A simple formula for the Casson-Walker invariant

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1 Introduction

The Casson-Walker invariant $\lambda_w(M)$ is one of the fundamental invariants of rational homology spheres. Its restriction to the class of integer homology spheres is an integer extension of the Rokhlin invariant [1, 16]. In the theory of finite type invariants of 3-manifolds it is the simplest $\mathbb{Q}$-valued invariant after $|H_1(M)|$ [3]. $\lambda_w(M)$ remains, however, in general quite difficult to calculate. While it is easy to do for a manifold M_L obtained from S^3 by surgery on a framed knot L, the same question for links remains quite complicated. In particular, for 2-component links satisfactory formulas only exist for some special cases [7, 5, 6]. Formulas from [8], although explicit, require enormously huge calculations, even in simplest cases (see Section 8).

Problem. Given a framed 2-component link $L = L_1 \cup L_2$ with the linking matrix $\mathbb{L}$, we want

1. Find simple explicit formulas for $\lambda_w(M_L)$;
2. Understand/separate the dependence of $\lambda_w(M_L)$ on L (considered as an unframed link) and on $\mathbb{L}$.

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2 Arrow diagrams

Let A be an oriented 3-valent graph whose edges are divided into two classes: *fat* and *thin*. The union of all vertices and all fat edges of A is called a *skeleton* of A.

Definition 1. *A is called an* arrow diagram*, if the skeleton of A consists of disjoint circles. Thin edges are called* arrows.

It follows from the definition that each arrow connects two vertices, which may lie in the same circle or in different circles. By a *based* arrow diagram we mean an arrow diagram with a marked point in the interior of one of its fat edges. See Fig. 1 for simple examples of based and unbased arrow diagrams.

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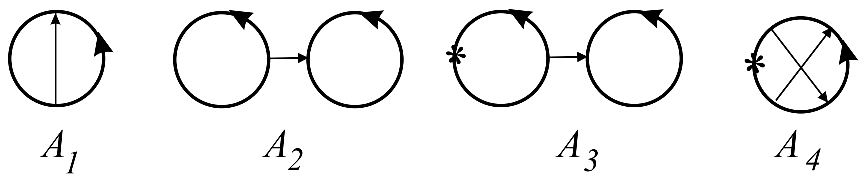

Figure 1: Simple examples of arrow diagrams.

3 Gauss diagrams

Recall that an n-component link diagram is a generic immersion of the disjoint union of $n \geq 1$ oriented circles to plane, equipped with the additional information on overpasses and underpasses at double points. Any link diagram can be presented numerically by its Gauss code, which consists of several strings of signed integers. The strings are obtained by numbering double points and traversing the components. Each time when we pass a double point number k, we write k if we are on the upper strand and $-k$ if on the lower one. If we prefer to distinguish knots and their mirror images or if we are considering a link with ≥ 2 components, then an additional string of $\varepsilon_i = \pm 1$ called *chiral signs* is needed. Here i runs over all double points a_i of the diagram and signs are determined by the right hand grip rule. See Fig. 2.

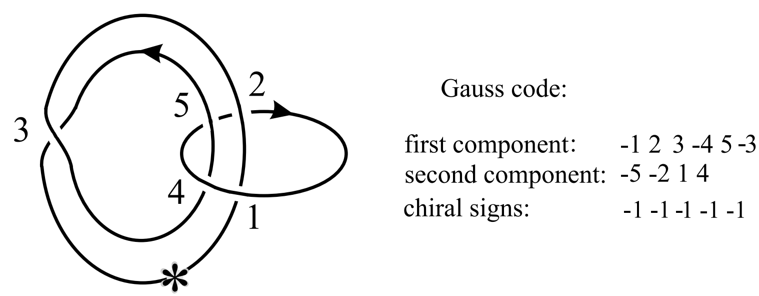

Figure 2: A 2-component link and its Gauss code

A convenient way to visualize a Gauss code is a *Gauss diagram* consisting of the oriented link components with the preimages of each double point connected with an arrow from the upper point to the lower one. Each arrow c is equipped with the chiral sign of the corresponding double point. The numbering of endpoints of arrows is not necessary anymore, see Fig. 3. We say that a link diagram is *based* if a non-double point in one of its components is chosen. An equivalent way of saying this consists in considering *long links* in R^3, when the base point is placed in infinity. If the link is based, then the corresponding Gauss diagram is based too. Note that forgetting signs converts any Gauss diagram into an arrow diagram, but not any arrow diagram (for example, a fat circle with two thin oriented diameters) can be realized by a Gauss diagram of a link. However, that is possible, if we allow virtual links. This is actually the main idea of the virtualization.

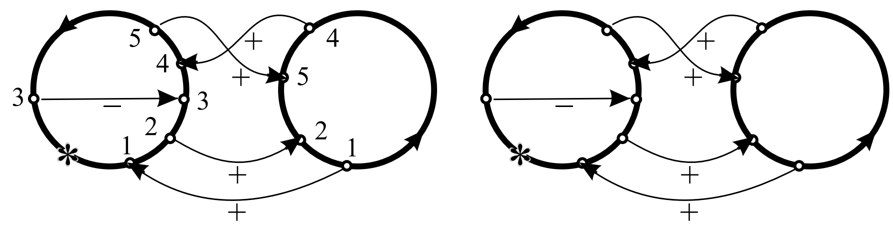

Figure 3: Two presentations of the Gauss diagram for the link diagram in Fig. 2

4 Arrow diagrams as functionals on Gauss diagrams

As described in [**?**, 10], any arrow diagram A defines an integer-valued function $\langle A, *\rangle$ on set of all Gauss diagrams. Let A be an n-component arrow diagram and G be an n-component Gauss diagram. By a *representation* of A in G we mean an embedding of A to G which takes the circles and arrows of A to the circles, respectively, arrows of G such that the orientations of all circles and arrows are preserved. If both diagrams are based, then representations must respect base points. For a given representation $\varphi\colon A \to G$ we define its *sign* by $\varepsilon(\varphi) = \prod \varepsilon(\varphi(a))$, where the product is taken over all arcs $a \in A$.

Definition 2. *Let A be an n-component arrow diagram. Then for any n-component arrow diagram G we set $\langle A, G\rangle = \sum \varepsilon(\varphi)$, where the sum is taken over all representations of A in G.*

Example 1. Let us describe functions for arrow diagrams $A_1 - A_4$ shown in Fig. 1. Evidently, A_1 determines the writhe of the link, which is defined as the sum of the chiral signs of all double points. Let G be a Gauss diagram of an oriented 2-component link $L = L_1 \cup L_2$. Then $\langle A_2, G\rangle = 2n$, where $n = \text{lk}\ (L_1, L_2)$ is the linking number of the components. Indeed, for any arrow of G we have exactly one representation of A_2 to G. Therefore, all double points contribute to $\langle A_3, G\rangle$ (not only points where the one preferred component is over the other). If we insert a base point into A_2 and a base point into G (thus fixing ordering of the two link component), we get n without doubling. It means that $\langle A_3, G\rangle = \text{lk}\ (L_1, L_2)$. The meaning of $\langle A_4, G\rangle$ is more complicated. If G is a Gauss diagram of a knot $K \subset S^3$, then $\langle A_4, G\rangle$ is the second coefficient v_2 of the Conway polynomial of K, which is often called the Casson invariant of K. See [**?**] for a diagrammatic description and properties of v_2.

Example 2. For arrow diagrams U_1, U_2 shown in Fig. 4 and the Gauss diagram G shown in Fig. 3, we have $\langle U_1, G\rangle = 0$ and $\langle U_2, G\rangle = -1$. The image of the unique representation of U_2 to G contains arrows 2,3,4.

Figure 4: Two arrow diagrams

Remark 1. Often it is convenient to extend Definition 2 by linearity to the free abelian group generated by arrow diagrams. Let $A = \sum_{i=1}^{m} k_i A_i$ be a linear combination of arrow diagrams. In general, the value $\langle A, G\rangle$ depends on the choice of the Gauss diagram G of a given link L as well as on the choice of the base point. However, for some carefully composed linear combinations of arrow diagrams the result does not depend on the above choices. This gives a link invariant $\langle A, G\rangle$, which we will denote by $A(L)$. It is easy to show that such an invariant is of finite type. Moreover, any finite type invariant of long knots can be presented in such a form. A similar result for links is unknown. See [10].

5 A useful finite type link invariant of order 3

Consider the following linear combination $U = U_1 + U_2 + U_3 + U_4$ of arrow diagrams (see Fig. 5).

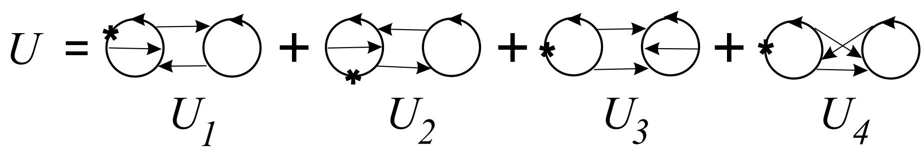

Figure 5: Remarkable linear combination of arrow diagrams

Example 3. For the link and Gauss diagrams shown in Fig. 2 and Fig. 3 we have $\langle U, G\rangle = -1+1 = 0$, since there are only two representations of summands of U in G: one representation of U_2 (see Example 2) and one representation of U_4 with arrows 2,4,5 in its image.

Example 4. For the link and Gauss diagrams shown in Fig. 6 we have $\langle U, G\rangle = -1-1 = -2$, since no summands of U have representation in G except the first one which has two representations described by arrow triples (2,5,7) and (4,5,7).

Let n be an integer and B an unknotted annular band having n negative full twists if $n \geq 0$ and $|n|$ positive full twists if $n < 0$, see Fig. 7 for $n = 5$. If we equip the components of ∂B by orientations induced by an orientation of B, then their linking number is equal to n.

Definition 3. *The 2-component link ∂B is called the* generalized Hopf link *and denoted $H(n)$.*

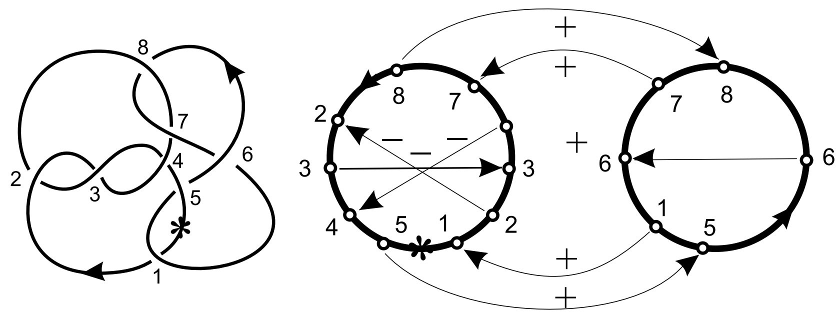

Figure 6: Link 8^2_{11} in the Alexander-Briggs-Rolfsen table [15].

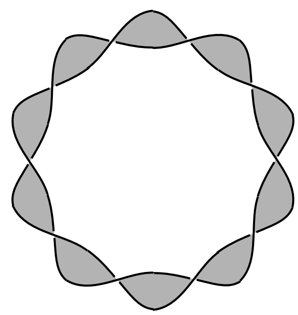

Figure 7: Generalized Hopf link $H(n)$ for $n = 5$.

Of course, $H(n)$ has many different diagrams. For example, two differently oriented diagrams of $H(n)$ for $n = 3$ are shown in Fig. 8. Note that the linking numbers of their components have opposite signs: 3 for the top diagram and -3 for the bottom one.

Example 5. The diagrams of $H(3)$ mentioned above differ only by orientation of one component. Nevertheless, their Gauss diagrams look quite different. For the top Gauss diagram G_t we have $\langle U, G_t \rangle = 0$, since no summands of U have representation in G_t. Of course, the same fact holds for any n. For the bottom diagram we get $\langle U, G_b \rangle = 4$. Indeed, in this case there are four representations of U_4 in G_b. They can be described by four triples of positive arrows (1,2,3),(1,2,5),(1,4,5), (3,4,5) contained in their images. Since $0 = \langle U, G_u \rangle \neq \langle U, G_l \rangle = 4$, we may conclude that the value of $\langle U, G \rangle$ depends of orientation of the components. Nevertheless, the following proposition shows that aside this phenomenon $\langle U, G \rangle$ is invariant.

Proposition 1. *U determines an invariant $U(L) = \langle U, G \rangle$ for oriented links of two numbered components.*

Proof. We will always assume that the base point is in the first component. It suffices to show that $\langle U, G \rangle$ is invariant with respect to

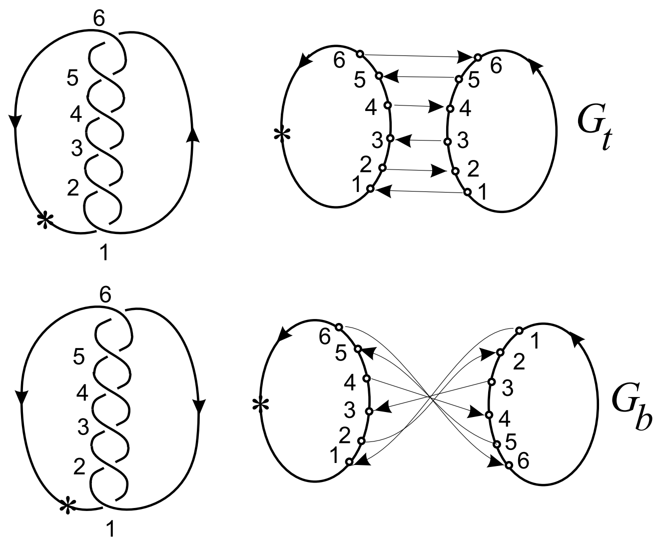

Figure 8: $H(3)$ with different orientations

1. Reidemeister moves far from the base point.

2. Replacing the base point within the first component.

In order to verify the invariance under Reidemeister moves, it suffices to show that $\langle U, R\rangle = 0$, where R runs over all relations of the Polyak algebra P defined in [4].

Invariance of $\langle U, G\rangle$ under replacing the base point within the same component follows from the observation that two link diagrams with the common base point in their first components are Reidemeister equivalent if and only if they are equivalent via Reidemeister moves performed far from the common base point. One can give a "folklore" reformulation of that fact by saying that the theory of links is equivalent to the theory of *long links* (having in mind that the base point is at infinity). ∎

Remark 2. One can show that $\langle U, G\rangle$ does not depend on the ordering of the components of L. We prefer to extract this result from our main theorem (see Corollary 1).

6 The Formula

Let $L = L_1 \cup L_2$ be an oriented 2-component framed link such that the corresponding 3-manifold M_L is a rational homology sphere, i.e. the first homology group of M_L is finite. Denote by a, b, and n the framings of L_1, L_2, and their linking number, respectively. Then the linking matrix $\mathbb{L} = \begin{pmatrix} a & n \\ n & b \end{pmatrix}$ of L has non-zero determinant $D = \mathrm{Det}(\mathbb{L})$. Note that the first homology group $H_1(M_L)$ is finite and its order $|H_1(M_L)|$ equals to $|D|$. It is easy to see that the signature $\sigma(\mathbb{L})$ of $\mathbb{L}$ can be found by the following rule:

$$\sigma(\mathbb{L}) = \begin{cases} 0 & \text{if } D < 0 \\ 2 & \text{if } D > 0 \text{ and } a+b > 0 \\ -2 & \text{if } D > 0 \text{ and } a+b < 0 \end{cases}$$

We point out that if $D > 0$, then a, b have the same sign. Therefore, the sign of σ is determined only by the sign, say, of a.

Let $\lambda_w(M_L)$ be the Casson-Walker invariant of M_L, see [16]. We normalize it so as to have $\frac{1}{2}\lambda_w(P_{120}) = 1$, where P_{120} is the positively oriented Poincaré homology sphere, which is obtained from S^3 by surgery along the right-handed trefoil with framing 1. Recall that $v_2(K)$ denotes the Casson invariant of a knot $K \subset S^3$. It coincides with the coefficient at z^2 of the Conway polynomial of K. It can also be extracted from the Alexander polynomial $\Delta_K(t)$ normalized so that $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$ as follows: $v_2(K) = \frac{1}{2}\Delta_K''(1)$. We introduce a function $F\colon \mathcal{L} \to Q$, where $\mathcal{L}$ is the set of all oriented framed two-component links, as follows.

Definition 4.

$$F(L) = \frac{1}{12}(n^3-n)+\frac{1}{24}(a+b)(n^2-D-2)+\frac{1}{8}\sigma(\mathbb{L})D+av_2(L_2)+bv_2(L_1)-U(L)$$

Theorem 3. *(Main) For any oriented framed 2-component link $L = L_1 \cup L_2$ we have $\frac{1}{2}D\lambda_w(M_L) = F(L)$.*

Example 6. Let us apply the main theorem for calculation of the Casson-Walker invariant λ_w for the manifold $M_{H(2,a,b)}$, where $H(2,a,b) = L_1 \cup L_2$ is the Hopf link $H(2)$ shown in Fig. 9 and a, b are the framings of its components.

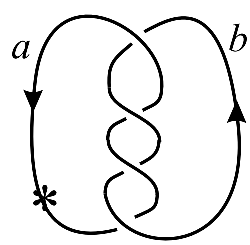

Figure 9: Generalized Hopf link $H(2)$ framed by a, b

We orient the components of $H(-2)$ such that their linking number is 2. It is easy to see that $U(H(2)) = 0$ (see Example 5, where we have shown that U=0 for $H(3)$). Taking into account that $n = 2$ and $v_2(L_1) = v_2(L_2) = 0$, we get

$$\frac{1}{2}\lambda_w(M_{H(2,a,b)}) = \frac{1}{24D}(12 + (a+b)(n^2 - D - 2) + \frac{1}{8}\sigma(\mathbb{L}),$$

where $D = ab - 4$ and $\sigma(\mathbb{L})$ is the signature of the matrix $\begin{pmatrix} a & 2 \\ 2 & b \end{pmatrix}$. For $a = 3, b = 1$ we get $\frac{1}{2}\lambda_w(M_{H(2,a,b)}) = -1$, which is not surprising, since $M_{H(2,3,1)}$ is the negatively oriented Poincaré homology sphere $-P_{120}$. Indeed, one Kirby destabilization move transforms $H(2,3,1)$ the left trefoil framed by -1.

The plan of proving the above theorem consists of two steps.

Step 1. We show that the correctness of equality $\frac{1}{2}D\lambda_w(M_L) = F(L)$ is preserved under self-crossing of a link component.

Step 2. We show that the equality is true for the Hopf links $H(n), n \neq 0$ framed by a, b. The orientations and ordering of components of $H(n)$ may be arbitrary. We do that by calculating $\lambda_w(H(n,a,b)$ by our and by Lescop's formula [8].

Steps 1,2 imply Theorem 3, since any 2-component link can be transformed by self-crossings of its components into a generalized Hopf link.

7 Behavior of λ_w with respect to self-crossings

Let us carry out Step 1. Suppose that a diagram G^- of a framed link $L^- = L_i^- \cup L_j$ is obtained from a diagram G^+ of a framed link $L^+ = L_i^+ \cup L_j$ by a single crossing change at a double point C of L_i^+ such that the chiral sign of C is 1 in L_i^+ and -1 in L_i^-. Set $i = 1, j = 2$ or $i = 2, j = 1$ thus fixing the ordering of the components or, equivalently, the based component. Note that L_i^+ can be considered as to consist of two loops (*lobes*) with endpoints in C. Denote by ℓ the linking number of the lobes, by k the linking number of one of the lobes with L_j, and by n the linking number of L_i and L_j. Note that the crossing change preserves the linking matrix $\mathbb{L}$ of L.

Lemma 1. *In the situation above we have*

$$\frac{1}{2}D(\lambda_w(M_{L^+}) - \lambda_w(M_{L^-})) = b\ell - k(n-k) = F(L^+) - F(L^-),$$

where $D = \mathrm{Det}(\mathbb{L})$ *and* b *is the framing of* L_j.

Proof. The first equality is a partial case (for 2-component links) of the crossing change formula, which is the main result of [5]. Let us prove the second one. Since the crossing change preserves the linking matrix, we have $F(L^+) - F(L^-) = bv_2(L_1^+) - bv_2(L_1^-) - \langle U, G^+\rangle + (\langle U, G^-\rangle)$. Recall that the equality $v_2(L_1^+) - v_2(L_1^-) = \ell$ is one of the main properties of v_2, see [4]. For the Arf-invariant $v_2$mod2 it was known long ago, see [9]. It follows that $bv_2(L_1^+) - bv_2(L_1^-) = b\ell$. Let us compute $\langle U, G^+\rangle - \langle U, G^-\rangle$.

Case 1. Assume that $i = 1, j = 2$. Then the base point of L^+ is in L_i^+ and according to Proposition 1 we my think that the base point of G^+ is placed just before the initial point of the arrow $a^+(C) \subset G^+$ which corresponds to C. See Fig. 10, where C is the double point number 1.

Then only U_1 contains an arrow a which can be mapped to the arrow $a^+(C)$ of G^+. Further, each such representation of U_1 in G^+ is determined by $a^+(C)$

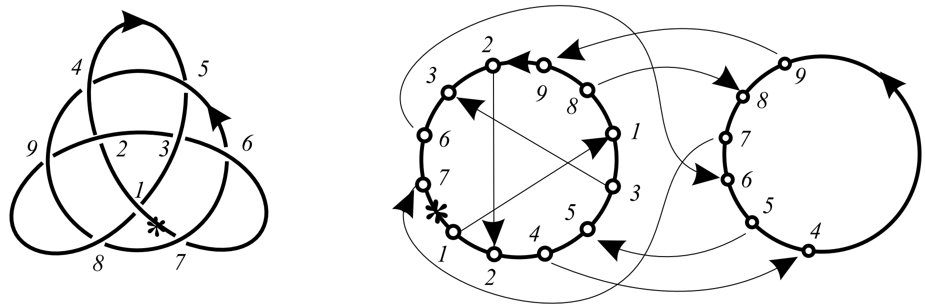

Figure 10: There is only one representation of U_1 to G^+. Arrows 1,8,5 are in its image

and two other arrows of G^+. One of them goes from the lobe containing the base point to L_j, the other connects L_j to the other lobe. Taking into account Example 1, we may conclude that the contribution to $\langle U_1, G^+\rangle$ of representations $U_1 \to G^+$ whose images contain $a^+(C)$ is $k(n-k)$, where k and $n-k$ are the linking numbers of L_j with the lobes.

Every other representation of $U_k, 1 \le k \le 4$, to $G^\pm$ takes no arrow to $a^\pm$ and hence makes no contribution to the difference $\langle U, G^+\rangle - \langle U, G^-\rangle$.

Case 2. Let $i = 2, j = 1$. Then the base point is in L_j while C is the self-crossing point of L_i. It follows that U_1, U_2, U_4 have no representations in $G^\pm$. Moreover, U_3 can have representations only in G^+ and each such representation is determined by two arrows from L_j to L_i such that there endpoints are in different lobes. As above, the total contribution of those representations to $\langle U, G^+\rangle - \langle U, G^-\rangle$ is $k(n-k)$. ▮

Example 7. *Let $L^+ = L_1^+ \cup L_2$ be the link shown in Fig. 10. We assume that the framings of L_1^+, L_2 are a, b. Note that $n = 3$ and k is either 1 or 2. Taking into account that $v_2(L_1^+) = 1$ and $v_2(L_2) = 0$, we get $F(L^+) - F(L^-) = b - 2$. Therefore, $\frac{1}{2}D\lambda_w(M_{L^+} = \frac{1}{2}D\lambda_w(M_{L^-}) + b - 2$.*

8 Casson-Walker invariant for model manifolds

Let $H(n, a, b)$ be a generalized Hopf link (see Definition 3) whose components are framed by a, b. We orient the components so as to have the linking number $-n$ as in the top of Fig. 8. Let $Q = Q(n, a, b)$ be the corresponding 3-manifold. Our goal is to calculate $\lambda_w(Q)$. To that end we introduce another framed link $S(A, B, C)$ shown in Fig. 11. It consists of four unknotted circles framed by $A = a + n, B = b + n, C = -n$, and 0 such that each of the first three circles links the forth circle framed be 0 exactly ones.

Lemma 2. *Manifolds obtained by Dehn surgery of S^3 along $H(n, a, b)$ and $S(A, B, C)$ are related by a homeomorphism which preserves orientations induced from the orientation of S^3.*

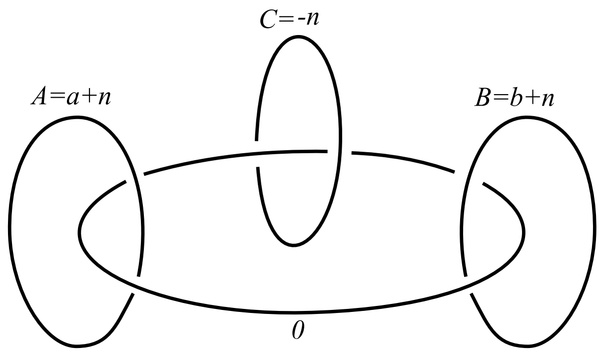

Figure 11: Framed link presenting a Seifert manifold

Proof. Adding the C-framed component to the components framed by A, B and then removing the 0-framed component together with the C-framed one, one can easily show that $S(A,B,C)$ is Kirby equivalent to $H(n,a,b)$. ∎

Remark 4. Let us denote by Q the manifold obtained from S^3 by Dehn surgery along $H(n,a,b)$ or $S(A,B,C)$. Performing surgery of S^3 along the 0-framed component of $S(A,B,C)$, we get $S^2 \times S^1$ such that the other three components are the fibers of the natural fibration $S^2 \times S^1 \to S^2$. It follows that Q is a Seifert manifold fibered over S^2 with three exceptional fibers of types $(A,1),(B,1),(C,1)$. The Euler number of Q is $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}$. The normalized parameters of the exceptional fibers are $(|X|, s(X)\mathrm{mod}|X|)$, where $X = A, B, C$ and $s(X)$ is the sign of X.

We also need the following technical lemma.

Lemma 3. *For any numbers A, B, C such that $D = AB + AC + BC \neq 0$, $K = ABC \neq 0$, and $e = D/K > 0$ we have $s(A) + s(B) + s(C) = 1 + \sigma$, where $s(X)$ denotes the sign of X and σ is the signature of the matrix $\mathbb{L} = \begin{pmatrix} a & n \\ n & b \end{pmatrix}$ for $a = A + C, b = B + C, n = -C$.*

Proof. One can extract from Lemma 2 that the matrices $\mathbb{L} = \begin{pmatrix} a & n \\ n & b \end{pmatrix}$ and $\begin{pmatrix} A & 0 & 0 & 1 \\ 0 & B & 0 & 1 \\ 0 & 0 & C & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ have the same signature. It follows that permutations of A, B, C do not affect the correctness of Lemma 3. So we may assume that $A \geq B \geq C$.

Suppose that $K > 0$. Then $D > 0$ and either $A \geq B \geq C > 0$ and $\sigma = 2s(a) = 2s(b) = 2$ or $A > 0, B < 0, C < 0$ and $\sigma = 2s(a) = 2s(b) = -2$. In both cases we get the conclusion of the lemma.

Now suppose that $K < 0$. Since $D < 0$, we have $\sigma = 0$ and exactly one negative number among A, B, C. Therefore, $s(A) + s(B) + s(C) = 1 + \sigma$. ∎

Let M be a rational homology sphere. Recall that the Lescop invariant $\lambda_L(M)$ is related to the Casson-Walker invariant $\lambda_w(M)$ by

$$\lambda_L(M) = \frac{1}{2}|H_1(M)|\lambda_w(M),$$

see [8, 14]. If M is presented by an oriented framed link L with linking matrix $\mathbb{L}$, then we can rewrite that formula as follows:

$$\varepsilon\lambda_L(M) = \frac{1}{2}D\lambda_w(M),$$

where D is the determinant of $\mathbb{L}$ and ε is the sign of D. We will use the Lescop formula ([8], page 97) for the Seifert manifold $M = (S^2; (a_1, b_1), \ldots, (a_m, b_m)(1, b))$ (in the original notation $M = (Oo0|b; (a_{k}, _k\ )_{k=1,\ldots,m}))$, where $0 < b_k < a_k$ and $e = b + \sum_{k=1}^m b_k/a_k$ is the Euler number:

$$\lambda_L(M) = \left(\frac{sign(e)}{24}\left(2 - m + \sum_{k=1}^m \frac{1}{a_k^2}\right) + \frac{e|e|}{24} - \frac{e}{8} - \frac{|e|}{2}\sum_{k=1}^m s(b_k, a_k)\right)|\prod_{k=1}^m a_k|,$$

where $s(b_k, a_k)$ are the Dedekind sums.

Let us recall the definition and properties of $s(b, a)$. If a, b are coprime integers, then $s(b, a)$ is defined by

$$s(b, a) = \sum_{k=1}^{|a|} \left(\left(\frac{k}{a}\right)\right)\left(\left(\frac{kb}{a}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases}$$

is the sawtooth function, see Fig. 12.

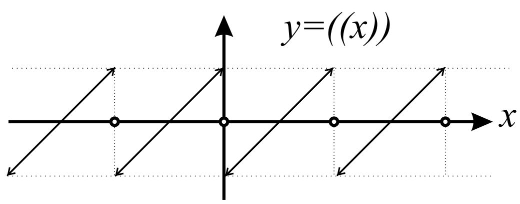

Figure 12: The sawtooth function

It follows from the definition that $s(b, a)$ possesses the property $s(b, a) = s(-b, -a) = -s(-b, a) = -s(b, -a) = s(b \pm a, a)$. In particular, $s(b, a)$ only depends on b mod a and a.

Proposition 2. *Let $Q = Q(n,a,b)$ be obtained by surgery of S^3 along the framed generalized Hopf link $H(n,a,b)$. Then $\frac{1}{2}D\lambda_w(M) = F(H(n,a,b))$, where*

$$F(H(n,a,b) = \frac{1}{12}(n^3-n) + \frac{1}{24}(a+b)(ab-2D-2) + \frac{1}{8}\sigma(\mathbb{L})D,$$

$D = ab - n^2$.

Proof. According to Lemma 2 and Remark 4, Q is a Seifert manifold fibered over S^2 with three exceptional fibers of types $(A,1),(B,1),(C,1)$, where $A = a+n, B = b+n, C = -n$. The Euler number of Q is $e = \frac{1}{A} + \frac{1}{B} + \frac{1}{C}$ and the normalized parameters of the exceptional fibers are $(|X|, s(X) \bmod |X|)$, where $X = A, B, C$ and $s(X)$ is the sign of X.

Note that reversing signs of n, a, b (and hence signs of A, B, C and e) does not affect the correctness of the conclusion of the lemma. So we may restrict ourselves to the case $e > 0$.

Let us introduce the following notations.

1. $K = ABC$. Then $e = \frac{D}{K}$. Since $e > 0$, K and D have the same sign. We denote it by ε.
2. $P = A^2B^2 + A^2C^2 + B^2C^2$. Then $\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} = \frac{P}{K^2}$.
3. $S = AC(B - s(B))(B - 2s(B)) + AB(C - s(C))(C - 2s(C)) + BC(A - s(A))(A - 2s(A))$. In order to explain the meaning of S, we recall that the Dedekind sums $s(1,\ell)$ can be calculated by the rule

   $$s(1,\ell) = \frac{1}{12\ell}(\ell - s(\ell))(\ell - 2s(\ell)).$$

   For the case $\ell > 0$ this formula is contained in [**?**], for the case $\ell < 0$ it can be easily obtained by using properties of $s(b,a)$ listed above. Therefore, $s(1,A) + s(1,B) + s(1,C) = \frac{S}{12K}$.
4. $\Sigma = A + B + C$.

Using this notation and applying the Lescop formula, we get

$$24\lambda_L(Q) = -|K| + \frac{1}{|K|}(P + D^2 - DS) - 3D$$

Simple calculation show that $P - D^2 = -2K\Sigma$ and $2D - S = K(3(s(A)+s(B)+s(C)) - \Sigma)$. It follows that

$$12D\lambda_w(Q) = 24\varepsilon\lambda_L(Q) = -K - 2\Sigma - D\Sigma + 3D(s(A) + s(B) + s(C) - 1),$$

where ε is the sign of D. Let us now substitute $a = A + C, b = B + C, n = -C$ to the expression for $F(H(n,a,b))$. We get

$$24F(H(n,a,b)) = -C^3 + \Sigma C^2 - \Sigma D - 2\Sigma + C^3 - CD + 3D\sigma.$$

Let us show that $12D\lambda_w(Q) - 24F(H(n,a,b)) = 0$. Performing the substraction, we get

$$12D\lambda_w(Q) - 24F(H(n,a,b)) = -K + 3D(s(A)+s(B)+s(C)-1) + C^3 - \Sigma C^2 + CD - 3D\sigma$$

or, taking into account that $-K + C^3 - \Sigma C^2 + CD = 0$,

$$\varepsilon\lambda_L(Q) - F(H(n,a,b)) = 3D(s(A) + s(B) + s(C) - 1 - \sigma).$$

It remains to note that $s(A) + s(B) + s(C) - 1 - \sigma = 0$ by Lemma 3. ∎

Proof of Main Theorem (which states that $\frac{1}{2}D\lambda_w(M_L) = F(L)$, see Theorem 3). We have realized the plan of the proof indicated in page 8. Using Lemma 1, we reduce the proof to the partial case of manifolds presented by generalized framed Hopf links. Then we use Proposition 2 for proving the theorem in this partial case. ■

Corollary 1. *Let $L = L_1 \cup L_2$ be an oriented 2-component link in S^3 and U the linear combination of arrow diagrams shown in Fig. 5. Then the invariant $U(L) = \langle U, G\rangle$ does not depend on the ordering of the components of L.*

Proof. Evident, since $\frac{1}{2}D\lambda_w(M_L)$ and all summands of $F(L)$ (maybe except $U(L)$) do not depend on the ordering. It follows from Theorem 3 that so is $U(L)$. ∎

9 Asymptotic behavior of the Casson-Walker invariant

Let $L = L_1 \cup L_2$ be an oriented framed 2-component link. Then $\lambda_w(M_L)$ depends on the underlying link and on the framing. Theorem 3 allows us to understand the contribution of those two ingredients. We use that for describing the asymptotic behavior of λ_w as the parameters of the framing tend to ∞. For simplicity we restrict ourselves to the simplest case when they have the form $a = a_0 t, b = b_0 t$ and $t \to \infty$.

Theorem 5. *Let a 3-manifold $M(t)$ be obtained by surgery of S^3 along a framed link $L = L_1 \cup L_2$ having linking matrix $\mathbb{L}(t) = \begin{pmatrix} a_0 t & n \\ n & b_o t \end{pmatrix}$ with determinant $D(t) = a_0 b_0 t^2 - n^2$ and signature*

$$\sigma(t) = \begin{cases} 0 & \text{if } D(t) < 0 \\ 2 & \text{if } D(t) > 0 \text{ and } (a_0 + b_0)t > 0 \\ -2 & \text{if } D(t) > 0 \text{ and } (a_0 + b_0)t < 0 \end{cases}$$

Case 1. *Suppose that $a_0 + b_0 \neq 0$ and $a_0 b_0 \neq 0$. Then*

$$\lambda_w(M_t) = -\frac{1}{12}(a_0 + b_0)t + \frac{1}{4}\sigma(t) + r(t),$$

where $r(t) \to 0$ as $t \to \pm\infty$.

CASE 2. *Suppose that $a_0 + b_0 = 0$ and $a_0 b_0 \neq 0$. Then*

$$\lambda_w(M_t) = 2\frac{v_2(L_1) - v_2(L_2)}{a_0} t^{-1} + r(t),$$

where $r(t)t \to 0$ as $t \to \pm\infty$.

CASE 3. *Suppose that $a_0 b_0 = 0$. In order to be definite, we assume that $b_0 = 0$. Then*

$$\lambda_w(M_t) = -\frac{1}{6n^2}(a_0(n^2 - 1 + 12v_2(L_2))t + n^3 - n - 12U(L))$$

Proof. Follows easily from Theorem 3. ∎

Theorem 5 shows that the asymptotic behavior of $\lambda_w(M_t)$ as $t \to \pm\infty$ only depends on a_0, b_0. Let us illustrate the behavior of $\lambda_w(M_t)$ graphically for $a_0 + b_0 \neq 0$, $a_0 b_0 \neq 0$, and $t \to \infty$. The right-hand sides of the expression for $\lambda_w(M_t)$ (see Theorem 3) and its approximation $-\frac{1}{12}(a_0+b_0)t + \frac{1}{4}\sigma(t)$ make sense for all (not necessarily integer) values of t. We show the graphs of these functions for the generalized framed Hopf link $H(n, a_0, b_0)$, where $n = 2, a_0 = 3, b_0 = 2$ (Fig. 13) and $n = 5, a_0 = -3, b_0 = 2$ (Fig. 14). Both graphs in Fig. 13 have singularities at $t = \frac{n^2}{\sqrt{a_0 b_0}} \approx 0.8$ (because of the jump of σ).

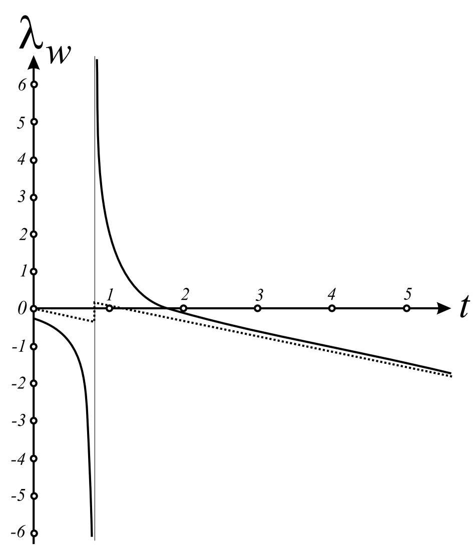

Figure 13: The behavior of $\lambda_w(M_t)$ and its approximation for $H(2,3,2)$

The following theorem shows the power series presentation of $\lambda_w(M_t)$.

Theorem 6. *Let a 3-manifold $M(t)$ be obtained by surgery of S^3 along a framed link $L = L(t)$ with linking matrix $\mathbb{L}(t) = \begin{pmatrix} a_0 t & n \\ n & b_o t \end{pmatrix}$. Suppose that*

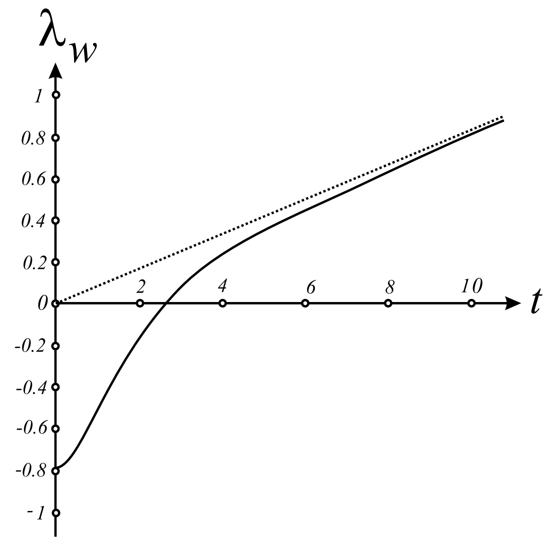

Figure 14: The behavior of $\lambda_w(M_t)$ and its approximation for $H(2, -3, 2)$

$a_0 b_0 \neq 0$. *Then for* $|t| > \frac{|n|}{\sqrt{|a_0 b_0|}}$ *we have*

$$\lambda_w(M_t) = \frac{1}{4}\sigma(\mathbb{L}) +$$

$$+\frac{1}{12}\left(-(a_0+b_0)t + \sum_{k=0}^{\infty}\left((C_1 - C_3(a_0+b_0))C_3^k t^{-(2k+1)} + C_2 C_3^k t^{-(2k+2)}\right)\right),$$

where $C_1 = \frac{(2n^2-2)(a_0+b_0)+24av_2(L_2)+24bv_2(L_1)}{a_0 b_0}$, $C_2 = \frac{-24U(L)+2n^3-2n}{a_0 b_0}$, *and* $C_3 = \frac{n^2}{a_0 b_0}$.

Proof. Follows form Theorem 3. The infinite series in the expression for $\lambda_w(M_t)$ arises after replacing $D^{-1} = (a_0 b_o t^2 - n^2)^{-1}$ by $\frac{t^{-2}}{a_0 b_0}\sum_{k=0}^{\infty}(\frac{n^2}{a_0 b_0}t^{-2})^k$. ∎

10 A skein-type relations for U and λ_w

A usual skein relation involves diagrams of three oriented links L^+, L^-, L^0. The diagrams are identical outside a small neighborhood of one positive crossing C of the diagram for L^+. The diagram of the second link L^- is obtained from the diagram of L^+ by a crossing change at C while the diagram of the oriented knot L^0 has no crossings at C.

We will consider the case when $L^+ = L_1^+ \cup L_2^+$, $L^- = L_1^- \cup L_2^-$ are oriented 2-component links and C is the crossing point of L_1^+ and L_2^+. Then L^0 is a knot obtained by coherent fusion of L_1^+ and L_2^+, see Fig. 15. We shall refer to any triple (L^+, L^-, L^0) of the above type as an *admissible skein triple*.

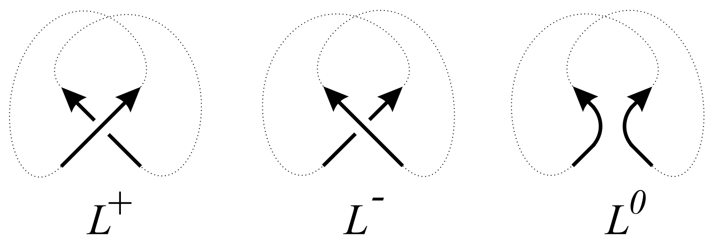

Figure 15: Three links participating in the skein-type relation

Proposition 3. *For any admissible skein triple* (L^+, L^-, L^0) *we have* $U(L^+) - U(L^-) = v_2(L^0)$.

Proof. Denote by G^+, G^-, G^0 Gauss diagrams corresponding to the diagrams of L^+, L^-, L^0. The Gauss diagrams are almost identical. The only difference between G^+ and G^- is that the arrows $a^+(C), a^-(C)$ corresponding to C have opposite orientations and signs. Knot diagram G^0 is obtained by coherent fusion of the circles of G^+ along $a^+(C)$. Chose a base point in the first circle of G^+ just before the initial point of $a^+(C)$. We may assume that there are no endpoints of other arrows on small arcs containing the endpoints of $a^+(C), a^-(C)$, and on arcs of L^0 obtained by their fusion. See Fig. 16, where those free-of-endpoints arcs are shown dotted and the complementary arcs are numbered by 1,2. For reader's conveniens we have also placed arrows diagrams for U and v_2.

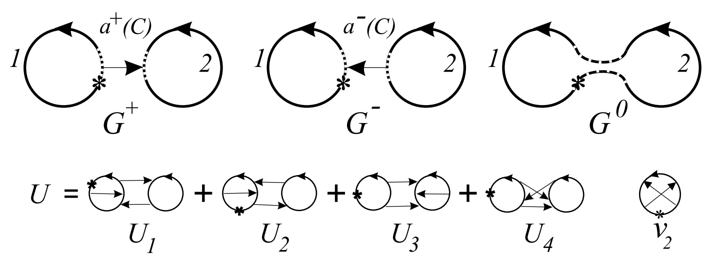

Figure 16: Skein triple of Gauss diagrams

Let us analyze representations of $U_k, 1 \leq k \leq 4$ in $G^{\pm}$. We call a representation $\varphi\colon U_k \to G^{\pm}$ *significant*, if its image contains $a^{\pm}(C)$. Otherwise φ is *insignificant*.

Step 1. Since $G^{\pm}$ are identical outside $a^{\pm}(C)$, there is a natural bijection between insignificant representations of U_k to G^+ and G^- such that the values of corresponding representations are equal. It follows that insignificant representations do not contribute to the difference $U(L^+) - U(L^-)$.

Step 2. The careful choice of the base point tells us that there are no significant representations of U_1 in $G^{\pm}$. The reason is that that the first arrow we meet traveling from the base point along the first circle of $G^{\pm}$ is not $a^{\pm}(C)$.

Step 3. By similar reason there are no significant representations of $U_k, 2 \leq$

$k \leq 4$ in G^-: the first arrow we meet is $a^-(C)$, which is directed from the second circle to the first, not vice-versa, as in $U_2 - -U_4$.

STEP 4. Let φ be a significant representation of $U_k, 2 \leq k \leq 4$ in G^+. Denote by a_k the first arrow of U_k we meet traveling from the base point along the first circle. Let us coherently fuse the circles of U_k along a_k. It is easy to see that we get an arrow diagram A for calculation of v_2, together with the corresponding representation $\varphi'\colon A \to L_0$. The values of φ and φ' are equal. Vice versa, any representation of $A \to L^0$ determines a representation $U_k \to G^+$ having the same value. It follows that $U(L^+) - U(L^-) = v_2(L^0)$. ∎

As a version for a skein-type relation for λ_w we suggest the following straightforward corollary of Proposition 3.

Corollary 2. *For any admissible skein triple* (L^+, L^-, L^0) *we have*

$$\frac{D^+}{2}(\lambda_w(L^+) - \frac{1}{4}\sigma(\mathbb{L}^+)) - \frac{D^-}{2}(\lambda_w(L^-) - \frac{1}{4}\sigma(\mathbb{L}^-)) = (n-1)\frac{3n+a+b+3}{12} - v_2(L^0)$$

11 An alternative Formula

Consider the following linear combination $U' = U_1 + U_2 + \frac{1}{2}(U_3 + U_3') + \frac{1}{2}(U_4 - U_4')$ of arrow diagrams, see Fig. 17.

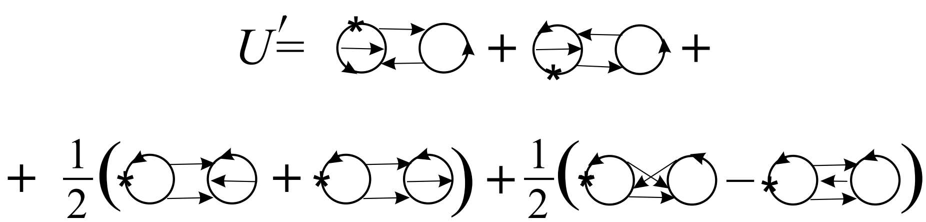

Figure 17: Another remarkable linear combination of arrow diagrams

Lemma 4. *Let $L = L_1 \cup L_2$ be an oriented 2-component link. Denote by L' the link $L_1 \cup L_2'$ obtained from L by reversing the orientation of the second component. Let G and G' be their based Gauss diagrams (assuming that the base points are in the first components). Then $\langle U_1, G\rangle = \langle U_1, G'\rangle$, $\langle U_2, G\rangle = \langle U_2, G'\rangle$, $\langle U_3, G\rangle = \langle U_3', G'\rangle$, $\langle U_4, G\rangle = -\langle U_4', G'\rangle$, and $\langle U', G\rangle = \frac{1}{2}(\langle U, G\rangle + \langle U, G'\rangle)$.*

Proof. Note that G and G' actually coincide. The only difference is that their second circles have opposite orientations and that arrows joining different circles have opposite signs. First two equalities of the conclusion of the lemma are evident, since any representation of $U_i, i = 1, 2$, to G determines a representation of U_i to G', and vice-versa. The images of those representations contains the same arrows. Since exactly two of these arrows join differen components, the values of the representations are equal. Similarly, any representation of $U_j, j = 3, 4$ determines a representation of U_j' to G', and vice-versa. The values of those representations are the same for $j = 3$ and have opposite signs for $j = 4$. This is because the number of arrows in the images of representations is 2 for $j = 3$ and 3 for $j = 4$. Taking the sums, we get $\langle U', G\rangle = \frac{1}{2}\left(\langle U, G\rangle + \langle U, G'\rangle\right)$. ∎

The following proposition and theorem are similar to Proposition 1 and Theorem 3.

Proposition 4. *U' determines an invariant $U'(L) = \langle U', G\rangle$ for non-oriented links of two numbered components.*

Proof. It follows from Lemma 4 that $\langle U', G\rangle = \frac{1}{2}\left(\langle U, G\rangle + \langle U, G'\rangle\right)$. Therefore, $U'(L) = \langle U', G\rangle$ is an invariant of oriented links by Proposition 1. On the other hand, $\langle U', G\rangle$ is invariant under reversing orientation of one of its components, since the above expression for it is symmetric. ∎

Theorem 7. *For any non-oriented framed 2-component link $L = L_1 \cup L_2$ we have $\frac{1}{2}D\lambda_w(M_L) = F'(L)$, where*

$$F'(L) = \frac{1}{24}(a+b)(n^2 - D - 2) + \frac{1}{8}\sigma(\mathbb{L})D + av_2(L_2) + bv_2(L_1) - U'(L)$$

Proof. Let us orient L and denote by L' the oriented framed link obtained from L by reversing orientation of L_2. Note that the linking number n of the components of L and the linking number n' of the components of L' have the same modules and different signs, that is, $n' = -n$. All other variables in the expressions for $F(L)$ and $F(L')$ (see Definition 4) except of $U(L), U(L')$ are the same. In other words, a, b, D, σ and both v_2 for L coincide with the corresponding variables for L'. It follows from Lemma 4 that $F'(L) = \frac{1}{2}(F(L) + F(L')$. Since $F(L) = \frac{1}{2}D\lambda_w(M_L) = \frac{1}{2}D\lambda_w(M_{L'}) = F(L')$ by Theorem 3, we get the conclusion. ∎

12 Results of computer experiments

The formula for $\lambda_w(M)$ from Theorem 3 is very convenient for calculation. If a diagram of a link framed by integers of reasonable size has about a dozen crossing points, then the manual calculation takes a few minutes. A simple computer program written by V. Tarkaev accepts Gauss codes and takes only seconds for calculating λ_w for framed links with thousands crossings. We present here a few results of calculation λ_w for all rational homology 3-spheres which can be presented by diagrams of 2-component (but not of 1-component) links with ≤ 9 crossings and black-board framings. We thank S. Lins, who kindly prepared for us a list of Gauss codes of those links.

- Number of different manifolds: 194
- Number of different values of $|\lambda_w|$ for these manifolds: 66
- Numbers of different manifolds having given values of λ_w are presented in Table 1.

$\lambda_\mathbf{w}$	$\#\mathbf{M^3}$	$\lambda_\mathbf{w}$	$\#\mathbf{M^3}$	$\lambda_\mathbf{w}$	$\#\mathbf{M^3}$	$\lambda_\mathbf{w}$	$\#\mathbf{M^3}$
0	19	9/64	1	11/36	4	25/44	1
1/64	1	4/27	2	5/16	7	16/27	1
1/36	4	5/32	1	9/28	2	3/5	1
1/32	1	11/64	1	11/32	2	23/36	6
1/28	2	5/28	1	9/26	1	11/16	3
1/26	1	3/16	10	13/36	2	25/36	3
3/64	1	5/26	1	10/27	2	23/32	1
1/16	8	7/36	1	3/8	5	29/36	1
3/44	1	1/5	4	25/64	1	15/16	3
2/27	2	13/64	1	2/5	1	1	7
5/64	1	9/44	1	13/32	1	19/16	1
3/32	2	2/9	5	19/44	1	3/2	1
1/10	2	15/64	1	7/16	5	2	16
5/44	2	1/4	5	4/9	2	3	2
3/26	1	13/44	2	13/28	1	4	6
1/8	4	8/27	2	17/36	4	–	–
5/36	2	3/10	1	1/2	6	–	–

Table 1: How many manifolds have a given value of λ_w

We see from the table that for the set of 194 manifolds under consideration the most popular values of $|\lambda_w|$ are 0 (19 manifolds), 2 (16 manifolds), 3/16 (10 manifolds), 1/16 (8 manifolds), 5/16 and 1 (7 manifolds each). Exactly 30 values of $|\lambda_w|$ are taken by only one manifold each. In other words, those manifolds are determined by $|\lambda_w|$. In average, each value is taken by only 3 different manifolds from the list. This shows that the Casson-Walker invariant is unexpectedly informative. For example, the number of different first homology groups of the manifolds under consideration is 17, so the average number of manifolds having a given group is about 11.4.

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