j,r_j)}})\right) = \varnothing$$

and therefore (5.23) gives:

$$(5.25) \qquad (K_{D_{r_j}^{(C_j)}} \cdot ((\pi|_{D_{r_j}^{(C_j)}})^*(G))) = \left((\varphi_{r_j}^{(j)})^*(K_{\tilde{D}_{r_j}^{(C_j)}})\right) \cdot \left((\pi|_{D_{r_j}^{(C_j)}})^*(G)\right)$$

Combining now (5.22), (5.24) and (5.25) we get:

$$((D_{r_j}^{(C_j)})^2 \cdot \mathbf{L}_Y) =$$
$$((\varphi_{r_j}^{(j)})^*(-2C_j + (\bar{\pi}_{r_j}^{(j)})^*(K_{C_j} + E_{r_j}^{(j)})|_{F_{r_j}^{(j)}}))) \cdot (\varphi_{r_j}^{(j)})^*\left(\sum_{i=1}^{\tau}\mathfrak{d}_i(\bar{\pi}_{r_j}^{(j)})^*(G_i))\right) =$$
$$= -2\left(\sum_{i=1}^{\tau}\mathfrak{d}_i(C_j \cdot (\bar{\pi}_{r_j}^{(j)})^*(G_i))\right) + \sum_{i=1}^{\tau}\mathfrak{d}_i(((\bar{\pi}_{r_j}^{(j)})^*(K_{C_j} + E_{r_j}^{(j)})|_{F_{r_j}^{(j)}}) \cdot (\bar{\pi}_{r_j}^{(j)})^*(G_i)) =$$
$$= -2\left(\sum_{i=1}^{\tau}\mathfrak{d}_i\right) = -2\deg(G) = -2\deg(\mathbf{L}_X|_{C_j}) = -2\deg(\mathcal{O}_{C_j}(\eta_X)),$$

because $\deg(\varphi_{r_j}^{(j)}) = 1$, $(\bar{\pi}_{r_j}^{(j)})^*(G_i) \sim F_{r_j}^{(j)}$, $\forall i$, $1 \le i \le \tau$, $(F_{r_j}^{(j)})^2 = 0$ and $(C_j \cdot F_{r_j}^{(j)}) = 1$.

(ii) (5.19) and (5.20) follow directly from (5.13) and (5.14). On the other hand,

$$(D_{r_j-1}^{(C_j)} \cdot D_{r_j}^{(C_j)} \cdot \mathbf{L}_Y) = ((D_{r_j-1}^{(C_j)}|_{D_{r_j-1}^{(C_j)} \cap D_{r_j}^{(C_j)}}) \cdot \mathbf{L}_Y|_{D_{r_j}^{(C_j)}}) = \deg(\mathbf{L}_X|_{C_j}) = \deg(\mathcal{O}_{C_j}(\eta_X)).$$

(iii) and (iv) are obvious. □

Recapitulating, one can verify that the formulae of theorems 5.1., 5.3. and 5.7., which have been proved above, cover *all* $\binom{h^{1,1}(Y)+2}{3}$ triples that can be formed from elements of $\mathcal{B}_Y$. We shall now mention two additional arithmetical relations which are fulfilled by the intersection numbers. (See Oguiso and Peternell [95, (1.1.)].)

Proposition 5.8. *Let $\pi : Y \to X$ be a toroidal crepant desingularization of X and $D \in \mathcal{E}(X)$. Then*

$$(D \cdot \mathbf{L}_Y^2)^2 \ge (D^2 \cdot \mathbf{L}_Y)(\mathbf{L}_Y^3) \tag{5.26}$$

$$(D \cdot \mathbf{L}_Y^2) \equiv (D^2 \cdot \mathbf{L}_Y) (\operatorname{mod} 2) \tag{5.27}$$

Proof. ([95]) Let m be a large *odd* number and S a general element of $|m\mathbf{L}_Y|$. By the base-point-freeness theorem, we can choose m in such a way, that S is a smooth irreducible surface. Hodge-index theorem implies:

$$(D|_S \cdot (\mathbf{L}_Y|_S))^2 \ge (D|_S)^2 (\mathbf{L}_Y|_S)^2 \Rightarrow (D \cdot \mathbf{L}_Y^2)^2 \ge (D^2 \cdot \mathbf{L}_Y)(\mathbf{L}_Y^3).$$

On the other hand, by Riemann-Roch theorem for smooth surfaces and by adjunction formula $K_S = S|_S$, we get:

$$\chi(S, \mathcal{O}_S(D|_S)) - \chi(S, \mathcal{O}_{D|_S}) = \frac{1}{2}((D|_S)^2 - (D|_S) \cdot K_S) = \frac{1}{2}(m(D^2 \cdot \mathbf{L}_Y) - m^2(D \cdot \mathbf{L}_Y^2)),$$

which means that $(D \cdot \mathbf{L}_Y^2) \equiv (D^2 \cdot \mathbf{L}_Y)(\operatorname{mod} 2)$, because $m \equiv 1 (\operatorname{mod} 2)$. □

Remark 5.9. Basically, Oguiso and Peternell showed that the above proof remains valid, even if $(X, \mathbf{L}_X)$ is an arbitrary polarized CY model, $(Y, \mathbf{L}_Y)$ a quasipolarized CY model and $\pi : Y \to X$ a partial crepant desingularization of X. Note that the congruence $(D_1^2 \cdot D_2) \equiv (D_1 \cdot D_2^2)(\operatorname{mod} 2)$ holds true for any divisor D_1 and D_2 of a *smooth* simply connected 3-dimensional variety Y with trivial canonical bundle and torsion-free cohomology groups, and that it was already known by C.T.C. Wall [122, thm. 5, p. 361]. Furthermore, for such a Y we have: $[c_2]_Y(D) \equiv -2D^3 (\mathrm{mod}\, 12)$, for all divisors D, because the first Pontrjagin class equals $-2c_2$.

Up to now we have calculated the evaluations of $q_Y^{\mathbb{Q}}$- and $[c_2]_Y^{\mathbb{Q}}$- forms with respect to fixed triangulations $\mathcal{S}_{P_1}, \ldots, \mathcal{S}_{P_\mu}, \mathcal{S}_{Q_1}, \ldots, \mathcal{S}_{Q_\lambda}$. As we know from prop. 4.3., two distinct toroidal crepant desingularizations of X differ from each other by finitely many (simple) flops. Hence, up to an "arrangement algorithm" for the 1-simplices within our triangle subdivisions, the alteration of $q_Y^{\mathbb{Q}}$ and $[c_2]_Y^{\mathbb{Q}}$ due to the choice of other triangulations will be clear if we describe just the "single-flop" case.

Theorem 5.10. *Let $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ be a well-formed, well-stratified q.s.c.i. with $\mathrm{am}(X) = 0$ and $\dim_{\mathbb{C}}(X) = 3$, and let $\pi_1 : Y_1 \to X$ be a toroidal crepant desingularization of X with $\mathrm{SSt}^0(X) \neq \varnothing$. Suppose that $P_i \in \mathrm{SSt}^0(X)$ is an isolated point of $\mathrm{Sing}(X)$, for which $\sharp(\Phi_{G_{P_i}}) \geq 4$, and that the corresponding triangulation $\mathcal{S}_{P_i}$ satisfies the following properties:*
(i) There exist vertices n_1, n_2, n_3, n_4 from $s_0 \cap N_{G_{P_i}}$, such that $s(n_1, n_2, n_3)$ and $s(n_1, n_2, n_4)$ are two twisted 2-simplices of $\mathcal{S}_{P_i}$, and
(ii) $s(n_1, n_2, n_3, n_4)$ forms a convex quadrilateral of $\mathcal{S}_{P_i}$.
If $\pi_2 : Y_2 \to X$ is the toroidal crepant desingularization of X with Y_2 obtained by Y_1 after flopping the curve $\mathcal{C}(n_1, n_2)$ (i.e: after applying the elementary transformation $\Sigma_0'(\tilde{\mathcal{S}}_{P_i})$ of $\Sigma_0'(\mathcal{S}_{P_i})$ w.r.t. $\sigma(s(n_1, n_2, n_3))$, $\sigma(s(n_1, n_2, n_4))$ and $\sigma(s(n_1, n_2))$) and if $D_{n_j}^{(\mathcal{S}_{P_i})}$, resp. $D_{n_j}^{(\tilde{\mathcal{S}}_{P_i})}$, denote the exceptional prime divisors associated to n_j in $\mathcal{S}_{P_i}$, resp. in $\tilde{\mathcal{S}}_{P_i}$, $\forall j$, $1 \leq j \leq 4$, then their intersection numbers and their images under the $[c_2]$-form are related as follows:

$$((D_{n_j}^{(\mathcal{S}_{P_i})})^2 \cdot D_{n'_j}^{(\mathcal{S}_{P_i})}) - ((D_{n_j}^{(\tilde{\mathcal{S}}_{P_i})})^2 \cdot D_{n'_j}^{(\tilde{\mathcal{S}}_{P_i})}) = (D_{n_j}^{(\mathcal{S}_{P_i})} \cdot (D_{n_{j'}}^{(\mathcal{S}_{P_i})})^2) - D_{n_j}^{(\tilde{\mathcal{S}}_{P_i})} \cdot (D_{n'_j}^{(\tilde{\mathcal{S}}_{P_i})})^2)$$

$$(5.28) \qquad = \begin{cases} -1, & \text{for } \ (j,j') \in \{(1,2),(2,1)\} \\ 1, & \text{for } \ (j,j') \in \{(3,4),(4,3)\} \\ 0, & \text{otherwise} \end{cases}$$

$\forall j, j'$, $1 \leq j, j' \leq 4$. In particular, for all $j \in \{1,2,3,4\}$, for which $n_j \in \mathrm{int}(s_0) \cap N_{G_{P_i}}$, we have:

$$(5.29) \qquad (D_{n_j}^{(\mathcal{S}_{P_i})})^3 - (D_{n_j}^{(\tilde{\mathcal{S}}_{P_i})})^3 = \begin{cases} -1, & \text{if } \ j \in \{1,2\} \\ 1, & \text{if } \ j \in \{3,4\} \end{cases}$$

Moreover,

$$[c_2]_{Y_1}(D_{n_j}^{(\mathcal{S}_{P_i})}) - [c_2]_{Y_2}(D_{n_j}^{(\tilde{\mathcal{S}}_{P_i})}) = -2(D_{n_j}^{(\mathcal{S}_{P_i})} \cdot \mathcal{C}(n_1, n_2)) =$$

$$(5.30) \qquad = 2(D_{n_j}^{(\tilde{\mathcal{S}}_{P_i})} \cdot \mathcal{C}(n_3, n_4)) = \begin{cases} 2, & \text{if } \ j \in \{1,2\} \\ -2, & \text{if } \ j \in \{3,4\} \end{cases}$$

(Note that if we assume that $\mathrm{INP}(X) \neq \varnothing$, Q_ι *is an individual point of* $\mathrm{Sing}(X)$, *and* Y_2 *comes from* Y_1 *just by an elementary transformation* $\Sigma_0'(\tilde{\mathcal{S}}_{Q_\iota})$ *of* $\Sigma_0'(\mathcal{S}_{Q_\iota})$ *as above, then the formula (5.28) remains true whenever one replaces* P_i *by* Q_ι*. Analogously (5.29) and (5.30) remain valid for all* $j \in \{1,2,3,4\}$*, for which* $n_j \in (s_0 \cap N_{G_{Q_\iota}}) \setminus \{e_1, e_2, e_3\}$*.)*

Proof. (5.28) follows from $n_1 + n_2 = n_3 + n_4$ and (3.2) or (5.9). Similarly, one gets (5.29) and (5.3) by using the formulae (5.7) and (5.8) (resp. (5.15) and (5.16)). □

Remark 5.11. Formulae (5.28), (5.29) and (5.30) can be viewed not only as realizations of our computational algorithm for this concrete construction, but also as special cases of more general formulae holding true for any flop along a rational $(-1,-1)$ -curve of an arbitrary smooth complex threefold. For such an approach, see Friedman [42, 7.4. and 7.5., p. 123].

Remark 5.12. It should be noted that, after having given the description of the strata of $\mathrm{Sing}(X)$, the main part of the desingularization method which was developed in § 4, does not depend intrinsically on the embedding of X's in $\mathbb{P}^{m-1}(\mathbf{w})$, and can be actually applied to any CY model being a V-variety with cyclic (or, more general, abelian) quotient singularities and globally known singular locus [24]. If, however, one considers the special case, in which our Y's can be represented as strict transforms of appropriate crepant desingularizations of $\mathbb{P}^{m-1}(\mathbf{w})$, then it is possible to determine not only the evaluations of $[c_2]_Y^{\mathbf{Q}}$ at a member of $\mathcal{B}_Y$, but also the second rational Chern class $c_2^{\mathbf{Q}}(Y) \in H^4(Y, \mathbb{Q})$ itself.
Let us explain this more closely. We can conceive the space $\mathbb{P}^{m-1}(\mathbf{w})$ itself as a toric complete variety $\mathbb{P}^{m-1}(\mathbf{w}) = Z(N(\mathbf{w}), \Sigma(\mathbf{w}))$ (in the notation of § 3) by setting $N(\mathbf{w}) := N_0/\mathbb{Z}w_0$, $N_0 = \Sigma_{i=1}^m \mathbb{Z}e_i$, $w_0 := \Sigma_{i=1}^m w_i e_i$ ($e_i := (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th place), $\Sigma(\mathbf{w}) := \{\{\sigma_i | 1 \le i \le m\}$, together with their faces $\}$, where

$$\sigma_i := \begin{cases} \mathbb{R}_{\ge 0} n(w_2) + \ldots + \mathbb{R}_{\ge 0} n(w_m), \ i = 1 \\ \mathbb{R}_{\ge 0} n(w_1) + \ldots + \mathbb{R}_{\ge 0} n(w_{i-1}) + \mathbb{R}_{\ge 0} n(w_{i+1}) + \ldots + \mathbb{R}_{\ge 0} n(w_m), \ 2 \le i \le m-1 \\ \mathbb{R}_{\ge 0} n(w_1) + \ldots + \mathbb{R}_{\ge 0} n(w_{m-1}), \ i = m \end{cases}$$

and $n(w_i) := e_i + \mathbb{Z}w_0$, $\forall i$, $1 \le i \le m$. If we assume that $\mathbb{P}^{m-1}(\mathbf{w})$ is *Gorenstein*, i.e. $\mathrm{lcm}(w_1, \ldots, w_m) | \Sigma_{i=1}^m w_i$ (cf. [10, cor. 6.B.10 (a)]), then, following Batyrev [7], we can always construct a projective maximal crepant (in general *partial*) $T_{N(\mathbf{w})}$ -equivariant resolution

$$\hat{\pi} : \hat{\mathbb{P}}^{m-1}(\mathbf{w}) := Z(N(\mathbf{w}), \hat{\Sigma}(\mathbf{w})) \to \mathbb{P}^{m-1}(\mathbf{w})$$

of singularities of $\mathbb{P}^{m-1}(\mathbf{w})$ (by means of suitable projective subdivisions of the SCRPC's of $\Sigma(\mathbf{w})$ gluing together to give $\hat{\Sigma}(\mathbf{w})$) such that

$$\hat{\pi}|_Y = \pi : Y \to X,$$

i.e. such that our Y appears as strict transform of X under $\hat{\pi}$. Under these assumptions we get:

Proposition 5.13. *Let* $\hat{\Sigma}(\mathbf{w})(1) = \{\hat{\rho}_1, \ldots, \hat{\rho}_\nu\}$, $\nu \geq \sharp(\Sigma(\mathbf{w})(1)) = m$, *be the set of 1-dimensional SCRPC's of the fan* $\hat{\Sigma}(\mathbf{w})$. *Then:*

$$c_2^{\mathbf{Q}}(Y) = \sum_{1 \leq i_1 < i_2 \leq \nu} \left\{ c_1^{\mathbf{Q}}(\mathcal{O}_Y(V(\hat{\rho}_{i_1}))) \smile c_1^{\mathbf{Q}}(\mathcal{O}_Y(V(\hat{\rho}_{i_2}))) \right\} \tag{5.31}$$

Proof. 1st case: If $\hat{\mathbb{P}}^{m-1}(\mathbf{w})$ is *smooth*, we use the normal bundle sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})}|_Y \to \mathcal{N}_{Y/\hat{\mathbb{P}}^{m-1}(\mathbf{w})} \to 0 \tag{5.32}$$

which gives $c^{\mathbf{Q}}(Y) \cdot c^{\mathbf{Q}}(\mathcal{N}_{Y/\hat{\mathbb{P}}^{m-1}(\mathbf{w})}) = c^{\mathbf{Q}}(\mathcal{T}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})}|_Y)$. Denoting the inclusion map of Y in $\hat{\mathbb{P}}^{m-1}(\mathbf{w})$ by $j : Y \hookrightarrow \hat{\mathbb{P}}^{m-1}(\mathbf{w})$ and making use of

$$c^{\mathbf{Q}}(\hat{\mathbb{P}}^{m-1}(\mathbf{w})) = \prod_{i=1}^{\nu} (1 + c_1^{\mathbf{Q}}(\mathcal{O}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})}(V(\hat{\rho}_i)))) \tag{5.33}$$

(see [71, p. 131]), $c_1^{\mathbf{Q}}(Y) = c_2^{\mathbf{Q}}(Y)(\mathcal{N}_{Y/\hat{\mathbb{P}}^{m-1}(\mathbf{w})}) = 0$,
$c^{\mathbf{Q}}(\mathcal{T}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})}|_Y) = c^{\mathbf{Q}}(j^*(\mathcal{T}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})})) = j^*(c^{\mathbf{Q}}(\mathcal{T}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})}))$ and of the usual multiplication rule, we get (5.31).

2nd case: If $\hat{\mathbb{P}}^{m-1}(\mathbf{w})$ is singular, then it admits at most Gorenstein $\mathbb{Q}$ -factorial terminal singularities and therefore $\mathrm{codim}_{\hat{\mathbb{P}}^{m-1}(\mathbf{w})}(\mathrm{Sing}(\hat{\mathbb{P}}^{m-1}(\mathbf{w}))) > 3$ (see [7]). Consequently, $\hat{\mathbb{P}}^{m-1}(\mathbf{w})$ is smooth in a neighbourhood of Y and (5.32) remains exact (see [46, B.7.2., p. 438]). Furthermore, since $\hat{\mathbb{P}}^{m-1}(\mathbf{w})$ is a "$\mathbb{Q}$-homology" variety, as coming from $\mathbb{P}^{m-1}(\mathbf{w})$ after *simplicial* subdivisions, it respects Poincaré duality over $\mathbb{Q}$ ([13, ch. V]). Thus (5.33) is true too.

□

§ 6. Topology change after flopping

As it follows from rem 4.5.(ii) and thm. 5.10., two distinct toroidal crepant desingularizations Y_1, Y_2 of an X do not respect, in general, the topological "triple couplings" and the $[c_2]$ -forms, although they have identical Hodge diamonds. This "typical phenomenon" leads to the conjecture that most of the pairs (Y_1, Y_2) will be equipped with different topologies. (A geometrical method for the determination of the number of all possible projective toroidal crepant desingularizations is described in [24].) We shall illustrate here just an indicative example and explain how the testing bilinear forms (cf. 1.4.) can be used in order to distinguish diffeomorphism (resp. homotopy) types according to lemma 1.5.

Let $X = X_{36} = \{[z_1, z_2, z_3, z_4, z_5] \in \mathbb{P}^4(1,2,3,12,18) | z_1^{36} + z_2^{18} + z_3^{12} + z_4^3 + z_5^2 = 0\}$ be the Fermat hypersurface of degree 36 (with delta genus $\Delta(X, \mathbf{L}_X) = 2$). It is $\text{Pic}(X) = \langle [\mathbf{L}_X] \rangle$, $\mathbf{L}_X = \mathcal{O}_X(\eta_X)$, with $\eta_X = \text{lcm}(1,2,3,6) = 6$, $\mathbf{L}_X^3 = 6$, $h^{1,1}(X) = b_2(X) = 1$, $h^{1,2}(X) = 182$ (by (2.17)) and $e(X) = 2(1 - h^{1,2}(X)) = -362$. The singular locus of X can be written as the union $\text{Sing}(X) = C_1 \cup C_2$ of two curves $C_1 := X_{\{1,3\}}$ and $C_2 := X_{\{1,2\}}$ having the individual point $Q := [0,0,0,-1,1]$ as their intersection locus $C_1 \cap C_2 = X_{\{1,2,3\}} = \{Q\}$. By prop. 2.30., we have the isomorphisms:

$$C_1 = (X_{36} \subset \mathbb{P}^2(2,12,18)) \cong (X_{18} \subset \mathbb{P}^2(1,6,9)) \cong (X_6 \subset \mathbb{P}^2(1,2,3)) = C_1',$$

because, in the notation of 2.2., $(\overline{2}, \overline{12}, \overline{18}) = (1,6,9)$, $\rho_1(2,12,18) = \gcd(6,9) = 3$, $\rho_2(2,12,18) = \rho_3(2,12,18) = 1$ and therefore $(2', 12', 18') = (1,2,3)$. Since $l_{C_1} = 2$ and $\text{am}(C_1') = 0$, the genus $g(C_1) = g(C_1')$ of C_1 equals $h^0(C_1, \mathcal{O}_{C_1}) = 1$ and by (5.12): $\deg(\mathcal{O}_{C_1}(6)) = \frac{1}{2} h^0(C_1', \mathcal{O}_{C_1'}(\theta(6;2,12,18))$. Using the notation of 2.6., we get: $\gamma_1(6;2,12,18) = \gamma_2(6;2,12,18) = \gamma_3(6;2,12,18) = 0$, $\varepsilon_1(6;2,12,18) = 2$, $\varepsilon_2(6;2,12,18) = 6$, $\varepsilon_3(6;2,12,18) = 6$. Thus, $\theta(6;2,12,18) = \frac{1}{3}(6-0) = 2$ and $\deg(\mathcal{O}_{C_1}(6)) = \frac{1}{2}(pt(2;1,2,3) - pt(-4;1,2,3)) = \frac{1}{2}(2-0) = 1$. Similarly we have:

$$C_2 = (X_{36} \subset \mathbb{P}^2(3,12,18)) \cong (X_{12} \subset \mathbb{P}^2(1,4,6)) \cong (X_6 \subset \mathbb{P}^2(1,2,3)) = C_2' \cong C_1',$$

$$\begin{aligned}
&(\overline{3}, \overline{12}, \overline{18}) = (1,4,6), \rho_1(3,12,18) = 2, \rho_2(3,12,18) = \rho_3(3,12,18) = 1,\\
&(3', 12', 18') = (1,2,3), l_{C_2} = 3, g(C_2) = g(C_2') = 1, \gamma_1(6;3,12,18) = \gamma_2(6;3,12,18) =\\
&\gamma_3(6;3,12,18) = 0, \varepsilon_1(6;3,12,18) = 3, \varepsilon_2(6;3,12,18) = \varepsilon_3(6;3,12,18) = 6,\\
&\theta(6;3,12,18) = \frac{1}{2}(6-0) = 3, \deg(\mathcal{O}_{C_2}(6)) = \frac{1}{3}(pt(3;1,2,3) - pt(-3;1,2,3)) =\\
&\frac{1}{3}(3-0) = 1.
\end{aligned}$$

Now the germ of a point $P \in \text{Sing}(X)$ is isomorphic to:

$$(X, P) \cong \begin{cases} ((\mathbb{C}^2 / \langle \text{diag}(\zeta_2, \zeta_2) \rangle) \times \mathbb{C},\ [0] \times \{0\}), & \text{if } \ P \in C_1 \setminus \{Q\} \\ ((\mathbb{C}^2 / \langle \text{diag}(\zeta_3, \zeta_3^2) \rangle) \times \mathbb{C},\ [0] \times \{0\}), & \text{if } \ P \in C_2 \setminus \{Q\} \\ (\mathbb{C}^3 / \langle \text{diag}(\zeta_6, \zeta_6^2, \zeta_6^3) \rangle,\ [0]), & \text{if } \ P = Q \end{cases}$$

We resolve an open neighbourhood of Q by means of one of the five toric crepant morphisms $Z(N_{G_Q}, \Sigma'_0(\mathcal{S}_i)) \to Z(N_{G_Q}, \Sigma_0)$, $G_Q \cong (\mathbb{Z}/6\mathbb{Z})$, corresponding to one of the five possible triangulations $\mathcal{S}_i$, $1 \le i \le 5$, of s_0, as they are drawn in figure 4. Since these morphisms are compatible with the usual blow-ups along $\mathcal{N}_{C_1/X}$, $\mathcal{N}_{C_2/X}$, we can construct five (global) toroidal crepant desingularizations $\pi_i : Y_i \to X$ of X with $a(Q) = 1$, $h^{1,1}(Y_i) = 1 + (2+3) - 1 = 5$, $h^{1,2}(Y_i) = h^{1,2}(X) + (2-1)\cdot 1 + (3-1)\cdot 1 = 182 + 3 = 185$ (cf. (4.21), (4.22)) and topological Euler-Poincaré characteristic $e(Y_i) = 2(h^{1,1}(Y_i) - h^{1,2}(Y_i)) = 2\cdot(5-185) = -360$. Over C_1, C_2 and Q are placed the exceptional prime divisors $D_1^{(i)}$, $\{D_2^{(i)}, D_3^{(i)}\}$ and $D_4^{(i)}$ respectively. (In the "toric picture" they are realized by D_{n_1}, $\{D_{n_2}, D_{n_3}\}$ and D_{n_4}, $\forall i$, $1 \le i \le 5$, with $n_1 = (\frac{1}{2}, 0, \frac{1}{2})$, $n_2 = (\frac{1}{3}, \frac{2}{3}, 0)$, $n_3 = (\frac{2}{3}, \frac{1}{3}, 0)$, $n_4 = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6})$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.)

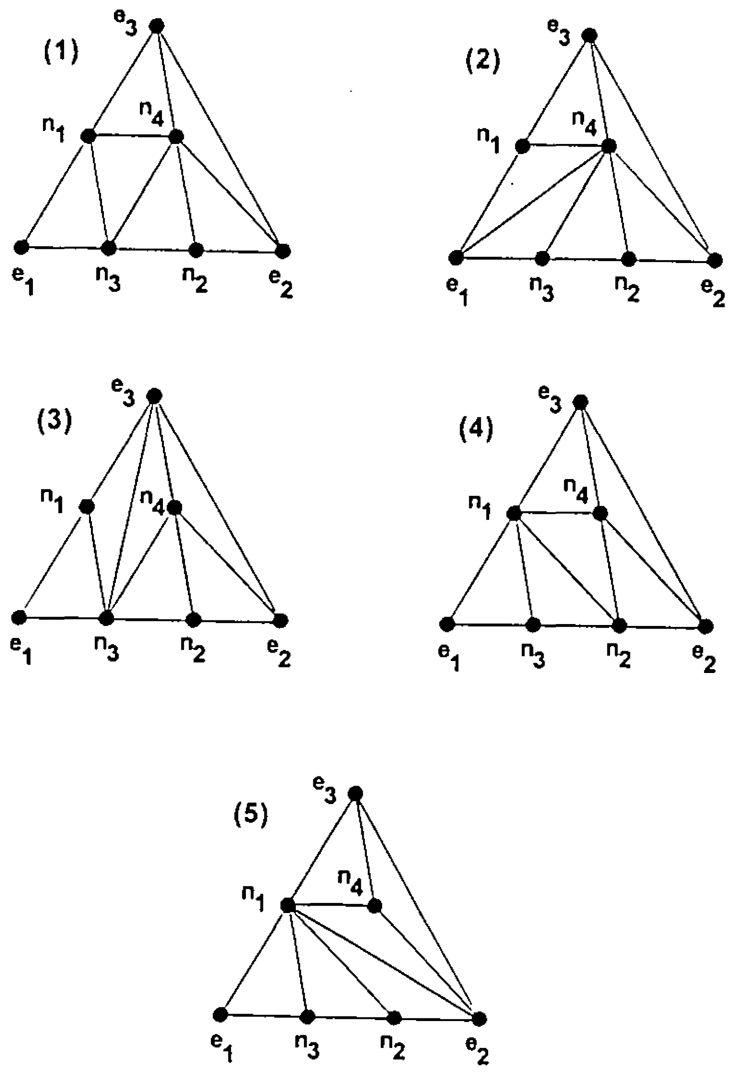


Fig.4

Thus,

$$\mathrm{Pic}_{\mathbb{Q}}(Y_i) \cong (\mathbb{Q}c_1(\mathrm{L}_{Y_i})) \oplus (\oplus_{j=1}^{4}\mathbb{Q}c_1(\mathcal{O}_{Y_i}(D_j^{(i)})).$$

Moreover, it is easy to see that all five desingularization spaces are projective, i.e. that Y_i is a CY threefold, $\forall i$, $1 \leq i \leq 5$. Note that Y_2, Y_3 and Y_4 are obtained from Y_1 after a single flop along the curves $\mathcal{C}(n_1, n_3)$, $\mathcal{C}(n_1, n_4)$ and $\mathcal{C}(n_3, n_4)$ respectively. Y_5 is nothing but Y_4 being flopped along $\mathcal{C}(n_2, n_4)$.

Proposition 6.1. *Y_i and $Y_{i'}$ do not have the same diffeomorphism (resp. homotopy) type, $\forall i, i'$, $1 \leq i, i' \leq 5$, $i \neq i'$.*

Proof. At first we compute the $5 \cdot \binom{7}{3} = 175$ intersection numbers which can be formed by triples of $\{\mathcal{B}_{Y_i} | 1 \leq i \leq 5\}$, as well as the images of the elements of these bases under $[c_2]_{Y_i}$, $1 \leq i \leq 5$, by using the formulae of theorems 5.1., 5.3., 5.7., and 5.10. All formulae, up to (5.19) and (5.20) for C_2, are now directly applicable if one takes account of the toric data of figure 4 and of the discussion preceding the formulation of prop. 6.1. For (5.19) and (5.20) we need, in addition, to compute $\deg(\mathcal{O}_{C_2'}(\theta(2-\alpha_2^{(C_2)};\ 3,12,18)))$ and $\deg(\mathcal{O}_{C_2'}(\theta(-1+\alpha_2^{(C_2)};\ 3,12,18)))$ respectively. Since $(\alpha_1^{(C_2)}, \alpha_2^{(C_2)}) = (1,2)$,
$\theta(0;3,12,18) = 0$, $\gamma_1(1;3,12,18) = 1$, $\gamma_2(1;3,12,18) = \gamma_3(1;3,12,18) = 0$,
$\varepsilon_1(1;3,12,18) = 0$, $\varepsilon_2(1;3,12,18) = \varepsilon_3(1;3,12,18) = 1$, we get $\theta(1;3,12,18) = \frac{1}{2}(1-1) = 0$,
which means that both of the regarded degrees vanish (cf. (5.12)).

The intersection numbers are given by the following table:

Nr.	int. numbers/i	1	2	3	4	5
(1)	$(D_1^{(i)})^3$	-1	0	0	-2	-3
(2)	$((D_1^{(i)})^2 \cdot D_2^{(i)})$	0	0	0	-1	0
(3)	$((D_1^{(i)})^2 \cdot D_3^{(i)})$	-1	0	-2	0	0
(4)	$((D_1^{(i)})^2 \cdot D_4^{(i)})$	-1	-2	0	0	1
(5)	$(D_2^{(i)})^3$	0	0	0	-1	0
(6)	$((D_2^{(i)})^2 \cdot D_1^{(i)})$	0	0	0	-1	-2
(7)	$((D_2^{(i)} \cdot D_3^{(i)})$	0	0	0	0	0
(8)	$((D_2^{(i)})^2 \cdot D_4^{(i)})$	-2	-2	-2	-1	0

(9)	$(D_3^{(i)})^3$	-1	0	-2	0	0
(10)	$((D_3^{(i)})^2 \cdot D_1^{(i)})$	-1	0	0	-2	-2
(11)	$((D_3^{(i)})^2 \cdot D_2^{(i)})$	0	0	0	0	0
(12)	$((D_3^{(i)})^2 \cdot D_4^{(i)})$	-1	-2	0	0	0
(13)	$(D_4^{(i)})^3$	7	6	8	8	9
(14)	$((D_4^{(i)})^2 \cdot D_1^{(i)})$	-1	0	0	-2	-3
(15)	$((D_4^{(i)})^2 \cdot D_2^{(i)})$	0	0	0	-1	0
(16)	$((D_4^{(i)})^2 \cdot D_3^{(i)})$	-1	0	-2	0	0
(17)	$\mathbf{L}_{Y_i}^3$	6	6	6	6	6
(18)	$((\mathbf{L}_{Y_i})^2 \cdot D_1^{(i)})$	0	0	0	0	0
(19)	$((\mathbf{L}_{Y_i})^2 \cdot D_2^{(i)})$	0	0	0	0	0
(20)	$((\mathbf{L}_{Y_i})^2 \cdot D_3^{(i)})$	0	0	0	0	0
(21)	$((\mathbf{L}_{Y_i})^2 \cdot D_4^{(i)})$	0	0	0	0	0
(22)	$(\mathbf{L}_{Y_i} \cdot (D_1^{(i)})^2)$	-2	-2	-2	-2	-2
(23)	$(\mathbf{L}_{Y_i} \cdot (D_2^{(i)})^2)$	-2	-2	-2	-2	-2
(24)	$(\mathbf{L}_{Y_i} \cdot (D_3^{(i)})^2)$	-2	-2	-2	-2	-2
(25)	$(\mathbf{L}_{Y_i} \cdot (D_4^{(i)})^2)$	0	0	0	0	0
(26)	$(D_1^{(i)} \cdot D_2^{(i)} \cdot D_3^{(i)})$	0	0	0	1	1
(27)	$(D_1^{(i)} \cdot D_2^{(i)} \cdot D_4^{(i)})$	0	0	0	1	0
(28)	$(D_1^{(i)} \cdot D_3^{(i)} \cdot D_4^{(i)})$	1	0	0	0	0
(29)	$(D_2^{(i)} \cdot D_3^{(i)} \cdot D_4^{(i)})$	1	1	1	0	0
(30)	$(\mathbf{L}_{Y_i} \cdot D_1^{(i)} \cdot D_2^{(i)})$	0	0	0	0	0
(31)	$(\mathbf{L}_{Y_i} \cdot D_1^{(i)} \cdot D_3^{(i)})$	0	0	0	0	0

(32)	$(\mathbf{L}_{Y_i} \cdot D_1^{(i)} \cdot D_4^{(i)})$	0	0	0	0	0
(33)	$(\mathbf{L}_{Y_i} \cdot D_2^{(i)} \cdot D_3^{(i)})$	1	1	1	1	1
(34)	$(\mathbf{L}_{Y_i} \cdot D_2^{(i)} \cdot D_4^{(i)})$	0	0	0	0	0
(35)	$(\mathbf{L}_{Y_i} \cdot D_3^{(i)} \cdot D_4^{(i)})$	0	0	0	0	0

Correspondingly, the images of the elements of the bases $\{\mathcal{B}_{Y_i},\ 1 \leq i \leq 5\}$ under $[c_2]_{Y_i}$ are given by the table:

Nr.	images/i	1	2	3	4	5
(1)	$[c_2]_{Y_i}(D_1^{(i)})$	2	0	0	4	6
(2)	$[c_2]_{Y_i}(D_2^{(i)})$	0	0	0	2	0
(3)	$[c_2]_{Y_i}(D_3^{(i)})$	2	0	4	0	0
(4)	$[c_2]_{Y_i}(D_4^{(i)})$	−2	0	−4	−4	−6
(5)	$[c_2]_{Y_i}(\mathbf{L}_{Y_i})$	72	72	72	72	72

In the next step we consider the testing bilinear forms of Y_i

$$\beta_{Y_i}^{\mathbf{Q}} : (\mathrm{Sym}^2(\mathrm{Pic}_{\mathbf{Q}}(Y_i)))^2 \to \mathbf{Q},\ 1 \leq i \leq 5.$$

$\mathrm{Sym}^2(\mathrm{Pic}_{\mathbf{Q}}(Y_i))$ has dimension $\frac{h^{1,1}(Y_i)(h^{1,1}(Y_i)+1)}{2} = \frac{5 \cdot 6}{2} = 15$.

Let $\mathcal{M}_i$ denote the symmetric matrix $\{\beta_{Y_i}^{\mathbf{Q}}(\mathfrak{b}_s^{(i)}, \mathfrak{b}_t^{(i)}) | 1 \leq s,\ t \leq 15\}$ defined by the ordered basis

$$\Big\{\mathfrak{b}_1^{(i)} := (\mathbf{L}_{Y_i}, \mathbf{L}_{Y_i}),\ \mathfrak{b}_2^{(i)} := (\mathbf{L}_{Y_i}, D_1^{(i)}),\ \mathfrak{b}_3^{(i)} := (\mathbf{L}_{Y_i}, D_2^{(i)}),\ \mathfrak{b}_4^{(i)} := (\mathbf{L}_{Y_i}, D_3^{(i)}),$$
$$\mathfrak{b}_5^{(i)} := (\mathbf{L}_{Y_i}, D_4^{(i)}), \mathfrak{b}_6^{(i)} := (D_1^{(i)}, D_1^{(i)}),\ \mathfrak{b}_7^{(i)} := (D_1^{(i)}, D_2^{(i)}),\ \mathfrak{b}_8^{(i)} := (D_1^{(i)}, D_3^{(i)}),$$
$$\mathfrak{b}_9^{(i)} := (D_1^{(i)}, D_4^{(i)}), \mathfrak{b}_{10}^{(i)} := (D_2^{(i)}, D_2^{(i)}), \mathfrak{b}_{11}^{(i)} := (D_2^{(i)}, D_3^{(i)}),\ \mathfrak{b}_{12}^{(i)} := (D_2^{(i)}, D_4^{(i)}),$$
$$\mathfrak{b}_{13}^{(i)} := (D_3^{(i)}, D_3^{(i)}),\ \mathfrak{b}_{14}^{(i)} := (D_3^{(i)}, D_4^{(i)}),\ \mathfrak{b}_{15}^{(i)} := (D_4^{(i)}, D_4^{(i)})\Big\}$$

of $\mathrm{Sym}^2(\mathrm{Pic}_{\mathbf{Q}}(Y_i))$. We compute $\mathcal{M}_i$'s by the above tables. For typographical reasons we write the entries of each of their lines between commas:

$$\mathcal{M}_1 = \begin{pmatrix} [1728,12,0,12,-12,-288,0,0,0,-288,144,0,-288,0,0], \\ [12,-288,0,0,0,-84,0,-76,-68,-4,2,0,-76,72,-72], \\ [0,0,-288,144,0,0,-4,2,0,-6,-4,-140,4,70,0], \\ [12,0,144,-288,0,-76,2,-76,72,-4,4,70,-84,-68,-72], \\ [-12,0,0,0,0,-68,0,72,-72,-140,70,0,-68,-72,504], \\ [-288,-84,0,-76,-68,-8,0,-8,-4,0,0,0,-8,4,0], \\ [0,0,-4,2,0,0,0,0,0,0,0,-4,0,2,0] \\ [0,-76,2,-76,72,-8,0,-8,4,0,0,2,-8,4,-8], \\ [0,-68,0,72,-72,-4,0,4,0,-4,2,0,4,-8,11] \\ [-288,-4,-6,-4,-140,0,0,0,-4,0,0,0,0,-4,6], \\ [144,2,-4,4,70,0,0,0,2,0,0,-4,0,4,-4], \\ [0,0,-140,70,0,0,-4,2,0,0,-4,6,4,-4,0] \\ [-288,-76,4,-84,-68,-8,0,-8,4,0,0,4,-8,-4,0], \\ [0,72,70,-68,-72,4,2,4,-8,-4,4,-4,-4,0,20], \\ [0,-72,0,-72,504,0,0,-8,11,6,-4,0,0,20,-56] \end{pmatrix}$$

$$\mathcal{M}_2 = \begin{pmatrix} [1728,0,0,0,0,-288,0,0,0,-288,144,0,-288,0,0], \\ [0,-288,0,0,0,-6,0,0,-144,0,0,0,0,0,0], \\ [0,0,-288,144,0,0,0,0,0,-6,0,-144,0,72,0], \\ [0,0,144,-288,0,0,0,0,0,0,0,72,0,-144,0], \\ [0,0,0,0,0,0,0,0,0,-144,72,0,144,0,432], \\ [-288,-6,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ [0,-144,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ [-288,0,-6,0,-144,0,0,0,0,0,0,0,0,0,0], \\ [144,0,0,0,72,0,0,0,0,0,0,0,0,0,0], \\ [0,0,-144,72,0,0,0,0,0,0,0,0,0,0,0], \\ [-288,0,0,0,-144,0,0,0,0,0,0,0,0,0,0], \\ [0,0,72,-144,0,0,0,0,0,0,0,0,0,0,0], \\ [0,0,0,0,432,0,0,0,0,0,0,0,0,0,0] \end{pmatrix}$$

$$\mathcal{M}_3 = \begin{pmatrix} [1728,0,0,24,-24,-288,0,0,0,-288,144,0,-288,0,0], \\ [0,-288,0,0,0,-6,0,-152,8,0,0,0,0,0,0], \\ [0,0,-288,144,0,0,0,0,0,-6,-8,-136,8,68,0], \\ [24,0,144,-288,0,-152,0,0,0,-8,8,68,-168,8,-144], \\ [-24,0,0,0,0,8,0,0,0,-136,68,0,8,-144,576], \\ [-288,-6,0,-152,8,0,0,0,0,0,0,0,-16,8,0], \\ [0,0,0,0,0,0,0,0,0,0,0,0,0,0,0], \\ [0,-152,0,0,0,0,0,-16,8,0,0,0,0,0,0], \\ [0,8,0,0,0,0,0,8,0,0,0,0,0,0,0], \\ [-288,0,-6,-8,-136,0,0,0,0,0,0,0,0,-8,16], \\ [144,0,-8,8,68,0,0,0,0,0,0,-8,0,8,-8], \\ [0,0,-136,68,0,0,0,0,0,0,-8,16,8,-8,0], \\ [-288,0,8,-168,8,-16,0,0,0,0,0,8,-32,8,-16], \\ [0,0,68,8,-144,8,0,0,0,-8,8,-8,8,-16,56], \\ [0,0,0,-144,576,0,0,0,0,16,-8,0,-16,56,-128] \end{pmatrix}$$

$$\mathcal{M}_4 = \left(\begin{array}{c}
[1728, 24, 12, 0, -24, -288, 0, 0, 0, -288, 144, 0, -288, 0, 0], \\
[24, -288, 0, 0, 0, -168, -76, 0, 8, -80, 76, 72, -152, 0, -144], \\
[12, 0, -288, 144, 0, -76, -80, 76, 72, -84, 4, -64, -4, -4, -72], \\
[0, 0, 144, -288, 0, 0, 76, -152, 0, 4, -4, -4, 0, 8, 0], \\
[-24, 0, 0, 0, 0, 8, 72, 0, -144, -64, -4, -72, 8, 0, 576], \\
[-288, -168, -76, 0, 8, -32, -16, 0, 8, -12, 8, 12, -16, 0, -16], \\
[0, -76, -80, 76, 72, -16, -12, 8, 12, -10, 4, 4, -4, -4, -6], \\
[0, 0, 76, -152, 0, 0, 8, -16, 0, 4, -4, -4, 0, 8, 0], \\
[0, 8, 72, 0, -144, 8, 12, 0, -16, 4, -4, -6, 8, 0, 26], \\
[-288, -80, -84, 4, -64, -12, -10, 4, 4, -8, 0, -2, 0, 0, 4], \\
[144, 76, 4, -4, -4, 8, 4, -4, -4, 0, 0, 0, 0, 0, 0], \\
[0, 72, -64, -4, -72, 12, 4, -4, -6, -2, 0, 4, 0, 0, 28], \\
[-288, -152, -4, 0, 8, -16, -4, 0, 8, 0, 0, 0, 0, 0, 0], \\
[0, 0, -4, 8, 0, 0, -4, 8, 0, 0, 0, 0, 0, 0, 0], \\
[0, -144, -72, 0, 576, -16, -6, 0, 26, 4, 0, 28, 0, 0, -128]
\end{array}\right)$$

$$\mathcal{M}_5 = \left(\begin{array}{c}
[1728, 36, 0, 0, -36, -288, 0, 0, 0, -288, 144, 0, -288, 0, 0], \\
[36, -288, 0, 0, 0, -252, 0, 0, 84, -156, 78, 0, -156, 0, -216], \\
[0, 0, -288, 144, 0, 0, -156, 78, 0, -6, 0, 12, 0, -6, 0], \\
[0, 0, 144, -288, 0, 0, 78, -156, 0, 0, 0, -6, 0, 12, 0], \\
[-36, 0, 0, 0, 0, 84, 0, 0, -216, 12, -6, 0, 12, 0, 648], \\
[-288, -252, 0, 0, 84, -72, 0, 0, 36, -24, 12, 0, -24, 0, -48], \\
[0, 0, -156, 78, 0, 0, -24, 12, 0, 0, 0, 12, 0, -6, 0], \\
[0, 0, 78, -156, 0, 0, 12, -24, 0, 0, 0, -6, 0, 12, 0], \\
[0, 84, 0, 0, -216, 36, 0, 0, -48, 12, -6, 0, 12, 0, 45], \\
[-288, -156, -6, 0, 12, -24, 0, 0, 12, 0, 0, 0, 0, 0, 0], \\
[144, 78, 0, 0, -6, 12, 0, 0, -6, 0, 0, 0, 0, 0, 0], \\
[0, 0, 12, -6, 0, 0, 12, -6, 0, 0, 0, 0, 0, 0, 0], \\
[-288, -156, 0, 0, 12, -24, 0, 0, 12, 0, 0, 0, 0, 0, 0], \\
[0, 0, -6, 12, 0, 0, -6, 12, 0, 0, 0, 0, 0, 0, 0], \\
[0, -216, 0, 0, 648, -48, 0, 0, 45, 0, 0, 0, 0, 0, -216]
\end{array}\right)$$

Using the computer programme MAPLE, we find their ranks:

$$\mathrm{rk}(\mathcal{M}_1) = \mathrm{rk}(\mathcal{M}_3) = 14,\ \mathrm{rk}(\mathcal{M}_2) = 10,\ \mathrm{rk}(\mathcal{M}_4) = \mathrm{rk}(\mathcal{M}_5) = 13.$$

For $i \in \{1, \ldots, 5\}$, let Ω_i denote the quadratic form being associated to $\mathcal{M}_i$, $(\mathrm{n}_+^{(i)}, \mathrm{n}_-^{(i)}, \mathrm{n}_0^{(i)})$ the signature data of Ω_i, $\mathrm{chp}_i(x) = \Sigma_{j=0}^{15} \mu_j^{(i)} x^j$ the characteristic polynomial of $\mathcal{M}_i$ and $\lambda_1^{(i)}, \ldots, \lambda_{15}^{(i)}$ the corresponding eigenvalues. Obviously, $\mathrm{n}_0^{(1)} = \mathrm{n}_0^{(3)} = 1$, $\mathrm{n}_0^{(2)} = 5$, $\mathrm{n}_0^{(4)} = \mathrm{n}_0^{(5)} = 2$. Since $\mathcal{M}_i$, $1 \le i \le 5$, are real symmetric matrices, they have only *real* eigenvalues. Furthermore, according to the spectral theorem, the triples $(\mathrm{n}_+^{(i)}, \mathrm{n}_-^{(i)}, \mathrm{n}_0^{(i)})$ give the numbers of positive, negative and vanishing eigenvalues of $\mathcal{M}_i$ respectively.

Let now $\lambda_{14}^{(4)}, \lambda_{15}^{(4)}, \lambda_{14}^{(5)}$ and $\lambda_{15}^{(5)}$ be the vanishing eigenvalues of $\mathcal{M}_4$ and $\mathcal{M}_5$. By Viète's root theorem, we get $\mu_2^{(4)} = -\lambda_1^{(4)} \cdots \lambda_{13}^{(4)}$ and $\mu_2^{(5)} = -\lambda_1^{(5)} \cdots \lambda_{13}^{(5)}$. If

$(n_+^{(4)}, n_-^{(4)}) = (n_+^{(5)}, n_-^{(5)})$, then we should have $\mathrm{sgn}(\mu_2^{(4)}) = (-1)^{n_-^{(4)}+1} = (-1)^{n_-^{(5)}+1} = \mathrm{sgn}(\mu_2^{(5)})$, which would contradict MAPLE's computations:

$$\mu_2^{(4)} = -576198190423640899584000 < 0$$
$$\mu_2^{(5)} = 585283053111058287820 80 > 0$$

Hence $\mathfrak{Q}_4$ and $\mathfrak{Q}_5$ (resp. $\beta^{\mathbb{R}}_{Y_4}$ and $\beta^{\mathbb{R}}_{Y_5}$, $\beta^{\mathbb{Q}}_{Y_4}$ and $\beta^{\mathbb{Q}}_{Y_5}$) are inequivalent. Unfortunately, $(n_+^{(1)}, n_-^{(1)}, n_0^{(1)}) = (n_+^{(3)}, n_-^{(3)}, n_0^{(3)}) = (7,7,1)$ and the above trick cannot be applied to $\mathcal{M}_1$ and $\mathcal{M}_3$. Nevertheless, we can write $\mathrm{Sym}^2(\mathrm{Pic}_{\mathbb{Q}}(Y_1)) = V_1 \oplus W_1, \mathrm{Sym}^2(\mathrm{Pic}_{\mathbb{Q}}(Y_3)) = V_3 \oplus W_3$, where W_1 and W_3 denote the kernels of the linear maps corresponding to $\mathcal{M}_1$ and $\mathcal{M}_3$, and V_1, V_3 the nondegeneracy loci, and compare the so arising nondegenerate quadratic forms $\tilde{\mathfrak{Q}}_1 := \mathfrak{Q}_1|_{V_1}$ and $\tilde{\mathfrak{Q}}_3 := \mathfrak{Q}_3|_{V_3}$ of rank 14 over $\mathbb{Q}$. Note that

$$W_1 = \langle(0,0,0,0,0,1,-4,-2,0,0,4,0,1,0,0)\rangle \quad \text{and}$$
$$W_3 = \langle(0,0,0,0,0,0,1,0,0,0,0,0,0)\rangle.$$

In fact, $\tilde{\mathfrak{Q}}_3$ corresponds to the symmetric (14×14) -matrix $\tilde{\mathcal{M}}_3$ coming from $\mathcal{M}_3$ after deletion of the 7-th column and of the 7-th row. If we regard $\{e_1, \ldots, e_6,$ the generator of $W_1, e_8, \ldots, e_{15}\}$ as a new basis for $\mathbb{Q}^{15}$, then the matrix of the quadratic form $\tilde{\mathfrak{Q}}_1$ w.r.t. it will be $S^T \mathcal{M}_1 S$, where S denotes the change of basis matrix. Deleting again the zero 7-th column and 7-th row we get the matrix:

$$\tilde{\mathcal{M}}_1 = \begin{pmatrix}
[1728, 12, 0, 12, -12, -288, 0, 0, -288, 144, 0, -288, 0, 0], \\
[12, -288, 0, 0, 0, -84, -76, -68, -4, 2, 0, -76, 72, -72], \\
[0, 0, -288, 144, 0, 0, 2, 0, -6, -4, -140, 4, 70, 0], \\
[12, 0, 144, -288, 0, -76, -76, 72, -4, 4, 70, -84, -68, -72], \\
[-12, 0, 0, 0, 0, -68, 72, -72, -140, 70, 0, -68, -72, 504], \\
[-288, -84, 0, -76, -68, -8, -8, -4, 0, 0, 0, -8, 4, 0], \\
[0, -76, 2, -76, 72, -8, -8, 4, 0, 0, 2, -8, 4, -8], \\
[0, -68, 0, 72, -72, -4, 4, 0, -4, 2, 0, 4, -8, 11], \\
[-288, -4, -6, -4, -140, 0, 0, -4, 0, 0, 0, 0, -4, 6], \\
[144, 2, -4, 4, 70, 0, 0, 2, 0, 0, -4, 0, 4, -4], \\
[0, 0, -140, 70, 0, 0, 2, 0, 0, -4, 6, 4, -4, 0], \\
[-288, -76, 4, -84, -68, -8, -8, 4, 0, 0, 4, -8, -4, 0], \\
[0, 72, 70, -68, -72, 4, 4, -8, -4, 4, -4, -4, 0, 20], \\
[0, -72, 0, -72, 504, 0, -8, 11, 6, -4, 0, 0, 20, -56]
\end{pmatrix}$$

If $\tilde{\mathfrak{Q}}_1$ and $\tilde{\mathfrak{Q}}_3$ were equivalent as $\mathbb{Q}$-quadratic forms, then $\det(\tilde{\mathcal{M}}_1)$ and $\det(\tilde{\mathcal{M}}_3)$ would be equal up to muliplication by the square of a number $\in (\mathbb{Q} \setminus \{0\})$. Luckily, by MAPLE we get

$$\det(\tilde{\mathcal{M}}_1) = -14286537432760320000,$$
$$\det(\tilde{\mathcal{M}}_3) = -13613985232597706342 4,$$

and

$$\left(\frac{\det(\tilde{\mathcal{M}}_1)}{\det(\tilde{\mathcal{M}}_3)}\right)^{\frac{1}{2}} = \frac{25}{788768}(2119)^{\frac{1}{2}}(49298)^{\frac{1}{2}} \notin \mathbb{Q},$$

which leads to the desirable contradiction. □

Remarks 6.2. (i) We should mention here, that if we wish (e.g. in another example) to compare two *nondegenerate rational* quadratic forms with *identical* signature data, whose ratio of determinants of their structure matrices w.r.t. our bases has *rational* square root, then we have to make use of additional *local invariants* involving the *Hilbert symbol* of numbers taken from the field $\mathbb{Q}_p$ of p-adic numbers. See Serre [109, Cor. of p. 44].

(ii) It is now obvious from the above that for a given c.i. $X = X_{\mathbf{d}} \subset \mathbb{P}^{m-1}(\mathbf{w})$ (as in § 5) with at least two distinct toroidal crepant desingularizations, one can develop a *formal comparison algorithm* (or, so to say, a *weak* C^∞ *-classification algorithm*) for all Y's. Let us describe it in broad outline:

Step 1: Find the number η_X by (2.24) and the singular locus Sing(X) of X by (4.1), as well as the type of the c.q.s. of each of its dissident points using lemma 4.2.

Step 2: Draw a picture for the "toric triangles" corresponding to the dissident points of X and determine on them all the "new" fixed vertices which are due to our group actions (cf. § 3). After that construct all the possible distinct subdivisions of these triangles with respect to these new vertices. (For *linear time algorithms* for the sorting of subdivisions of a plane triangle or, more general, of a simple polygon into smaller triangles with prescribed vertices, see Clarkson et al. [19], Chazelle [18] and further references given in these articles.)

Step 3: Consider an arbitrary pair (Y_1, Y_2) consisting of two distinct toroidal crepant desingularizations (4.16) of X. Use step 1, the first part of step 2 and the formulae of § 5, in order to specify the entries of the symmetric $\left(\frac{h^{1,1}(h^{1,1}+1)}{2} \times \frac{h^{1,1}(h^{1,1}+1)}{2}\right)$ -matrices, say $\mathcal{M}_1$ and $\mathcal{M}_2$, coming from the evaluations of the testing bilinear forms $\beta^{\mathbf{Q}}_{Y_1}$, $\beta^{\mathbf{Q}}_{Y_2}$ at the pairs formed by members of the natural ordered bases of $\mathrm{Sym}^2(\mathrm{Pic}_{\mathbf{Q}}(Y_i))$, $i = 1, 2$. If $\mathcal{M}_1$, $\mathcal{M}_2$ have different ranks or different signature data or - in the nondegenerate case - different discriminants, then Y_1 and Y_2 will be non diffeomorphic. In the case, where the above invariants are identical, try to use the "determinant trick" or p-adics (as it is explained in (i)). If this is still not enough to distinguish the diffeomorphism types of Y_1 and Y_2, then try to make use of *another* testing (real or rational) quadratic form and compare again the corresponding invariants.

Step 4: If none of the criteria being introduced in step 3 is able to give a definitive answer to the question, if Y_1 and Y_2 are of different type or not, then throw (Y_1, Y_2) into the "basket" of the "undecided cases". (We do not know any example of a pair

(Y_1, Y_2) belonging to the undecided cases, and we conjecture that the above basket is probably empty !)

Step 5: Repeat for all pairs (Y_1, Y_2) of the second part of step 2 the procedures of the other steps and close the flow chart of our algorithm.

§ 7. Appendix: On the Combinatorics concerning the Weighted Partitions and the Counting of Integral Points of a Polyhedron. From Euler's "Partitio Numerorum" to Ehrhart Polynomials.

Let $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{N}^m$ be again a system of "weights". For a fixed $n \in \mathbb{N}_0$ we define:

$$\mathrm{PT}(n; \mathbf{w}) := \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}_0^m \mid \sum_{i=1}^{m} \lambda_i w_i = n\}$$

$$\mathrm{PT}^+(n; \mathbf{w}) := \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m \mid \sum_{i=1}^{m} \lambda_i w_i = n\}$$

and $pt(n; \mathbf{w}) := \sharp(\mathrm{PT}(n; \mathbf{w}))$, $pt^+(n; \mathbf{w}) := \sharp(\mathrm{PT}^+(n; \mathbf{w}))$. Obviously, $pt^+(n; \mathbf{w}) = pt(n - \sum_{i=1}^{m} w_i; \mathbf{w})$.

The elements of $\mathrm{PT}(n; \mathbf{w})$ and $\mathrm{PT}^+(n; \mathbf{w})$ can always be found by means of standard polynomial time algorithms within the framework of the theory of integer linear programming (see Schrijver [106]). Nevertheless, the *precise* determination of $pt(n; \mathbf{w})$ or $pt^+(n; \mathbf{w})$ as a "closed" functional expression of n and $\mathbf{w}$ is indeed a very subtle problem. $pt(n; \mathbf{w})$ has the following equivalent interpretations:

(a) *arithmetical-combinatorial interpretation*: $pt(n; \mathbf{w})$ equals the number of non-negative integral solutions of a linear diophantine equation and expresses the *denumerant* of the *weighted partitions* of n w.r.t. $w_1, \ldots, w_m$.

(b) *geometrical-combinatorial interpretation*: $pt(n; \mathbf{w})$ gives the number of the integral points of the rational polyhedron

$$\text{(7.1)} \qquad \mathbf{P}(n; \mathbf{w}) := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} w_i x_i = n,\ x_i \geq 0,\ \forall i,\ i \leq i \leq m\}$$

with vertices $(\frac{n}{w_1}, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, \frac{n}{w_m})$. Note that $\mathbf{P}(n; \mathbf{w})$ can be represented as the *dilation* $\mathbf{P}(n; \mathbf{w}) = n \cdot \Pi(\mathbf{w})$ of the polyhedron $\Pi(\mathbf{w})$ by the factor n, where:

$$\text{(7.2)} \qquad \Pi(\mathbf{w}) := \{(y_1, \ldots, y_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} w_i y_i = 1,\ y_i \geq 0,\ \forall i,\ 1 \leq i \leq m\}$$

with vertices $(\frac{1}{w_1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \frac{1}{w_m})$.

Since $(1 - x^{w_i})^{-1} = \sum_{\nu=0}^{\infty} x^{w_i \nu}$, $pt(n; \mathbf{w})$ is exactly the n-th coefficient of the generating function of $\mathbf{F}(x) := \prod_{i=1}^{m} (1 - x^{w_i})^{-1}$ which was introduced by the formula

(2.19) in § 2. If we define $\widetilde{pt}(n; \mathbf{w})$ to be $pt(n; \mathbf{w})$ for $n \in \mathbb{N}_0$, and to be given via the combinatorial identity ([111, p. 206])

$$\sum_{n=1}^{\infty} \widetilde{pt}(-n; \mathbf{w}) x^n = -\mathbf{F}(x^{-1}), \quad \text{for} \quad n \in \mathbb{N}, \tag{7.3}$$

(which means that we choose an extension of $pt(n; \mathbf{w})$ on the whole $\mathbb{Z}$ *different* from the one introduced in 2.26.), then we get the *reciprocity relation*:

$$pt^{+}(n; \mathbf{w}) = (-1)^{m-1} \widetilde{pt}(-n; \mathbf{w}) \quad (\text{cf.}[35]). \tag{7.4}$$

We are basically interested in the pt-functions, because they constitute the "combinatorial cornerstones" of our formulae (2.20), (4.22), (5.2), (5.3), (5.12), (5.18), (5.19), (5.20) and (5.21). (Furthermore, $\mathrm{PT}(d; \mathbf{w}) \setminus \{\mathbf{0}\}$ is nothing but the parameter space of all quasihomogeneous monomials of degree d w.r.t. $\mathbf{w}$.) The purpose of this appendix is to emphasize the complexity of pt's, to make some brief historical remarks, to remind certain (mostly forgotten) combinatorial formulae for their computation, and to connect them with recent developments of the modern theory of geometric invariants.

Let us regard the above interpretation (a) as our starting-point. The pt-functions were first considered 1748 by Euler in his famous work [40]. He placed pt's among the most central themes of his "Partitio Numerorum". Euler himself studied the case where $w_i = i$, $\forall i$, $1 \le i \le m$ (and which, from now on, will be referred as *Eulerian case*) and gave some preliminary computational rules. During the 19th century, the investigations of these functions played a crucial role in number theory and in invariant and partition theory. (For extensive historical comments for this period the reader is referred to Dickson's renowned treatment [29], Ch. III. Books which devote substantial extracts to pt's or related functions, from the point of view not only of the classical but also of the modern partition theory, are, among others, those of Riordan [100], Comtet [20], Andrews [3] and Stanley [111].)

Euler's researches were mainly continued by Cayley [17] (1856), Sylvester [115], [116] (1857, 1882), Laguerre [80] (1876-7), Weihrauch [123], [124] (1875, 1877) and Glaisher [48] (1909). One of their very first results is that $pt(n; \mathbf{w})$ (resp. $pt^{+}(n; \mathbf{w})$) can be written as a *quasipolynomial* of degree $m-1$:

$$pt(n; \mathbf{w}) = \sum_{k=1}^{m-1} \mathbf{c}_{m-1-k}(n; \mathbf{w}) n^{m-1-k}, \tag{7.5}$$

which means that the coefficients $\mathbf{c}_{m-1-k}(n, \mathbf{w})$ are *periodic functions* (with integers periods) or, equivalently, that there exists an $n \in \mathbb{N}$ and polynomials $f_0, f_1, \ldots, f_{N-1}$, such that $pt(n; \mathbf{w}) = f_j(n; \mathbf{w})$ whenever $n \equiv j (\operatorname{mod} N)$.

They also used the splitting

$$pt(n; \mathbf{w}) = \Phi(n'; \mathbf{w}) + \Psi(n; \mathbf{w}) \tag{7.6}$$

where $\Phi(n'; \mathbf{w})$ denotes a polynomial in the variable $n' := n - \frac{1}{2} \sum_{i=1}^{m} w_i$ (with *constant* coefficients) and $\Psi(n; \mathbf{w})$ the *purely periodic part* of $pt(n; \mathbf{w})$.

The asymptotics of pt's are described by the following:

Theorem 7.1. *(Laguerre, Schur) The pt-function behaves asymptotically like:*

$$\lim_{n\to\infty} \frac{pt(n;\mathbf{w})}{n^{m-1}} = \mathbf{c}_{m-1}(n;\mathbf{w}) = \frac{1}{(m-1)!(\prod_{i=1}^{m} w_i)} \tag{7.7}$$

Theorem 7.2. *(Erdös-Lehner [39], Szekeres [117]) In the Eulerian case, for big n's and $m = o(\sqrt[3]{n})$ (or even for $m = o(\sqrt{n})$), we have:*

$$pt(n;1,2,\dots,m) \sim \frac{1}{m!}\binom{n-1}{m-1} \sim \frac{n^{m-1}}{(m-1)!m!} \tag{7.8}$$

In special cases, one can compute $pt(n;\mathbf{w})$ or $pt^+(n;\mathbf{w})$ very easily. For example, if $w_1 = \dots = w_m = 1$, we get directly the binomial coefficients:

$$pt(n;\mathbf{1}) = \binom{n+m-1}{n}, \quad pt^+(n;\mathbf{1}) = \binom{n-1}{n-m}.$$

On the other hand for $n = m$, we have:

Proposition 7.3. *(Formula of Fergola (1863) and Sardi (1865)). If $n = m$, the number $pt^+(n;\mathbf{w})$ is given by the formula:*

$$pt^+(n;\mathbf{w}) = \frac{1}{n!}\det(M), \tag{7.9}$$

where M denotes the $(n-1)\times(n-1)$ -matrix:

$$\begin{pmatrix} s_1s_{n-1}+s_n & -s_1 & -s_2 & -s_3 & \dots & -s_{n-3} & -s_{n-2} \\ s_1s_{n-2}+s_{n-1} & n-1 & -s_1 & -s_2 & \dots & -s_{n-4} & -s_{n-3} \\ s_1s_{n-3}+s_{n-2} & 0 & n-2 & -s_1 & \dots & -s_{n-5} & -s_{n-4} \\ s_1s_{n-4}+s_{n-3} & 0 & 0 & n-3 & \dots & -s_{n-6} & -s_{n-5} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s_1s_2+s_3 & 0 & 0 & 0 & \dots & 3 & -s_1 \\ s_1^2+s_2 & 0 & 0 & 0 & \dots & 0 & 2 \end{pmatrix}$$

while s_i is the sum of those divisors of i which occur among $w_1,\dots,w_n$. (In the Eulerian case, s_i becomes the sum of all divisors of i, $1 \le i \le n$).

The first *general* computational method for the *pt*'s, due to Cayley, Sylvester and Glaisher, is based on the decomposition of $\mathbf{F}(x)$ into *partial fractions*. $pt(n;\mathbf{w})$ is written as a sum of "waves" giving the coefficients of $\frac{1}{x}$ in the development (in ascending powers of x) of certain fractions depending on various roots of unity. The purely periodic part $\Psi(n;\mathbf{w})$ of $pt(n;\mathbf{w})$ in (7.6) is described in terms of "circulating functions", which have been introduced by Herschel in [62]. The "calculus" with these circulators seems to be extremely complicated and belongs without doubt to the (partially undecoded) "19-th century mystics". For an introduction to it we refer to the "Lehrbuch der Combinatorik" of Netto [92, § 84-95, pp. 140-158].

Weihrauch's computational technique [123], when w_i's are pairwise coprime, was somehow different and was complemented 90 years later by Ehrhart [35], [37], who discovered some beautiful trogonometric expressions for $\Psi(n;\mathbf{w})$.

Theorem 7.4. *(Formulae of Weihrauch and Ehrhart). Let $w_1, \dots, w_m$ be pairwise coprime. Then the first summand $\Phi(n'; \mathbf{w})$ of $pt(n; \mathbf{w})$ in the expression (7.6) is a polynomial of degree $m-1$. In particular, if $2 \le m \le 6$, we have:*

For $m = 2: \quad \Phi(n'; \mathbf{w}) = (w_1 w_2)^{-1} n'$

For $m = 3: \quad \Phi(n'; \mathbf{w}) = (2 w_1 w_2 w_3)^{-1}(n'^2 - \frac{1}{12}(\sum_{i=1}^{3} w_i^2))$

For $m = 4: \quad \Phi(n'; \mathbf{w}) = (6\, w_1 w_2 w_3 w_4)^{-1}(n'^3 - \frac{1}{4}(\sum_{i=1}^{4} w_i^2) n')$

For $m = 5: \quad \Phi(n'; \mathbf{w}) =$

$$(24 \prod_{i=1}^{5} w_i)^{-1} \left\{ n'^4 - (\frac{1}{2}\sum_{i=1}^{5} w_i^2) n'^2 + \frac{1}{24}\left[\frac{1}{2}(\sum_{i=1}^{5} w_i^2)^2 + \frac{1}{5}(\sum_{i=1}^{5} w_i^4)\right]\right\}$$

For $m = 6: \quad \Phi(n'; \mathbf{w}) =$

$$(120 \prod_{i=1}^{6} w_i)^{-1} \left\{ n'^5 - (\frac{5}{6}\sum_{i=1}^{6} w_i^2) n'^3 + \frac{5}{24}\left[\frac{1}{2}(\sum_{i=1}^{6} w_i^2)^2 + \frac{1}{5}(\sum_{i=1}^{6} w_i^4)\right] n' \right\}$$

Moreover, the second summand in (7.6) can be written as

$$\Psi(n; \mathbf{w}) = \sum_{i=1}^{m} \psi_{w_i}(n),$$

$$\text{where} \quad \psi_{w_i}(n) := \frac{1}{2^{m-2} w_i} \sum_{\rho=1}^{\frac{1}{2}(w_i - 1)} \kappa(\rho) \quad (\text{for } w_i \text{ odd})$$

$$\psi_{w_i}(n) := \frac{(-1)^n}{2^{m-1} w_i} + \frac{1}{2^{m-2} w_i} \sum_{\rho=1}^{\frac{1}{2}(w_i - 2)} \kappa(\rho) \quad (\text{for } w_i \text{ even})$$

are functions of period w_i and

$$\kappa(\rho) := \cos\left[(\frac{m-1}{2} - \frac{1}{w_i}(2n + \sum_{\substack{j=1 \\ j \ne 1}}^{m} w_j)\rho)\pi\right]\left[\prod_{\substack{j=1 \\ j \ne i}}^{m} \sin(\frac{\pi w_j}{w_i}\rho)\right]^{-1}.$$

On the other hand, $pt^{+}(n; \mathbf{w})$ obeys to the reciprocity law:

$$pt^{+}(n; \mathbf{w}) = (-1)^{m-1}\Phi(\sum_{i=1}^{m} w_i - n'; \mathbf{w}) + (-1)^{m-1}\Psi(-n; \mathbf{w}).$$

Unfortunately, the above condition which has to be satisfied by the weights is very restrictive as it covers, for instance, only the case in which we compute $h^0(X, \mathcal{O}_X(n))$ (cf. (2.20)) for *smooth* complete intersections X.

In the most general case, where w_i's are arbitrary, one has to take into consideration the *divisibility relations* between them. Csorba [23], following a remark of Weihrauch concerning the dependence of the coefficients $\mathbf{c}_{m-1-k}(n; \mathbf{w})$ of $pt(n; \mathbf{w})$ (cf. (7.5)) on the Bernoulli numbers, gave formulae for this general case, which reduce the computation of $\mathbf{c}_{m-1-k}(n; \mathbf{w})$'s to the solutions of finitely many systems of linear congruence equations with at most $m-1$ unknowns. (Similar formulae for $pt^+(n; \mathbf{w})$ were found independently by Vahlen [121].) To present them, let us first introduce some useful notations.

(i) For $1 \leq k \leq m-1$, $1 \leq l \leq k$ and an index-set $\{i_1, \ldots, i_l\} \subset \{1, \ldots, m\}$ of "length l" we set:

$$e(i_1, \ldots, i_l) := \gcd(w_j | 1 \leq j \leq m,\ j \notin \{i_1, \ldots, i_l\})$$

$$e^{(\varepsilon)}(i_1, \ldots, i_l) := e(i_1, \ldots, i_{\varepsilon-1}, i_{\varepsilon+1}, \ldots, i_m),\ \forall \varepsilon,\ 1 \leq \varepsilon \leq l$$

$$\mu(l; i_1, \ldots, i_l) := \left(\prod_{\varepsilon=1}^{l} e^{(\varepsilon)}(i_1, \ldots, i_l) \right) (e(i_1, \ldots, i_l))^{1-l}$$

$$T(k; i_1, \ldots, i_l) := \{(t_{i_1}, \ldots, t_{i_l}) | 1 \leq t_{i_s} \leq k,\ \forall s,\ 1 \leq s \leq l, \quad \text{and} \quad \sum_{s=1}^{l} t_{i_s} = k\}$$

$$\Xi^{(\varepsilon)}(i_1, \ldots, i_l) :=$$
$$\left\{ \xi^{(\varepsilon)}_{i_1, \ldots, i_l} | 0 \leq \xi^{(\varepsilon)}_{i_1, \ldots, i_l} < \frac{e(i_1, \ldots, i_l)}{e^{(\varepsilon)}(i_1, \ldots, i_l)} : \sum_{\varepsilon=1}^{l} w_{i_\varepsilon} \xi^{(\varepsilon)}_{i_1, \ldots, i_l} \equiv n (\operatorname{mod} e(i_1, \ldots, i_l)) \right\}$$

Furthermore, we define $\mathbf{I}(l)$ as the set of the following index-sets:

$$\mathbf{I}(l) := \{(i_1, \ldots, i_l) | 1 \leq i_s \leq m,\ \forall s,\ 1 \leq s \leq l \quad \text{and} \quad i_1 < i_2 < \ldots < i_l\}.$$

(We shall notice that $\sharp(T(k; i_1, \ldots, i_l)) = \binom{k-1}{k-l}$,
$\sharp(\Xi^{(\varepsilon)}(i_1, \ldots, i_l)) = \frac{(e(i_1, \ldots, i_l))^{l-1}}{\sum_{\varepsilon=1}^{l} e^{(\varepsilon)}(i_1, \ldots, i_l)}$ and $\sharp(\mathbf{I}(l)) = \binom{m}{l}$.)

(ii) The *Bernoulli numbers* are defined by the series

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{\mathrm{B}_1 x^2}{2!} - \frac{\mathrm{B}_2 x^4}{4!} + \frac{\mathrm{B}_3 x^6}{6!} - \ldots$$

and are easily computable, as we have $\mathrm{B}_1 = \frac{1}{6}$, $\mathrm{B}_2 = \frac{1}{30}$, $\mathrm{B}_3 = \frac{1}{42}$, $\mathrm{B}_4 = \frac{1}{30}$, $\mathrm{B}_5 = \frac{5}{66}$ and, in general, for $j \geq 1$:

$$\binom{2j+1}{2}2^2\mathbf{B}_1-\binom{2j+1}{4}2^4\mathbf{B}_2+\binom{2j+1}{6}2^6\mathbf{B}_3-\ldots+(-1)^{j-1}(2j+1)2^{2j}\mathbf{B}_j=2j.$$

(iii) For $1 \le k \le m-1$, $1 \le l \le k$, a fixed index-set $(i_1,\ldots,i_l) \in \mathbf{I}(l)$, and $(t_{i_1},\ldots,t_{i_l}) \in T(k;i_1,\ldots,i_l)$, $\xi^{(\epsilon)}_{i_1,\ldots,i_l} \in \Xi(i_1,\ldots,i_l)$, we define the function:

$$f_{t_{i_\epsilon}}(\xi^{(\epsilon)}_{i_1,\ldots,i_l}) := \sum_{v=0}^{t_{i_\epsilon}} \mathbf{G}_v \binom{t_{i_\epsilon}}{v} \left(\frac{e(i_1,\ldots,i_l)}{e^{(\epsilon)}(i_1,\ldots,i_l)}\right)^v (\xi_{i_1,\ldots,i_l})^{t_{i_\epsilon}-v}$$

where

$$\mathbf{G}_0 := 1,\ \mathbf{G}_1 := -\frac{1}{2},\ \mathbf{G}_v := \begin{cases} 0, & \text{if } v = 2\nu+1,\ \nu > 0 \\ (-1)^{\nu-1}\mathbf{B}_{2\nu-1}, & \text{if } v = 2\nu,\ \nu > 0 \end{cases}$$

Theorem 7.5. *(Formulae of Csorba (1914))*
The coefficients $\mathbf{c}_{m-1-k}(n;\mathbf{w})$ of $pt(n;\mathbf{w})$ in (7.5), for $k > 0$, are given by the following formulae:

(7.10)

$$\mathbf{c}_{m-1-k}(n;\mathbf{w}) =$$

$$\frac{(-1)^k}{(m-1-k)!\prod_{\nu=1}^m w_\nu} \sum_{l=1}^{k} \sum_{(i_1,\ldots,i_l)\in\mathbf{I}(l)} \mu(l;i_1,\ldots,i_l) \sum_{\xi^{(\epsilon)}_{i_1,\ldots,i_l}\in\Xi^{(\epsilon)}_{(i_1,\ldots,i_l)}} \cdots\cdots$$

$$\cdots\cdots \sum_{(t_{i_1},\ldots,t_{i_l})\in T(k;i_1,\ldots,i_l)} \prod_{\epsilon=1}^{l} \frac{w_{i_\epsilon}}{t_{i_\epsilon}!} f_{t_{i_\epsilon}}(\xi^{(\epsilon)}_{i_1,\ldots,i_l})$$

In the following years, Israilov [66] was eventually the only one who carried on the tradition of "Cayley-Sylvester era". Combining the expansion of $\mathrm{F}(x)$ into partial fractions with Möbius inversion law, he derived a "mammoth algorithmic formula" consisting of subroutine summations, which reduces the computation of $pt(n;\mathbf{w})$ to the determination of elements of certain PT's corresponding to the Eulerian case. To write it down in a "compact form", let us introduce some special extra notations.

(i) For $1 \le l \le m$ and an index-set $\{j_1,\ldots,j_l\} \subset \{1,\ldots,m\}$ let $e(j_1,\ldots,j_l)$ denote again

$$e(j_1,\ldots,j_l) := \gcd(w_i | 1 \le i \le m,\ i \notin \{j_1,\ldots,j_l\}).$$

For $1 \le i \le m$, $1 \le \nu \le m$, we define successively:

$$R_1^{(i)} := \{\{1, e(j_1)\} | 1 \le j_1 \le m,\ j_1 \ne i\}$$
$$R_2^{(i)} := R_1^{(i)} \cup \{e(j_1, j_2) | 1 \le j_1 < j_2 \le m,\ i \notin \{j_1, j_2\}\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$R_\nu^{(i)} := R_{\nu-1}^{(i)} \cup \{e(j_1, \ldots, j_\nu) | 1 \le j_1 < j_2 < \ldots < j_\nu \le m,\ i \notin \{j_1, \ldots, j_\nu\}\}$$

and $\mathbf{R}_\nu := \bigcup_{i=1}^{m} R_\nu^{(i)}$.

(ii) For $1 \le i \le m$, $\nu \in \mathbb{N}$, $k \in \mathbb{N}_0$, $\{j_1, \ldots, j_l\} \subset \{1, \ldots, m\}$, and $s(j_1, \ldots, j_l)$ a divisor of $e(j_1, \ldots, j_l)$, we define:

$$V(\nu; i; k; s(j_1, \ldots, j_l)) := \begin{cases} \frac{1}{k+1} w_i^{[k+1]}, & \text{if } i \notin \{j_1, \ldots, j_l\} \\ 1 - \zeta_{s(j_1, \ldots, j_l)}^{-\nu w_i}, & \text{if } k = 0 \text{ and } i \in \{j_1, \ldots, j_l\} \\ -w_i^{[k]} \zeta_{s(j_1, \ldots, j_l)}^{-\nu w_i}, & \text{if } k > 0 \text{ and } i \in \{j_1, \ldots, j_l\} \end{cases}$$

$$U(t; \nu; s(j_1, \ldots, j_l)) := \sum_{\substack{0 \le k_1, \ldots, k_m \le t \\ k_1 + \ldots + k_m = t}} \frac{t!}{k_1! \ldots k_m!} \left(\prod_{q=1}^{m} V(\nu; q; k_q; s(j_1, \ldots, j_l)) \right)$$

and for $1 \le p \le m - 1$,

$$\Gamma(p; \nu; s(j_1, \ldots, j_l)) :=$$
$$(-1)^{p-1} \sum \frac{(-1)^r r!}{r_1! \ldots r_{p-1}!} \left(\prod_{i=1}^{m} V(\nu; i; 0; s(j_1, \ldots, j_l)) \right)^{-1} \left(\prod_{t=1}^{p-1} \left(\frac{U(t; \nu; s(j_1, \ldots, j_l))}{t!} \right)^{r_t} \right),$$

where the summation runs over all $(r_1, \ldots, r_{p-1}) \in \mathrm{PT}(p-1; 1, 2, \ldots, p-1)$ and $r := r_1 + \ldots + r_p$.

(iii) Correspondingly, for $1 \le i \le m$, $1 \le t \le i - 1$, we set:

$$Y(t) := \sum_{\substack{0 \le \rho_1, \ldots, \rho_m \le t \\ \rho_1 + \ldots + \rho_m = t}} \frac{t!}{(\rho_1 + 1)! \ldots (\rho_m + 1)!} w_1^{[\rho_1 + 1]} \ldots w_m^{[\rho_m + 1]}, \quad \text{and}$$
$$A_i := (-1)^{i-1} \sum \frac{(-1)^r r!}{(w_1 \ldots w_m)^{r+1}} \left(\prod_{t=1}^{i-1} \frac{1}{r_t!} \left(\frac{Y(t)}{t!} \right)^{r_t} \right),$$

where the sum runs again over all $(r_1, \ldots, r_{i-1}) \in \mathrm{PT}(i-1; 1, 2, \ldots, i-1)$ and $r := r_1 + \ldots + r_{i-1}$.

Theorem 7.6. *(Formulae of Israilov (1981))*
(i) If the weights $w_1, \dots, w_m$ are pairwise coprime, then we have:

(7.11)

$$pt(n;\mathbf{w}) = \sum_{i=1}^{m} \left\{ A_i \frac{(n+m-i)^{[m-i]}}{(m-i)!} + \sum_{\mu=1}^{w_i-1} \zeta_{w_i}^{\mu n} w_i^{-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{m} (1-\zeta_{w_i}^{-\mu w_j}) \right)^{-1} \right\}$$

(ii) In the general case (in which we may assume that $\mathbf{w}$ *is reduced), the number of the weighted partitions of* n *is given by the formula:*

(7.12)

$$pt(n;\mathbf{w}) = \sum_{i=1}^{m} A_i \frac{(n+m-i)^{[m-i]}}{(m-i)!} +$$

$$+ \sum_{j=1}^{m} \sum_{\nu_j=1}^{e(j)-1} \sum_{p=1}^{m-1} \Gamma(p;\nu_j;e(j)) \frac{(n+m-p-1)^{[m-p-1]}}{(m-p-1)!} \zeta_{e(j)}^{\nu_j n} +$$

$$+ \sum_{l=2}^{m-2} \sum_{1\leq j_1<\dots<j_l\leq m} \cdots$$

$$\cdots \sum_{(1)} \sum_{(2)} \sum_{p=1}^{m-l} \Gamma(p;\nu;e(s_{j_1},\dots,s_{j_l})) \frac{(n+m-l-p)^{[m-l-p]}}{(m-l-p)!} \zeta_{s(j_1,\dots,j_l)}^{\nu n} +$$

$$+ \sum_{i=1}^{m} \sum_{(3)} \sum_{(4)} \zeta_{s(i)}^{\sigma_i n} w_i^{-1} \left(\prod_{\substack{\alpha=1\\ \alpha\neq i}}^{m} (1-\zeta_{s(i)}^{-\sigma_\alpha w_\alpha}) \right)^{-1}$$

By $\sum_{(1)}$ *we mean the sum running over all divisors* $(j_1, \dots, j_l)$ *of* $e(j_1, \dots, j_l)$ *with* $s(j_1, \dots, j_l) \nmid \mathbf{R}_{l-1}$. $\sum_{(2)}$ *denotes the sum of all* $1 \leq \nu \leq s(j_1, \dots, j_l)$, *for which* $\gcd(\nu, s(j_1, \dots, j_l)) = 1$. $\sum_{(3)}$ *denotes the sum of all divisors* $s(i)$ *of the weight* w_i *with* $s(i) \nmid R_{m-2}^{(i)}$. *Finally, by* $\sum_{(4)}$ *is meant the summation over all indices* $1 \leq \sigma_i \leq s(i)$, *for which* $\gcd(\sigma_i, s(i)) = 1$.

Remarks 7.7. (i) As the right hand sides of the formulae of theorems 7.4 and 7.6. contain trigonometric and transcendental functions in their periodic parts, the computation of the denumerants has to be made by using suitable approximation procedures. Sometimes it is enough to consider the "nearest integer function" or other standard inequalities, but in general the minimization of possible errors demands more sophisticated arithmetical methods. For some simple concrete examples see [20, pp. 109-115], [34], [38], [66, pp. 268-272], [111, p. 211] and [100,

pp. 117 -123].
(ii) For the Eulerian case, Gupta [57] gave the following denumerant bounds:

$$\frac{1}{m!}\binom{n-1}{m-1} \le pt(n;1,2,\dots,m) \le \frac{1}{m!}\binom{n+\frac{1}{2}m(m-1)}{m-1} \tag{7.13}$$

For the general case, Lambe [81] derived the upper bound:

(7.14)

$$pt(n;\mathbf{w}) \le \binom{n+u(m;\mathbf{w})}{m-1} \gcd(w_1,\dots,w_m) \prod_{i=1}^{m} \frac{1}{w_i},$$

$$u(m;\mathbf{w}) := m + \frac{w_1 w_2}{\gcd(w_1,w_2)} - 2 + \sum_{i=3}^{m} \left[\frac{w_i \cdot \gcd(w_1,\dots,w_{i-1})}{2\gcd(w_1,\dots,w_i)}\right],$$

where $[\cdot]$ denotes here the truncation to the nearest integer. (Note that $u(m;\mathbf{w})$ depends on the enumeration of the weights and that the above bound makes strange "jumps".)

Remark 7.8. Lee proved in [82] that, in fact, PT's are enough to describe completely the power series expansion of any rational function in one variable. For related topics see Stanley [111, Ch. 4].

Remark 7.9. Another reason which made the occupation with *pt*-functions very popular, not only among the mathematicians but also among the bank clerks and cashiers, was that these functions gave the answer to the *money changing problem.* (See Wilf [125, p. 87]). $pt(n;\mathbf{w})$ represents namely the number of the ways one can change an amount of money, say n, into coins or banknotes of denominations $w_1,\dots,w_m$. An indicative example is that one given by Luckey [84] in 1933, who defends the introduction of the "4-Pfennig" coin (100 Pfennig = 1 german Mark) by using the argument that, for instance, 30 Pfennig can be changed in $pt(30;1,2,4,5,10) = 285$ ways, if one makes use of the "4-Pfennig" coin, and in only $pt(30;1,2,5,10) = 98$ ways if not.

Let us now proceed to the interpretation (b) of *pt*'s, which was mentioned at the beginning of this section. By 1875 Weihrauch had already pointed out [123, pp. 99-100] that the enumeration of non-negative solutions of linear diophantine equations can be made "auf geometrischem Wege". During the period 1955-1975 Ehrhart ([34], [35], [36], [37], [38]) developed a whole theory dealing with polyhedral enumerators.

Let $\mathbf{P} \subset \mathbb{R}^m$ be a q-dimensional rational polyhedron. (By a polyhedron we mean, as before, a *bounded convex* polyhedron, i.e. a *convex polytope*, which can always be represented as the convex hull of finitely many points.) We define:

$$E(n;\mathbf{P}) := \sharp(\mathbb{Z}^m \cap n\mathbf{P}), \quad E^+(n;\mathbf{P}) := \sharp(\mathbb{Z}^m \cap \mathrm{int}(n\mathbf{P}))$$

and the *Ehrhart series*:

$$E_{\mathbf{P}}(x) := \sum_{n=0}^{\infty} E(n;\mathbf{P})x^n, \quad \text{with} \quad E(0;\mathbf{P}) = 1, \quad \text{and}$$

$$E^+_{\mathbf{P}}(x) := \sum_{n=0}^{\infty} E^+(n;\mathbf{P})x^n, \quad \text{with} \quad E(0;\mathbf{P}) = 0, \quad \text{respectively.}$$

Theorem 7.10. *(Ehrhart (1967)) If $q > 0$, then:*
(i) $E_{\mathbf{P}}(x)$ is a rational function and there is a quasipolynomial f of degree q with

$$E(n;\mathbf{P}) = f(n), \quad \textit{for all} \quad n \in \mathbb{N}_0.$$

(ii) It is $E^+(n;\mathbf{P}) = (-1)^q E(-n;\mathbf{P})$, for all $n \in \mathbb{N}$, where $E(-n;\mathbf{P}) := f(-n)$, and $E^+_{\mathbf{P}}(x) = (-1)^{q+1}E_{\mathbf{P}}(x^{-1})$.

Remarks 7.11. (i) f is called the *Ehrhart quasipolynomial* of $\mathbf{P}$. In particular, if $\mathbf{P}$ is an *integral* polyhedron (i.e. if all the coordinates of its vertices are integers), then f has *constant* (rational) coefficients and we call it the *Ehrhart polynomial* of $\mathbf{P}$.
(ii) In the notation of (7.1) and (7.2) we have obviously $q = m - 1$, $pt(n;\mathbf{w}) = E(n;\pi(\mathbf{w}))$ and $pt^+(n;\mathbf{w}) = E^+(n;\pi(\mathbf{w}))$, $\forall n \in \mathbb{N}_0$, while $\widetilde{pt}(n;w) = E(n;\pi(\mathbf{w}))$, $\forall n \in \mathbb{Z}$ (cf. (7.4)).

Ehrhart's work was extended to various directions by Macdonald [85], [86], Stanley (see [110], [111] and the other references given there), Frumkin [43] and Betke-McMullen [11]. They did not only consider quasipolynomials arising from arbitrary systems of linear diophantine equations, but they also made use of techniques which allow a precise study of the properties of general $E(n;\mathbf{P})$'s.
Especially Stanley connected $E(n,\mathbf{P})$'s with "magic labelings" of certain graphs and with a whole "corpus" of interesting invariants appearing in the abstract commutative algebra.

Even more recently, and parallel to algorithmic investigations of the counting of integer points in polyhedra, like those of Dyer [33], Cook et al. [21], Bánáry et al. [4] and Barvinok [6], combinatorialists and algebraic geometers attempted to find expressions for the coefficients a_j of the Ehrhart polynomials

$$(7.15) \qquad E(n;\mathbf{P}) = 1 + \mathrm{a}_1(\mathbf{P})n + \mathrm{a}_2(\mathbf{P})n^2 + \ldots + \mathrm{a}_{q-1}(\mathbf{P})n^{q-1} + \mathrm{a}_q(\mathbf{P})n^q$$

of a q-dimensional *integral* polyhedron $\mathbf{P}$ in terms of the geometry of $\mathbf{P}$ by means of the theory of toric varieties.

It is well-known (see [25, § 5.8]) that one can associate every integral polyhedron $\mathbf{P}$ in a q-dimensional lattice M with a complete toric variety $Z(N,\Sigma_{\mathbf{P}})$ (w.r.t. to its dual lattice N) by defining the corresponding fan $\Sigma_{\mathbf{P}}$ as follows: If F is a face of $\mathbf{P}$, let σ_F in M denote the cone consisting of all vectors $\lambda \cdot (x - x')$, where λ is a nonnegative rational number, $x \in \mathbf{P}$ and $x' \in F$. Then (in the notation of § 3) we set $\Sigma_{\mathbf{P}} := \{\check{\sigma}_F | F \quad \text{is a face of} \quad \mathbf{P}\}$.

Applying Hirzebruch's version of Riemann-Roch theorem [63, p. 155], [46, p. 288], to the line bundle $\mathcal{O}_{Z(N,\Sigma_{\mathbf{P}})}(D)$ of a T_N- Cartier divisor D being generated by its sections [47, p. 110], we get:

$$\chi(Z(N,\Sigma_{\mathbf{P}}),\ \mathcal{O}_{Z(N,\Sigma_{\mathbf{P}})}(nD)) = E(n;\mathbf{P}) \tag{7.16}$$

and consequently

$$\mathbf{a}_j(\mathbf{P}) = \frac{1}{j!}\deg(D^j \frown \mathrm{Td}_j(Z(N,\Sigma_{\mathbf{P}}))) \tag{7.17}$$

where $\mathrm{Td}(Z(N,\Sigma_{\mathbf{P}}))$ denotes the homology Todd class of $Z(N,\Sigma_{\mathbf{P}})$.

In fact, (7.17) is enough to show that $\mathbf{a}_j(\mathbf{P})$ is nothing but a linear combination of the volumes of the intersection of $\mathbf{P}$ with the corresponding translations of the subspaces which are perpendicular to the j-codimensional cones of $\Sigma_{\mathbf{P}}$ ([25, pp. 134-135], [47, pp. 112-113]). Therefore, what one needs, is a geometric characterization of the rational (not always uniquely defined) coefficients of this linear combination.

The last two coefficients of $E(n;\mathbf{P})$ are actually easily computable, because by the classical Pick's theorem [96] (1870) we get:
(i) $\mathbf{a}_q(\mathbf{P}) = \mathrm{Vol}(\mathbf{P})$,
(ii) $\mathbf{a}_{q-1}(\mathbf{P})$ equals half the sum of volumes of the $(q-1)$ -dimensional faces (By the volume of a j-dimensional face of $\mathbf{P}$ is meant the relative volume w.r.t. the j-dimensional lattice in the j-plane containing it.)

For $q \geq 3$, however, the description of $\mathbf{a}_1(\mathbf{P}), \ldots, \mathbf{a}_{q-2}(\mathbf{P})$ by means of the "local geometry" of $\Sigma_{\mathbf{P}}$ (resp. of $\mathbf{P}$) turned out to be much more complicated. (Even for $q = 3$, $\mathbf{a}_1(\mathbf{P})$ cannot be given by only using the 1-dimensional faces of $\mathbf{P}$.) The determination of these remaining coefficients of $E(n;\mathbf{P})$ became possible only after the proof of "finer" versions of *combinatorial Riemann-Roch theorem* and after further analysis of the corresponding Todd classes, due to Brion [14], Khovanskii-Pukhlikov [71] [72], Pommersheim [97], Kantor-Khovanskii [69], Morelli [89] and Capell-Shaneson [16]. For $q = 3$, Pommersheim derived a formula for $\mathbf{a}_1(\mathbf{P})$ in terms of the lattice volumes of 1- and 2-dimensional faces of $\mathbf{P}$ and of functions depending on certain Dedekind sums. He generalized in this way a beautiful formula due to Mordell [88] (1951). Kantor and Khovanskii discussed the 4-dimensional case. For another approach to the most general case, see Morelli [89, p. 208].

Completing this appendix, we shall recall the formulae of Capell and Shaneson [16] as they lead to concrete computations and they connect, in a certain sense, Sylvester's "waves" with the "RR-arithmetics". The latter was originally introduced by Hirzebruch in his "Grundlehren"-monograph [63, ch. I, § 1, and ch. II, § 9] and involved many useful properties of hyperbolic tangent and cotangent functions relating the Todd classes with the L-classes. Some extra notations will be again unavoidable.

Let $\mathbf{P}$ be a q-simplex with vertices in $M \cong \mathbb{Z}^q$ and N be the dual lattice of M. For each face R of $\mathbf{P}$ we set: $\mathcal{F}_R :=$ {faces of $\mathbf{P}$ of codimension one containing R} and $\mathcal{H}_R := \mathcal{F}_\varnothing \setminus \mathcal{F}_R$. For a simplicial q-dimensional cone σ in N, generated by

$n_1, \dots, n_q$, let $n'_1, \dots, n'_q$ be the unique primitive elements of N with $n'_i \cdot n_j = 0$ for $i \neq j$, and $\xi_i := n'_i \cdot n_i > 0$, $N' := \Sigma_{i=1}^q \mathbb{Z} n'_i$, $G_\sigma := N/N'$. Furthermore, for a $g = n_0 + N'$, let $\lambda_{n_j}(g)$ be the number

$$\lambda_{n_j}(g) := \exp(2\pi \sqrt{-1} \quad \gamma_{n_j}(g)),$$

where $\gamma_{n_j}(g) := \frac{n_0 \cdot n_j}{\xi_j}$, and

$$G^0_\sigma := \{g \in G_\sigma | \lambda_{n_j}(g) \neq 1, \forall j, 1 \leq j \leq q\}$$

(G^0_σ consists of the elements of G_σ having the form $n_0 + N'$ with n_0 lying in the interior of the cone spanned by $n'_1, \dots, n'_q$.)

Keeping now these notations, as well as their "relative" analogues for all the cones of $\Sigma_{\mathbf{P}}$, in mind, we have:

Theorem 7.12. *(Formulae of Capell and Shaneson (1994)).*
Let $\mathbf{P}$ be a q-simplex with vertices in the lattice M. For $0 \leq j \leq q$ let $\mathbf{r}_j$ denote the coefficient of x^j in the power series

$$\sum_{R \prec \mathbf{P}} \frac{1}{|G_{\breve{\sigma}_R}|} \left\{ \sum_{F \prec R} \mathbf{y}(F) \prod_{H \in \mathcal{H}_F} \frac{(\mathrm{Vol}(H)x)}{\tanh(\mathrm{Vol}(H)x)} \right\} \sum_{g \in G^0_{\breve{\sigma}_R}} \mathbf{t}(R; g),$$

where $\mathbf{y}(F) := |G_{\breve{\sigma}_F}| \cdot \prod_{h \in \mathcal{F}_F} (\mathrm{Vol}(H)x)$ and

$$\mathbf{t}(R; g) := \prod_{H \in \mathcal{F}_R} \coth\{\pi \sqrt{-1} \quad \gamma^R_H(g) + \mathrm{Vol}(H)x\}.$$

(γ^R_H's are "measured" again via the sublattices corresponding to H's.) Furthermore, for any $R \prec \mathbf{P}$ with $\dim(R) = j$, let

$$\mathbf{s}_j := \frac{\mathrm{Vol}(R)}{2^{q-j} |G_{\breve{\sigma}_R}| \cdot \prod_{H \in \mathcal{F}_R} \mathrm{Vol}(H)}.$$

Then the Ehrhart polynomial of $\mathbf{P}$ is given by

$$E(n; \mathbf{P}) = \sum_{j=0}^{q} \mathbf{r}_j \mathbf{s}_j n^j. \tag{7.18}$$

Applications 7.13. The formulae of Capell and Shaneson can be applied for special n's in our specific cases. Let $\mathbf{w} := (w_1, \dots, w_m)$ be a system of weights.
(i) If $\mathrm{lcm}(w_1, \dots, w_m) | n$, i.e. if $n = k \cdot \mathrm{lcm}(w_1, \dots, w_m)$ for some $k \in \mathbb{N}$, then

$$pt(n; \mathbf{w}) = E(k; \mathbf{P}'(\mathbf{w})), \tag{7.19}$$

where $\mathbf{P}'(\mathbf{w}) := \{(x_1, \dots, x_m) \in \mathbb{R}^m | \sum_{i=1}^m w_i x_i = \mathrm{lcm}(w_1, \dots, w_m)\}$ with vertices $(\frac{1}{w_1}\mathrm{lcm}(w_1, \dots, w_m), 0, \dots, 0), \dots, (0, \dots, 0, \frac{1}{w_m}\mathrm{lcm}(w_1, \dots, w_m))$.

(ii) For an $n \in \mathbb{N}$, let $\tilde{n} := n + \sum_{i=1}^{m} w_i$. If $\mathrm{lcm}(w_1, \dots, w_m) | \tilde{n}$, i.e. if $\tilde{n} = \lambda \cdot \mathrm{lcm}(w_1, \dots, w_m)$ for some $\lambda \in \mathbb{N}$, and if for a weight, say w_m, we have $w_m = 1$, then:

(7.20)
$$pt(n; \mathbf{w}) = pt^+(\tilde{n}; \mathbf{w}) = E^+(\tilde{n}; \Pi(w_1, \dots, w_{m-1}, 1)) = E^+(\lambda; \tilde{\mathbf{P}}(w_1, \dots, w_{m-1})),$$

where $\tilde{\mathbf{P}}(w_1, \dots, w_{m-1})$ denotes the convex hull of the points $(0, \dots, 0)$, $(\frac{1}{w_1}\mathrm{lcm}(w_1, \dots, w_{m-1}), 0, \dots, 0), \dots, (0, \dots, 0, \frac{1}{w_{m-1}}\mathrm{lcm}(w_1, \dots, w_{m-1}))$. Thus, $pt(n; \mathbf{w})$ can be found by the reciprocity law of theorem 7.10. (ii) and the formula (7.18) of Capell and Shaneson.
(iii) If $X = X_d \subset \mathbb{P}^{m-1}(\mathbf{w})$ is a well-formed *BP-like* (cf. 2.16.) quasismooth hypersurface, then (2.20) gives:

$$h^0(X, \mathcal{O}_X(\mathrm{am}(X))) = pt(d - \sum_{i=1}^{m} w_i; \mathbf{w}) = pt^+(d; \mathbf{w}) \tag{7.21}$$

If one of the weights happens to be 1, (7.21) can be computed by (7.20).
(iv) In the special case in which $m = 4$, $w_4 = 1$, $\gcd(w_1, w_2, w_3) = 1$, $\tilde{n} := n + w_1 + w_2 + w_3$ and $\tilde{n} = \lambda \cdot \mathrm{lcm}(w_1, w_2, w_3)$ for some $\lambda \in \mathbb{N}$, one obtains (7.20) via Pommersheim's formula for the tetrahedron $\tilde{\mathbf{P}}(w_1, w_2, w_3)$ ([97, thm. 5, p. 17]) after having substracted the lattice points on its faces. The result is the following:

(7.22)
$$\begin{aligned} pt(n; w_1, w_2, w_3, 1) &= E^+(\lambda; \tilde{\mathbf{P}}(w_1, w_2, w_3)) = \\ &= \frac{1}{6}(w_1^2 w_2^2 w_3^2 T^3)\lambda^3 - \frac{1}{4}[w_1 w_2 w_3 (w_1 + w_2 + w_3 + 1) T^2]\lambda^2 + \\ &+ \Big\{ \frac{1}{12}(w_1^2 + w_2^2 + w_3^2 + 1)T + \frac{1}{4}[(w_1 w_2 + w_2 w_3 + w_3 w_1)T + w_1^* + w_2^* + w_3^*] \\ &- w_1^* \cdot \mathrm{DS}(w_1, \gcd(w_2, w_3)) - w_2^* \cdot \mathrm{DS}(w_2, \gcd(w_1, w_3)) \\ &- w_3^* \cdot \mathrm{DS}(w_3, \gcd(w_1, w_2)) \Big\} \lambda - 1 \end{aligned}$$

In (7.22) we use the abbreviations:

$$T := \left(\prod_{i<j} \gcd(w_i, w_j) \right)^{-1}, \quad w_i^* := w_i \left(\prod_{j \neq i} \gcd(w_i, w_j) \right)^{-1}, \quad \forall i, \quad 1 \leq i \leq 3.$$

Moreover, by $\mathrm{DS}(\mu, \nu)$ we denote the *Dedekind sum* of two coprime integers μ and ν being defined by

$$\mathrm{DS}(\mu, \nu) := \sum_{i=1}^{\nu-1} \left(\left(\frac{i}{\nu} \right) \right) \left(\left(\frac{\mu i}{\nu} \right) \right) \tag{7.23}$$

where for an $x \in \mathbb{Q}$:

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{cases}$$

and $\lfloor x \rfloor$ is the greatest integer $\leq x$.
(v) Formulae similar to (7.22), when $m = 5$, can be derived by the results of Kantor and Khovanskii [69].

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