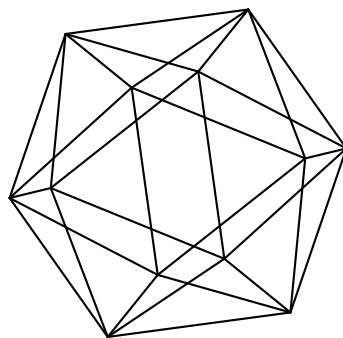# Max-Planck-Institut für Mathematik Bonn

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by

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EQUIVARIANT CONTROL DATA AND NEIGHBORHOOD DEFORMATION RETRACTIONS

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In memory of John Mather

ABSTRACT. In this article we study Whitney (B) regular stratified spaces with the action of a compact Lie group G which preserves the strata. We prove an equivariant submersion theorem and use it to show that such a G-stratified space carries a system of G-equivariant control data. As an application, we show that if $A \subset X$ is a closed G-stratified subspace which is a union of strata of X, then the inclusion $i : A \hookrightarrow X$ is a G-equivariant cofibration. In particular, this theorem applies whenever X is a G-invariant analytic subspace of an analytic G-manifold M and $A \hookrightarrow X$ is a closed G-invariant analytic subspace of X.

1. INTRODUCTION

Mather's concept of control data [10] has crystallized as an indispensible tool for the proof of Thom's first and second isotopy lemmata and more generally for the proof of the topological stability theorem which was originally conjectured by Thom [14] and finally proved by Mather [10] Moreover, control data are a powerful tool in stratified Morse theory [4], to prove triangulability of stratified spaces fulfilling Whitney's condition (B) [5], and to verify de Rham theorems in intersection homology theory [2]. A further topological application of the concept of control data is that it allows for a transparent proof that every submersed stratified subspace A of a (B) regular stratified space X is a neighborhood deformation retract (NDR) or equivalently that $i : A \hookrightarrow X$ is a cofibration. The assumption that A is a closed submersed stratified subspace of X hereby means that A is a union of connected components of strata of X; see Appendix A.

In this article we extend the existence of control data and the latter result to the G-equivariant case, where G is a compact Lie group. More precisely, we show in Theorem 2.11 that if M is a smooth G-manifold and $X \subset M$ a (B) regular stratified subspace such that G leaves the strata invariant, then there exists a system of G-equivariant control data on X. We use this observation in Section 4 to prove that for every G-invariant closed submersed stratified subspace $A \subset X$ the inclusion $i : A \hookrightarrow X$ is a G-cofibration. More precisely, we prove the following which is the main result of our paper.

Theorem 1.1. *Let X be a G-invariant (B) regular stratified space in a G-manifold M and A a G-invariant closed submersed stratified subspace of X. The there exists a G-invariant open*

neighborhood U of A in X and a stratified G-equivariant strong deformation retraction $r : X \times I \to$
X onto A such that $U_s := r(U, s)$ for each $s \in I = [0, 1]$ satisfies

(a) U_s *is open in X for all $s \in [0, 1)$,*
(b) $\overline{U_t} \subset U_s$ *for all $0 \leqslant s < t \leqslant 1$, and*
(c) $U_s = \bigcup_{s<t\leqslant 1} U_t$ *and $\overline{U_s} = \bigcap_{0\leqslant t<s} U_t$ for all $s \in (0, 1)$.*

In particular $i : A \hookrightarrow X$ is a G-equivariant cofibration.

We then consider the situation where X and A are G-invariant analytic subspaces of an analytic G-manifold M with $A \subset X$ being a closed subspace. Using methods by Wall [16] we show in Theorem 3.1 that X possesses a G-invariant (B) regular stratification such that A is a union of strata. Hence our main result applies to a such a G-invariant analytic pair (X, A).

2. Control data compatible with a group action

Mather proved in [10] that every (B) regular stratified subspace of a smooth manifold carries a system of control data. In this section we extend his result to the G-equivariant case. To this end we first introduce G-equivariant versions of stratifications, tubular neighborhoods, their isomorphisms and diffeotopies. Afterwards we prove a G-equivariant submersion theorem. This will be used to derive uniqueness and existence results for equivariant tubular neighborhoods. These tools then entail the main result of this section.

2.1. **Equivariant versions of stratifications and tubular neighborhoods.**

Definition 2.1. Suppose that a compact Lie group G acts on the total space X of a stratified space $(X, \mathcal{S})$. The stratification $\mathcal{S}$ is called a *G-stratification* or *G-invariant* and $(X, \mathcal{S})$ a *G-stratified space* if for all $g \in G$ and $x \in X$ the set germs $g\mathcal{S}_x$ and $\mathcal{S}_{gx}$ coincide and if for each open neighborhood U with an $\mathcal{S}$-inducing decomposition $\mathcal{Z}$ the map from a piece $R \in \mathcal{Z}$ to gR given by the g-action is a diffeomorphism of smooth manifolds. We also say in this situation that the G-action on X is compatible with the stratification.

Example 2.2. The orbit type stratification of a G-manifold M is G-invariant since the group action leaves the orbit types of the points of M invariant.

Proposition 2.3. *Each stratum of a G-stratified space $(X, \mathcal{S})$ is preserved by the G-action. Moreover, G acts smoothly on the strata of X.*

Proof. Let x be a point of a stratum S of X. Choose a decomposition $\mathcal{Z}$ of an open neighborhood U of x inducing the stratification $\mathcal{S}$ over U. Then gU is an open neighborhood of gx, and $g\mathcal{Z}$ a decomposition of gU. Moreover, if $y \in U$ and R_y is the piece of $\mathcal{Z}$ through y, then gR_y is the piece of $g\mathcal{Z}$ through gy. Hence $g\mathcal{Z}$ induces the stratification $\mathcal{S}$ over gU. But that means that gx has the same depth as x and that the dimension of the piece in which gx lies has the same dimension as the piece of $\mathcal{Z}$ through x. So gx and x lie in the same stratum. So we have proved that G acts on each stratum of X. This action is smooth since it is smooth locally by definition. The claim is proved. $\qquad\square$

Definition 2.4. By *G-equivariant system of control data* on a stratified space $(X, \mathcal{S})$ with a compatible G-action we understand a family $\mathcal{T} = (T_S, \pi_S, \varrho_S)_{S \in \mathcal{S}}$ of triples called *tubes* consisting for each $S \in \mathcal{S}$ of an open neighborhood T_S of S, a continuous retraction $\pi_S : T_S \to S$ called *projection* and a continuous map $\varrho_S : S \to [0, \infty)$ called *tubular function* such that the following control conditions hold true:

(CC1) For any $S \in \mathcal{S}$ the neighborhood T_S is G-invariant, the projection π_S is G-equivariant, and the tubular function ϱ_S is G-invariant.

(CC2) For any $S \in \mathcal{S}$ the tubular function ϱ_S satisfies $S = \rho_S^{-1}(0)$.

(CC3) For any $R, S \in \mathcal{S}$ with $R < S$, the map

$$(\pi_{R,S}, \rho_{R,S}) : T_{R,S} \to R \times (0, \infty)$$

is a smooth submersion, where $T_{R,S} := T_R \cap S$, $\pi_{R,S} := \pi_R|_{T_{R,S}}$ and $\rho_{R,S} = \rho_R|_{T_{R,S}}$.

(CC4) For any $Q, R, S \in \mathcal{S}$ with $Q < R < S$

$$\pi_{Q,R} \circ \pi_{R,S}|_{T_{Q,R,S}} = \pi_{Q,S}|_{T_{Q,R,S}} \quad \text{and} \quad \rho_{Q,R} \circ \pi_{R,S}|_{T_{Q,R,S}} = \rho_{Q,S}|_{T_{Q,R,S}},$$

where $T_{Q,R,S} = \pi_{R,S}^{-1}(T_{Q,R})$.

If G is the trivial group one recovers the original definition of a system of control data by Mather [10]. A stratified space $(X, \mathcal{S})$ together with some control data $\mathcal{T}$ will be called a *Thom-Mather stratified space*. If $(X, \mathcal{S})$ carries a compatible G-action and the control data are G-equivariant in the sense defined above, we call $(X, \mathcal{S}, \mathcal{T})$ a *Thom-Mather stratified space* with a *compatible G-action* or briefly a *Thom-Mather G-stratified space*.

We will now introduce some further language where G always denotes a compact Lie group and G a smooth G-manifold. Let $S \subset M$ be a G-invariant smooth manifold. By a *G-equivariant tubular neighborhood* of S in M we understand a triple $\mathsf{T} = (E, \varepsilon, \varphi)$ where $\pi_E : E \to S$ is a G-vector bundle over S carrying a G-invariant inner product $\langle -, - \rangle : S \to E \otimes E$, $\varepsilon : S \to (0, \infty)$ is a G-invariant smooth map, and φ is a G-equivariant diffeomorphism from $B(\varepsilon, E) := \{v \in E \mid \langle v, v \rangle < \varepsilon(\pi_E(v))\}$ onto an open neighborhood T_S of S such that the composition of φ with the zero section of E coincides with the identical embedding of S into M. Note that by the requirements T_S is a G-invariant open neighborhood of S in M.

Following Mather [10] we define the *projection* $\pi_S : T_S \to S$ as the composition $\pi_E \circ \varphi^{-1}$ and the *tubular function* $\varrho_S : T_S \to [0, \infty)$ as the function which maps a point $x \in T_S$ to $\langle \varphi^{-1}(x), \varphi^{-1}(x) \rangle$.

If N is a a second G-manifold and $f : M \to N$ a G-equivariant smooth map, then a tubular neighborhood $\mathsf{T} = (E, \varepsilon, \varphi)$ is said to be *compatible* with f if $f \circ \pi_S = f|_{T_S}$.

Example 2.5. Let η be a G-invariant riemannian metric on M and $\pi^N : N \to S$ the normal bundle of S in M. Identify N with the orthogonal complement of TS in T_SM via η. Then there exists an open neighborhood U of the zero section of N such that the exponential function $\exp : U \to M$ is a G-equivariant diffeomeorphism onto an open neighborhood of S in M. Since G is compact there exists a G-invariant continuous function ε such $B(\varepsilon, N) := \{v \in E \mid \langle v, v \rangle < \varepsilon(\pi_N(v))\} \subset U$ The inner product $\langle -, - \rangle$ on N hereby is the one induced by the riemannian metric η. The triple $\mathsf{T} = (N, \varepsilon, \exp|_{B(\varepsilon, N)})$ now is a G-equivariant tubular neighborhood of S in M.

Given two tubular neighborhoods $\mathsf{T} = (E, \varepsilon, \varphi)$ and $\mathsf{T}' = (E', \varepsilon', \varphi')$ an *isomorphism* between T cand T' consists of an isometric vector bundle isomorphism $\psi : E \to E'$ and a continuous map $\delta : S \to (0, \infty)$ such that $\delta \leqslant \min(\varepsilon, \varepsilon')$ and such that $\varphi' \circ \psi|_{B(\delta, E)} = \varphi|_{B(\delta, E)}$. We denote such an isomorphism briefly by $\psi : \mathsf{T} \sim \mathsf{T}'$.

In addition to isomorphisms of equivariant tubular neighborhoods there is a corresponding equivariant version of diffeotopies on a G-manifold M. By a G-equivariant diffeotopy on M we understand a smooth map $H : M \times I \to M$, where I denotes the interval $[0, 1]$, such that each of the maps $H_t : M \to M$, $x \mapsto H(x, t)$ with $t \in I$ is G-equivariant diffeomorphism and such that $H_0 = \mathrm{id}_M$. If in addition a G-equivariant map $f : M \to N$ is given, then a G-equivariant diffeotopy $H : M \times I \to M$ is said to be *compatible* with f if $f(H(x, t)) = f(x)$ for all $x \in M$ and $t \in I$. For every diffeotopy $H : M \times I \to M$ one calls the set $\mathrm{supp}\, H := \overline{\{x \in M \mid \exists t \in I : H(x, t) \neq x\}}$ the *support* of H.

If $h : (M, S) \to (M', S')$ is a G-equivariant diffeomorphism between pairs of G-manifolds and G-submanifolds one defines the push-forward tubular neighborhood $h_* \mathsf{T}$ of S' in M' of a tubular neighborhood $\mathsf{T} = (E, \varepsilon, \varphi)$ of S in M by $h_* \mathsf{T} = \big((h^{-1})^* E, \varepsilon \circ h^{-1}, h \circ \varphi\big)$.

2.2. **The Equivariant Submersion Theorem.** Locally in charts, every submersion looks like a linear projection. The following equivariant versions of this result appear to be folklore.

Proposition 2.6. *Let $f : M \to P$ be a G-equivariant submersion between G-manifolds M and P. Let $x \in M$ be a point, $H = G_x$ be the isotropy group of x and $K = G_{f(x)} \supset H$ the one of $f(x)$. Then there exist a finite dimensional orthogonal K-representation space N, a finite dimensional orthogonal H-representation space W, a K-invariant open convex neighborhood of the origin $B \subset N$, an H-invariant open convex neighborhood of the origin $C \subset W$, G-equivariant open embeddings $\Theta : G \times_H (B \times C) \hookrightarrow M$ and $\Psi : G \times_K B \hookrightarrow P$ such that $\Theta\big([e, 0]_{G \times_H (B \times C)}\big) = x$, $\Psi\big([e, 0]_{G \times_K B}\big) = f(x)$ and such that the following diagram commutes, where $\pi : B \times C \to C$ is projection onto the first factor and $\overline{\mathrm{id}_G \times \pi}$ maps $[g, (v, w)]_{G \times_H (B \times C)}$ to $[g, v]_{G \times_K B}$.*

(2.1)
$$\begin{array}{ccc} G \times_H (B \times C) & \xrightarrow{\overline{\mathrm{id}_G \times \pi}} & G \times_K B \\ {\scriptstyle \Theta}\big\downarrow & & \big\downarrow{\scriptstyle \Psi} \\ M & \xrightarrow{\quad f \quad} & P \end{array}$$

Moreover, the dimensions of the manifolds and representation spaces fulfill the relations $\dim P = \dim G - \dim K + \dim N$ *and* $\dim M = \dim G - \dim H + \dim N + \dim W$.

Proof. First choose a G-riemannian metric ϱ on P, identify for every $p \in P$ the normal space $N_p := T_p P / T_p G p$ with the orthogonal complement of $T_p G p$ in $T_p P$, put $N := N_{f(x)}$ and let $B \subset N$ be an open ball around the origin of radius smaller than the injectivity radius of the exponential function $\exp^\varrho$ (with respect to ϱ) over the orbit $G f(x)$. Then, N is an orthogonal K-representation space and, by the Slice Theorem [7, p. 139], [11], & [3, Sec. II.4], the subset $Z = \exp^\varrho(B) \subset P$ is a K-invariant submanifold and the map

$$\Psi : G \times_K B \to P, \; [g,v]_{G \times_K D} \mapsto g \exp^\varrho v$$

a G-equivariant diffeomorphism onto an open neighborhood of $G f(x)$ in P. Following [3, Sec. II.4], we call such a $Z \subset P$ a *slice* through $f(x)$. Next choose a G-equivariant bundle $\mathcal{H} \to M$ complementary to the vertical bundle $\mathcal{V} = \ker Tf \to M$ in TM and call it the horizontal bundle. For each $v \in B$ let $\widetilde{\gamma}_v : [0,1] \to M$ be the horizontal lift of the geodesic $\gamma_v : [0,1] \to P$, $t \mapsto \exp^\varrho(tv)$ such that $\widetilde{\gamma}_v(0) = x$. Let $\widetilde{Z} = \{\widetilde{\gamma}_v(1) \mid v \in B\}$. Then $\widetilde{Z}$ is a submanifold of M with a chart given by the inverse of $B \to \widetilde{Z}$, $v \mapsto \widetilde{\gamma}_v(1)$. Moreover, $\widetilde{Z}$ is H-invariant since the action by $h \in H$ maps the geodesic γ_v for $v \in B$ to the geodesic γ_{hv} and the lift of γ_v to the lift of γ_{hv}. After possibly shrinking B and with it Z and $\widetilde{Z}$ the map $G \times_H \widetilde{Z} \to M$ becomes a G-equivariant embedding and the composition $G \times_H \widetilde{Z} \to M \xrightarrow{f} P$ a submersion. Now consider the fiber $F \subset M$ of f through the point x. Then $F \subset M$ is a submanifold which is invariant under the action of K. Choose a K-invariant riemannian metric η on F and let $W := T_x F / T_x K x$ be the normal space at x and identify it with the orthogonal complement of $T_x K x$ in $T_x F$. Observe that W is an orthogonal H-representation space since $H = K_x$. For a sufficiently small open ball $C \subset W$ around the origin the set $Y := \exp^\eta(C)$ is a slice of the K-manifold F through x. Now let $\widetilde{\gamma}_{y,v}$ for each $y \in Y$ and $v \in B$ be the horizontal lift of γ_v such that $\widetilde{\gamma}_{y,v}(0) = y$. Then, after possibly shrinking B and C (together with the corresponding slices) the map $\theta : B \times C \hookrightarrow M$, $(v,w) \mapsto \widetilde{\gamma}_{\exp^\eta w, v}(1)$ is an embedding since $T_0 \theta$ is the linear injection $N \times W \hookrightarrow \mathcal{H}_x \times \mathcal{V}_x \cong T_x M$ which maps (v,w) to the unique pair $(v^{\mathrm{h}}, w) \in \mathcal{H}_x \times \mathcal{V}_x$ such that $T_x(v^{\mathrm{h}}) = v$. Moreover, the embedding θ is H-equivariant since the horizontal lift is H-equivariant. Hence the set $X = \{\widetilde{\gamma}_{y,v}(1) \mid v \in B, \, y \in Y\}$ is an H-invariant submanifold of M transversal to the orbit $G x$.

Consider now the G-manifold $G \times_H (B \times C)$ and define

$$\Theta : G \times_H (B \times C) \to M, \; [g, (v,w)]_{G \times_H (B \times C)} \mapsto g \theta(v,w) \, .$$

This map is well-defined by equivariance of θ. The restriction of Θ to the zero section of $G \times_H (B \times C)$, which is canonically diffeomorphic to G/H, is a diffeomeorphism onto to the orbit Gx since $H = G_x$. Moreover, the image of $\{e\} \times (B \times C)$ under Θ is the manifold X, so the image of Θ equals GX. Now recall that the orbit Gx and X are transverse. By G-equivariance of Θ it follows that $T_{gx}\Theta$ is an isomorphism for all $g \in G$. After possibly shrinking B and C the map Θ therefore is a diffeomorphism onto an open neighborhood of the orbit Gx by the Implicit Function Theorem. In other words, X is a slice of M at x.

Finally let $\psi : B \to P$ be the embedding $v \mapsto \exp^\varrho v$. The image of ψ then is the slice Z, and $\psi^{-1} \circ f \circ \theta = \pi$ by definition of θ via horizontal lifts. This entails that the diagram commutes and the proposition is proved. The dimension relation follows from the definition of N and W and the Slice Theorem. $\qquad\square$

Corollary 2.7. *Let $f : M \to P$ be a G-equivariant map between G-manifolds M and P, and let $S \subset M$ be a G-invariant submanifold such that the restriction $f|_S : S \to P$ is a submersion. Let $x \in S$ be a point, $H = G_x$ be the isotropy group of x and $K = G_{f(x)} \supset H$ the one of $f(x)$. Then there exist an orthogonal K-representation space N, orthogonal H-representation spaces W and W', a K-invariant open convex neighborhood of the origin $B \subset N$, H-invariant open convex neighborhoods of the origin $C \subset W$ and $D \subset W'$, and finally G-equivariant open embeddings $\Phi : G \times_H (B \times C \times D) \hookrightarrow M$ and $\Psi : G \times_K B \hookrightarrow P$ such that $\Phi\big([e, 0]_{G \times_H (B \times C \times D)}\big) = x$, $\Phi\big([e, 0]_{G \times_K B}\big) = f(x)$ and such that the following two properties hold true.*

(1) *The map*
$$\Theta : G \times_K (B \times C) \to M, \ [g, (v, w)]_{G \times_K (B \times C)} \to \Psi\big([v, w, 0]_{G \times_K (B \times C \times D)}\big)$$

has image in S and comprises a G-equivariant diffeomorphism onto an open neighborhood of the orbit Gx in S.

(2) *With $\Pi : B \times C \times D \to B$ denoting projection onto the first factor the diagram below commutes.*

(2.2)
$$
\begin{array}{ccc}
G \times_H (B \times C \times D) & \xrightarrow{\overline{\mathrm{id}_G \times \Pi}} & G \times_K B \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Psi} \\
M & \xrightarrow{\quad f \quad} & P
\end{array}
$$

Proof. Choose a G-invariant riemannian metric ϱ on P and a G-equivariant horizontal bundle $\mathcal{H} \to P$ such that the restriction $\mathcal{H}|_S$ is a G-equivariant subbundle of TS. Let $N = T_{f(x)}P/T_{f(x)}Gf(x)$ and choose $B \subset N$ as in the proof of the proposition. Then $Z = \exp^\varrho B$ is a slice and $\Psi : G \times_K B \to B$ defined as above a G-equivariant diffeomorphism. Now let $F \subset M$ be the fiber of the submersion f through x. Then F is a K-invariant submanifold of M and $F \cap S$ a K-invariant submanifold of S. Choose a K-invariant riemannian metric η on F. Put $W = T_x(F \cap S)/T_xKx$ and identify it with the orthogonal complement of T_xKx in $T_x(F \cap S)$. Let W' be the orthogonal complement of $T_x(K \cap S)$ in T_xF and choose small enough open convex neighborhoods of the origin $C \subset W$ and

$D \subset W'$. Then $Y = \exp^\eta(C \times D)$ is a slice through x of the K-manifold F. Let γ_v for $v \in B$ the geodesics as in the proof of the proposition and denote for $y \in Y$ and $v \in B$ by $\tilde{\gamma}_{y,v}$ the horizontal lift of γ_v in M which starts at y. Now define $\varphi : B \times C \times D \to M$ by $(v, w, z) \mapsto \tilde{\gamma}_{\exp^\eta(w,z),v}$. Then

$$\Phi : G \times_H (B \times C \times D) \hookrightarrow M, \quad [g, (v, w, z)]_{G \times_H (B \times C \times D)} \mapsto g\varphi(v, w, z)$$

is a G-invariant diffeomorphism onto an open neighborhood of Gx in M and the diagram (2.2) commutes by the proof of the proposition. By assumptions on the horizontal bundle $\mathcal{H}$ and the construction of Θ property (1) holds true. $\qquad\square$

2.3. Uniqueness and existence of equivariant tubular neighborhoods.

For the construction of G-equivariant control data one needs stronger versions of existence and uniqueness results of G-equivariant tubular neighborhoods. In the following we prove equivariant versions of [10, Prop. 6.1] and [10, Prop. 6.2].

Theorem 2.8 (Uniqueness of equivariant tubular neighborhoods)**.** *Let M, P be smooth G-manifolds, $S \subset M$ a closed G-invariant smooth submanifold, and $f : M \to P$ a G-equivariant smooth map such that the restriction $f|_S : S \to P$ is a submersion. Assume that T_0 and T_1 are two G-equivariant tubular neighborhoods of S in M and that they are compatible with f. Further assume that $U \subset S$ is a G-invariant relatively open subset and that $\psi_0 : \mathsf{T}_0|_U \to \mathsf{T}_1|_U$ is an isomorphism of G-equivariant tubular neighborhhoods over U. Let $A, Z \subset S$ be two G-invariant relatively closed subsets such that $A \subset U$ and let $V \subset M$ be a G-invariant open neighborhood of Z in M. Then there exists a G-equivariant diffeotopy $H : M \times I \to M$ which leaves S invariant, is compatible with f and has support in $V \backslash A$ such that the tubular neighborhoods $(H_1)_*\big(\mathsf{T}_0|_{A \cup Z}\big)$ and $\mathsf{T}_1|_{A \cup Z}$ are isomorphic. If $O \subset M \times M$ is a G-invariant neighborhood of the diagonal, one can choose H such that $(H_t(x), x) \in O$ for all $t \in I$ and $x \in M$. Finally, the isomorphism $\psi : (H_1)_*\big(\mathsf{T}_0|_{A \cup Z}\big) \to \mathsf{T}_1|_{A \cup Z}$ is G-equivariant and can be constructed so that $\psi|_A = \psi_0|_A$.*

Proof. Our proof adapts Mather's argument in [10, Proof of Prop. 6.1] to the G-equivariant case.

Step 1. We first consider the local G-equivariant case as stated in Corollary 2.7. So we assume for now the following:

(1) P is of the form $G \times_{G_{f(s)}} B$, where $G_{f(s)} \subset G$ is the isotropy group of some point $f(s)$ with $s \in S$ and B is an open convex neighborhood of the origin of some euclidean space $\mathbb{R}^k$ carrying an orthogonal $G_{f(s)}$-representation. The point $f(s)$ is then identified with $[e, 0] \in G \times_{G_{f(s)}} B$.

(2) S is equivariantly diffeomorphic to an associated bundle of the form $G \times_{G_s} (B \times C)$, where $G_s \subset G$ is the isotropy group of $s \in S$ and C is an open convex neighborhood of the origin of some euclidean space $\mathbb{R}^l$ carrying an orthogonal G_s-representation. Under the corresponding diffeomorphism the point s can be identified with $[e, 0] \in G \times_{G_s} (B \times C)$. Note that $G_s \subset G_{f(s)}$.

(3) M is equivariantly diffeomorphic to an associated bundle of the form $G \times_{G_s} (B \times C \times D)$, where D is an open convex neighborhood of the origin of a euclidean space $\mathbb{R}^m$ with an orthogonal G_s-representation and where the G_s-action on $B \times C \times D$ is the diagonal action.

(4) Under these identifications $f : M \to P$ coincides with the G-equivariant map $G \times_{G_s} (B \times C \times D) \to G \times_{G_{f(s)}} B$ which maps $[g, (v, w, z)]$ to $[g, v]$. So for every $x = [g, (v, w, z)] \in M$ the fiber through x in M coincides with $F_x := \left[\{g\} \times \{v\} \times C \times D \right]$, the image of $\{g\} \times \{v\} \times C \times D$ in $G \times_{G_s} (B \times C \times D)$.

In addition to this we also assume for the moment that Z is compact.

Since G is compact, there exists a bi-invariant riemannian metric μ on G. The spaces B, C, D all carry natural invariant metrics induced by the ambient euclidean spaces. Denote by η and ϱ the induced G-invariant riemannian metrics on M and P, respectively. With these metrics, f then becomes a riemannian submersion. Actually, the fibers of this riemannian submersion are even totally geodesic by construction of η. Now assume that x and y are points of M which are both in the same fiber F_x. Then $x = [g, (v, w, z)]$ and $y = [g, (v, w', z')]$ for some $g \in G$, $v \in B$, $w, w' \in C$ and $z, z' \in D$. The unique geodesic connecting x with y then is given by

$$\gamma_{x,y}(t) = [g, (v, (1 - t)w + tw', (1 - t)z + tz')] \quad \text{for all } t \in I, \text{ where } I = [0, 1].$$

Note for later that $\gamma_{x,y}$ completely runs within the fiber F_x.

Denote by $\pi_N : N \to S$ the normal bundle of S in M that is $N_x = T_x M / T_x S \cong T_0 D \cong \mathbb{R}^m$ for all $x \in S$. Via the riemannian metric η one can identify N with the subbundle of $T_S M$ orthogonal to TS. By assumptions and construction of the riemannian metric η one has $N \subset \ker T_S f$.

Now observe that for $i = 0, 1$ the map

$$\alpha_i : E_i \to N, \quad v \mapsto T\varphi_i(v) + T_{\pi_{E_i}(v)} S$$

is a vector bundle isomorphism. Hereby we have identified E_i with the vertical subbundle of TE_i restricted to the zero section. By assumptions α_i is an isomorphism of G-bundles, hence $\alpha := \alpha_1^{-1} \circ \alpha_0 : E_0 \to E_1$ is one, too. Note that for $x \in U$, $\alpha_x : E_{0,x} \to E_{1,x}$ coincides with $\psi_{0,x} : E_{0,x} \to E_{1,x}$. By uniqueness of the polar decomposition there exists a unique G-equivariant vector bundle automorphism $\beta : E_1 \to E_1$ such that for every $x \in S$ the linear map $\beta_x : E_{1,x} \to E_{1,x}$ is positive definite and $\psi_x := \beta_x \circ \alpha_x : E_{0,x} \to E_{1,x}$ an orthogonal transformation. Then

$$\xi_t := (1 - t)\alpha + t\psi : E_0 \to E_1$$

is an isomorphism for every $t \in I$ which over U coincides with ψ_0. After possibly lessening ε_1 and ε_0, where both stay G-invariant and positive, the set $T := T_{0,S} \cap T_{1,S}$ is a G-invariant open neighborhood of S in M over which the maps

$$q_t : T \to M, \ x \mapsto \varphi_1 \circ \xi_t \circ \varphi_0^{-1}(x)$$

are well-defined and open G-equivariant embeddings for every $t \in I$. Note that over S each q_t is the identical embedding and that each q_t acts as identity over some open G-invariant neighborhood $U' \subset T$ of A. Moreover, each q_t is compatible with f since both T_0 and T_1 are compatible with f. Put $V_1 = T \cap V$ and observe that $Z \subset V_1$. By compactness of Z there exists a G-invariant open neighborhood V_2 of Z which is relatively compact in V_1 and which satisfies $V_2 \subset q_t(V_1)$ for all $t \in I$.

Next choose a smooth G-invariant function $\chi : M \to I$ with compact support in V_2 such that χ is identically 1 over a G-invariant neighborhood of Z in V_2. Define $Q_{s,t} : M \to M$ for $s, t \in I$ by

$$Q_{s,t}(x) = \begin{cases} \gamma_{x, q_t \circ q_s^{-1}(x)}\big(\chi(x)\big) & \text{if } x \in V_2, \\ x & \text{if } x \in M \backslash V_2, \end{cases}$$

Since the q_t are compatible with f, the geodesic $\gamma_{x, q_t \circ q_s^{-1}(x)}$ is well-defined for every $x \in V_2$. Hence the $Q_{s,t}$ are well-defined as well and also compatible with f. Next observe that by construction $Q_{t,t}$ is the identity map for all $t \in I$ and that there is a compact G-invariant subset containing the support of $Q_{s,t}$ for all $s, t \in I$. Hence there is some $\delta > 0$ such that $Q_{s,t}$ is a diffeomorphism for all s, t with $|s - t| < \delta$. From here on we can follow Mather's treatment of the local case in [10, Prop. 6.1] almost literally. Choose a positive integer n such that $\frac{1}{n} < \delta$. Put

$$\widetilde{H}_t = Q_{0, \frac{t}{n}} \circ Q_{\frac{t}{n}, \frac{2t}{n}} \circ \ldots \circ Q_{\frac{(n-1)t}{n}, t} .$$

Then $\widetilde{H}$ is a G-equivariant diffeotopy, compatible with f, and leaves S fixed. Since the q_t acts as identity over U', $\widetilde{H}_t$ does so, too. Moreover, $\widetilde{H}$ coincides by construction with $q_1 \circ q_0^{-1}$ over some sufficiently small G-invariant open neighborhood of Z in V_2. Hence $\widetilde{H}$ coincides with $q_1 \circ q_0^{-1}$ over $U' \cup V'$. Furthermore $\widetilde{H} \circ q_0 \circ \varphi_0 = q_1 \circ \varphi_0 = \varphi_1 \circ \psi$ over the G-invariant neighborhood $\varphi_0^{-1}(U' \cup V')$ of $A \cup Z$ in E_0. Therefore, ψ is an isomorphism between $(\widetilde{H}_1 q_0)_* \mathsf{T}_0|_{A \cup Z}$ and $\mathsf{T}_1|_{A \cup Z}$. Moreover, the support of $\widetilde{H}$ is contained in $V_2 \subset V$ by construction. Finally, by requiring that the cut-off function χ has support in a sufficiently small G-invariant open neighborhood of Z one can achieve that with regard to the compact-open topology $\widetilde{H}_t$ is uniformly in $t \in I$ as close to the identity map as one wishes.

By the following step there exists, after possibly shrinking U' and V', a G-equivariant diffeotopy $\widehat{H}$ of M which is compatible with f and leaves S invariant such that for all $t \in I$ the diffeomorphisms $\widehat{H}_t$ act as identity over U' and such that $\widehat{H}_t = q_0$ over V'. The map $H : M \times I \to M$, $(x, t) \mapsto \widetilde{H}_t \circ \widehat{H}_t(x)$ then is a G-equivariant diffeotopy with all the required properties.

Step 2. Here we show that there exists a G-equivariant diffeotopy $\widehat{H}$ of M with compact support which is compatible with f, leaves S invariant, acts as identity over a sufficiently small open G-invariant neighborhood U' of A and coincides over a sufficiently small open G-invariant neighborhood V' of Z with q_0. Note that for the non-equivariant case the existence of such a diffeotopy $\widehat{H}$ has been claimed in the proof of [10, Prop. 6.1] with the argument left to the reader. Since the equivariant case is more subtle, we present a proof here which obviously covers Mather's claim, too.

Observe that for all $x \in T$ the image $q_0(x)$ lies in the fiber F_x by construction. Hence the geodesic $\gamma_{x, q_0(x)}$ is well-defined and fully runs in F_x. Now put $K(x, t) = \gamma_{x, q_0(x)}(t)$ for all $x \in T$ and $t \in I$. After possibly shrinking T, $\widehat{K} : T \times I \to M \times I$, $(x, t) \mapsto \big(K(x, t), t\big)$ is an open embedding since q_0 acts as identity over S, one has $T_x q_0 = \text{id}_{T_x M}$ for all $x \in S$ and finally since I is compact. Moreover, K_t acts as identity for all $x \in U'$ where U' is a G-invariant open neighborhood of A in T over which q_0 acts as identity. Finally, K is compatible with f since q_0 is. Now define the time-dependent

vector field $X_K : T \times I \to TM$ by

$$X_K(x,t) = \left. \frac{\partial}{\partial s} \right|_{s=0} K(x, t+s) \ .$$

Note that over $U' \times I$ the vector field X_K vanishes and that X_K is G-equivariant. Next choose a sufficiently small relatively compact G-invariant open neighborhood V' of Z and a non-negative G-invariant smooth function $\delta : M \to [0,1]$ which is identical to 1 over V' and has compact support in T. Define the time-dependent vector field $X : M \times I \to TM$ by

$$(x,t) \mapsto \begin{cases} \delta(x) \, X_K(x,t) & \text{for } x \in T, \\ 0 & \text{for } x \in M \backslash T. \end{cases}$$

Then X is a time-dependent G-equivariant vector field on M with compact support. By [6, Chap. 8, Thm. 1.1] it generates a diffeotopy $\widehat{H} : M \times I \to M$. The diffeotopy is G-equivariant since X is, and is compatible with f since X is tangent to the fibers of f by construction. Since δ has compact support, $\widehat{H}$ has so too. Over V' the diffeotopy $\widehat{H}$ coincides with K, hence one obtains in particular that $\widehat{H}_1|_{V'} = q_0|_{V'}$. Over U', each $\widehat{H}_t$ acts as identity for every $t \in I$. This finishes Step 2.

Step 3. Let us pass to the general case, now. Here we follow closely [10, Prop. 6.1]. By the Equivariant Submersion Theorem and Corollary 2.7 there exists for every $x \in S$ an open relatively compact G-invariant open neighborhood W_x of x in M together with G-equivariant open embeddings called *equivariant charts* $\Phi_x : W_x \hookrightarrow G \times_{G_x} \mathbb{R}^{p+k+l}$ and $\Psi_x : f(W_x) \hookrightarrow G \times_{G_{f(x)}} \mathbb{R}^p$, where $\mathbb{R}^p$ carries an orthogonal $G_{f(x)}$-representation and $\mathbb{R}^k$ and $\mathbb{R}^l$ orthogonal G_x-representations, such that the following conditions hold true:

(1) The image of Φ_x is of the form $G \times_{G_x} (B \times C \times D)$ with $B \subset \mathbb{R}^p$, $C \subset \mathbb{R}^k$, and $D \subset \mathbb{R}^l$ open convex neighborhoods of the origin, and $\Psi_x\big(f(W_x)\big) = G \times_{G_{f(x)}} B$.

(2) One has $W_x \cap S = \Phi_x^{-1}\big(G \times_{G_x} (B \times C \times \{0\})\big) = \Phi_x^{-1}\big(G \times_{G_x} \mathbb{R}^{p+k}\big)$.

(3) The diagram

$$
\begin{array}{ccc}
W_x & \xrightarrow{\ \ \Phi_x\ \ } & G \times_{G_x} \mathbb{R}^{p+k+l} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{id}_G \times \Pi}} \\
f(W_x) & \xrightarrow[\ \ \Psi_x\ \ ]{} & G \times_{G_{f(x)}} \mathbb{R}^p
\end{array}
$$

commutes, where Π is projection onto the first p coordinates.

After possibly shrinking the W_x one can achieve that

(2.3)
$$
\begin{aligned}
W_x \cap A \neq \varnothing &\implies W_x \subset V \\
W_x \cap Z \neq \varnothing &\implies W_x \cap S \subset U \ .
\end{aligned}
$$

The family $M \backslash S \cup \{W_x\}_{x \in S}$ then is covering of M by G-invariant open subsets. Since the orbit space M/G is separable and paracompact one can find a locally finite countable refinement $M \backslash S \cup \{W_i\}_{i \in \mathbb{N}^*}$ with each W_i being G-invariant, open in M and contained in W_{x_i} for some $x_i \in S$. Moreover, the W_i are so that there exist equivariant charts $\Phi_i : W_i \to G \times_{G_i} \mathbb{R}^{p+k+l}$ and $\Psi_i : f(W_i) \to G \times_{G_{f(i)}} \mathbb{R}^p$ fulfilling conditions (1) to (3). Following Mather we discard all W_i for which

$W_i \cap Z \neq \varnothing$ or $W_i \cap A = \varnothing$, and reindex the remaining W_i's again by the positive integers. By (2.3) we then have $A \subset U \cup \bigcup_{i \in \mathbb{N}^*} W_i$ and $W_i \subset V$ for all $i \in \mathbb{N}^*$. Next, choose G-invariant closed subsets $W_i' \subset W_i \cap S$ such that $A \subset U \cup \bigcup_{i \in \mathbb{N}^*} W_i'$. Since the W_x are relatively compact, all W_i' are compact. Finally put for all $j \in \mathbb{N}$

$$U_j = \varphi_0 \big(\pi_{E_0}^{-1}(U) \cap B(\varepsilon_0, E_0) \big) \cup W_1 \cup \ldots \cup W_j \; .$$

Note that the U_j are then G-invariant and open and that $U_0 \cap S = U$.

We now construct inductively G-equivariant diffeotopies $H^0, H^1, H^2, \ldots$ of M together with a sequence $\psi_0, \psi_1, \psi_2, \ldots$ of G-equivariant isomorphims of tubular neighborhoods. We start with defining H_t^0 to be the identity map for all $t \in I$ and let ψ_0 be the ismorphism from the statement of the theorem.

For the induction step we assume to be given diffeotopies $H^0, H^1, \ldots, H^{i-1}$ of M together with G-equivariant isomorphisms $\psi_0, \ldots, \psi_{i-1}$ of tubular neighborhoods having the following properties:

(a) The diffeotopies $H^0, H^1, \ldots, H^{i-1}$ and isomorphisms $\psi_0, \ldots, \psi_{i-1}$ are G-equivariant and compatible with f.

(b) The diffeotopies $H^0, H^1, \ldots, H^{i-1}$ leave S pointwise fixed.

(c) For each $j = 0, \ldots, i-1$ the diffeotopy K^j of M defined by $K_t^j := H_t^j \circ H_t^{j-1} \circ \ldots \circ H_t^0$ for $t \in I$ has support in $U_j \cap V$.

(d) One has $\big(K_t^j(x), x \big) \in O$ for all $x \in M$, $t \in I$, and $j = 0, \ldots, i-1$.

(e) For each $j = 0, \ldots, i-1$ there exist G-invariant relatively compact open neighborhoods U_j^* of $A \cup W_1' \cup \ldots \cup W_j'$ in S such that $\overline{U}_j^* \subset U_{j-1}^* \cup W_j$ when $j > 0$ and such that ψ_j is an isomorphism of tubular neighborhoods $k_*^j \mathsf{T}_0 |_{\overline{U}_j^*} \to \mathsf{T}_1 |_{\overline{U}_j^*}$, where $k^j : M \to M$ is the diffeomorphism K_1^j.

By the local G-equivariant case from *Step 1* there exist a G-equivariant diffeotopy H^i on M together with an isomorphism of tubular neighborhoods ψ_i such that the conditions of the induction are satisfied. Let us provide a detailed argument by adapting Mather's argument to the G-equivariant case. First choose a G-equivariant relatively compact open subset W_i^0 of W_i with $W_i' \subset W_i^0$. Then let U_i^* be a G-equivariant open neighborhood of $A \cup W_1' \cup \ldots \cup W_i'$ in S with closure being compact and in $U_{i-1}^* \cup W_i^0$. By the local G-equivariant case there exists a diffeotopy H^j of W_i which is G-equivariant and compatible with f, has support in W_i^0 and leaves $S \cap W_i$ invariant. Moreover, since $Z_i := \overline{U}_i^* - U_{i-1}^*$ is a G-invariant and compact subset of W_i and $k_*^{i-1} \mathsf{T}_0 |_{\overline{U}_{i-1}^* \cap W_i} \sim \mathsf{T}_1 |_{\overline{U}_{i-1}^* \cap W_i}$, the diffeotopy H^i can be chosen so that there exists a G-equivariant isomorphism of tubular neighborhoods

$$\psi_i : (H_1^i)_* k_*^{i-1} \mathsf{T}_0 |_{\overline{U}_i^* \cap W_i} \to \mathsf{T}_1 |_{\overline{U}_i^* \cap W_i}$$

which fulfills $\psi_i |_{\overline{U}_i^* \cap W_i \cap \overline{U}_{i-1}^*} = \psi_{i-1} |_{\overline{U}_i^* \cap W_i \cap \overline{U}_{i-1}^*}$. Finally, one can even achieve that the H_t^i with $t \in I$ are arbitrarily and uniformly close to the identity. Since the support of the diffeotopy H^i is a compact G-invariant subset of W_i, one can extend H^i by the identity outside W_i to a G-invariant diffeotopy on M which has support in W_i and is compatible with f. By putting $\psi_i |_{\overline{U}_{i-1}^*} = \psi_{i-1} |_{\overline{U}_{i-1}^*}$,

the isomorphism ψ_i can be extended to $\overline{U}_i^*$ and the thus extended isomorphism has all the desired properties. This completes the induction step.

Since the Lie group G is compact, one can shrink the G-invariant open neighborhood $O \subset M \times M$ of the diagonal so that the projection $\mathrm{pr}_2 : \overline{O} \to M$ onto the second factor is proper. The sequences $\left(K_t^i\right)_{i \in \mathbb{N}}$ and $\left(\psi_i\right)_{i \in \mathbb{N}}$ then eventually become locally constant. Hence the maps

$$H : \ M \times I \to M, \quad (x,t) \mapsto \lim_{i \to \infty} K_t^i(x) \quad \text{and} \quad \psi : \ M \to \mathrm{Hom}(E_0, E_1), \quad x \mapsto \lim_{i \to \infty} \psi_i(x)$$

By construction, H then is a G-equivariant diffeotopy of M compatible with f and $\psi : (H_1)_* \mathsf{T}_0 \sim \mathsf{T}_1$ a G-equivariant isomorphism of tubular neighborhoods compatible with f as well. Moreover, H and ψ have the properties claimed in the theorem. $\qquad\square$

Theorem 2.9 (Existence of equivariant tubular neighborhoods)**.** *Let M, N be G-manifolds, $S \subset M$ a G-invariant smooth submanifold, and $f : M \to N$ a G-equivariant smooth map which is submersive over S. Let $U \subset S$ be relatively open G-invariant subset, and $A \subset U$ relatively closed and G-invariant. Assume that T_0 is a G-equivariant tubular neighborhood of U in M compatible with $f|_{\mathsf{T}_0}$. Then there exists a G-equivariant tubular neighborhood T of S compatible with f such that $\mathsf{T}|_A$ and $\mathsf{T}_0|_A$ are G-equivariantly isomorphic.*

Proof. Step 1. The Equivariant Submersion Theorem entails existence of tubular neighborhoods in the local equivariant case. Let us explain this. The global case will be considered in the following step. Assume that M is of the form $G \times_H (B \times C \times D)$, P is equivariantly diffeomorphic to $G \times_K B$, and under these identifications S has the form $G \times_H (B \times C)$ and f the form $\overline{\mathrm{id}_G \times \Pi}$. Hereby, $H \subset K \subset G$ are closed subgroups, $B \subset \mathbb{R}^p$, $C \subset \mathbb{R}^k$, and $D \subset \mathbb{R}^l$ are open convex neighborhoods of the origin, where $\mathbb{R}^p$ carries an orthogonal K-repersentation, and $\mathbb{R}^k$, $\mathbb{R}^l$ orthogonal H-representations, and Π is projection onto the third factor. Now let E be the bundle $G \times_H (B \times C \times \mathbb{R}^l) \to S \cong G \times_H (B \times C)$, $\varepsilon : S \to (0, \infty)$ a constant map such that the ball of radius ε in $\mathbb{R}^l$ is contained in D, and $\varphi : B(\varepsilon, E) \hookrightarrow M$ the identical embedding. Then $\mathsf{T} = (E, \varepsilon, \varphi)$ is a G-equivariant tubular neighborhood compatible with f.

Step 2. We adapt Mather's argument in the proof of [10, Prop. 6.2] to the equivariant case. Without loss of generality we can assume that S is closed in M. Now choose G-invariant relatively compact open neighborhoods W_i, $i \in \mathbb{N}^*$ together with equivariant charts $\Phi_i : W_i \hookrightarrow G \times_{G_i} \mathbb{R}^{p+k+l}$ and $\Psi_i : W_i \hookrightarrow G \times_{G_i} \mathbb{R}^{p+k+l}$ fulfilling conditions (1) to (3) in *Step 3* of the preceding proof such that the family $(W_i)_{i \in \mathbb{N}^*}$ is a locally finite covering of S. Next choose G-invariant closed subsets $W_i' \subset S \cap W_i$ such that the family $(W_i')_{i \in \mathbb{N}^*}$ covers S as well. Put $U_0 := T_0 = \varphi_0\big(B(\varepsilon_0, E_0\big)$ and define inductively $U_i := W_i \cup U_{i-1}$ for $i \in \mathbb{N}^*$. Furthermore put $U_0' := A$ and $U_i' := W_i' \cup U_{i-1}'$ for $i \in \mathbb{N}^*$. Finally let U_0'' be a G-invariant relatively open neighborhhood of A in S such that $\overline{U}_i'' \subset U$ and then choose inductively for all $i \in \mathbb{N}^*$ relatively open neighborhoods U_i'' of U_i' in S such that U_i'' is contained in $W_i \cup U_{i-1}''$ and such that U_i'' can be decomposed into G-invariant relatively open subsets $X_i, Y_i \subset U_i''$ so that $\overline{X}_i \subset W_i \backslash U_{i-1}'$, $\overline{Y}_i \subset U_{i-1}''$ and so that $X_i \cap Y_i$ is relatively compact in W_i.

Now we inductively construct G-equivariant tubular neighborhoods T_i of U_i'' in M. The tubular neighborhood T_0 is the given one. Assume that for some $i \in \mathbb{N}^*$ a G-equivariant tubular neighborhood T_{i-1} of U_{i-1}'' in M has been constructed and that it is compatible with f. By *Step 1* there exists a G-equivariant tubular neighborhood T_i' of $W_i \cap S$ in W_i which is compatible with f. So we have two G-equivariant tubular neighborhoods over the G-invariant subset $U_i'' \cap W_i \cap S$, the corresponding restrictions of T_{i-1} and T_i'. By Theorem 2.8 there exists a G-quivariant diffeomorphism h of M which is compatible with f and has support within a sufficiently small relatively compact neighborhood of $\overline{X_i \cap Y_i}$ such that $h_* \mathsf{T}_{i-1}|_{X_i \cap Y_i} = \mathsf{T}_i'|_{X_i \cap Y_i}$. By h having a sufficiently small support we in particular mean that h is the identity in a neighborhood of U_{i-1}'. One can now glue together $h_* \mathsf{T}_{i-1}$ and T_i' to a G-equivariant tubular neighborhood T_i over $U_i'' = X_i \cup Y_i$. By construction T_i is compatible with f.

Since for all $i \in \mathbb{N}^*$ the tubular neighborhoods T_{i-1} and T_i are isomorphic over a small neighborhood of U_{i-1}' in S there exists a G-equivariant tubular neighborhood T of S in M such that $\mathsf{T}|_{U_i'} \sim \mathsf{T}_i|_{U_i'}$ for all i. This tubular neighborhood is compatible with f since all the T_i are and fulfills the claim. The theorem is proved. $\qquad\square$

2.4. **Existence of equivariant control data.** Before proving the existence of G-equivariant control data in Theorem 2.11 below, we first need the following equivariant analog of [10, Lem. 7.3]. Given a stratum S, a tubular neighbourhood $\mathsf{T} = (E, \varepsilon, \varphi)$ and a smooth function $\varepsilon' : S \to \mathbb{R}_{>0}$, define $T_{\varepsilon'}^\circ := \varphi(B_\varepsilon \cap B_{\varepsilon'})$.

Lemma 2.10. *Let R and S be disjoint submanifolds of M which are preserved by G, such that the pair (S, R) satisfies condition (B). Let T be a G-equivariant tubular neighbourhood of R in M. Then there exists a G-invariant smooth function $\varepsilon' : R \to \mathbb{R}_{>0}$ such that the mapping*

$$(\pi_T, \varrho_T) : S \cap T_{\varepsilon'}^\circ \to R \times (0, \infty)$$

is a smooth submersion.

Proof. Since G is compact, then the result follows from the non-equivariant version in [10, Lem. 7.3] by averaging over the G-orbits in R. $\qquad\square$

Theorem 2.11. *Let G be a compact Lie group and M, N smooth G-manifolds. Assume that $(X, \mathcal{S})$ is a (B) regular stratified subspace of M, that X is invariant under the G-action and that the induced G-action on X is compatible with the stratification $\mathcal{S}$. Assume further that $f : X \to N$ is a G-equivariant smooth stratified submersion. Then there exists a system of G-equivariant control data $\mathcal{T} = (T_S, \pi_S, \varrho_S)_{S \in \mathcal{S}}$ on $(X, \mathcal{S})$ compatible with f.*

Proof. The proof is by induction on the dimension of the strata, following the strategy of [10, Sec. 7]. Let $\mathcal{S}_k$ be the subset of $\mathcal{S}$ consisting of strata of dimension less than or equal to k, and let X_k be the union of all strata in $\mathcal{S}_k$.

Since the strata in $\mathcal{S}_0$ all have dimension zero, then there exists a system of G-equivariant control data $\mathcal{T}_0 = (T_S, \pi_S, \varrho_S)_{S \in \mathcal{S}_0}$ on $(X_0, \mathcal{S}_0)$ which is compatible with $f|_{X_0}$.

Now suppose that there exists a system of G-equivariant control data $\mathcal{T}_{k-1} = (T_S, \pi_S, \varrho_S)_{S \in \mathcal{S}_{k-1}}$ on $(X_{k-1}, \mathcal{S}_{k-1})$ which is compatible with $f|_{X_{k-1}}$.

Let S be a stratum of dimension k, and for each $\ell = 0, \ldots, k$, define

$$U_\ell := \bigcup_{Y < S, \dim Y \geqslant \ell} T_Y, \quad S_\ell := U_\ell \cap S.$$

For each ℓ, we will construct a tubular neighbourhood T_ℓ of S_ℓ which satisfies the equivariant control data relations (CC1)–(CC4). Using the approach of [10, Proof of Prop. 7.1], we will do this by descending induction on ℓ. Note that it is sufficient to construct T_ℓ separately for each stratum Y of dimension ℓ, since if Y, Y' both have dimension ℓ then $T_Y \cap T_{Y'} = \varnothing$.

For the base case $\ell = k$, note that $S_k = \varnothing$, and so there is nothing to prove.

Now suppose that we have constructed $T_{\ell+1}$ such that $\varrho_{\ell+1} : T_{\ell+1} \to \mathbb{R}$ is G-invariant, $\pi_{\ell+1} : T_{\ell+1} \to S_{\ell+1}$ is G-equivariant, and if $Y < S$, $\dim Y \geqslant \ell + 1$, $m \in T_{\ell+1} \cap T_Y$, then

(2.4)
$$\varrho_Y \circ \pi_{\ell+1}(x) = \varrho_Y(x)$$
$$\pi_Y \circ \pi_{\ell+1}(x) = \pi_Y(x).$$

If necessary, shrink the neighbourhood $T_{\ell+1}$ so that $x \in T_{\ell+1}$ implies that there exists a stratum $Z < S$ with $\dim Z \geqslant \ell + 1$ such that if x is also in T_Z then $\pi_{\ell+1}(x) \in T_Z$.

Given $x \in T_{\ell+1} \cap T_Y$ such that $\pi_{\ell+1}(x) \in T_Y$, then there exists $Z < S$ with $\dim Z \geqslant \ell + 1$, $x \in T_Z$ and $\pi_{\ell+1}(x) \in T_Z$. Therefore $\pi_{\ell+1}(x) \cap T_Y \cap T_Z$ and so $T_Y \cap T_Z$ is non-empty, hence $Y < Z$. Note that the relations (CC1)–(CC4) hold for the pair (Y, Z) by the inductive hypothesis, and also that since $\dim Z \geqslant \ell + 1$ then (2.4) holds with Y replaced by Z. Therefore we have

$$\varrho_Y \circ \pi_{\ell+1}(x) = \varrho_Y \circ \pi_Z \circ \pi_{\ell+1}(x) = \varrho_Y \circ \pi_Z(x) = \varrho_Y(x)$$
$$\pi_Y \circ \pi_{\ell+1}(x) = \pi_Y \circ \pi_Z \circ \pi_{\ell+1}(x) = \pi_Y \circ \pi_Z(x) = \pi_Y(x).$$

Again, since $\dim Y < k$, then we can further suppose from (CC3) that $(\rho_Y, \pi_Y) : T_Y \cap S \to \mathbb{R} \times Y$ is a submersion, and from (CC1) that ϱ_Y is G-invariant and π_Y is G-equivariant.

Therefore we have constructed a tubular neighbourhood $T_{\ell+1} \cap T_Y \to S_{\ell+1} \cap T_Y$ and so it only remains to extend it to a neighbourhood $T_{S,Y} \to S \cap T_Y$ and then to a neighbourhood $T_S \to S$.

Now if $S_{\ell+1}^\circ$ is an open subset of S whose closure lies in $S_{\ell+1}$, then Theorem 2.9 shows that there exists a tubular neighbourhood $T_{S,Y}$ of $T_Y \cap S$ such that

$$\varrho_Y \circ \pi_{S,Y}(x) = \varrho_Y(x)$$
$$\pi_Y \circ \pi_{S,Y}(x) = \pi_Y(x),$$

the map $\pi_{S,Y}$ is G-equivariant and the function $\varrho_{S,Y}$ is G-invariant, and such that the restriction of $T_{S,Y}$ to $|T_Y| \cap S_{\ell+1}^\circ$ is isomorphic to the restriction of $T_{\ell+1}$.

Now in the same way as the second step of [10, Proof of Prop. 7.1], we can inductively extend the tubular neighbourhood to a neighbourhood T_S of all of S, which is compatible with the submersion f, where we use Theorem 2.9 and Lemma 2.10 in place of [10, Prop. 6.2 & Lem. 7.3] in order to

guarantee that the tubular neighbourhoods are G-equivariant. This completes the inductive step, and hence also the proof of the theorem. □

3. A stratification compatible with a given set of subvarieties

In this section we use a construction due to Wall [16] to prove the following theorem.

Theorem 3.1. *Let G be a Lie group acting smoothly on a nonsingular affine variety M, and let $(A_r)_{r=1}^n$ be a finite family of subvarieties, each of which is preserved by the action of G. Then there is a (B) regular stratification of M in which each A_r is a finite union of G-invariant strata.*

As a preliminary to the proof of Theorem 3.1, we prove the following results.

Lemma 3.2. *Let M be a nonsingular variety, and let G be a connected Lie group acting smoothly on M. If X is a subvariety of M preserved by G, then the singular set X_{sing} is also preserved by G.*

Proof. Given $p \in X$, let $\{f_1, \ldots, f_n\}$ be the equations defining X in a neighbourhood of p. Any $g \in G$ defines a diffeomorphism $\psi_g : M \to M$. In particular, since X is preserved by the action of G then $\{f_1 \circ \psi_g^{-1}, \ldots, f_n \circ \psi_g^{-1}\}$ defines X in a neighbourhood of $g \cdot p$. Then the Jacobian of these equations is $df \circ d\psi_g^{-1}$, which has the same rank as df. Therefore $g \cdot p$ is a singular point if and only if p is singular, and so X_{sing} is preserved by the action of G. □

Lemma 3.3. *Let M be a nonsingular variety, and let G be a group acting continuously on M. If $X \subset M$ is any subset preserved by the action of G, then $\overline{X}$ and $\overline{X}\backslash X$ are also preserved by the action of G.*

Proof. Given $p \in \overline{X}\backslash X$, let $(p_n)_{n \in \mathbb{N}} \subset X$ be a sequence in X converging to p. Since the action of G is continuous, then for any $g \in G$ the sequence $(g \cdot p_n)_{n \in \mathbb{N}} \subset X$ converges to $g \cdot p$. Since G preserves X and $p \notin X$ then $g \cdot p \notin X$ also. Therefore $g \cdot p \in \overline{X}\backslash X$ for all $g \in G$, and therefore $\overline{X}\backslash X$ is also preserved by the action of G, hence so is $\overline{X}$. □

Lemma 3.4. *Let G be a Lie group acting smoothly on a nonsingular affine variety M, and let X and Y be two disjoint strata in a stratification of M. Then G preserves the set of points $x \in X \cap \overline{Y}$ where (X, Y) is (B) regular.*

Proof. Given $g \cdot x$, let $\psi_g : M \to M$ denote the diffeomorphism associated to the action of $g \in G$. Since G acts smoothly on M, then for each $g \in G$ a chart $\varphi : U \to \mathbb{R}^d$ around $x \in M$ determines a chart $\varphi \circ \psi_g^{-1} : g(U) \to \mathbb{R}^d$ around $g \cdot x \in M$. Since Whitney's condition (B) is independent of the choice of chart (cf. [12, Lem. 1.4.4]) then (X, Y) is (B) regular at $x \in X$ if and only if (X, Y) is (B) regular at $g \cdot x$. □

We can now use the above results to prove the main theorem of the section.

Proof of Theorem 3.1. We closely follow the proof of the corresponding result of Wall [16] when the group action is trivial, and use the above results to show that the construction extends to the equivariant setting.

Suppose that there exists a filtration $T_i \subset T_{i+1} \subset \cdots \subset T_m = M$ such that

- each T_j is a semivariety closed in M,
- each T_j is preserved by the action of G,
- for each $j = i+1, \ldots, m$, the set $S_j = T_j \backslash T_{j-1}$ is a j-dimensional manifold, and
- each $A_r \cap S_j$ is a union of components of S_j.

The above conditions are clearly satisfied for $T_m = M$, thus giving us the base case for the induction. Define

$$B_1 = \begin{cases} (T_i)_{\text{sing}} & \text{if } \dim T_i = i \\ T_i & \text{if } \dim T_i < i \end{cases}$$

and

$$B_2 = \bigcup_r \left(A_r \cap \overline{(T_i)_{\text{reg}} \backslash A_r} \right),$$

Lemma 3.2 shows that B_1 is preserved by G, therefore so is $(T_i)_{reg}$ and so together with Lemma 3.3 this implies that B_2 is also preserved by G. Therefore $S_i' := T_i \backslash (B_1 \cup B_2)$ is also preserved by G.

To finish the proof, define the set B_3 of points where some higher-dimensional stratum fails to be (B) regular. Lemma 3.4 shows that this is preserved by G. Then define $T_{i-1} := T_i \backslash (B_1 \cup B_2 \cup B_3)$ and $S_i = T_i \backslash T_{i-1}$. The above argument shows that these sets are both G-invariant. Moreover, Wall [16] shows that this defines a regular stratification by semivarieties, and so we can continue inductively to define a (B) regular G-invariant stratification by semivarieties such that each A_r is a finite union of strata.   □

4. Constructing the equivariant neighbourhood deformation retract

Let M be a smooth manifold equipped with the action of a compact Lie group G, and let $A \subset X \subset M$ be closed subsets with inclusion map denoted $i : A \hookrightarrow M$. Suppose that X carries a (B) regular G-invariant Whitney stratification $\{S\}_{S \in \mathcal{S}}$, which restricts to a (B) regular G-invariant Whitney stratification $\{S\}_{S \in \mathcal{S}_A}$ of A. Theorem 2.11 shows that there exists a system of G-equivariant control data on $(X, \mathcal{S})$ and Theorem 3.1 shows that these assumptions are satisfied when A and X are G-invariant analytic subvarieties of M with $A \subset X$.

In this section we prove Theorem 4.4 which shows that the inclusion $A \hookrightarrow X$ is an equivariant cofibration of stratified spaces. In particular, the result of Corollary 4.5 shows that the homotopy equivalences in the Morse theory of [18] can be chosen to be G-equivariant.

Using Theorem 2.11, construct a system of G-equivariant control data $(T_S, \pi_S, \rho_S)_{S \in \mathcal{S}}$ for X. Since A is a G-invariant stratified subspace of X then $(T_S, \pi_S, \rho_S)_{S \in \mathcal{S}_A}$ is a system of G-equivariant control data for A. On restricting to a small enough open neighbourhood of A, we can assume that

(1) if $S \subset X$ is a stratum of lowest dimension, then $S \subset A$, and

(2) if $S \subset X$ is a stratum of X then $\bar{S} \cap A \neq \varnothing$.

First we set up some notation and prove some preliminary results. On each tubular neighbourhood T_S, fix a radial vector field $\frac{\partial}{\partial \rho_S}$ as in [12, Cor. 3.7.4]. Since ρ_S is G-invariant then $\frac{\partial}{\partial \rho_S}$ is G-equivariant and so is its integral flow. Using the integral flow of radial vector fields, for each stratum S and each $x \in S$, there exists a neighbourhood $U_x \subset T_S$ and a real number $r > 0$ together with an isomorphism of stratified spaces

$$(4.1) \qquad U_x \cong (\rho_S^{-1}(r) \cap U_x) \times [0, r] / \sim,$$

where $(y_1, 0) \sim (y_2, 0)$ if and only if $\pi_S(y_1) = \pi_S(y_2)$. Equivalently, U_x is homeomorphic to the mapping cylinder of $\pi_S|_{U_x \cap \rho_S^{-1}(r)}$ and this homeomorphism is determined by the flow of the radial vector field $\frac{\partial}{\partial \rho_S}$.

Given any stratum $S_0 \in \mathcal{S}_A$ and a sequence of strata $S_0 < S_1 < \cdots < S_k \subset A$ of increasing height, define

$$(4.2) \qquad T_{S_0,\ldots,S_k} := T_{S_0} \cap T_{S_1} \cap \cdots \cap T_{S_k} \setminus (S_0 \cup S_1 \cup \cdots \cup S_k).$$

Given any $x \in T_{S_0,\ldots,S_k}$, there exists a neighbourhood U of x such that U is contained in a trivialisation for each of $\pi_{S_0}, \pi_{S_1}, \ldots, \pi_{S_k}$. Therefore there exist $r_0, \ldots, r_k$ and $\varepsilon > 0$ such that

$$V := \bigcap_{\ell=0}^{k} \rho_{S_\ell}^{-1}((r_\ell - \varepsilon, r_\ell + \varepsilon)) \subset U.$$

Let $Y := \bigcap_{\ell=0}^{k} \rho_{S_\ell}^{-1}(r_\ell)$. Again using the integral flow of radial vector fields, we have

$$(4.3) \qquad V \cong Y \times \prod_{\ell=0}^{k} (r_\ell - \varepsilon, r_\ell + \varepsilon),$$

and for any $x = (y, t_0, \ldots, t_k) \in V$ we have $\rho_{S_\ell}(x) = t_\ell$ for each $\ell = 0, \ldots, k$.

For each stratum S_ℓ, let φ_{S_ℓ} be the integral flow of the radial vector field on the tubular neighbourhoood T_{S_ℓ}. Recall that these flows have the following properties

- φ_{S_ℓ} preserves the tubular distance functions ρ_{S_j} for each $j \neq \ell$,
- φ_{S_ℓ} is G-equivariant,
- the flow on the cylinder $\rho_{S_\ell}^{-1}(r_\ell) \times (r_\ell - \varepsilon, r_\ell + \varepsilon)$ is given by $\varphi_{S_\ell}((y_\ell, t_\ell), t) = (y_\ell, t_\ell + t)$, and
- the flow preserves strata.

Therefore, on the neighbourhood $V = \bigcap_{\ell=0}^{k} \rho_{S_\ell}^{-1}((r_\ell - \varepsilon, r_\ell + \varepsilon)) \cong Y \times \prod_{\ell=0}^{k}(r_\ell - \varepsilon, r_\ell + \varepsilon)$ the flow is given by

$$\varphi_{S_\ell}((y, t_0, \ldots, t_k), t) = (y, t_0, \ldots, t_\ell + t, \ldots, t_k).$$

In particular, φ_{S_ℓ} is the flow of the vector field $\frac{\partial}{\partial t_\ell}$ on $Y \times \prod_{\ell=0}^{k}(r_\ell - \varepsilon, r_\ell + \varepsilon)$, the vector fields $\{\frac{\partial}{\partial t_\ell}\}_{\ell=0,\ldots,k}$ are linearly independent, and the flows φ_{S_ℓ} for $\ell = 0, \ldots, k$ all commute and preserve strata. Moreover, even though the above calculations have been done with respect to the local

neighbourhood V, these vector fields and flows are well-defined and G-equivariant on the entire neighbourhood $T_{S_0,\dots,S_k}$, since the radial vector fields are well-defined and G-equivariant on $T_{S_0,\dots,S_k}$.

Given functions $a_\ell : V \to \mathbb{R}$ for each $\ell = 0, \dots, k$, the vector field

$$\chi(y, t_0, \dots, t_k) = \sum_{\ell=0}^{k} a_\ell(y, t_0, \dots, t_k) \frac{\partial}{\partial t_\ell}$$

is also tangent to strata, and so the flow preserves strata. Moreover, if the functions a_ℓ are independent of y, then this vector field is G-equivariant and hence the flow is G-equivariant, since the G-action preserves the radial distance functions ρ_{S_ℓ}.

The next lemma is used in the proof of Theorem 4.4.

Lemma 4.1. *Let* $Q = [-1, 1]$, $B = [0, 1] \times [0, 1]$ *and* $C = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{0\}) \subset B$. *Then there exists a proper continuous mapping* $H : Q \times [0, 1] \to B$ *such that*

$$H(Q \times (0, 1)) \subset B \setminus C, \quad H(Q \times \{0\}) = C$$

and $H|_{Q \times (0,1)}$ *is a diffeomorphism onto its image.*

Proof. Choose a smooth monotone function $\phi : [0, \frac{\pi}{2}] \to [0, \pi]$ such that $\phi(\theta) = \theta$ if $0 \leqslant \theta \leqslant \frac{\pi}{3}$ and $\phi(\theta) = \theta + \frac{\pi}{2}$ if $\frac{2\pi}{3} \leqslant \theta \leqslant \frac{\pi}{2}$. For notation, let $P : \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0, (x, y) \neq (0, 0)\} \to \mathbb{R}_{>0} \times [0, \pi]$ be the polar coordinate homeomorphism. Then the map $h : B \to [-1, 1] \times \mathbb{R}_{\geqslant 0}$ given by

$$h \circ P^{-1}(r, \theta) := P^{-1}(r, \phi(\theta)), \quad h(0, 0) = (0, 0)$$

is a homeomorphism onto its image, which restricts to a diffeomorphism of $B \setminus C$ onto $h(B \setminus C)$. Moreover (in Cartesian coordinates), the image of h contains $[-1, 1] \times [0, \frac{1}{2}]$. Now define $H : Q \times [0, 1] \to B$ by $H(q, t) = h^{-1}(q, \frac{t}{2})$. $\qquad\square$

Let $W \subset B$ be the image of $H|_{Q \times [0,1)}$. The previous lemma shows that H restricts to a diffeomorphism $Q \times (0, 1) \cong W \backslash C$. Using the homeomorphism H, for any $w \in W$ we can write $w = H(q(w), s(w))$, where $(q(w), s(w)) \in Q \times (0, 1)$. Define a flow $\varphi : W \times [0, \infty) \to W$ by

$$\varphi(w, t) = \begin{cases} H(q(w), e^{-t} s(w)) & w \notin C \\ w & w \in C \end{cases}$$

Taking the vector field associated to this flow gives us the following lemma.

Lemma 4.2. *There exist non-negative smooth functions* $a, b : W \to \mathbb{R}_{\geqslant 0}$ *such that the vector field*

$$X(x, y) = -a(x, y) \frac{\partial}{\partial x} - b(x, y) \frac{\partial}{\partial y}$$

defined on W *satisfies the boundary conditions* $X(x, 0) = 0 = X(0, y)$,

(4.4) $X(x, 1) = -x \dfrac{\partial}{\partial x}$ *for all* $x \in \left[0, \frac{1}{2}\right]$, $\quad X(1, y) = -y \dfrac{\partial}{\partial y}$ *for all* $y \in \left[0, \frac{1}{2}\right]$,

and the flow of X *defines a smooth map*

$$\varphi : W \times [0, \infty) \to W$$

such that $\lim_{t\to\infty} \varphi((x,y),t) \in C$ *for all* $(x,y) \in W$.

Now define the sets

$$W_{1/2} := H(Q \times \{\tfrac{1}{2}\}) \cong Q$$
$$W_{\leqslant 1/2} := H(Q \times [0, \tfrac{1}{2}])$$
$$W_{<1/2} := W_{\leqslant 1/2} \backslash W_{1/2}.$$

Note that the flow φ of the vector field X from the Lemma 4.2 defines a deformation retract of $W_{\leqslant 1/2}$ onto C. Moreover, given such a vector field, for any $w \in W\backslash C$ there exists a unique $t = t(w) \in [0, \infty)$ such that $\varphi(w, -t(w)) \in W_{1/2}$.

Definition 4.3. Given $\varepsilon_1, \varepsilon_2 > 0$, identify $W_{1/2} \cong Q \cong [-1, 1]$ and choose a smooth monotone function $f : [-1, 1] \to \mathbb{R}$ such that $f(-1) = \varepsilon_1$ and $f(1) = \varepsilon_2$. The *modified radial distance* $\tilde{\rho} : W \to [0, 1]$ is given by

$$\tilde{\rho}(x, y) = \begin{cases} e^{-t(w)} f(\varphi(w, -t(w))) & \text{if } w \in W\backslash C \\ 0 & \text{if } w \in C. \end{cases}$$

Now let h be the maximal height of a stratum in A. For each $\ell = 0, \ldots, h$, let $\hat{S}_\ell \subset A$ denote the union of all the strata $S \in \mathcal{S}_A$ such that $\text{ht}(S) \leqslant \ell$. Consider a pair (U, φ_ℓ) consisting of an open set $U \subset X$ containing $\hat{S}_\ell$ and a flow φ_ℓ defined on U. We say that (U, φ_ℓ) *has property* (R_ℓ) if all of the following are satisfied.

(1) φ_ℓ is continuous.
(2) $\lim_{t\to\infty} \varphi_\ell(x, t) \in \hat{S}_\ell$.
(3) $\varphi(x, t) = x$ for all $x \in \hat{S}_\ell$.
(4) For any stratum $S \in \mathcal{S}$, if $x \in S$ then $\varphi(x, t) \in S$ for all $t \in [0, \infty)$.
(5) φ_ℓ is G-equivariant.
(6) For each $t \in [0, 1)$, define $U_t := \varphi_\ell(U, -\log(1 - t))$, and define $U_1 := A$. Then U_t satisfies the following conditions
   (a) U_s is open in X for all $s \in [0, 1)$,
   (b) $\overline{U_t} \subset U_s$ for all $t > s$,
   (c) $U_s = \bigcup_{t>s} U_t$ and $\overline{U_s} = \bigcap_{t<s} U_t$ for all $s \in (0, 1)$.

The following theorem is the main result of this section.

Theorem 4.4. *Let M be a smooth manifold equipped with the action of a compact Lie group G, and let $A \subset X \subset M$ be closed subsets with inclusion map denoted $i : A \hookrightarrow M$. Suppose that X carries a G-invariant (B) regular Whitney stratification $\{S\}_{S\in\mathcal{S}}$ and that there exists a subset $\mathcal{S}_A \subset \mathcal{S}$ such that $A = \bigcup_{S\in\mathcal{S}_A} S$, therefore $\{S\}_{S\in\mathcal{S}_A}$ is a G-invariant (B) regular Whitney stratification of A.*

Then there exists a G-stratified space $\tilde{A}$, a proper continuous map $\eta : \tilde{A} \to A$ an open neighbourhood U of A in X and a G-equivariant homeomorphism of U onto the mapping cylinder

$\psi : U \to Z_\eta = (\tilde{A} \times [0,1])/\sim$ *such that* $\psi|_A$ *is the identity and* $\psi|_{A \times (0,1]}$ *is a homeomorphism of stratified spaces.*

Proof. The proof reduces to showing that there exists a pair (U, φ_h) which has property (R_h). We inductively construct such a pair as follows. First consider the neighbourhood $U^{(0)} = \bigcup_{\mathrm{ht}(S)=0} T_S$ of $\hat{S}_0$, and define the vector field $X_0 = -\rho_S \frac{\partial}{\partial \rho_S}$ (note that the vector field is well-defined as the tubular neighbourhoods do not overlap since the strata S all have the same height). Since the radial distance functions ρ_S are G-invariant and the radial vector field $\frac{\partial}{\partial \rho_S}$ is G-equivariant, then X_0 is also G-equivariant and so the flow is G-equivariant. It is easy to check the first four conditions of property (R_ℓ). Since the flow is continuous and the tubular distance function ρ_S is strictly decreasing, then the remaining condition of property (R_ℓ) is also satisfied.

Now suppose that we have a vector field $X_{\ell-1}$ defined on a G-invariant neighbourhood $U^{(\ell-1)}$ of $\hat{S}_{\ell-1}$ with G-invariant tubular distance function $\tilde{\rho}_{\ell-1}$ and G-invariant tubular size function $\tilde{\varepsilon}_{\ell-1}$ such that $U^{(\ell-1)} = \{\tilde{\rho}_{\ell-1}(x) < \tilde{\varepsilon}_{\ell-1}(x)\}$ and such that the flow $\varphi_{\ell-1}$ of $X_{\ell-1}$ satisfies property $(R_{\ell-1})$. In analogy with the non-equivariant case studied by Verona [15] (see also [12, Sec. 3.9]), we define a G-invariant neighbourhood $U^{(\ell)}$ of $\hat{S}_\ell$ and a vector field X_ℓ satisfying property (R_ℓ) by "smoothing the corner" using Lemma 4.2 as follows. First we define X_ℓ on $U^{(\ell-1)} \cup \bigcup_{\mathrm{ht}(S)=\ell} T_S$ by

(1) On the subset $U^{(\ell-1)} \setminus \left(\bigcup_{\mathrm{ht}(S)=\ell} T_S \right)$, define $X_\ell = X_{\ell-1}$.

(2) For each stratum S with $\mathrm{ht}(S) = \ell$, on the subset $(T_S \cap \{\rho_S(x) < \varepsilon_S(x)\}) \setminus U^{(\ell-1)}$ define $X_\ell = -\rho_S(x) \frac{\partial}{\partial \rho_S}$.

(3) For each stratum S with $\mathrm{ht}(S) = \ell$, on the subset $(T_S \cap \{\rho_S(x) < \varepsilon_S(x)\}) \cap U^{(\ell-1)}$ define

$$X_\ell(x) = -a\left(\frac{\rho_S(x)}{\varepsilon_S(x)}, \frac{\tilde{\rho}_{\ell-1}(x)}{\tilde{\varepsilon}_{\ell-1}(x)}\right) \frac{\partial}{\partial \rho_S} - b\left(\frac{\rho_S(x)}{\varepsilon_S(x)}, \frac{\tilde{\rho}_{\ell-1}(x)}{\tilde{\varepsilon}_{\ell-1}(x)}\right) X_{\ell-1}(x)$$

where the functions a and b are given by Lemma 4.2.

Now restrict to the subset

$$U' := \left(U^{(\ell-1)} \cap \{\tilde{\rho}_{\ell-1}(x) < \tilde{\varepsilon}_{\ell-1}(x)\} \right) \cup \bigcup_{\mathrm{ht}(S)=\ell} (T_S \cap \{\rho_S(x) < \varepsilon_S(x)\}).$$

The result of Lemma 4.2 shows that the vector field X_ℓ is smooth on U'. By setting $\varepsilon_1(x) = \varepsilon_S(x)$ and $\varepsilon_2(x) = \tilde{\varepsilon}_{\ell-1}(x)$ (both of which are G-invariant), we can glue the modified radial distance $\tilde{\rho}_\ell$ of Definition 4.3 with the radial distance $\tilde{\rho}_{\ell-1}$ on $U^{(\ell-1)} \setminus \left(\bigcup_{\mathrm{ht}(S)=\ell} T_S \right)$ and the radial distance ρ_S on $T_S \setminus U^{(\ell-1)}$ for each stratum S of height ℓ. Since ρ_S and $\tilde{\rho}_{\ell-1}$ are both G-invariant then this gives us a smooth G-invariant radial distance function $\tilde{\rho}_\ell : U' \to \mathbb{R}_{\geqslant 0}$, together with a G-invariant size function $\tilde{\varepsilon}_\ell : \hat{S}_\ell \to \mathbb{R}_{>0}$ such that $\{\tilde{\rho}_\ell(x) < \tilde{\varepsilon}_\ell(x)\} \subset U'$. Moreover, for each stratum of height ℓ, on the subset $T_S \setminus U^{(\ell-1)}$ we have $\tilde{\rho}_\ell = \rho_S$ and $\tilde{\varepsilon}_\ell = \varepsilon_S$, and on the subset $U^{(\ell-1)} \setminus \bigcup_{\mathrm{ht}(S)=\ell} T_S$ we have $\tilde{\rho}_\ell = \tilde{\rho}_{\ell-1}$ and $\tilde{\varepsilon}_\ell = \tilde{\varepsilon}_{\ell-1}$.

Now define

$$U^{(\ell)} := \{x \in U' \ : \ \tilde{\rho}_\ell(x) < \tilde{\varepsilon}_\ell(x)\}.$$

It only remains to verify that the conditions of property (R_ℓ) are satisfied. Since the construction of X_ℓ only depends on the G-invariant functions $\tilde{\rho}_{\ell-1}$, $\tilde{\varepsilon}_{\ell-1}$, ρ_S and ε_S, as well as the G-equivariant vector fields $X_{\ell-1}$ and $\frac{\partial}{\partial \rho_S}$ then X_ℓ is G-equivariant. Since the vector fields $X_{\ell-1}$ and $\frac{\partial}{\partial \rho_S}$ commute and their flows preserve strata, then the flow of X_ℓ also preserves strata. Since the vector field from Lemma 4.2 satisfies the remaining conditions (1)–(3) and (6) of property (R_ℓ), then X_ℓ also satisfies these conditions.

Therefore we can inductively construct a vector field X_h whose flow φ_h has property (R_h). $\square$

This immediately gives us the following result, which shows that the main theorem of Morse theory from [18, Thm. 1.1] can be made to work in the equivariant setting.

Corollary 4.5. *Let M be a smooth manifold equipped with the action of a compact Lie group G, and let $A \subset X \subset M$ be closed subsets with inclusion map denoted $i : A \hookrightarrow M$. Suppose that X carries a G-invariant (B) regular Whitney stratification $\{S\}_{S \in \mathcal{S}}$, which restricts to a G-invariant (B) regular Whitney stratification $\{S\}_{S \in \mathcal{S}_A}$ of A.*

Then there exists a neighbourhood U of A in X and a G-equivariant flow $\varphi : U \times [0,1] \to X$ defining a deformation retract of U onto A such that $U_t := \varphi(U, t)$ satisfies the following conditions

(1) U_s is open in X for all $s \in [0, 1)$,

(2) $\overline{U_t} \subset U_s$ for all $t > s$,

(3) $U_s = \bigcup_{t > s} U_t$ and $\overline{U_s} = \bigcap_{t < s} U_t$ for all $s \in (0, 1)$.

Appendix A. Stratified spaces in the sense of Mather

In this paper we use stratified space in the sense of Mather [9]. Let us briefly recall the definition; for further details see [12, Sec. 1.2].

By a *prestratification* or *decomposition* of a separable locally compact (Hausdorff) space X one understands a partition $\mathcal{Z}$ of X into locally closed subspaces $S \subset X$ each carrying the structure of a smooth manifold such that the decomposition is locally finite and fulfills the *condition of frontier*. The latter means that for each pair $R, S \in \mathcal{Z}$ with the closure of S meeting R the relation $R \subset \overline{S}$ holds true. The elements of $\mathcal{Z}$ are called the *pieces* or *strata* of the decomposition. If R, S are two strata of X one calls R *incident* to S if $R \subset \overline{S}$ and denotes this by $R \leqslant S$ respectively by $R < S$ if in addition R is not equal to S.

A *stratification* of a locally compact X now is a map $\mathcal{S}$ which assigns to every point x of X a set germ $\mathcal{S}_x$ at x such that there exists for each $x \in X$ an open neighborhood U of x and a decomposition $\mathcal{Z}$ of U with the property that for every point y in U the set germ $\mathcal{S}_y$ coincides with the set germ $[R]_y$ at y of the piece $R \in \mathcal{Z}$ containing y. One calls such a decomposition $\mathcal{Z}$ a decomposition *inducing* the stratification $\mathcal{S}$ over U or a local $\mathcal{S}$-decomposition around x.

By a *stratified space* we understand a pair $(X, \mathcal{S})$ consisting of a separable locally compact space X called the *total space* together with a stratification $\mathcal{S}$ on it. In the following $(X, \mathcal{S})$ will always denote a stratified space.

Given an element x of a stratified space $(X, \mathcal{S})$ one defines its *depth* $\mathrm{dp}(x)$ as the maximal number d such that there exist pieces $S_0, S_1, \ldots, S_d$ of a local $\mathcal{S}$-decomposition $\mathcal{Z}$ around x which fulfill

$$x \in S_0 < \ldots < S_d \ .$$

The depth of x is actually not dependent on a local $\mathcal{S}$-decomposition $\mathcal{Z}$ around x, see [9, Lem. 2.1] or [12, Lem. 1.2.5]. The depth function is locally constant on each stratum of a local decomposition. It allows to define a global decomposition of X inducing the stratification $\mathcal{S}$. Namely for each pair of natural numbers d, m let $S_{d,m}$ be the set of points $x \in X$ of depth d and for which the dimension of the set germ $\mathcal{S}_x$ equals m. Then $S_{d,m}$ is a smooth manifold and the set $\{S_{d,m} \mid d, m \in \mathbb{N}\}$ is a global decomposition of X inducing $\mathcal{S}$. It is the coarsest decomposition with that property, see [12, Prop. 1.2.7]. We denote this decomposition by the symbol $\mathcal{S}$ also and call its pieces the strata of $(X, \mathcal{S})$. We often write $S \in \mathcal{S}$ to denote that S is a stratum of $(X, \mathcal{S})$. The supremum of all depths $\mathrm{dp}(x)$, where x runs through the points of X, will be called the *depth* of the stratified space $(X, \mathcal{S})$. It can be infinite. Note that the depth is constant on each stratum so it is clear what is meant by the *depth* of a stratum. It is denoted $\mathrm{dp}(S)$.

Closely related to the depth is the *height* $\mathrm{ht}(R)$ of a stratum R. It is defined as the maximal natural number h such that there exists strata $R_0 < \ldots < R_h$ with $R = R_h$.

If $(X, \mathcal{S})$ and $(Y, \mathcal{R})$ denote stratified spaces, a continuous map $f : X \to Y$ is called *stratified*, if $f(\mathcal{S}_x) \subset \mathcal{R}_{f(x)}$ for all $x \in X$ and if the restriction of f to each connected component S of a stratum of $(X, \mathcal{S})$ is a smooth map from S to the stratum R_S of $(Y, \mathcal{R})$ containing $f(S)$. If in addition all the restrictions $f_{|S} : S \to R_S$ are immersions (resp. submersions), one calls f a *stratified immersion* (resp. *stratified submersion*).

A subspace A of a stratified space $(X, \mathcal{S})$ is called a *stratified subspace* if the map $\mathcal{S}_A$ which associates to each point $x \in A$ the set germ $A \cap S$ is a stratification of A. In this case $(A, \mathcal{S}_A)$ becomes a stratified space and the canonical injection $i : A \hookrightarrow X$ is a stratified immersion. If in addition i is a stratified submersion we call $(A, \mathcal{S}_A)$ a *submersed stratified subspace*. A subspace $A \subset X$ is a closed submersed stratified subspace of $(X, \mathcal{S})$ if and only if it is a union of connected components of strata of X.

Whitney's regularity conditions (A) and (B) play a crucial role in stratification theory in particular in Mather's proof of Thom's isotopy lemmata [10]. They describe properties how a stratum of a stratified space embedded in a smooth manifold M can approach an incident stratum near its frontier. Let us recall the Whitney conditions following [12, 1.4.3]. A pair (R, S) of smooth submanifolds of M is said to fulfill *Whitney's condition* (A) at $x \in R$ or that (R, S) is (A) *regular* at x if the following holds.

(A) Let $(y_k)_{k \in \mathbb{N}}$ be a sequence of points of S converging to x such that the sequence $T_{y_k}S$, $k \in \mathbb{N}$, of tangent spaces converges in the Graßmannian bundle of $\dim S$-dimensional subspaces of TM to some $\tau \subset T_xM$. Then $T_xR \subset \tau$.

The pair (R, S) is said to fulfill *Whitney's condition* (B) at $x \in R$ or that (R, S) is (B) *regular* at x if for some chart $\chi : U \to \mathbb{R}^d$ of M around x the following is satisfied.

(B) Let $(y_k)_{k\in\mathbb{N}}$ be a sequence in S and $(x_k)_{k\in\mathbb{N}}$ a sequence in R such that both converge to x and such that $x_k \neq y_k$ for all $k \in \mathbb{N}$. Assume that the sequence of lines $\overline{\chi(x_k)\chi(y_k)}$, where k is large enough so that $x_k, y_k \in U$, converges in projective space $\mathbb{RP}^{d-1}$ to some line ℓ. Assume further that the sequence of tangent spaces $T_{y_k}S$, $k \in \mathbb{N}$, converges to some subspace $\tau \subset T_x M$. Then $\ell \subset \tau$.

By [12, Lem.1.4.4], Whitney's condition (B) does not depend on the choice of the chart φ around x. A stratified subspace $(X, \mathcal{S})$ of a smooth manifold M is said to be (A) respectively (B) regular if every pair of strata (R, S) with R incident to S is (A) respectively (B) regular at each point $x \in R$. (B) regularity implies (A) regularity but in general not vice versa. Complex algebraic varieties [17], orbit spaces of compact Lie group actions [12] and of proper Lie groupoids [13], analytic varieties [8], and subanalytic sets [1] all possess (B) regular stratifications.

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