To Joseph A. Wolf with admiration

Ind–varieties of generalized flags as homogeneous spaces for classical ind–groups

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Abstract

The purpose of the present paper is twofold: to introduce the notion of a generalized flag in an infinite dimensional vector space V (extending the notion of a flag of subspaces in a vector space), and to give a geometric realization of homogeneous spaces of the ind–groups $SL(\infty)$, $SO(\infty)$ and $Sp(\infty)$ in terms of generalized flags. Generalized flags in V are chains of subspaces which in general cannot be enumerated by integers. Given a basis E of V, we define a notion of E–commensurability for generalized flags, and prove that the set $\mathcal{F}\ell(\mathcal{F}, E)$ of generalized flags E–commensurable with a fixed generalized flag $\mathcal{F}$ in V has a natural structure of an ind–variety. In the case when V is the standard representation of $G = SL(\infty)$, all homogeneous ind–spaces G/P for parabolic subgroups P containing a fixed splitting Cartan subgroup of G, are of the form $\mathcal{F}\ell(\mathcal{F}, E)$. We also consider isotropic generalized flags. The corresponding ind–spaces are homogeneous spaces for $SO(\infty)$ and $Sp(\infty)$. As an application of the construction, we compute the Picard group of $\mathcal{F}\ell(\mathcal{F}, E)$ (and of its isotropic analogs) and show that $\mathcal{F}\ell(\mathcal{F}, E)$ is a projective ind–variety if and only if $\mathcal{F}$ is a usual, possibly infinite, flag of subspaces in V.

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Introduction

Flag varieties play a fundamental role both in representation theory and algebraic geometry. There are two standard approaches to flag varieties: the group–theoretic one, where a flag variety is defined as G/P for a classical algebraic group G and a parabolic subgroup P, and the geometric one, where a flag variety is defined as the set of all chains of subspaces of fixed dimensions in a finite dimensional vector space, which in addition are assumed isotropic in the presence of a bilinear form. The very existence of these two approaches is in the heart of the interplay between representation theory and geometry.

The main topic of this paper is a purely geometric construction of homogeneous spaces for the classical ind–groups $SL(\infty)$, $SO(\infty)$ and $Sp(\infty)$. Despite the fact that many phenomena related to inductive limits of classical groups have been studied (see for instance,

[O1], [O2], [SV], [VK1], [VK2], many natural questions remain unanswered. In particular, the only approach to homogeneous spaces of classical ind–groups discussed in the literature is a representation–theoretic one, and has been introduced by Joseph A. Wolf and his collaborators, [NRW], [DPW].

The difficulty in the purely geometric approach is that the consideration of flags, i.e. chains of subspaces enumerated by integers, is no longer sufficient. To illustrate the problem, let, more specifically, G denote the ind–group $SL(\infty)$ over a field of characteristic 0, and P be a parabolic subgroup of G. By definition, G is the union of a standard system of nested algebraic groups $SL(n)$ and P is the union of parabolic subgroups. If V is the natural representation of G, all P–invariant subspaces in V form a chain $\mathcal{C}$ of subspaces of V. In general, the chain $\mathcal{C}$ has a rather complicated structure and is not necessarily a flag, i.e. cannot be indexed by integers. We show, however, that $\mathcal{C}$ always contains a canonical subchain $\mathcal{F}$ of subspaces of V with the property that every element of $\mathcal{F}$ is either the immediate predecessor F' of a subspace $F'' \in \mathcal{F}$ or the immediate successor F'' of a subspace $F' \in \mathcal{F}$, and, in addition, each nonzero vector $v \in V$ belongs to a difference $F'' \backslash F'$. These two properties define generalized flags. (Maximal generalized flags already appeared in [DP] in a related but somewhat different context.) If, in addition, the vector space V is equipped with a non–degenerate bilinear (symmetric or antisymmetric) form, we introduce the notion of an isotropic generalized flag.

Informally we think of two (possibly isotropic) generalized flags being commensurable if they only differ in a finite dimensional subspace of V in which they reduce to flags of the same type. The precise definition is given in Section 4. The main result of this paper is the construction of the ind–varieties of commensurable generalized flags and their identification with homogeneous ind–spaces G/P for classical locally linear ind–groups G isomorphic to $SL(\infty)$, $SO(\infty)$, or $Sp(\infty)$, and corresponding parabolic subgroups P.

The paper is completed by providing two applications: an explicit computation of the Picard group of any ind–variety of commensurable generalized flags X and a criterion for projectivity of X. We show that the Picard group of X admits a description very similar to the classical one; however, X is projective if and only if it is an ind–variety of usual flags.

The "flag realization" of the ind–varieties G/P given in the present paper opens the way for a detailed and explicit study of the geometry of G/P, which should play a role as prominent as the geometric representation theory of the classical algebraic groups.

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Conventions
$\mathbb{N}$ stands for $\{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The ground field is a field k of characteristic 0 which will be assumed algebraically closed only when explicitly indicated in the text. As usual, $k^\times$ is the multiplicative group of k. The superscript * denotes dual vector space.

The signs $\varinjlim$ and $\varprojlim$ stand respectively for direct and inverse limit over a direct or inverse system of morphisms parametrized by $\mathbb{N}$ or $\mathbb{Z}_+$. $\Gamma(X, \mathcal{L})$ denotes the global sections of a sheaf $\mathcal{L}$ on a topological space X. All orders are assumed linear and strict, and all partial orders are assumed to have the additional property that the relation "neither $x \prec y$ nor $y \prec x$" is an equivalence relation.

1 Preliminaries

An *ind–variety* (over k) is a set X with a filtration

$$X_0 \subset X_1 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots \tag{1}$$

such that $X = \cup_{n \in \mathbb{Z}_+} X_n$, each X_n is a Noetherian algebraic variety, and the inclusions $X_n \subset X_{n+1}$ are closed immersions of algebraic varieties. An ind–variety X is automatically a topological space: a subset $U \subset X$ is *open* in X if and only if, for each n, $U \cap X_n$ is an open subvariety of X_n. The *sheaf of regular functions on* X, or *the structure sheaf* $\mathcal{O}_X$ of X, is the inverse limit $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_n}$ of the sheaves of regular functions $\mathcal{O}_{X_n}$ on the X_n. An ind–variety $X = \cup_{n \in \mathbb{Z}_+} X_n = \varinjlim X_n$ is *proper* if and only if all the varieties X_n are proper, is *affine* if and only if all the X_n are affine. A *morphism* from an ind–variety X to an ind–variety Y is a map $\varphi : X \to Y$ such that, for every $n \geq 0$, the restriction $\varphi|_{X_n}$ is a morphism of X_n into Y_m for some $m = m(n)$. An *isomorphism* of ind–varieties is a morphism which admits an inverse morphism. An *ind–subvariety* Z of X is a subset $Z \subset X$ such that $Z \cap X_n$ is a subvariety of X_n for each n. Finally, an *ind–group* is by definition a group object in the category of ind–varieties. In this paper we consider only ind–groups G which are *locally linear*, i.e. ind–varieties G with an ind–variety filtration $G_0 \subset G_1 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$, such that all G_n are linear algebraic groups and the inclusions are group morphisms.

Let V be a vector space of countable dimension. Fix an integer $l \geq 1$. The set $Gr(l; V)$ of all l–dimensional subspaces of V has a canonical structure of proper ind–variety: any filtration $0 \subset V_l \subset V_{l+1} \subset \ldots \subset V = \cup_{r \geq 0} V_{l+r}$, $\dim V_{l+r} = l + r$, induces a filtration

$$Gr(l; V_l) \subset Gr(l; V_{l+1}) \subset \ldots \subset Gr(l; V),$$

and the associated ind–variety structure on $Gr(l; V)$ is independent of choice of filtration on V. For $l = 1$, $\mathbb{P}(V) := Gr(1; V)$ is by definition the *projective ind–space associated to* V.

An *invertible sheaf* on an ind–variety X is a sheaf of $\mathcal{O}_X$–modules locally isomorphic to $\mathcal{O}_X$. The set of isomorphism classes of invertible sheaves on X is an abelian group (the group structure being induced by the operation of tensor product over $\mathcal{O}_X$ of invertible sheaves). By definition, the latter is the *Picard group* $\operatorname{Pic} X$ *of* X. It is an easy exercise to show that $\operatorname{Pic} X = \varprojlim \operatorname{Pic} X_n$ for any filtration (1). If $X = \mathbb{P}(V)$, then $\operatorname{Pic} X \cong \mathbb{Z}$. The preimage of 1 under this isomorphism is the class of the standard sheaf $\mathcal{O}_{\mathbb{P}(V)}(1)$ where, by definition, $\mathcal{O}_{\mathbb{P}(V)}(1) := \varprojlim \mathcal{O}_{\mathbb{P}(V_n)}(1)$.

An invertible sheaf $\mathcal{L}$ on a proper ind–variety X is *very ample* if, for some filtration (1), its restrictions $\mathcal{L}_n$ on X_n are very ample for all n, and all restriction maps $\Gamma(X_n; \mathcal{L}_n) \to \Gamma(X_{n-1}, \mathcal{L}_{n-1})$ are surjective. A very ample invertible sheaf defines a closed immersion of X into $\mathbb{P}(\varinjlim \Gamma(X_n, \mathcal{L}_n)^*)$ as for each n the restrictions $\mathcal{L}_n$ and $\mathcal{L}_{n-1}$ define a commutative diagram of closed immersions

$$\begin{array}{ccc} X_{n-1} & \hookrightarrow & \mathbb{P}(\Gamma(X_{n-1}, \mathcal{L}_{n-1})^*) \\ \cap & & \cap \\ X_n & \hookrightarrow & \mathbb{P}(\Gamma(X_n, \mathcal{L}_n)^*). \end{array}$$

Conversely, given a closed immersion $X \hookrightarrow \mathbb{P}(V)$, the inverse image of $\mathcal{O}_{\mathbb{P}(V)}(1)$ on X is a very ample invertible sheaf on X. Therefore, a proper ind–variety X is projective, i.e. X admits a closed immersion into a projective ind–space, if and only if it admits a very ample invertible sheaf.

2 Generalized flags: definition and first properties

Let V be a vector space over k. A *chain of subspaces in V* is a set $\mathcal{C}$ of pairwise distinct subspaces of V such that for any pair F', $F'' \in \mathcal{C}$, either $F' \subset F''$ or $F'' \subset F'$. Every chain of subspaces $\mathcal{C}$ is ordered by proper inclusion. Given $\mathcal{C}$, we denote by $\mathcal{C}'$ (respectively, by $\mathcal{C}''$) the subchain of $\mathcal{C}$ which consists of all $C \in \mathcal{C}$ with an immediate successor (respectively, an immediate predecessor). A *generalized flag in V* is a chain of subspaces $\mathcal{F}$ which satisfies the following conditions:

(i) each $F \in \mathcal{F}$ has an immediate successor or an immediate predecessor, i.e. $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$;

(ii) $V \backslash \{0\} = \cup_{F' \in \mathcal{F}'} F'' \backslash F'$, where $F'' \in \mathcal{F}''$ is the immediate successor of $F' \in \mathcal{F}'$.

Given a generalized flag $\mathcal{F}$ and a subspace $F'' \in \mathcal{F}''$ (respectively, $F' \in \mathcal{F}'$), we will always denote by F' (resp., by F'') its immediate predecessor (resp., immediate successor). Furthermore, condition (ii) implies that each nonzero vector $v \in V$ determines a unique pair $F'_v \subset F''_v$ of subspaces in $\mathcal{F}$ with $v \in F''_v \backslash F'_v$.

Example 1.

(i) We define a *flag in V* to be a chain of subspaces $\mathcal{F}$ satisfying (ii) and which is isomorphic as an ordered set to a subset of $\mathbb{Z}$. A flag can be equivalently defined as a chain of subspaces $\mathcal{F}$ for which there exists a strictly monotonic map of ordered sets $\varphi : \mathcal{F} \to \mathbb{Z}$ and, in addition, $\cap_{F \in \mathcal{F}} F = 0$ and $\cup_{F \in \mathcal{F}} = V$. There are four different kinds of flags: a finite flag of length k $\mathcal{F} = \{0 = F_1 \subset F_2 \subset \ldots \subset F_{k-1} \subset F_k = V\}$; an infinite accending flag $\mathcal{F} = \{0 = F_1 \subset F_2 \subset F_3 \ldots\}$, where $\cup_{i \geq 1} F_i = V$; an infinite descending flag $\mathcal{F} = \{\ldots \subset F_{-3} \subset F_{-2} \subset F_{-1} = V\}$, where $\cap_{i\leq -1}=0$; and a two sided infinite flag $\mathcal{F}=\{\ldots\subset F_{-1}\subset F_0\subset F_1\subset\ldots\}$, where $\cap_{i\in\mathbb{Z}}F_i=0$ and $\cup_{i\in\mathbb{Z}}F_i=V$.

(ii) One of the simplest examples of a generalized flag in V which is not a flag is a generalized flag with both an infinite accending part and an infinite descending part, i.e. $\mathcal{F}=\{0=F_1\subset F_2\subset F_3\subset\ldots\subset F_{-3}\subset F_{-2}\subset F_{-1}=V\}$, where $\cup_{i\geq 1}F_i=\cap_{j\leq -1}F_j$.

(iii) Let V be a countable dimensional vector space with basis $\{e_q\}_{q\in\mathbb{Q}}$. Set $F'(q):=\mathrm{span}\{e_r\,|\,r<q\}$ and $F''(q):=\mathrm{span}\{e_r\,|\,r\leq q\}$. Then $\mathcal{F}:=\cup_{q\in\mathbb{Q}}\{F'(q),F''(q)\}$ is a generalized flag in V. $\mathcal{F}$ is not a flag, and moreover, no $F\in\mathcal{F}$ has both an immediate predecessor and an immediate successor.

The following Proposition shows that each of the subchains $\mathcal{F}'$ and $\mathcal{F}''$ reconstructs $\mathcal{F}$.

Proposition 1 *Let $\mathcal{F}$ be a generalized flag in V. Then*
(i) *for every $F'\in\mathcal{F}'$, $F'=\cup_{G''\in\mathcal{F}'',G''\subset F'',G''\neq F''}G''$;*
(ii) *for every $F''\in\mathcal{F}''$, $F''=\cap_{G'\in\mathcal{F}',G'\supset F',G'\neq F'}G'$.*

Proof. (i) The inclusion $F'\supset\cup_{G''\in\mathcal{F}'',G''\subset F'',G''\neq F''}G''$ is obvious. Assume now that $v\in F'$. Let $v\in H''\backslash H'$ for some $H'\in\mathcal{F}'$ and its immediate successor $H''\in\mathcal{F}''$. Then $H'\subset F'$ and hence $H''\subset F'$, i.e. $v\in\cup_{G''\in\mathcal{F}'',G''\subset F'',G''\neq F''}G''$ which proves that $F'\subset\cup_{G''\in\mathcal{F}'',G''\subset F'',G''\neq F''}G''$. Assertion (ii) is proved in a similar way. □

Any chain $\mathcal{C}$ of subspaces in V determines the following partition of V:

$$V=\sqcup_{v\in V}[v]_{\mathcal{C}},\qquad\text{where}\qquad[v]_{\mathcal{C}}:=\{w\in V\,|\,w\in F\Leftrightarrow v\in F,\ \forall F\in\mathcal{C}\}.\tag{2}$$

Consider this correspondence as a map π from the set of chains of subspaces in V into the set of partitions of V. This map is not injective, for $\pi(\mathcal{C}')=\pi(\mathcal{C})$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by adding arbitrary intersections and unions of elements of $\mathcal{C}$. As we show in Proposition 2 below, the notion of a generalized flag provides us with a natural right inverse of π, i.e. with a map γ (defined on the image of π) such that $\pi\circ\gamma=\mathrm{id}$. This explains the special role of generalized flags among arbitrary chains of subspaces in V. Namely, every generalized flag in V is a natural representative of the class of chains of subspaces in V which yield the same partition of V.

Proposition 2 *Given a chain $\mathcal{C}$ of subspaces in V, there exists a unique generalized flag $\mathcal{F}$ in V for which $\pi(\mathcal{C})=\pi(\mathcal{F})$.*

Proof. To prove the existence, set $F'_v:=\cup_{W\in\mathcal{C},v\notin W}W$ and $F''_v:=\cap_{W\in\mathcal{C},v\in W}W$, and put $\mathcal{F}:=\cup_{v\in V\backslash\{0\}}\{F'_v,F''_v\}$. It is obvious from the definition of $\mathcal{F}$ that $\pi(\mathcal{C})=\pi(\mathcal{F})$. To show that $\mathcal{F}$ is a generalized flag, notice that, for any pair of nonzero vectors $u,v\in V$, exactly one of the following three possibilities holds:

$F'_u=F'_v$, and hence $F''_u=F''_v$;

$F'_u \subset F'_v$, and hence $F''_u \subset F'_v$;

$F'_u \supset F'_v$, and hence $F'_u \supset F''_v$.

Indeed, if, for every $W \in \mathcal{C}$, $u \in W$ if and only if $v \in W$, then $F'_u = F'_v$ and $F''_u = F''_v$. Assume now, that there exists $W \in \mathcal{C}$ such that $u \in W$ but $v \notin W$. Then $F''_u \subset W \subset F'_v$. Similarly, if there exists $W \in \mathcal{C}$ such that $u \notin W$ but $v \in W$, we have $F''_v \subset W \subset F'_u$. The existence of $\mathcal{F}$ is now established.

The uniqueness follows from the fact that $[v]_{\mathcal{C}} = (\cap_{W\in\mathcal{C}, v\in W} W)\backslash(\cup_{W\in\mathcal{C}, v\notin W} W)$, while, for a generalized flag $\mathcal{F}$, $[v]_{\mathcal{F}} = F''_v \backslash F'_v$. $\square$

We now define the map γ by setting $\gamma(\pi(\mathcal{C})) := \mathcal{F}$, and put $fl := \gamma \circ \pi$. In the example below we determine the preimages under fl of the generalized flags introduced in Example 1. The computation is based on the following simple fact: if $\bar{\mathcal{C}}$ is any chain in $fl^{-1}(\mathcal{F})$, then every nonzero subspace $\bar{C} \in \bar{\mathcal{C}}$ is the union of spaces from $\mathcal{F}$.

Example 2. The cases (i), (ii) and (iii) below refer to the corresponding cases in Example 1.

(i) If $\mathcal{F}$ is a flag in V then $fl^{-1}(\mathcal{F})$ consists of $\mathcal{F}$ and the chains obtained from $\mathcal{F}$ by adding 0, V or both, in case 0 and/or V do not belong to $\mathcal{F}$.

(ii) In this case $fl^{-1}(\mathcal{F})$ consists of two chains: $\mathcal{F}$ itself and the chain obtained by adding $\cup_{i\geq 1} F_i = \cap_{j\leq -1} F_j$ to $\mathcal{F}$.

(iii) In this case there are infinitely many chains $\bar{\mathcal{C}}$ with $fl(\bar{\mathcal{C}}) = \mathcal{F}$. Set $F'(x) := \mathrm{span}\{e_r \,|\, r < x\}$ for any $x \in \mathbb{R}$, and let $\mathcal{C}$ denote the chain $\{F'(x) \,|\, x \in \mathbb{R}\} \cup \{F''(q) \,|\, q \in \mathbb{Q}\} \cup \{0, V\}$. It is easy to check that $fl(\mathcal{C}) = \mathcal{F}$ and that any chain in $fl^{-1}(\mathcal{F})$ is a subchain of $\mathcal{C}$. To characterize explicitly all chains in $fl^{-1}(\mathcal{F})$, for any subchain $\bar{\mathcal{C}} \subset \mathcal{C}$, set $\mathbb{R}_{\bar{\mathcal{C}}} := \{x \in \mathbb{R} \,|\, F'(x) \in \bar{\mathcal{C}}\}$ and $\mathbb{Q}_{\bar{\mathcal{C}}} := \{q \in \mathbb{Q} \,|\, F''(q) \in \bar{\mathcal{C}}\}$. Then $fl(\bar{\mathcal{C}}) = \mathcal{F}$ if and only if, for any $r \in \mathbb{Q}$, we have $r \in \mathbb{Q}_{\bar{\mathcal{C}}}$ or $r = \inf\{x \in \mathbb{R}_{\bar{\mathcal{C}}} \cup \mathbb{Q}_{\bar{\mathcal{C}}} \,|\, r < x\}$, and $r \in \mathbb{R}_{\bar{\mathcal{C}}}$ or $r = \sup\{x \in \mathbb{R}_{\bar{\mathcal{C}}} \cup \mathbb{Q}_{\bar{\mathcal{C}}} \,|\, x < r\}$.

A generalized flag $\mathcal{F}$ in V is *maximal* if it is not properly contained in another generalized flag in V. It is easy to see that the generalized flags introduced in Example 1(ii) and (iii) are maximal. More generally, a generalized flag $\mathcal{F}$ is maximal if and only if $\dim(F''_v/F'_v) = 1$ for every nonzero $v \in V$. Indeed, assume $\dim(F''_{v_0}/F'_{v_0}) > 1$ for some v_0. Let $F \subset V$ be a subspace with proper inclusions $F'_{v_0} \subset F \subset F''_{v_0}$. Then the generalized flag $\mathcal{F} \cup \{F\}$ properly contains $\mathcal{F}$. Conversely, if $\dim(F''_v/F'_v) = 1$ for every nonzero $v \in V$, and if $\mathcal{G}$ is a generalized flag which contains $\mathcal{F}$, then $F'_v \subset G'_v \subset G''_v \subset F''_v$. Hence $F'_v = G'_v$, $F''_v = G''_v$, i.e. $\mathcal{F} = \mathcal{G}$.

The map fl establishes a bijection between maximal chains of subspaces in V and maximal generalized flags in V. More precisely, if $\mathcal{C}$ is a maximal chain, $fl(\mathcal{C})$ is the unique maximal generalized flag which is a subchain of $\mathcal{C}$. Conversely, $\mathcal{C}$ is the unique maximal chain containing $fl(C)$. These latter statements are essentially equivalent to Theorem 9 in [DP]. For example, if $\mathcal{F}$ is the maximal generalized flag from Example 1 (iii), its corresponding maximal chain $\mathcal{C}$ is described in Example 2 (iii).

We conclude this section by introducing isotropic generalized flags. Let $w : V \times V \to V$ be a non–degenerate symmetric or skew–symmetric bilinear form on V. Denote by $U^\perp$ the w–orthogonal complement of a subspace $U \subset V$. A generalized flag $\mathcal{F}$ in V is *w–isotropic* if there exists an involution $\tau : \mathcal{F} \to \mathcal{F}$ such that $(F')^\perp = \tau(F)''$ for every $F' \in \mathcal{F}'$. If $\mathcal{F}$ is w–isotropic, τ is determined by w and is a decreasing map. If τ has a fixed point, the latter is unique and will be denoted by F_τ. If τ has no fixed point, we introduce $F'_\tau \notin \mathcal{F}$ by setting $F'_\tau := \cup_{F \subset \tau(F)} F$. One checks immediately that $F'_\tau = \cap_{\tau(F) \subset F} F$.

3 Compatible bases

If V is finite dimensional, any ordered basis determines a maximal flag in V. Conversely, a maximal flag in V determines a set of compatible bases in V. More generally, if V is any vector space, $\mathcal{F}$ is a generalized flag in V and $\{e_\alpha\}_{\alpha \in A}$ is a basis of V, we say that $\mathcal{F}$ and $\{e_\alpha\}_{\alpha \in A}$ are *compatible* if there exists a strict partial order $\prec$ on A (satisfying the condition stated in the Conventions) such that $F'_{e_\alpha} = \mathrm{span}\{e_\beta \,|\, \beta \prec \alpha\}$ and $\mathcal{F} = \mathit{fl}(\{F'_{e_\alpha}\}_{\alpha \in A})$.

Not every generalized flag admits a compatible basis. Indeed, let $V := \mathbb{C}[[x]]$ be the space of formal power series in the indeterminate x and let $\mathcal{F}$ denote the flag $\ldots \subset F_n \subset F_{n-1} \subset \ldots \subset F_1 \subset F_0 = V$, where $F_n := x^n V$. Clearly, $\mathcal{F}$ is a maximal flag in V as $\dim(F_{n-1}/F_n) = 1$ for all $n > 0$. However, as V is uncountable dimensional, no basis of V can be compatible with the countable flag $\mathcal{F}$.

The following proposition shows that the uncountability of $\dim V$ is crucial in the above example.

Proposition 3 *If V is countable dimensional, every generalized flag $\mathcal{F}$ in V admits a compatible basis.*

Proof. Assume first that $\mathcal{F}$ is a maximal generalized flag in V. Let $\{l_i\}_{i \in \mathbb{N}}$ be a basis of V. Define inductively a basis $\{e_i\}_{i \in \mathbb{N}}$ of V as follows. Put $e_1 := l_1$. Assuming that $e_1, \ldots, e_n$ have been constructed, choose e_{n+1} of the form $l_{n+1} + c_1 e_1 + \ldots + c_n e_n$ so that $F'_{e_{n+1}}$ is not among $F'_{e_1}, \ldots, F'_{e_n}$. Then, obviously,

$$\mathrm{span}\{l_1, \ldots, l_n\} = \mathrm{span}\{e_1, \ldots, e_n\} \tag{3}$$

for every n and the subspaces F'_{e_n} are pairwise distinct. Furthermore, as it is not difficult to check, for every $F' \in \mathcal{F}'$, the set $F'' \backslash F'$ contains exactly one element of the basis $\{e_i\}_{i \in \mathbb{N}}$, and hence $\mathbb{N}$ is linearly ordered in the following way: $i \prec j$ if and only if $F'_{e_i} \subset F'_{e_j}$. This proves that $\mathcal{F}$ is compatible with $\{e_i\}_{i \in \mathbb{N}}$.

For a not necessarily maximal generalized flag $\mathcal{F}$, it is enough to consider a basis compatible with a maximal generalized flag $\mathcal{G}$ containing $\mathcal{F}$. Such a basis is automatically compatible with $\mathcal{F}$. $\square$

Let V be a finite or countable dimensional vector space and w be a non–degenerate symmetric or skew–symmetric bilinear form on V. Define a basis of V of the form $\{e_n, e^n\}$ to be *of type C* if $w(e_i, e_j) = w(e^i, e^j) = 0$ and $w(e_i, e^j) = \delta_{i,j}$ for a skew–symmetric w. A basis of V of the form $\{e_0 = e^0, e_n, e^n\}$ (respectively $\{e_n, e^n\}$) is *of type B* (resp. *of type D*) if $w(e_i, e_j) = w(e^i, e^j) = 0$ and $w(e_i, e^j) = \delta_{i,j}$ for a symmetric w. For uniformity we will always label a basis of type B, C or D simply as $\{e_i, e^i\}$ where we assume that in the case of B, $e_0 = e^0$ and i runs over $\mathbb{Z}_+$ when V is countable dimensional, or over a finite subset of $\mathbb{Z}_+$ when V is finite dimensional, while in the cases of C and D, i runs over $\mathbb{N}$ or over a finite subset of $\mathbb{N}$. A *w–isotropic basis of* V is by definition a basis of V admitting an order which makes it a basis of type B, C or D. If V is finite dimensional, V admits a basis of type B, C or D if and only if w is symmetric and V is odd dimensional, w is skew–symmetric and then V is necessarily even dimensional, or w is symmetric and V is even dimensional respectively. If V is countable dimensional, by an appropriate modification of the Gram–Schmidt orthogonalization process, one shows that if w is symmetric, then V admits both a basis of type B and a basis of type D, and if w is skew–symmetric, then V admits a basis of type C.

In the rest of the paper we assume that the dimension of V is countable.

Proposition 4 *Let $\mathcal{F}$ be an w–isotropic generalized flag in V.*
(i) *Assume that w is skew–symmetric. Then V admits a basis of type C compatible with $\mathcal{F}$. In particular, the vector space F''_τ/F'_τ is even dimensional or infinite dimensional.*
(ii) *Assume that k is algebraically closed and w is symmetric. If the vector space F''_τ/F'_τ is odd dimensional or infinite dimensional, then V admits a basis of type D compatible with $\mathcal{F}$. If F''_τ/F'_τ is even dimensional or infinite dimensional, then V admits a basis of type B compatible with $\mathcal{F}$.*

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ be a basis of V compatible with $\mathcal{F}$. Set $U_{F'} := \operatorname{span}\{l_i \mid F'_{l_i} = F'\}$ for $F' \in \mathcal{F}'$. Then $F' = \oplus_{G'\in\mathcal{F}', G'\subset F', G'\neq F'} U_{G'}$ and $F'' = F' \oplus U_{F'}$ for every $F' \in \mathcal{F}'$. It is clear therefore that the restriction of w on $U_{F'} \times U_{\tau(F')}$ is a non–degenerate bilinear form for every $F' \in \mathcal{F}'$. Furthermore, if τ has a fixed point, the restriction of w on $U_{F'_\tau} \times U_{F'_\tau}$ is a non-degenerate skew–symmetric (respectively symmetric) bilinear form. If w is skew–symmetric, this implies, in particular, that $U_{F'_\tau}$, and hence F''_τ/F'_τ is even dimensional or infinite dimensional. Then $U_{F'_\tau}$ admits a basis of type C, B, or D depending on whether w is skew–symmetric or symmetric and on the dimension of $U_{F'_\tau}$. Denote such a basis by $\{l'_i, l'^i\}$. Let, furthermore, $\{l''_i\}$ be the subset of $\{l_n\}$ consisting of all l_n with $F'_{l_n} \subset F'_\tau$. Finally, relabel the set $\{l'_i\} \cup \{l''_i\}$ and denote the resulting set by $\{g_n\}$, where $g_0 := l'_0 = l'^0$ if $\{l'_i, l'^i\}$ is of type B.

We are now ready to construct inductively the desired w–isotropic basis $\{e_n, e^n\}$. Assume that e_i, e^i have been constructed for $i \leq n$. Put $e_{n+1} := g_{n+1} - \sum_{i=1}^n (w(e_i, g_{n+1})e^i + w(g_{n+1}, e^i)e_i)$. There exists $g \in U_{\tau(F'_{g_{n+1}})}$ such that $w(e_{n+1}, g) = 1$. Set then $e^{n+1} :=$

$g - \sum_{i=1}^{n}(w(e_i, g)e^i + w(g, e^i)e_i)$. One checks immediately that the basis $\{e_n, e^n\}$ is w–isotropic and compatible with $\mathcal{F}$. $\square$

4 Ind–varieties of generalized flags

For a finite dimensional V, two flags belong to the same connected component of the variety of all flags in V if and only if their types coincide, i.e. if the dimensions of the subspaces in the flags coincide. If V is infinite dimensional the notion of type is in general not defined, and flags, or generalized flags, can be compared using a notion of commensurability. Such notions are well–known in the special case of subspaces of V, i.e. of flags of the form $0 \subset W \subset V$, see [T] and Chapter 7 of [PS]. Below we introduce a notion of commensurability for generalized flags which in the case of subspaces reduces to a refinement of Tate's notion of commensurability, [T].

In the rest of the paper we fix a basis $E = \{e_n\}$ of V. In the presence of a bilinear form w on V we fix a w–isotropic basis $E = \{e_n, e^n\}$, and whenever other bases of V or generalized flags in V are considered they are automatically assumed to be w–isotropic. We call a generalized flag $\mathcal{F}$ *weakly compatible with* E if $\mathcal{F}$ is compatible with a basis L of V such that $E \backslash (E \cap L)$ is a finite set. Furthermore, we define two generalized flags $\mathcal{F}$ and $\mathcal{G}$ in V to be *E–commensurable* if both $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E and there exists an inclusion preserving bijection $\varphi : \mathcal{F} \to \mathcal{G}$ and a finite dimensional subspace $U \subset V$, such that for every $F \in \mathcal{F}$

(i) $F \subset \varphi(F) + U$ and $\varphi(F) \subset F + U$;

(ii) $\dim(F \cap U) = \dim(\varphi(F) \cap U)$.

It follows immediately from the definition that any two E–commensurable generalized flags are isomorphic as ordered sets, and that two flags in a finite dimensional space are E–commensurable if and only if their types coincide. (In the latter case the condition of weak compatibility with E is empty.) Furthermore, E–commensurability is an equivalence relation. Indeed, it is obviously reflexive and symmetric. It is also transitive. To see this, note first that, in the definition of E–commensurability, one can replace (ii) by

(ii′) $\dim(F/(F \cap \varphi(F))) = \dim(\varphi(F)/(F \cap \varphi(F)))$.

Consider now $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$, such that $\mathcal{F}$ is E–commensurable with $\mathcal{G}$ and $\mathcal{G}$ is E–commensurable with $\mathcal{H}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be the respective bijections, and U and W be the finite dimensional subspaces of V corresponding to φ and ψ respectively. Then $\mathcal{F}$ and $\mathcal{H}$ satisfy (i) and (ii′) with $\psi \circ \varphi : \mathcal{F} \to \mathcal{H}$ and $U + W$.

Example 3.

(i) Let $\mathcal{F} = \{0 \subset F \subset V\}$ and $\mathcal{G} = \{0 \subset G \subset V\}$. If F and G are finite dimensional, then $\mathcal{F}$ and $\mathcal{G}$ are automatically weakly compatible with E. Furthermore, $\mathcal{F}$ and $\mathcal{G}$ are E–commensurable if and only if $\dim F = \dim G$. If, however, F and G are infinite dimensional, the condition that $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E is not automatic. For example, if $F = \operatorname{span}\{e_2, e_3, \ldots\}$ and $G = \{e_2 - e_1, e_3 - e_1, \ldots\}$, then $\mathcal{F}$ is weakly compatible with E but $\mathcal{G}$ is not, and consequently, $\mathcal{F}$ and $\mathcal{G}$ are not E–commensurable. Finally, if F and G are both of finite codimension in V, and $\mathcal{F}$ and $\mathcal{G}$ are weakly E–compatible, then $\mathcal{F}$ and $\mathcal{G}$ are E–commensurable if and only if $\operatorname{codim}_V F = \operatorname{codim}_V G$.

(ii) Let $\mathcal{F} = \{0 = F_1 \subset F_2 \subset F_3 \subset \ldots\}$ and $\mathcal{G} = \{0 = G_1 \subset G_2 \subset G_3 \subset \ldots\}$ be two finite or infinite accending flags in V compatible with E. If all subspaces F_i and G_i are finite dimensional, then $\mathcal{F}$ and $\mathcal{G}$ are E–commensurable if and only if $\dim F_i = \dim G_i$ for every i, and $F_n = G_n$ for large enough n. If, however, there are infinite dimensional spaces among F_i and G_i, the above conditions are still necessary for $\mathcal{F}$ and $\mathcal{G}$ to be E–commensurable but they are not always sufficient. The exact sufficient conditions can be derived as a consequence of the proof of Proposition 5 below.

Given a generalized flag $\mathcal{F}$ weakly compatible with E, we denote by $\mathcal{F}\ell(\mathcal{F}, E)$ the set of all generalized flags in V E–commensurable with $\mathcal{F}$. For the rest of the paper we fix the following notations: $E_n = \{e_i\}_{i \le n}$, $V_n = \operatorname{span} E_n$, $E_n^c := \{e_i\}_{i>n}$ and $V_n^c := \operatorname{span} E_n^c$. If $\mathcal{F}$ is a w–isotropic generalized flag in V, $\mathcal{F}\ell(\mathcal{F}, w, E)$ stands for the set of all w–isotropic generalized flags E–commensurable with $\mathcal{F}$. If $\mathcal{G}$ is an isotropic generalized flag in V, the involution τ is an order reversing involution on $\mathcal{G}$ considered as an ordered set. In this case $E_n = \{e_i, e^i\}_{i \le n}$, $V_n = \operatorname{span} E_n$, $E_n^c := \{e_i, e^i\}_{i>n}$ and $V_n^c := \operatorname{span} E_n^c$. Since all generalized flags in $\mathcal{F}\ell(\mathcal{F}, w, E)$ are isomorphic as ordered sets, we will use the same letter τ to denote the involution on any $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, w, E)$.

Proposition 5 *$\mathcal{F}\ell(\mathcal{F}, E)$, as well as $\mathcal{F}\ell(\mathcal{F}, w, E)$, has a natural structure of an ind–variety.*

Proof. We present the proof in the case of $\mathcal{F}\ell(\mathcal{F}, E)$ only. The reader will supply a similar proof for $\mathcal{F}\ell(\mathcal{F}, w, E)$. For any $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$ choose a positive integer $n_{\mathcal{G}}$ such that $\mathcal{F}$ and $\mathcal{G}$ are compatible with bases containing $E_{n_{\mathcal{G}}}^c$, and $V_{n_{\mathcal{G}}}$ contains a finite dimensional subspace U which (together with the corresponding φ) makes $\mathcal{F}$ and $\mathcal{G}$ E–commensurable. Obviously we can pick $n_{\mathcal{F}}$ so that $n_{\mathcal{F}} \le n_{\mathcal{G}}$ for every $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$. Set also

$$\mathcal{G}_n := \{G \cap V_n \,|\, G \in \mathcal{G}\} \tag{4}$$

for $n \ge n_{\mathcal{G}}$.

The type of the flag $\mathcal{F}_n$ yields a sequence of integers $0 = d_{n,0} < d_{n,1} < \ldots < d_{n,s_{n-1}} < d_{n,s_n} = n$, and $\mathcal{F}\ell(\mathcal{F}_n, E_n)$ is the usual flag variety $\mathcal{F}\ell(d_n; V_n)$ of type $d_n = (d_{n,1}, \ldots, d_{n,s_n - 1})$ in V_n. Notice that $s_{n+1} = s_n$ or $s_{n+1} = s_n + 1$. Furthermore, in both cases an integer j_n is determined as follows: in the former case, $d_{n+1,i} = d_{n,i}$ for $0 \le i < j_n$ and $d_{n+1,i} = d_{n,i} + 1$ for $j_n \le i < s_n$, and in the latter case $d_{n+1,i} = d_{n,i}$ for $0 \le i < j_n$ and $d_{n+1,i} = d_{n,i-1} + 1$ for $j_n \le i < s_n$.

Now we define a map $\iota_n : \mathcal{F}\ell(d_n; V_n) \to \mathcal{F}\ell(d_{n+1}, V_{n+1})$ for every $n \geq n_{\mathcal{F}}$. Given $\mathcal{G}_n = \{0 = G_0^n \subset G_1^n \subset \ldots \subset G_{s_n}^n = V_n\} \in \mathcal{F}\ell(d_n; V_n)$, put $\iota_n(\mathcal{G}_n) = \mathcal{G}_{n+1} := \{0 = G_0^{n+1} \subset G_1^{n+1} \subset \ldots \subset G_{s_{n+1}}^{n+1} = V_{n+1}\}$, where

$$
G_i^{n+1} := \begin{cases} G_i^n & \text{if} \quad 0 \leq i < j_n \\ G_i^n \oplus ke_{n+1} & \text{if} \quad j_n \leq i \leq s_{n+1} \quad \text{and} \quad s_{n+1} = s_n \\ G_{i-1}^n \oplus ke_{n+1} & \text{if} \quad j_n \leq i \leq s_{n+1} \quad \text{and} \quad s_{n+1} = s_n + 1. \end{cases} \tag{5}
$$

It is clear that ι_n is a closed immersion of algebraic varieties, and hence $\varinjlim \mathcal{F}\ell(d_n; V_n)$ is an ind–variety. Let $\psi_n : \mathcal{F}\ell(d_n; V_n) \to \varinjlim \mathcal{F}\ell(d_n; V_n)$ denote the canonical embedding corresponding to the direct system $\{\iota_n\}$.

To endow $\mathcal{F}\ell(\mathcal{F}, E)$ with an ind–variety structure we construct a bijection $\mathcal{F}\ell(\mathcal{F}, E) \to \varinjlim \mathcal{F}\ell(d_n; V_n)$. Set

$$
\theta : \mathcal{F}\ell(\mathcal{F}, E) \to \varinjlim \mathcal{F}\ell(d_n; V_n), \qquad \theta(\mathcal{G}) := \varinjlim \mathcal{G}_n,
$$

see (4). Checking that θ is injective is straightforward. To check that θ is surjective, fix $\widetilde{\mathcal{G}} = \varinjlim \widetilde{\mathcal{G}_n} \in \varinjlim \mathcal{F}\ell(d_n; V_n)$, an integer $\tilde{n}$ and a flag $\widetilde{\mathcal{G}_{\tilde{n}}} \in \mathcal{F}\ell(d_{\tilde{n}}; V_{\tilde{n}})$ with $\psi_{\tilde{n}}(\widetilde{\mathcal{G}_{\tilde{n}}}) = \widetilde{\mathcal{G}}$. Denote by $\varphi_{\tilde{n}}$ the inclusion preserving bijection $\varphi_{\tilde{n}} : \mathcal{F}_{\tilde{n}} \to \widetilde{\mathcal{G}_{\tilde{n}}}$. For every $F \in \mathcal{F}$, put $\varphi(F) := \varphi_{\tilde{n}}(F \cap V_{\tilde{n}}) \oplus (F \cap V_{\tilde{n}}^c)$. It is clear that $\mathcal{G} := \{\varphi(F)\}_{F \in \mathcal{F}}$ is a generalized flag in V E–commensurable with $\mathcal{F}$ via φ and $V_{\tilde{n}}$. Furthermore, using (5), one verifies that $\theta(\mathcal{G}) = \widetilde{\mathcal{G}}$. Hence θ is surjective. $\square$

Example 4. Let $\mathcal{F} = \{0 \subset F \subset V\}$, see Example 3 (i). If F is a finite dimensinal subspace of V of dimension l, then, regardless of E, $\mathcal{F}\ell(\mathcal{F}, E)$ is nothing but the ind–variety $Gr(l; V)$ introduced in Section 1. If F is infinite dimensional subspace of V of codimension l, then as a set $\mathcal{F}\ell(\mathcal{F}, E)$ depends on the choice of E. However, the isomorphisms between $Gr(l; V_n)$ and $Gr(n - l; V_n)$ extend to an ind–variety isomorphism between $\mathcal{F}\ell(\mathcal{F}, E)$ and $Gr(l; V)$ which depends on E. The latter isomorphism is a particular case of the following general duality. Let $\mathcal{F}$ be an arbitrary generalized flag in V. Assume that E is compatible with $\mathcal{F}$, and for every $F \in \mathcal{F}$ set $F^c := \mathrm{span}\{e \in E \mid e \notin F\}$. Then $\mathcal{F}^c := \{F^c \mid F \in \mathcal{F}\}$ is a generalized flag in V compatible with E and $\mathcal{F}\ell(\mathcal{F}^c, E)$ is isomorphic to $\mathcal{F}\ell(\mathcal{F}, E)$.

We complete this section by defining big cells in $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$. Let $L = \{l_n\}_{n \in \mathbb{N}}$ be a basis of V compatible with $\mathcal{F}$ and such that $E \backslash (E \cap L)$ is a finite set, and let $U_{F'} = \mathrm{span}\{l \in L \mid F'_l = F'\}$ for any $F' \in \mathcal{F}'$. Denote by $\Phi = \{\Phi_{F'}\}_{F' \in \mathcal{F}'}$ a set of linear maps of finite rank $\Phi_{F'} : F' \to U_{F'}$, such that $\Phi_{F'} \neq 0$ for finitely many subspaces $F'_1 \subset \ldots \subset F'_p$ only. Given Φ, define

$$
\begin{array}{ll} \Gamma_{F'} : F' \to F'', & \Gamma_{F'}(v) := v + \Phi_{F'}(v), \\ \Gamma : V \to V, & \Gamma(v) := \Gamma_{F'_p} \circ \ldots \circ \Gamma_{F'_{i+1}}(v), \end{array}
$$

where i is the largest integer with $v \notin F'_i$. Put $\Phi(\mathcal{F}) := \mathit{fl}(\{\mathit{\Gamma}(F')\}_{F' \in \mathcal{F}'})$. Then define the *big cell* $C(\mathcal{F}, E; L)$ *of* $\mathcal{F}\ell(\mathcal{F}, E)$ *corresponding to the basis* L by setting

$$
C(\mathcal{F}, E; L) := \{\Phi(\mathcal{F}) \mid \text{ for all possible } \Phi\}.
$$

To define the big cell $C(\mathcal{F}, w, E; L)$ in $\mathcal{F}\ell(\mathcal{F}, w, E)$, we start with a w–isotropic basis $L = \{l_n, l^n\}$ of V compatible with $\mathcal{F}$ and such that $E\backslash(E \cap L)$ is a finite set, and repeat the above construction of $\Gamma(F')$ for all $F' \in \mathcal{F}'$ with $F' \subset \tau(F')$. As a result we obtain subspaces $\Gamma(F')$ for $F' \subset \tau(F')$ and set $\Phi(\mathcal{F}) := fl(\{\mathit{\Gamma}(F'), (\mathit{\Gamma}(F'))^{\perp}\}_{F' \in \mathcal{F}, F' \subset \tau(F')})$. Then, we set

$$C(\mathcal{F}, w, E; L) := \{\Phi(\mathcal{F}) \,|\, \text{ for all possible } \Phi\}.$$

Note that the role of $\mathcal{F}$ in defining big cells is not special and that big cells $\mathcal{C}(\mathcal{G}, E; L)$, or $\mathcal{C}(\mathcal{G}, w, E; L)$, are well defined for every $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$, or respectively $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, w, E)$.

Proposition 6

(i) *The big cell $C(\mathcal{F}, E; L)$ (respectively, $C(\mathcal{F}, w, E; L)$) is an affine open ind–subvariety of $\mathcal{F}\ell(\mathcal{F}, E)$ (resp. of $\mathcal{F}\ell(\mathcal{F}, w, E)$).*

(ii) *We have*

$$\mathcal{F}\ell(\mathcal{F}, E) = \cup_L \; C(\mathcal{F}, E; L) \tag{6}$$

and

$$\mathcal{F}\ell(\mathcal{F}, w, E) = \cup_L \; C(\mathcal{F}, w, E; L), \tag{7}$$

where the unions run over all bases (respectively w–isotropic bases) L of V compatible with $\mathcal{F}$ and such that $E\backslash(E \cap L)$ is a finite set.

Proof. We discuss the case of $\mathcal{F}\ell(\mathcal{F}, E)$ only. The argument for the case of $\mathcal{F}\ell(\mathcal{F}, w, E)$ is similar. Put $L_n := \{l_i\}_{i \leq n}$ and $W_n := \operatorname{span} L_n$. Let $\mathcal{F}_n$ and $\mathcal{F}\ell(d_n; W_n)$ be as in the proof of Proposition 5 above. Set $\mathcal{C}(d_n; W_n; L_n) := \{\Phi(\mathcal{F})_n \,|\, \text{for all } \Phi \text{ such that, for every } F' \in \mathcal{F}', \Phi_{F'}(W_n) \subset W_n \text{ and } \Phi_{F'}(l_i) = 0 \text{ for } i > n\}$. Obviously, $\mathcal{C}(d_n; W_n; L_n)$ is a big cell in $\mathcal{F}\ell(d_n; W_n)$, and hence is an affine open subset. Therefore, the inclusion $\iota_n(\mathcal{C}(d_n; W_n; L_n)) \subset \mathcal{C}(d_{n+1}; W_{n+1}; L_{n+1})$ and the equality $\varinjlim \mathcal{C}(d_n; W_n; L_n) = C(\mathcal{F}, E; L)$ show that $C(\mathcal{F}, E; L)$ is an affine open ind–subvariety of $\mathcal{F}\ell(\mathcal{F}, E)$. The fact that the set of cells $\{C(\mathcal{F}, E; L) \,|\, L \text{ is a basis of } V \text{ compatible with } \mathcal{F}, \text{ such that } L\backslash(E \cap L) \text{ is a finite set}\}$ is a covering of $\mathcal{F}\ell(\mathcal{F}, E)$ is an easy consequence of the definition of E–commensurability. $\square$

5 Ind–varieties of generalized flags as homogeneous ind–spaces

Let $G(E)$ be the group of automorphisms g of V such that $g(e) = e$ for all but finitely many $e \in E$ and in addition $\det g = 1$. Recall that $E_n = \{e_i\}_{i \leq n}$ and $V_n = \operatorname{span} E_n$. The natural inclusion

$$G(E_n) \subset G(E_{n+1}), \qquad g \mapsto \kappa_n(g),$$

where $\kappa_n(g)_{|V_n} = g$ and $\kappa_n(g)(e) = e$ for $e \in E_{n+1} \backslash E_n$, is a closed immersion of algebraic groups. Furthermore, $G(E) = \cup_{n \in \mathbb{N}} G(E_n)$. In particular $G(E)$ is a locally linear ind–group, and $G(E) = G(L)$ for any basis L of V such that $E \backslash (E \cap L)$ is a finite set.

Similarly, when E is a w–isotropic basis of V, let $G^w(E) := \{g \in G(E) \,|\, w(g(u), g(v)) = w(u,v)$ for any $u, v \in V\}$. There are natural closed immersions $G^w(E_n) \subset G^w(E_{n+1})$, and $G^w(E) = \cup_{n \in \mathbb{N}} G^w(E_n)$, where in this case $E_n := \{e_i, e^i\}_{i \leq n}$.

The ind–group $G(E)$ (respectively, $G^w(E)$) is immediately seen to be isomorphic to the classical ind–group $A(\infty)$ (resp., $B(\infty)$, $C(\infty)$ or $D(\infty)$ if E is a w–isotropic basis of type B, C or D). The ind–groups $A(\infty)$, $B(\infty)$, $C(\infty)$ and $D(\infty)$ are discussed in detail in [DPW]. An alternative notation for $A(\infty)$ is $SL(\infty)$, and $B(\infty) \cong D(\infty)$ and $C(\infty)$ are also denoted respectively by $SO(\infty)$ and $Sp(\infty)$.

In the rest of the paper the letter G will denote one of the groups $G(E)$ or $G^w(E)$, and G_n will denote respectively $G(E_n)$ or $G^w(E_n)$. The basis E equips G with a subgroup H, consisting of all diagonal automorphisms of V in G, i.e. of the elements $g \in G$ such that $g(e) \in ke$ for every $e \in E$. We call H a *splitting Cartan subgroup* (in the terminology of [DPW], H is a Cartan subgroup of G). Following [DPW], for the purposes of the present paper, we define a *parabolic* (respectively, a *Borel*) *subgroup of* G to be an ind–subgroup P (resp., B) of G such that its intersection with G_n for every n is a parabolic (resp., a Borel) subgroup of G_n for some, or equivalently any, order on E.

If $\mathcal{F}$ is a generalized flag in V compatible with E (and w–isotropic, whenever E is w–isotropic), we denote by $P_{\mathcal{F}}$ the stabilizer of $\mathcal{F}$ in G.

Proposition 7
(i) *$P_{\mathcal{F}}$ is a parabolic subgroup of G containing H.*
(ii) *The map $\mathcal{F} \mapsto P_{\mathcal{F}}$ establishes a bijection between generalized flags in V compatible with E and parabolic subgroups of G containing H.*

Proof. The inclusion $H \subset P_{\mathcal{F}}$ follows directly from the definition of H and $P_{\mathcal{F}}$. Furthermore, $P_{\mathcal{F}} \cap G_n$ is a parabolic subgroup of G_n as it is the stabilizer of $\mathcal{F}_n$ in G_n. Hence $P_{\mathcal{F}}$ is a parabolic subgroup of G. If, conversely, $P = \cup_n P_n$ is a parabolic subgroup of G containing H, denote by $\mathcal{F}(n)$ the flag in V_n whose stabilizer is P_n. Note that $\mathcal{F}(n)$ maps into $\mathcal{F}(n+1)$. More precisely, for $G = G(E)$, $\mathcal{F}(n+1) = \iota_n(\mathcal{F}(n))$, see (5); and for $G = G^w(E)$, the corresponding map is the w–isotropic analog of ι_n which we leave to the reader to reconstruct. In both cases we define $\mathcal{F}$ as $\theta^{-1}(\varinjlim \mathcal{F}(n))$. A direct checking shows that $P = P_{\mathcal{F}}$. □

Proposition 7 further justifies our consideration of generalized flags, see the discussion before Proposition 2 above. Indeed, it is clear that if $P \subset G$ is the stabilizer of a chain $\mathcal{C}$ of subspaces in V, then P depends only the partition $\pi(\mathcal{C})$, see (2), and not on $\mathcal{C}$ itself. Therefore, the generalized flag $\mathcal{F}$ emerges as a representative of the class of all chains $\mathcal{C}$ which have P as a stabilizer in G. Moreover, Proposition 3 together with Proposition 7 (respectively, Proposition 4 and Proposition 7 for w–isotropic flags), imply that the stabilizer in G of any generalized flag (resp. isotropic generalized flag) compatible with E is a parabolic subgroup of G. Finally, maximal generalized flags in V correspond to Borel subgroups under the above bijection.

Note that, for any order on E and for any generalized flag $\mathcal{F}$ compatible with E, $G/P_{\mathcal{F}} = \cup_n(G_n/P_n)$, where $P_n := P_{\mathcal{F}} \cap G_n$. In particular, $G/P_{\mathcal{F}}$ is an ind–variety. Moreover, any other order on E, for which E is isomorphic to $\mathbb{N}$ as an ordered set, defines an isomorphic ind–variety. We are now ready to exhibit the homogeneous ind–space structure on $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$.

Theorem 1 *For any E and $\mathcal{F}$ as above there is a respective isomorphism of ind–varieties $\mathcal{F}\ell(\mathcal{F}, E) \cong G/P_{\mathcal{F}}$ or $\mathcal{F}\ell(\mathcal{F}, w, E) \cong G/P_{\mathcal{F}}$.*

Proof. Given $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$ (or, respectively, $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, w, E)$), let $U \subset V$ be the corresponding to $\mathcal{G}$ finite dimensional subspace. We may assume that $U = V_n = \operatorname{span} E_n$ for some n. Since $\mathcal{F}_n$ and $\mathcal{G}_n$ are flags of the same type in the finite dimensional space V_n, there exists $g_n \in G_n$, so that $g(\mathcal{F}_n) = \mathcal{G}_n$. We extend g_n to an element $g \in G$ by setting $g(e) = e$ for $e \in E \backslash E_n$. Now

$$f : \mathcal{F}\ell(\mathcal{F}, E) \to G/P_{\mathcal{F}} \text{ (or, respectively, } f : \mathcal{F}\ell(\mathcal{F}, w, E) \to G/P_{\mathcal{F}}), \quad f(\mathcal{G}) := gP$$

is a well–defined map and it is easy to check that it is an isomorphism of ind–varieties. □

6 Picard group and projectivity

The interpretation of $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$ as homogeneous ind–spaces $G/P_{\mathcal{F}}$ provides us with a representation theoretic description of the Picard groups of $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$. Namely, $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E)$, as well as $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, w, E)$, is naturally isomorphic to the group of integral characters of the Lie algebra of the ind–group $P_{\mathcal{F}}$.

Consider $\mathcal{F}\ell(\mathcal{F}, E)$. There is a canonical isomorphism of abelian groups $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E) = \operatorname{Hom}(P_{\mathcal{F}}, k^\times)$. To see this, notice that $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E) = \varprojlim \operatorname{Pic} \mathcal{F}\ell(d_n; V_n) = \varprojlim \operatorname{Pic} G_n/(P_{\mathcal{F}})_n$. It is a classical fact that $\operatorname{Pic} G_n/(P_{\mathcal{F}})_n = \operatorname{Hom}((P_{\mathcal{F}})_n, k^\times)$ for every n, and an immediate verification shows that the diagram

$$\begin{array}{ccc} \operatorname{Pic}(G_{n+1}/(P_{\mathcal{F}})_{n+1}) & \cong & \operatorname{Hom}((P_{\mathcal{F}})_{n+1}, k^\times) \\ \downarrow & & \downarrow \\ \operatorname{Pic}(G_n/(P_{\mathcal{F}})_n) & \cong & \operatorname{Hom}((P_{\mathcal{F}})_n, k^\times) \end{array} \tag{8}$$

is commutative. Hence $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E) \cong \operatorname{Hom}(P, k^\times)$, and $\operatorname{Hom}(P_{\mathcal{F}}, k^\times)$ is nothing but the group of integral characters of the Lie algebra of $P_{\mathcal{F}}$. In the case of $\mathcal{F}\ell(\mathcal{F}, w, E)$ the desired isomorphism is established by replacing $\operatorname{Hom}((P_{\mathcal{F}})_n, k^\times)$ and $\operatorname{Hom}((P_{\mathcal{F}})_{n+1}, k^\times)$ in diagram (8) with the groups of integral characters of the Lie algebras of $P_{\mathcal{F}_n}$ and $P_{\mathcal{F}_{n+1}}$ respectively.

In the rest of this section we give a purely geometric description of $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},E)$ and $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},w,E)$. Consider the corresponding covering (6) or (7). Let L and M be two bases compatible with $\mathcal{F}$ for which $E\backslash(E\cap L)$ and $E\backslash(E\cap M)$ are finite sets. Denote by $g_{L,M}$ the automorphism of V such that $g_{L,M}(l_i) = m_i$ for $\mathcal{F}\ell(\mathcal{F},E)$, and $g_{L,M}(l_i) = m_i$, $g_{L,M}(l^i) = m^i$ for $\mathcal{F}\ell(\mathcal{F},w,E)$. It has a well–defined determinant and, moreover, it induces an automorphism of F''/F'. Denote the determinant of this latter automorphism by $\det_{L,M}(F''/F')$. In this way we obtain an invertible sheaf $\mathcal{L}_{F'}$ with transition functions $\det_{L,M}(F''/F')$ on $C(\mathcal{F},E;L)\cap C(\mathcal{F},E;M)$ or $C(\mathcal{F},w,E;L)\cap C(\mathcal{F},w,E;M)$ respectively. Finally, let $\gamma_{F'}\in \mathrm{Pic}\,\mathcal{F}\ell(d_n;V_n)$, respectively $\gamma_{F'}\in \mathrm{Pic}\ \mathcal{F}\ell(d_n,w;V_n)$, denote the class of $\mathcal{L}_{F'}$.

Proposition 8 *There are canonical isomorphisms of abelian groups* $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},E)\cong (\Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}))/(\mathbb{Z}\,\Pi_{F'\in\mathcal{F}'}\gamma_{F'})$ *and* $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},w,E)\cong \Pi_{F'\in\mathcal{F}',F'\subset\tau(F'),F'\neq F'_\tau}(\mathbb{Z}\gamma_{F'})$.

Proof. Consider the case of $\mathcal{F}\ell(\mathcal{F},E)$ first. Let $\gamma_{F',n}$ the class of the restriction $(\mathcal{L}_{F'})_n$ of $\mathcal{L}_{F'}$ to $\mathcal{F}\ell(d_n;V_n)$. Then $\gamma_{F',n}=0$ unless $F''\cap V_n\neq F'\cap V_n$. Define the group homomorphism $\varphi_n : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'})\to \mathrm{Pic}\,\mathcal{F}\ell(d_n;V_n)$ via $\varphi_n(\Pi_{F'\in\mathcal{F}'}m_{F'}\gamma_{F'}) := \sum_{F'\in\mathcal{F}'}m_{F'}\gamma_{F',n}$. The sum $\sum_{F'\in\mathcal{F}'}m_{F'}\gamma_{F',n}$ makes sense because $\gamma_{F',n}=0$ for all but finitely many $F'\in\mathcal{F}'$. Clearly $\varphi_n = r_n\circ\varphi_{n+1}$, where $r_n : \mathrm{Pic}\,\mathcal{F}\ell(d_{n+1};V_{n+1})\to\mathrm{Pic}\,\mathcal{F}\ell(d_n;V_n)$ is the restriction map. Therefore, by the universality property of $\varprojlim$, there is a homomorphism

$$\varphi : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'})\to \mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},E) = \varprojlim \mathrm{Pic}\,\mathcal{F}\ell(d_n;V_n).$$

Furthermore φ is surjective as φ_n is surjective for each n.

To compute $\ker\varphi$, note that $\ker\varphi = \cap\ker\varphi_n$. We have $\ker\varphi_n = (\mathbb{Z}(\Pi_{F'\in\mathcal{F}'}\gamma_{F'}))\times \Pi_{F'\in\mathcal{F}',F'\cap V_n=F''\cap V_n}(\mathbb{Z}\gamma_{F'})$ and therefore $\ker\varphi = \mathbb{Z}(\Pi_{F'\in\mathcal{F}'}\gamma_{F'})$, i.e. $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},E)\cong (\Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}))/(\mathbb{Z}\,\Pi_{F'\in\mathcal{F}'}\gamma_{F'})$.

In the case of $\mathcal{F}\ell(\mathcal{F},w,E)$ homomorphisms $\varphi_n : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'})\to\mathrm{Pic}\,\mathcal{F}\ell(d_n,w;V_n)$ and $\varphi : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'})\to \mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},w,E)$ are defined in a similar way. Here $\ker\varphi_n = \Pi_{F'\in\mathcal{F}',F'\subset\tau(F')}\ (\mathbb{Z}(\gamma_{F'}+\gamma_{\tau(F')'}))\times\Pi_{F'\in\mathcal{F}',F'\subset\tau(F'),F'\cap V_n=F''\cap V_n}(\mathbb{Z}\gamma_{F'})$, and consequently $\ker\varphi = \Pi_{F'\in\mathcal{F}',F'\subset\tau(F')}\ (\mathbb{Z}(\gamma_{F'}+\gamma_{\tau(F')'}))$ i.e. $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},w,E)\cong\Pi_{F'\in\mathcal{F}',F'\subset\tau(F'),F'\neq F'_\tau}(\mathbb{Z}\gamma_{F'})$. $\square$

We complete this paper by an explicit criterion for the projectivity of $\mathcal{F}\ell(\mathcal{F},E)$ and $\mathcal{F}\ell(\mathcal{F},w,E)$. The following Proposition is a translation of Proposition 15.1 in [DPW] into the language of generalized flags.

Proposition 9 *$\mathcal{F}\ell(\mathcal{F},E)$ or $\mathcal{F}\ell(\mathcal{F},w,E)$ is projective if and only if $\mathcal{F}$ is a flag.*

Proof. Consider the case of $\mathcal{F}\ell(\mathcal{F},E)$ (the case of $\mathcal{F}\ell(\mathcal{F},w,E)$ is similar). $\mathcal{F}\ell(\mathcal{F},E)$ is projective if and only if it admits a very ample invertible sheaf. An immediate verification shows that an invertible sheaf $\mathcal{L}$, whose class in $\mathrm{Pic}\,\mathcal{F}\ell(\mathcal{F},E)$ is the image of $\Pi_{F'\in\mathcal{F}'}m_{F'}\gamma_{F'}$, is very ample if and only if the map $c:\mathcal{F}'\to\mathbb{Z}$, $F'\mapsto m_{F'}$ is strictly increasing. Indeed,

$\mathcal{F}\ell(\mathcal{F}, E) = \varinjlim \mathcal{F}\ell(d_n; V_n)$, and $\mathcal{L}$ is very ample if and only if its restrictions $\mathcal{L}_n$ onto $\mathcal{F}\ell(d_n; V_n)$ are very ample for all n. Consider the map $c_n : \mathcal{F}'_n \to \mathbb{Z}$, defined via $c_n((\mathcal{F}_n)'_v) := c((\psi_n(\mathcal{F}_n))'_v)$ for every nonzero $v \in V_n$. (As the reader will check, c_n is well defined, i.e. if $(\mathcal{F}_n)'_{v_1} = (\mathcal{F}_n)'_{v_2}$, then $(\psi_n(\mathcal{F}_n))'_{v_1} = (\psi_n(\mathcal{F}_n))'_{v_2}$.) According to the classical Bott–Borel–Weil Theorem, $\mathcal{L}_n$ is very ample if and only if the map c_n is strictly increasing. Hence, $\mathcal{L}$ is very ample if and only if c is strictly increasing. This enables us to conclude that $\mathcal{F}\ell(\mathcal{F}, E)$ is projective if and only if there exists a strictly increasing map $\mathcal{F}' \to \mathbb{Z}$, i.e. if and only if $\mathcal{F}$ is a flag. $\square$

Propositions 8 and 9 allow us to make some initial remarks concerning the isomorphism classes of the ind–varieties $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$. For example, if $\mathcal{F}$ is a flag of finite length in V, and $\mathcal{G}$ is a flag (or generalized flag) in V of length different from the length of $\mathcal{F}$ (finite or infinite), then $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{G}, L)$ are not isomorphic because their Picard groups are not isomorphic. Furthermore, if $\mathcal{F}$ is a flag in V but $\mathcal{G}$ is not, then $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{G}, L)$ are not isomorphic because the former ind–variety is projective and the latter is not. Finally, a recent result of J. Donin and the second named author, [DoP], implies that if $\mathcal{F} = \{0 \subset F \subset V\}$ with F both infinite dimensional and of infinite codimension in V, then the "ind–grassmannian" $\mathcal{F}\ell(\mathcal{F}, E)$ is not isomorphic to $Gr(l; V)$ for any l, cf. Example 4.

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