The Dilogarithm and volumes of hyperbolic polytopes

Ruth Kellerhals

(Preliminary version of a chapter in the book "Properties of Polylogarithms".)

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3, FRG

Mathematisches Institut
der Universität
Wegelerstraße 10
D–5300 Bonn 1, FRG

MPI/90-97

Max-Planck-Institut
für Mathematik

Buchungsbeleg Nr. ....

Kassenbuchbeleg Nr. ....

AUSZAHLUNGSANORDNUNG

Empfänger: ....

....

Titel .... DM ....

....

....

Grund der
Zahlung: ....

....

Auszahlungsbetrag in DM: ....

i.W.: ....

Skonto .... MWSt ....

Bankleitzahl .... Bankkonto ....

Bankverbindung ....

Rechnungs Nr. .... Rechnungsdatum ....

Inventar Nr. ....

FESTGESTELLT	SACHLICH RICHTIG	ZUR ZAHLUNG ANGEWIESEN
....	....	....

ausgezahlt durch Bank/Kasse

Bonn, ...

13. THE DILOGARITHM AND VOLUMES OF HYPERBOLIC POLYTOPES

by

Ruth Kellerhals

Table of Contents

13.0. Introduction

In this chapter, we consider the Euler dilogarithm $\mathrm{Li}_2(z)$ in connection with the problem of calculating volumes of non-euclidean polytopes. In contrast to the euclidean case, where the volume of an arbitrary simplex $S \subset E^n$, $n \geq 3$, spanned by vectors $p_0, \ldots, p_n$, is given by the "elementary" formula

$$\mathrm{vol}_n(S) = \frac{1}{n!}|\det(p_0, \ldots, p_n)| \quad , \tag{13.1}$$

the corresponding volume problem for non-euclidean n-simplexes is considerably more difficult, and - in full generality - an unsolved problem.
However, in 1852, Schläfli proved a very beautiful formula for the volume differential $d\mathrm{vol}_n$ on the set of spherical n-simplexes S (cf. [26], p.227 ff):

$$d\mathrm{vol}_n(S) = \frac{1}{n-1} \sum_{1 \leq j < k \leq n} \mathrm{vol}_{n-2}(S_j \cap S_k)\, d\alpha_{jk} \quad , \quad \mathrm{vol}_0 := 1 \quad , \tag{13.2}$$

where α_{jk} is the dihedral angle formed by the facets S_j, S_k of S. This formula, the three-dimensional hyperbolic version of which was already known to Lobachevsky, was

very elegantly reproved and extended to the hyperbolic case by H. Kneser (see [16] and [4, §5.1]).
However, the remaining single integration cannot be carried out even in the simplest case of a three-dimensional simplex. Consequently, one has to look for polyhedral objects whose geometry allows one to simplify the last but most difficult step.
We consider the class of d-truncated orthoschemes R_d, $0 \leq d \leq 2$, which are convex polytopes bounded by $n+d+1$ hyperplanes $H_0, \dots, H_{n+d}$ such that

$$H_i \perp H_j \quad \text{for} \quad 2 \leq |i-j| \leq n$$

(for $d = 2$, indices are taken modulo $n+3$). For $d = 0$, these polytopes are the ordinary orthoschemes R, first introduced by Schläfli. They are determined (up to isometry) by their n non-right dihedral angles.
For spherical orthoschemes, Schläfli derived a variety of results concerning his volume function f_n. Independently of him, in 1836, Lobachevsky expressed the volume of a three-dimensional hyperbolic orthoscheme $R = R(\alpha_1, \alpha_2, \alpha_3)$ in the form (see [4, (18), p.250])

$$\begin{aligned} \mathrm{vol}_3(R) = \frac{1}{4}\{ & Л(\alpha_1+\theta) - Л(\alpha_1-\theta) + Л(\frac{\pi}{2}+\alpha_2-\theta) + Л(\frac{\pi}{2}-\alpha_2-\theta) + \\ & + Л(\alpha_3+\theta) - Л(\alpha_3-\theta) + 2Л(\frac{\pi}{2}-\theta)\} \quad , \end{aligned} \tag{13.3}$$

where $\quad Л(\omega) := -\int\limits_0^\omega \log|2\sin t|\,dt \quad$ denotes the Lobachevsky function, related to the dilogarithm by

$$Л(\omega) = \frac{1}{2}\mathrm{Im}\big(\mathrm{Li}_2(e^{2i\omega})\big) \quad ,$$

and where $\quad 0 \leq \theta := \arctan \frac{\sqrt{\cos^2\alpha_2 - \sin^2\alpha_1 \sin^2\alpha_3}}{\cos\alpha_1 \cos\alpha_3} < \frac{\pi}{2} \quad .$

About 1935, Coxeter [8] reformulated and combined the results of Lobachevsky and Schläfli by introducing the function

$$S(\alpha_1,\alpha_2,\alpha_3) := \sum_{r=1}^{\infty} \frac{(-X)^r}{r^2}\big(\cos 2r\alpha_1 - \cos 2r\alpha_2 + \cos 2r\alpha_3 - 1\big) - \alpha_1^2 + \alpha_2^2 - \alpha_3^2 \quad , \tag{13.4}$$

where

$$X = \frac{\sin\alpha_1 \sin\alpha_3 - D}{\sin\alpha_1 \sin\alpha_3 + D} \quad \text{with} \quad D = \sqrt{\cos^2\alpha_1 \cos^2\alpha_3 - \cos^2\alpha_2} \quad .$$

He showed that

$$S(\frac{\pi}{2}-\alpha_1, \alpha_2, \frac{\pi}{2}-\alpha_3) = \frac{\pi^2}{2} f_3(\alpha_1,\alpha_2,\alpha_3) \quad ,$$

and, for hyperbolic orthoschemes $R = R(\alpha_1, \alpha_2, \alpha_3)$,

$$\frac{i}{4} S(\frac{\pi}{2} - \alpha_1, \alpha_2, \frac{\pi}{2} - \alpha_3) = \mathrm{vol}_3(R) \quad .$$

In 1962, Böhm [4, §5] analyzed Coxeter's method and generalized it to spaces of non-zero constant curvature of arbitrary dimension without, however, solving the higher dimensional volume problem.
We shall see that the Böhm-Coxeter method is even applicable to d-truncated orthoschemes in H^3 and that the volume formula (13.3) remains valid (up to a minor modification in one case).

This chapter is organized as follows:
In paragraph **13.1.**, we collect some basic material about real hyperbolic space. Then, we introduce the notion of schemes due to Schläfli and Vinberg [28, 29] to describe polytopes with many right dihedral angles and, in particular, orthoschemes of degree (of truncation) d. In **13.2.**, we discuss the volume problem for d-truncated orthoschemes in H^3 (cf. [14]). For this, a "schematic" version of Schläfli's differential formula is presented for the polytopes under consideration. By Schläfli's differential formula, one can see that there is a fundamental difference in the volume problems of even and odd dimensions; the Reduction formula in **13.2.2.** shows that the first problem can be reduced to the second one (cf. [15])! Next, we generalize the integration method of Böhm and Coxeter to derive explicit volume formulae for d-truncated orthoschemes that are analogous to (13.3) (Theorem 13.5 and Theorem 13.6). In section **13.3.**, we discuss some applications; in particular, the volumes of all Coxeter orthoschemes of degree d, forming fundamental polyhedra for hyperbolic Coxeter groups, are determined. We append the corresponding list for the ten ordinary Coxeter orthoschemes. By means of dissection into truncated orthoschemes, we calculate the volumes of the totally asymptotic regular simplexes in $\overline{H^n}$ for $n = 2, 3, 4, 6$, as well as the volumes of the four totally asymptotic regular polyhedra, the tetrahedron $S^{\infty}_{reg}(\frac{\pi}{3})$, the hexahedron $H^{\infty}_{reg}(\frac{\pi}{3})$, the octahedron $O^{\infty}_{reg}(\frac{\pi}{4})$ and the dodecahedron $D^{\infty}_{reg}(\frac{\pi}{3})$. Also by dissection, we derive some interesting functional equations for the Lobachevsky function $Л(\omega)$.
In **13.4.**, a few further aspects are considered. We survey results concerning small elements in the volume spectrum of hyperbolic 3-space forms. By a result of Borel, volumes of arithmetic 3-folds are computable in terms of Dedekind zeta functions, which we demonstrate with two examples. Finally, in **13.4.2.**, we give a very lay introduction to the fascinating circle of ideas around Hilbert's Third Problem concerning scissors congruence. Following the paper [9] of Dupont and Sah, we summarize definitions and properties of the different

scissors congruence groups in hyperbolic space and describe how these groups admit a more general homological treatment. By the works of Bloch, Wigner and Dupont, Sah, we see at the end how geometrical notions such as volume and Dehn's invariant can be unified on the level of scissors congruence.

13.1. A particular class of hyperbolic polytopes

13.1.1. Hyperbolic space

Let X^n denote either the n-dimensional euclidean space E^n, the n-sphere S^n or the n-dimensional hyperbolic space $\overline{H^n}$. Let S^n be embedded in E^{n+1}, and use for H^n the model in the Lorentz space $E^{n,1}$ of signature $(n, 1)$, i.e.: If $E^{n,1}$ denotes the $(n+1)$-dimensional real vector space $\mathbf{R}^{n+1}$, together with the bilinear form

$$\langle x, y \rangle := -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n \quad , \quad \forall x\,,\, y \in \mathbf{R}^{n+1} \quad ,$$

of signature $(n, 1)$, then H^n can be interpreted as

$$H^n = \{ \, x \in E^{n,1} \mid \langle x, x \rangle = -1, x_0 > 0 \, \}.$$

Or, in the projective model, H^n is the interior $IQ_{n,1}$ of real projective space P^n with respect to the quadric

$$Q_{n,1} := \{ \, [x] \in P^n \mid \langle x, x \rangle = 0 \, \} \, .$$

The closure $\overline{H^n}$ of H^n in P^n represents the natural compactification of H^n. Points of the boundary $\partial H^n = \overline{H^n} - H^n$ are called *points at infinity* of H^n. Points in P^n lying outside $\overline{H^n}$ are said to be *ideal points* of H^n relative to $Q_{n,1}$, and the set of all such points is denoted by $AQ_{n,1}$.

To every point in P^n corresponds a hyperplane in P^n and vice versa: Let $P = [x] \in P^n$. A point $[y] \in P^n$ is said to be *conjugate to* $[x]$ *relative to* $Q_{n,1}$ if $\langle x, y \rangle = 0$ holds. The set of all points which are conjugate to $P = [x]$ form a projective hyperplane

$$\Pi_P := \{ \, [y] \in P^n \mid \langle x, y \rangle = 0 \, \} \quad ,$$

the *polar hyperplane to* P. P is called the *pole to* Π_P, and is denoted by $\mathrm{Pole}(\Pi_P)$. The map *pole* $\mapsto$ *polar hyperplane* is a bijection between the points and hyperplanes of P^n known as the duality principle of the projective space P^n (see [8], §4E). It has the following properties (see [8], §4):

(a) $P \in AQ_{n,1}$, $P \in Q_{n,1}$ or $P \in IQ_{n,1}$ if and only if Π_P intersects, touches or avoids the quadric $Q_{n,1}$.

(b) If two lines g, h in P^2 intersect at $S := g \cap h$, then Π_S is the line determined by Pole(g), Pole(h).

(c) If a line g in P^2 contains the point Pole(h) of the line h, then $g \perp h$ holds.

13.1.2. The scheme of a polytope

Let $P \subset \overline{H^n}$ denote a convex polytope bounded by finitely many hyperplanes H_i, $i \in I$, which are characterized by unit normal vectors $e_i \in E^{n,1}$ directed outwards with respect to P, say, i.e. (for basic notations and properties, see [28, §1]):

$$H_i = e_i^{\perp} := \{ x \in H^n \mid \langle x, e_i \rangle = 0 \} \quad \text{with} \quad \langle e_i, e_i \rangle = 1 \quad .$$

We always assume that P is acute-angled (i.e., all dihedral angles $\neq \frac{\pi}{2}$ are of measure strictly less than $\frac{\pi}{2}$) and of finite volume.

The Gram matrix $G(P) := (\langle e_i, e_j \rangle)_{i,j \in I}$ of the vectors e_i, $i \in I$, associated to P is an indecomposable symmetric matrix of signature $(n, 1)$ with entries $\langle e_i, e_i \rangle = 1$ and $\langle e_i, e_j \rangle \leq 0$ for $i \neq j$, having the following geometrical meaning (see [28, §1]):

$$-\langle e_i, e_j \rangle = \begin{cases} 0 & \text{if } H_i \perp H_j \quad , \\ \cos \alpha_{ij} & \text{if } H_i, H_j \text{ intersect on } P \text{ at the angle } \alpha_{ij} \quad , \\ 1 & \text{if } H_i, H_j \text{ are parallel} \quad , \\ \cosh l_{ij} & \text{if } H_i, H_j \text{ admit a common perpendicular of length } l_{ij} \quad . \end{cases}$$

On the other hand, if $G = (g_{ij})$ is an indecomposable symmetric $m \times m$ matrix of rank $n+1$ with $g_{ii} = 1$ and $g_{ij} \leq 0$, for $i \neq j$, then G can be realized as Gram matrix $G(P)$ of an acute-angled polytope $P \subset X^n$ of finite volume in the following way (see [28, §2]):

1. If G is positive definite (G is *elliptic*), then $m = n + 1$, and G is the Gram matrix of a simplex in S^n uniquely defined up to a motion.
2. If G is positive semidefinite (G is *parabolic*), then $m = n + 2$, and G is the Gram matrix of a simplex in E^{n+1} uniquely defined up to a similarity.
3. If G is of signature $(n, 1)$ (G is *hyperbolic*), then G is the Gram matrix of a convex polytope with m facets (faces of codimension 1) in $\overline{H^n}$ uniquely defined up to a motion.

The Gram matrix $G(P)$ reflects combinatorial and metrical properties of an acute-angled polytope $P \subset \overline{H^n}$ (see [28, §3-4]). In particular, every ordinary vertex p of P is characterized by an elliptic principal submatrix of $G(P)$ of order n describing the spherical vertex polytope P_p (intersection of P with the surface of a sufficiently small ball around p) of dimension $n-1$ associated to p. To every vertex q at infinity corresponds a parabolic principal submatrix of $G(P)$ of rank $n-1$ indicating that the vertex polytope P_q is euclidean of dimension $n-1$.

For the geometrical description of polytopes with many right dihedral angles, the language of schemes is much more convenient (see [29, §3]). A scheme Σ is a weighted graph (see [29, §2]) whose nodes n_i, n_j are joined by an edge with positive weight c_{ij} or are not joined at all; the last fact will be indicated by $c_{ij} = 0$. A subscheme of Σ is a subgraph of Σ with each pair of nodes connected by the same weighted edge as in Σ. The number $|\Sigma|$ of nodes is called the order of Σ. To every scheme Σ of order m corresponds a symmetric matrix $A(\Sigma) = (a_{ij})$ of order m with $a_{ii} = 1$ on the diagonal and non-positive entries $a_{ij} = -c_{ij} \leq 0$, $i \neq j$, off it. Σ is connected if and only if $A(\Sigma)$ is indecomposable. Rank, determinant, permanent and character of definiteness of Σ are defined to be the corresponding ones of $A(\Sigma)$. Furthermore, Σ is said to be either elliptic, or parabolic, or hyperbolic if either all its components are elliptic, or apart from elliptic components there is at least one parabolic component, or exactly one component is hyperbolic.

In particular, if $\Sigma(1,\dots,m)$ denotes a linear scheme with nodes $1,\dots,m$ and weights $c_i := c_{i,i+1}$, $1 \leq i \leq m-1$, then the following useful recursion formulae hold for $\det\Sigma(1,\dots,m)$ and $\mathrm{per}\Sigma(1,\dots,m)$:

LEMMA 13.1. *Let $m \geq 3$. Then*

$$\det\Sigma(1,\dots,m) = \det\Sigma(1,\dots,m-1) - c_{m-1}^2\det\Sigma(1,\dots,m-2) \quad , \tag{13.5}$$

$$\mathrm{per}\Sigma(1,\dots,m) = \mathrm{per}\Sigma(1,\dots,m-1) + c_{m-1}^2\mathrm{per}\Sigma(1,\dots,m-2) \quad . \tag{13.6}$$

Proof. Equation (13.5) is well-known (cf. [26, p.258]). To prove (13.6), denote by

$$C := A(\Sigma(1,\dots,m)) = (-c_{ij})$$

the symmetric $m \times m$ matrix associated to $\Sigma(1,\dots,m)$ with diagonal elements $-c_{ii} := 1$. Then, by definition,

$$\mathrm{per}\Sigma(1,\dots,m) = \mathrm{per}C = \sum (-1)^m c_{1\sigma(1)} \cdots c_{m\sigma(m)} \quad ,$$

where σ runs through all permutations of $\{1,\dots,m\}$. Since the permanent of every symmetric $m \times m$ matrix $M = (\mu_{ij})$ satisfies (cf. [23, Theorem 1.1(c) and (1.4)])

$$\mathrm{per}M = \sum_{k=1}^{m} \mu_{km}\mathrm{per}M(k|m) = \sum_{k=1}^{m} \mu_{mk}\mathrm{per}M(m|k) \quad , \tag{13.7}$$

where $M(i|j)$ is the matrix which is obtained from M by deleting row i and column j, we obtain

$$\mathrm{per}C = -c_{m-1}\mathrm{per}C(m-1|m) + \mathrm{per}C(m|m) \quad . \tag{13.8}$$

Applying (13.7) to the $(m-1)\times(m-1)$ matrix $C_m := C(m-1|m)$ we get

$$\mathrm{per}C_m = -c_{m-1}\mathrm{per}C_m(m-1|m-1) \quad . \tag{13.9}$$

Since $C_m(m-1|m-1) = A(\Sigma(1,\dots,m-2))$, (13.8) together with (13.9) imply (13.6).
Q.E.D.

The scheme $\Sigma(P)$ of an acute-angled polytope $P \subset X^n$ is defined to be the scheme whose matrix $A(\Sigma)$ coincides with the Gram matrix $G(P)$, i.e. whose nodes i correspond to the bounding hyperplanes $H_i = e_i^{\perp}$ (or equivalently to their normal vectors e_i) of P and whose weights equal $-\langle e_i, e_j\rangle_{X^n}$, $i,j \in I$.
Two acute-angled polytopes $P_1, P_2 \subset \overline{H^n}$ are said to be *of the same schematic type* if their schemes $\Sigma(P_1), \Sigma(P_2)$ are of the same graphical type (i.e., their underlying graphs as one-dimensional simplicial complexes are homeomorphic) and if corresponding weights c_{ij}^1 of $\Sigma(P_1)$ and c_{ij}^2 of $\Sigma(P_2)$ satisfy:

$$c_{ij}^1 \begin{cases} > \\ = \\ < \end{cases} 1 \iff c_{ij}^2 \begin{cases} > \\ = \\ < \end{cases} 1 \quad .$$

It follows that polytopes of the same schematic type are of the same combinatorial type (see [3]).
For the schemes of Coxeter polytopes $P_C \subset X^n$ (all dihedral angles are of the form $\frac{\pi}{p}$, $p \in \mathbb{N}$, $p \geq 2$) the usual conventions are adopted and - for convenience - used sometimes even in the non-Coxeter case: If two nodes are related by the weight $\cos\frac{\pi}{p}$, then they are joined by a $(p-2)$-fold line for $p = 3, 4$ and by a single line marked p (or $\alpha = \frac{\pi}{p}$) for $p \geq 5$. If two bounding hyperplanes of $P_C \subset X^n$, $X^n \neq S^n$, are parallel, then the corresponding nodes are joined by a line marked ∞; if they are divergent (occurring at most in the hyperbolic case), then their nodes are joined by a dotted line and the weight ≥ 1 is dropped.

13.1.3. Orthoschemes of degree d

The simplest examples of schemes are the linear and cyclic ones. One class of acute-angled hyperbolic polytopes described by such schemes is the following (see [13] and [14]):

Definition. An *n-dimensional orthoscheme of degree d*, $0 \leq d \leq 2$, is a convex polytope in $\overline{H^n}$, $n \geq 2$, denoted by R_d such that its scheme $\Sigma_d := \Sigma(R_d)$ is connected and linear of length $n+d+1$ for $d = 0,1$ or cyclic of order $n+3$ for $d = 2$.

Hence, orthoschemes of degree d in H^n are bounded by $n+d+1$ hyperplanes $H_0, \ldots, H_{n+d}$ such that

$$H_i \perp H_j \quad \text{for} \quad j \neq i-1, i, i+1, \tag{13.10}$$

where, for $d = 2$, indices are taken modulo $n+3$.

Orthoschemes of degree d allow the following geometrical description:

For $d = 0$, they constitute the class of (ordinary) orthoschemes introduced by Schläfli (see [26, p.243]): An orthoscheme in X^n ($n \geq 1$) is a simplex bounded by $n+1$ hyperplanes $H_0, \ldots, H_n$ such that $H_i \perp H_j$ for $2 \leq |i-j| \leq n$. Or, equivalently, it has vertices $P_0, \ldots, P_n$ numbered in such a way that $\operatorname{span}(P_0, \ldots, P_i) \perp \operatorname{span}(P_i, \ldots, P_n)$ for $1 \leq i \leq n-1$. The initial and final vertices P_0, P_n of the orthogonal edge-path $P_0P_1, \ldots, P_{n-1}P_n$ are called principal vertices, since they are distinguished in several ways. E.g. in $\overline{H^n}$, at most the principal vertices may be points at infinity (see [4, Satz 15, p.188]).

In the projective model for H^n (see **13.1.1.**), orthoschemes of degree $d > 0$ can be derived from ordinary ones by allowing d of its principal vertices (and with them possibly further vertices) to lie outside the quadric $Q_{n,1}$, and then by cutting off the ideal vertices by means of the polar hyperplanes $H_{n+1} := \Pi_{P_n}$ resp. $H_{n+2} := \Pi_{P_0}$ (inasmuch as they lie outside $Q_{n,1}$). Hence, orthoschemes of degree d are d-times (polarly) truncated orthoschemes bounded by hyperplanes $H_0, \ldots, H_{n+d}$ with the property (13.10) (cf. **13.1.1.**).

Remark. By adjoining to the bounding hyperplanes $H_0, \ldots, H_n$ the polar hyperplanes associated to the principal vertices of an orthoscheme, viewed as an object in projective space, the configuration of the corresponding $n+3$ outer normal vectors in $E^{n,1}$ form a Napier cycle in $E^{n,1}$. These were introduced and in the crystallographic case classified by Im Hof [13].

By construction, orthoschemes R_d of degree d are of finite volume (cf. also [28, Theorem 4.1]).Furthermore, they have at most $n+3$ non-right dihedral angles (or *essential* angles) $\alpha_1, \ldots, \alpha_m$, $m \leq n+3$, and all of them are acute, i.e., $\alpha_i < \frac{\pi}{2}$ for $i = 1, \ldots, m$ (see [4, §4.8, Hilfssatz 2] and the definition).

Let $d > 0$, and denote by Σ_d the scheme of R_d. Then, removing d non-connected nodes in

Σ_d leaves two disjoint components σ_1, σ_2 of Σ_d, which satisfy (see [13, Proposition 1.4])

$$\sigma_1 \quad \begin{cases} \text{elliptic,} \\ \text{parabolic,} \\ \text{hyperbolic,} \end{cases} \quad \Longleftrightarrow \quad \sigma_2 \quad \begin{cases} \text{hyperbolic,} \\ \text{parabolic,} \\ \text{elliptic.} \end{cases} \tag{13.11}$$

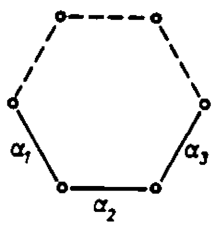
Therefore, if $n \geq 4$, Σ_d always contains a connected hyperbolic subscheme of order $n+1$ all of whose weights $c_1, \ldots, c_n$ are of the form $0 < c_i = \cos\alpha_i < 1$, $1 \leq i \leq n$ (see [13, Proposition 1.8]). Such schemes are said to be *of type A*.

In dimension three, however, the situation is principally different. First, if $R_d \subset H^3$ is compact, then, by the Euler equation for compact polyhedra, the number m of essential angles equals three. Moreover, all but one of the different schemes Σ_d are of type A; according to the degree d, the schemes of type A are of the form and likewise characterized by angle inequalities as follows (see (13.11) and **13.1.2.**):

$$\Sigma_0 \; : \quad \circ \overset{\alpha_1}{\text{———}} \circ \overset{\alpha_2}{\text{———}} \circ \overset{\alpha_3}{\text{———}} \circ \qquad , \alpha_1 + \alpha_2 > \tfrac{\pi}{2}\,, \; \alpha_2 + \alpha_3 > \tfrac{\pi}{2}\,, \tag{13.12}$$

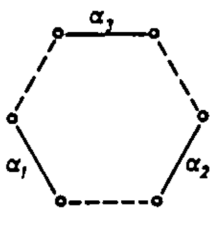
$$\Sigma_1 \; : \quad \circ \overset{\alpha_1}{\text{———}} \circ \overset{\alpha_2}{\text{———}} \circ \overset{\alpha_3}{\text{———}} \circ \cdots\cdots \circ \qquad , \alpha_1 + \alpha_2 < \tfrac{\pi}{2}\,, \; \alpha_2 + \alpha_3 > \tfrac{\pi}{2}\,, \tag{13.13}$$

$\Sigma_2 \;:$



$$, \alpha_1 + \alpha_2 < \tfrac{\pi}{2}\,, \; \alpha_2 + \alpha_3 < \tfrac{\pi}{2}\,. \tag{13.14}$$

The exceptional scheme is said to be *of type B*. By means of (13.11), it is given by



$$, 0 < \alpha_1, \alpha_2, \alpha_3 < \tfrac{\pi}{2}\,. \tag{13.15}$$

The corresponding polyhedron is a Lambert cube, i.e., a cube bounded by pairs of opposite Lambert quadrangles with equal angle α_j, $1 \leq j \leq 3$, (see Figure 13.1).

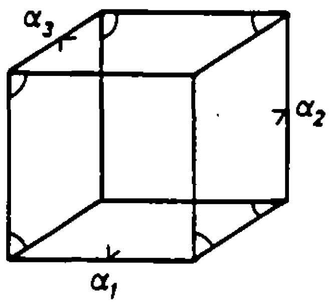


Figure 13.1

13.2. The volume of a d-truncated orthoscheme

13.2.1. The Schläfli differential formula

For $n \geq 1$, let Σ denote the elliptic linear scheme of order $n+1 \geq 2$ associated to a spherical n-orthoscheme R. The normalized volume function

$$f_n(\Sigma) = f_n := c_n \mathrm{vol}_n(R) \quad \text{with} \quad c_n = \frac{2^{n+1}}{\mathrm{vol}_n(S^n)} = \frac{2^n}{\pi^{\frac{n+1}{2}}} \Gamma\left(\frac{n+1}{2}\right) , \; f_0 := 1 , \quad (13.16)$$

is called the function of Schläfli (see [26, Nr.23, p.238]). The function f_n is proportional to $\mathrm{vol}_n(R)$ such that $f_n = 1$ for the orthoscheme with all dihedral angles equal to $\frac{\pi}{2}$. Moreover, the function of Schläfli satisfies the following factorization property (see [26, Nr.23, p.238]):

LEMMA 13.2. *Let Σ denote an elliptic linear scheme of order $n+1 \geq 2$ consisting of disjoint components $\sigma_1, \ldots, \sigma_r$ of orders $n_1+1, \ldots, n_r+1 \geq 1$. Then*

$$f_n(\Sigma) = f_{n_1}(\sigma_1) \cdots f_{n_r}(\sigma_r) \quad . \qquad (13.17)$$

For spherical Coxeter orthoschemes, Schläfli determined explicitly all possible values of f_n (cf. [26, Nr.30, p.268 ff]); in particular, he found

$$f_n(A_{n+1}) = \frac{2^{n+1}}{(n+2)!} \quad , \qquad (13.18)$$

where A_{n+1} denotes the scheme

$$\circ\!-\!\!-\!\circ\!-\!\!- \quad \cdots \quad -\!\!-\!\circ\!-\!\!-\!\circ$$

of order $n+1$. Interpreting hyperbolic n-space H^n as upper half of the pseudosphere of radius $i = \sqrt{-1}$ in E^{n+1}, the notion of Schläfli's function can be carried over to orthoschemes $R_d \subset H^n$ of degree d, $0 \leq d \leq 2$, with graph Σ_d:

The function

$$F_n(\Sigma_d) := i^n c_n \mathrm{vol}_n(R_d) \quad \text{with} \quad i^2 = -1 \quad , \quad F_0 := 1 \quad , \tag{13.19}$$

where the constant c_n is defined as in (13.16), is called the Schläfli function of R_d. Thus, for even dimensions,

$$F_{2n}(\Sigma_d) = (-1)^n \left(\frac{2}{\pi}\right)^n \cdot \prod_{p=1}^{n} (2p-1) \cdot \mathrm{vol}_{2n}(R_d) \quad , \quad n \geq 1 \quad ,$$

is a real-valued function.

Denote by $\mathcal{R}_\varsigma$ the set of compact d-truncated orthoschemes in H^n of schematic type ς (cf. **13.1.2.**). Since every element of $\mathcal{R}_\varsigma$ is acute-angled (see **13.1.2.**), its congruence class is uniquely determined by its dihedral angles (see [3, §3, Uniqueness Theorem]). Therefore, Schläfli's volume function $F_n = F_n|\mathcal{R}_\varsigma$ restricted on $\mathcal{R}_\varsigma$ may be expressed as a function of the dihedral angles. The differential of F_n depending on the dihedral angles can be represented by Schläfli's formula as follows (see [15, **2.**]):

THEOREM 13.3. (Schläfli's differential formula). *Let F_n, $n \geq 2$, be the Schläfli function on the set $\mathcal{R}_\varsigma$ of compact d-truncated orthoschemes in H^n of schematic type ς with essential angles $\alpha_1, \ldots, \alpha_{m(\varsigma)}$ and with scheme Σ_ς. Denote by $F_{n-2}(k)$ the Schläfli function of the apex of codimension 2 associated to the essential angle α_k of measure $f_1(k) := \frac{2}{\pi}\alpha_k$, $1 \leq k \leq m(\varsigma)$. Then*

$$dF_n(\Sigma_\varsigma) = \sum_{k=1}^{m(\varsigma)} F_{n-2}(k)\, df_1(k) \quad . \tag{13.20}$$

Schläfli discovered this formula for spherical simplexes, and separately for the more basic orthoschemes. Much later, H. Kneser found a different, very elegant proof for both, spherical and hyperbolic simplexes (see [16]). As Schläfli already remarked (cf. [26, Nr. 25, p.246 ff, Nr.32, p.272 ff], and [28, Corollary, p.48]), the differential formula for orthoschemes can be generalized to arbitrary acute-angled polytopes by means of subdivision into orthoschemes (cf. [15, **2.2**]).

13.2.2. The reduction formula
As can be read off from Schläfli's differential formula, there is a fundamental difference in calculating volumes of polytopes of even or odd dimensions. In fact, as the two-dimensional case already indicates, the volume of an even-dimensional simplex can be expressed in terms of the volumes of certain lower dimensional spherical ones. This reduction principle was proved by Schläfli in the spherical case and extended by Hopf to the hyperbolic case by means of analytic continuation (cf. [12, p.134ff]). In principle, for every class of even-dimensional polytopes, an appropriate formula can be derived as soon as their schematic type is known, namely by an inductive argument using Schläfli's differential formula. In terms of Schläfli's function, the reduction formula for orthoschemes $R_d \subset \overline{H^{2n}}$, $n \geq 1$, of degree d, $0 \leq d \leq 2$, reads as follows (see [15, **3.**]):

THEOREM 13.3. (Reduction formula). *Denote by $R_d \subset \overline{H^{2n}}$, $0 \leq d \leq 2, n \geq 1$, a $2n$-dimensional orthoscheme of degree d with scheme Σ_d. Then*

$$F_{2n}(\Sigma_d) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum_{\sigma} f_{2n-(2k+1)}(\sigma) \quad , \quad \sum f_{-1} := 1 \quad , \tag{13.21}$$

where σ runs through all elliptic subschemes of order $2(n-k)$ of Σ_d all of whose components are of even order.

Thus, in order to calculate volumes of non-euclidean polytopes, it is sufficient to consider the volume problem for polytopes of odd dimensions.

13.2.3. The principal parameter and the fundamental relations
Let $R_d \subset H^3$, $0 \leq d \leq 2$, denote a compact d-truncated orthoscheme with essential angles $\alpha_1, \alpha_2, \alpha_3$ and scheme Σ_d. For the integration of its Schläfli differential, the principal parameter θ is of fundamental importance. To introduce this notion, we begin with the following

Definition. A connected subscheme σ of order 4 of Σ_d is called *maximal* if its number of weights having the form $\cos\alpha$ is maximal.

Hence, if Σ_d is of type A, then

$$\circ\overset{\alpha_1}{\text{———}}\circ\overset{\alpha_2}{\text{———}}\circ\overset{\alpha_3}{\text{———}}\circ$$

is the unique maximal subscheme of Σ_d, whereas in case B, exactly the schemes

$$\circ\overset{\alpha_i}{\text{———}}\circ\cdots\circ\overset{\alpha_j}{\text{———}}\circ \quad , \quad 1 \leq i < j \leq 3 \quad ,$$

are maximal. Furthermore, since every connected subscheme of order 4 of Σ_d is hyperbolic, a maximal subscheme of Σ_d has negative determinant (see [13, Proposition 1.2]).

Definition. The *principal parameter* θ of R_d is given by

$$0 \le \theta := \arccos\left(\frac{\mathrm{per}\sigma + \det\sigma - 2}{\mathrm{per}\sigma - \det\sigma - 2}\right)^{\frac{1}{2}} \le \frac{\pi}{2} \quad , \tag{13.22}$$

where σ is a maximal subscheme of Σ_d.

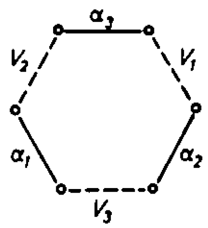
In the non-degenerate case, $\det\sigma < 0$ and $\mathrm{per}\sigma + \det\sigma > 2$. Consequently, the quotient in (13.22) is positive and less than 1. Thus, if Σ_d is of type A, θ is well-defined and takes the form

$$\cos^2\theta = \frac{\cos^2\alpha_1 \cos^2\alpha_3}{\cos^2\alpha_1 - \sin^2\alpha_2 + \cos^2\alpha_3} \quad , \text{ or}$$

$$\cos^2\theta(\cos^2\theta - \sin^2\alpha_2) = (\cos^2\theta - \cos^2\alpha_1)(\cos^2\theta - \cos^2\alpha_3) \quad . \tag{13.23}$$

In case B, it remains to check that θ is independent of the chosen maximal subscheme of Σ_d:

LEMMA 13.3. *Let Σ denote the scheme*



of type B describing a Lambert cube in H^3. For every maximal subscheme σ_k, $1 \le k \le 3$, of Σ with weights $\cos\alpha_{k-1}, \cosh V_k, \cos\alpha_{k+1}$ (indices are taken modulo 3), set

$$\theta_k := \arccos\left(\frac{\mathrm{per}\sigma_k + \det\sigma_k - 2}{\mathrm{per}\sigma_k - \det\sigma_k - 2}\right)^{\frac{1}{2}} \quad , \quad 0 \le \theta_k \le \frac{\pi}{2} \quad .$$

Then, $\theta_k = \theta_l$ for $1 \le k < l \le 3$.

Proof: We prove that $\cos^2\theta_1 = \cos^2\theta_3$, i.e.:

$$\frac{\cos^2\alpha_1 \cos^2\alpha_2}{\cos^2\alpha_1 + \sinh^2 V_3 + \cos^2\alpha_2} = \frac{\cos^2\alpha_2 \cos^2\alpha_3}{\cos^2\alpha_2 + \sinh^2 V_1 + \cos^2\alpha_3} \quad , \quad \text{or}$$

$$\cos^2\alpha_1(\cos^2\alpha_2 + \sinh^2 V_1) = \cos^2\alpha_3(\cos^2\alpha_2 + \sinh^2 V_3) \quad . \tag{13.24}$$

Since Σ is of signature (3,1), the extensions

$$\overline{\sigma_1} \;:\; \circ\overset{\alpha_3}{\text{———}}\circ\overset{V_1}{\cdots\cdots}\circ\overset{\alpha_2}{\text{———}}\circ\overset{V_3}{\cdots\cdots}\circ \quad , \quad \overline{\sigma_3} \;:\; \circ\overset{\alpha_1}{\text{———}}\circ\overset{V_3}{\cdots\cdots}\circ\overset{\alpha_2}{\text{———}}\circ\overset{V_1}{\cdots\cdots}\circ$$

of σ_1 and σ_3 in Σ have vanishing determinant. Hence, by Lemma 13.1, (13.5), we obtain

$$\det(\overline{\sigma_1}) = \sinh^2 V_3(\cosh^2 V_1 - \sin^2 \alpha_3) - \sin^2 \alpha_3 \cos^2 \alpha_2 = 0 \quad ,$$
$$\det(\overline{\sigma_3}) = \sinh^2 V_1(\cosh^2 V_3 - \sin^2 \alpha_1) - \sin^2 \alpha_1 \cos^2 \alpha_2 = 0 \quad ,$$

and therefore

$$\cos^2 \alpha_1 \cos^2 \alpha_2 + \sinh^2 V_1 \sinh^2 V_3 = \cos^2 \alpha_2 - \cos^2 \alpha_1 \sinh^2 V_1 \quad ,$$
$$\cos^2 \alpha_2 \cos^2 \alpha_3 + \sinh^2 V_1 \sinh^2 V_3 = \cos^2 \alpha_2 - \cos^2 \alpha_3 \sinh^2 V_3 \quad .$$

Subtraction yields

$$\cos^2 \alpha_2(\cos^2 \alpha_1 - \cos^2 \alpha_3) = \cos^2 \alpha_3 \sinh^2 V_3 - \cos^2 \alpha_1 \sinh^2 V_1 \quad ,$$

which proves (13.24). By the same procedure, one derives the remaining equalities.

Q.E.D.

This lemma implies that the principal parameter θ of a Lambert cube satisfies

$$\cos^2 \theta = \frac{\cos^2 \alpha_{k-1} \cos^2 \alpha_{k+1}}{\cos^2 \alpha_{k-1} + \sinh^2 V_k + \cos^2 \alpha_{k+1}} \quad , \quad \text{and}$$

$$\cos \theta^2(\cos^2 \theta + \sinh^2 V_k) = (\cos^2 \theta - \cos^2 \alpha_{k-1})(\cos^2 \theta - \cos^2 \alpha_{k+1}) \quad , \tag{13.25}$$

where indices are taken modulo 3.

In the asymptotic cases, which form the transitions from one degree to another, the principal parameter θ is realizable as a dihedral angle in $R_d \subset \overline{H^3}$. E.g., if R_0 is an ordinary orthoscheme, then

$$R_0 \quad \text{simply asymptotic} \quad \Longleftrightarrow \quad \theta = \frac{\pi}{2} - \alpha_2 \quad , \tag{13.26}$$

$$R_0 \quad \text{doubly asymptotic} \quad \Longleftrightarrow \quad \theta = \alpha_1 = \frac{\pi}{2} - \alpha_2 = \alpha_3 \quad . \tag{13.27}$$

Let R_0 denote a compact ordinary orthoscheme. Then, the following relationship holds between the measures of the essential angles α_k and the corresponding apices V_k, $k = 1, 2, 3$, (see [19] or [4, (11), p.229]):

$$\tanh V_k = \tan\theta \cdot \tan\overline{\alpha}_k \quad \text{or} \quad V_k = \frac{1}{2}\log\frac{\cos(\theta - \overline{\alpha}_k)}{\cos(\theta + \overline{\alpha}_k)} \quad , \quad k = 1,2,3 \quad , \tag{13.28}$$

where

$$\overline{\alpha}_k := \begin{cases} \alpha_2, & \text{k=2} \quad , \\ \frac{\pi}{2} - \alpha_k, & \text{k=1,3} \quad . \end{cases} \tag{13.29}$$

Now, similar relations hold for orthoschemes R_d of degree $d > 0$. In fact, in the projective model for H^3, a maximal subscheme of Σ_d describes an ordinary orthoscheme in P^3 with d ideal principal vertices lying outside the quadric $Q_{3,1}$. Since hyperbolic geometry admits a complex continuation to the space $AQ_{3,1}$ of ideal points of H^3 such that the distance between pole and polar line equals $i\frac{\pi}{2}$ (cf. [24, Sect. 5]), the fundamental relations for d-truncated orthoschemes can be summarized as follows (see [14, **3.3**]):

PROPOSITION 13.1. *Let R_d denote a compact orthoscheme of degree d, $0 \leq d \leq 2$, with essential angles α_k, $0 < \alpha_k < \frac{\pi}{2}$, corresponding apices of lengths V_k, $k = 1,2,3$, and principal parameter θ. Then,*

$$\tanh V_k = \frac{1}{2} \log \left| \frac{\cos(\theta - \overline{\alpha}_k)}{\cos(\theta + \overline{\alpha}_k)} \right| \quad , \quad k = 1,2,3 \quad , \quad \text{with} \tag{13.30}$$

$$\overline{\alpha}_k := \begin{cases} \alpha_2 \, , & \text{if } R_d \text{ of type A and } k = 2, \\ \frac{\pi}{2} - \alpha_k \, , & \text{else.} \end{cases} \tag{13.31}$$

Proposition 13.1 induces the following important identity

$$\frac{\partial V_k}{\partial \theta} = \frac{\sin \alpha_k \cos \alpha_k}{\cos^2 \theta - \sin^2 \overline{\alpha}_k} \qquad \text{for} \quad k = 1,2,3 \quad , \tag{13.32}$$

where $\overline{\alpha}_k$ satisfies (13.31).

The principal parameter characterizes also degenerations of R_d in the following way:

PROPOSITION 13.2. *Let R_d, $0 \leq d \leq 2$, denote a d-truncated orthoscheme with graph Σ_d, essential angles $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq \frac{\pi}{2}$ and principal parameter θ. If $\theta = 0$ or $\theta = \frac{\pi}{2}$, then $vol_3(R_d) = 0$.*

Proof: (i) Let $\theta = 0$. Then, by (13.22), $\det\sigma = 0$ for every maximal subscheme σ of Σ_d. Let Σ_d be of type A. First, if $0 < \alpha_1, \alpha_2, \alpha_3 < \frac{\pi}{2}$, Proposition 13.1 implies that the corresponding apices are of lengths zero. Hence, R_d is point-shaped. Second, the condition $\theta = 0$ implies that

$$\cos \alpha_2 = \sin \alpha_1 \sin \alpha_3 \quad .$$

Thus,

$$\alpha_2 = 0 \;\Leftrightarrow\; \alpha_1 = \alpha_3 = \frac{\pi}{2} \quad \text{or} \quad \alpha_2 = \frac{\pi}{2} \;\Leftrightarrow\; \alpha_1 \text{ (and/or } \alpha_3) = 0 \quad .$$

In both cases, at least one vertex triangle of R_d degenerates, and therefore $\mathrm{vol}_3(R_d) = 0$. If Σ_d is of type B, $\theta = 0$ reads as

$$\cosh V_k = \sin\alpha_{k-1}\sin\alpha_{k+1} = 1 \quad \text{for} \quad k \bmod 3 \quad ,$$

where V_k denotes the length of the apex associated to α_k. Hence, $\alpha_k = \frac{\pi}{2}$ and $V_k = 0$ for $k = 1, 2, 3$, and R_d is point-shaped.

(ii) Let $\theta = \frac{\pi}{2}$. By (13.22), $\mathrm{per}\sigma + \det\sigma - 2 = 0$ implies that at least one essential angle equals $\frac{\pi}{2}$. Let Σ_d be of type A. If α_1 and/or $\alpha_3 = \frac{\pi}{2}$ then $\mathrm{vol}_3(R_d) = 0$, since at least one facet degenerates in dimension. Hence, it remains to consider the case $\alpha_2 = \frac{\pi}{2}$. If $d > 0$, then at least one vertex triangle degenerates, and therefore $\mathrm{vol}_3(R_d) = 0$. If $d = 0$, and if $0 < \alpha_1, \alpha_3 < \frac{\pi}{2}$, then Proposition 13.1 yields $V_1 = V_3 = 0$ and therefore $\mathrm{vol}_3(R_d) = 0$. In the other case, where e.g. $\alpha_1 = 0$ (or $\frac{\pi}{2}$), one vertex triangle (or one facet) degenerates.
If Σ_d is of type B, at least two opposite Lambert quadrangles of the cube R_d degenerate to a point, from which $\mathrm{vol}_3(R_d) = 0$ follows.

Q.E.D.

13.2.4. The Euler dilogarithm and Lobachevsky's function
As will be seen later, the volume of a hyperbolic polyhedron is expressible in terms of the Euler dilogarithm and the Lobachevsky function related to it.

Let $z \in \mathbf{C}$, $|z| \leq 1$. Denote by

$$\mathrm{Li}_2(z) := \sum_{r=1}^{\infty} \frac{z^r}{r^2} = -\frac{1}{2}\int_0^z \frac{\log(1-z)}{z}\,dz$$

the Euler dilogarithm function. Splitting $\mathrm{Li}_2(z)$ up into real and imaginary parts, one deduces for

$$\mathrm{Li}_2(r,\phi) := \mathrm{Re}(\mathrm{Li}_2(re^{i\phi})) = \sum_{s=1}^{\infty} \frac{r^s\cos(s\phi)}{s^2} = -\int_0^r \frac{\log(1-2t\cos\phi+t^2)}{t}\,dt$$

the following properties (see [18, §5]):

$$\mathrm{Li}_2(r,0) = \mathrm{Li}_2(r) \quad , \quad \mathrm{Li}_2(r,\pi) = \mathrm{Li}_2(-r) \quad ; \tag{13.33}$$

$$\mathrm{Li}_2(r,\pi+\phi) = \mathrm{Li}_2(-r,\phi) \quad , \quad \mathrm{Li}_2(r,2n\pi\pm\phi) = \mathrm{Li}_2(r,\phi) \quad ; \tag{13.34}$$

$$\mathrm{Li}_2(\frac{1}{r},\phi) + \mathrm{Li}_2(r,\phi) = -\frac{1}{2}\log^2 r + \frac{1}{2}(\pi-\phi)^2 - \frac{\pi^2}{6} \quad ; \tag{13.35}$$

$$\mathrm{Li}_2(r,\frac{\pi}{3}) = \frac{1}{6}\mathrm{Li}_2(-r^3) - \frac{1}{2}\mathrm{Li}_2(-r) \quad . \tag{13.36}$$

For $z = e^{i\phi}$, $0 \le \phi \le 2\pi$,

$$\mathrm{Li}_2(e^{i\phi}) = \sum_{r=1}^{\infty} \frac{\cos(r\phi)}{r^2} + i \sum_{r=1}^{\infty} \frac{\sin(r\phi)}{r^2}$$
$$= \frac{\pi^2}{6} - \frac{\phi(2\pi - \phi)}{4} - i \int_0^{\phi} \log|2\sin\frac{t}{2}|\,dt \quad .$$

The *Lobachevsky function* is defined by

$$Л(z) := -\int_0^z \log 2\sin t\,dt \quad .$$

Thus, for $-\frac{\pi}{2} < \mathrm{Re}(z) \le \frac{\pi}{2}$, one obtains

$$Л(z) = \frac{i}{2}\{\mathrm{Li}_2(e^{2iz}) + z(\pi - z)^2 - \frac{\pi^2}{6}\} \quad . \tag{13.37}$$

Furthermore, if $0 < \frac{\pi}{2} - \beta < \alpha, \gamma < \frac{\pi}{2}$, then (13.37) together with (13.33)-(13.35) yield the following relation, which will be of use later (cf. **13.0.**, (13.4)):

$$\begin{aligned} S(\frac{\pi}{2} - \alpha, \beta, \frac{\pi}{2} - \gamma) =& \mathrm{Li}_2(r, 2\alpha) - \mathrm{Li}_2(r, 2\beta) + \mathrm{Li}_2(r, 2\gamma) - \mathrm{Li}_2(-r) - \\ & - (\frac{\pi}{2} - \alpha)^2 + \beta^2 - (\frac{\pi}{2} - \gamma)^2 \\ =& \mathrm{Re}(\frac{1}{i}\{Л(\alpha + i\tau) - Л(\alpha - i\tau) - Л(\frac{\pi}{2} - \beta + i\tau) + \\ & + Л(\frac{\pi}{2} - \beta - i\tau) + Л(\gamma + i\tau) - Л(\gamma - i\tau) + 2Л(\frac{\pi}{2} - i\tau)\}) \quad , \end{aligned} \tag{13.38}$$

where

$$\rho := \frac{\sqrt{\sin^2\alpha \sin^2\gamma - \cos^2\beta}}{\cos\alpha\cos\gamma} > 0 \quad , \quad r := \frac{1 - \rho}{1 + \rho} \quad , \quad \tau := -\frac{1}{2}\log r \quad . \tag{13.39}$$

Let $\omega \in \mathbf{R}$. Then

$$\begin{aligned} Л(\omega) &= \frac{1}{2}\mathrm{Re}(\mathrm{Li}_2(e^{2i\omega})) = \frac{1}{2}\sum_{r=1}^{\infty} \frac{\sin(2r\omega)}{r^2} \\ &= -\int_0^{\omega} \log|2\sin t|\,dt = (\frac{\pi}{2} - \omega)\log 2 + \int_0^{\frac{\pi}{2} - \omega} \log|\cos t|\,dt \quad , \end{aligned} \tag{13.40}$$

which is closely related to the Clausen function (see [18, §4])

$$\mathrm{Cl}_2(\omega) := \sum_{r=1}^{\infty} \frac{\sin(r\omega)}{r^2} = -\int_0^{\omega} \log|\sin\frac{t}{2}|\,dt$$

according to

$$Л(\omega) = \frac{1}{2}\,\mathrm{Cl}_2(2\omega) \quad , \quad \forall \omega \in \mathbf{R} \quad ,$$

and which satisfies the following

Properties. (see [18, §4])

(a) $Л(\omega)$ is well-defined and continuous for all $\omega \in \mathbf{R}$. $Л(\omega)$ is odd and π–periodic. It assumes its maximum value at $\omega_k = \frac{\pi}{6} + k\pi$, $k \in \mathbf{Z}$.

(b) $Л(\omega)$ satisfies the following distribution law:

$$Л(n\omega) = n \sum_{k \bmod n} Л(\omega + \tfrac{k\pi}{n}) \quad , \quad \forall n \in \mathbf{N}, \forall \omega \in \mathbf{R} \quad .$$

In particular for $n = 2$, this relation yields the duplication law

$$Л(2\omega) = 2Л(\omega) + 2Л(\tfrac{\pi}{2} + \omega) \quad , \quad \forall \omega \in \mathbf{R} \quad .$$

(c) For actual computation, the following representation of $Л(\omega)$ is very useful (cf. [31, Appendix 1, p.294]):

$$Л(\pi t) = \pi t\{9 - \log|2\sin \pi t| - \sum_{k=1}^{4} \left(c_k t^{2k+1} + k \log\frac{k+t}{k-t}\right) + \varepsilon\}$$

with

$$c_1 = 0.14754863716 \quad , \quad c_2 = 0.00142852188 \quad ,$$
$$c_3 = 0.00002919407 \quad , \quad c_4 = 0.00000076258 \quad ,$$

and $|\varepsilon| < 1.2 \times 10^{-11}$ for $|t| \leq \frac{1}{2}$.

13.2.5. The volume formula for R_d

Let R_d denote a compact orthoscheme of degree d, $0 \leq d \leq 2$, with scheme Σ_d, with essential angles α_k and corresponding apices of length V_k, $k = 1, 2, 3$. Then, by Schläfli's volume differential formula (see Theorem 13.3, **13.2.1.**),

$$d\mathrm{vol}_3(R_d) = -\frac{1}{2}\sum_{k=1}^{3} V_k\, d\alpha_k \quad , \tag{13.41}$$

where, for $k = 1, 2, 3$, the coefficients V_k are given by (see Proposition 13.1, **13.2.3.**)

$$V_k = \frac{1}{2}\log\left|\frac{\cos(\theta - \overline{\alpha}_k)}{\cos(\theta + \overline{\alpha}_k)}\right| \quad \text{with} \quad \overline{\alpha}_k := \begin{cases} \alpha_2\,, & \text{if } R_d \text{ of type A and } k = 2, \\ \frac{\pi}{2} - \alpha_k\,, & \text{else.} \end{cases} \tag{13.42}$$

Here, θ denotes the principal parameter of R_d given by

$$0 \le \theta = \arccos\left(\frac{\mathrm{per}\sigma + \det\sigma - 2}{\mathrm{per}\sigma - \det\sigma - 2}\right)^{\frac{1}{2}} \le \frac{\pi}{2} \quad ,$$

where σ is a maximal subscheme of Σ_d; θ depends on $\alpha_1, \alpha_2, \alpha_3$. Thus, V_k are very complicated expressions in the essential angles of R_d. However, by regarding $\alpha_1, \alpha_2, \alpha_3, \theta =: \alpha_4$ as four parameters independent from each other, the differential (13.41) suitably extended by $d\alpha_4$ can be integrated and thereafter identified with the volume of R_d. In this context, set

$$\widehat{V}_k(\alpha_1, \ldots, \alpha_4) := \frac{1}{2}\log\left|\frac{\cos(\alpha_4 - \overline{\alpha}_k)}{\cos(\alpha_4 + \overline{\alpha}_k)}\right| \quad , \quad k = 1, 2, 3 \quad . \tag{13.43}$$

Then,

$$\widehat{V}_k\big|_{\alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)} = V_k(\alpha_1, \alpha_2, \alpha_3) \quad , \quad k = 1, 2, 3 \quad .$$

Consider the region

$$G := \{(\alpha_1, \ldots, \alpha_4) \in \mathbf{R}^4 \mid 0 \le \alpha_1, \ldots, \alpha_4 \le \frac{\pi}{2}\,;\ \alpha_4 \ne \frac{\pi}{2} - \overline{\alpha}_k\,,\ k = 1, 2, 3\} \quad ,$$

and on G the following differential form

$$\Omega := \sum_{k=1}^{4} W_k\, d\alpha_k \quad , \quad \text{with} \tag{13.44}$$

$$W_k(\alpha_1, \ldots, \alpha_4) := -\frac{1}{2}\widehat{V}_k(\alpha_1, \ldots, \alpha_4) \quad , \quad k = 1, 2, 3 \quad , \tag{13.45}$$

and where $W_4 \in C^1(G)$ is determined by

(I) W_4 satisfies the integrability conditions $\dfrac{\partial W_i}{\partial \alpha_k} = \dfrac{\partial W_k}{\partial \alpha_i}$ for $1 \le i < k \le 4$.

(II) $W_4 = 0$ for $\alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)$.

By definition, $W_k(\alpha_1, \ldots, \alpha_4)$ depends only on α_k and α_4, i.e.:

$$\frac{\partial W_k}{\partial \alpha_i} = \frac{\partial W_i}{\partial \alpha_k} = 0 \qquad \text{for} \quad 1 \le i < k \le 3 .$$

On the other hand, **13.2.3.**, (13.32), yields

$$\frac{\partial W_k}{\partial \alpha_4} = -\frac{1}{2}\,\frac{\sin\alpha_k \cos\alpha_k}{\cos^2\alpha_4 - \sin^2\overline{\alpha}_k} \qquad \text{for} \quad k = 1,2,3\,,$$

where $\overline{\alpha}_k$ is determined by (13.42). Hence, according to the type of Σ_d, W_4 is given as follows (see [4, (11), p.237] for the case $d = 0$):

$$\begin{aligned} \text{Type A:} \quad W_4 &= \frac{1}{4}\log\frac{(\cos^2\alpha_4 - \sin^2\alpha_2)\cos^2\alpha_4}{(\cos^2\alpha_4 - \cos^2\alpha_1)(\cos^2\alpha_4 - \cos^2\alpha_3)}\,, \\ \text{Type B:} \quad W_4 &= \frac{1}{4}\log\frac{\sin^2\alpha_4\cos^4\alpha_4}{(\cos^2\alpha_1 - \cos^2\alpha_4)(\cos^2\alpha_2 - \cos^2\alpha_4)(\cos^2\alpha_3 - \cos^2\alpha_4)}\,. \end{aligned} \tag{13.46}$$

It is obvious that $W_4 \in C^1(G)$, and that it satisfies (I). By means of **13.2.3.**, (13.23) and (13.25), one easily derives (II). Therefore, the differential form Ω restricted to the hypersurface

$$\alpha_4 = \theta(\alpha_1,\alpha_2,\alpha_3) \qquad \text{in } \mathbf{R}^4$$

is identical with Schläfli's volume differential (13.41). Since Ω satisfies (I), it is exact and path-independently integrable in every connected component of G. To perform the integration of Ω, we distinguish between R_d with scheme Σ_d of type A or B.

A. Let R_0 denote an ordinary orthoscheme with essential angles $0 < \alpha_1,\alpha_2,\alpha_3 < \frac{\pi}{2}$; by **13.1.3.**, (13.12), these satisfy

$$\alpha_1 + \alpha_2 > \frac{\pi}{2} \quad , \quad \alpha_2 + \alpha_3 > \frac{\pi}{2} \quad ,$$

and thus, by **13.2.3.**, (13.22),

$$0 \le \theta < \alpha_1, \frac{\pi}{2} - \alpha_2, \alpha_3 < \frac{\pi}{2} \quad .$$

Consider the convex region

$$G_0 := \{\,(\alpha_1,\ldots,\alpha_4) \in G \mid \alpha_1+\alpha_2\,,\, \alpha_2+\alpha_3 > \frac{\pi}{2}\,,\, 0 \le \alpha_4 < \alpha_1, \frac{\pi}{2} - \alpha_2, \alpha_3 < \frac{\pi}{2}\,\}\,,$$

and on G_0, the differential form

$$\Omega = \sum_{k=1}^{4} W_k\, d\alpha_k \quad ,$$

where, for $k = 1,2,3$,

$$W_k = -\frac{1}{4}\log\left|\frac{\cos(\alpha_4 - \overline{\alpha}_k)}{\cos(\alpha_4 + \overline{\alpha}_k)}\right| \quad , \quad W_4 = \frac{1}{4}\log\frac{(\cos^2\alpha_4 - \sin^2\alpha_2)\cos^2\alpha_4}{(\cos^2\alpha_4 - \cos^2\alpha_1)(\cos^2\alpha_4 - \cos^2\alpha_3)} \quad .$$

Take an arbitrary point $P := (\alpha_1, \ldots, \alpha_4) \in G_0$ and integrate Ω along the line from $(\alpha_1, \alpha_2, \alpha_3, 0)$ to P. Since $W_k(\alpha_1, \alpha_2, \alpha_3, 0) = 0$ for $k = 1, 2, 3$, the integral

$$\begin{aligned}\widehat{V} :&= \int_0^{\alpha_4} W_4(\alpha_1, \alpha_2, \alpha_3, t)\, dt \\ &= \sum_{k=1}^{3} \frac{(-1)^k}{4} \{ Л(\frac{\pi}{2} + \alpha_4 + \overline{\alpha}_k) + Л(\frac{\pi}{2} + \alpha_4 - \overline{\alpha}_k) \} + \frac{1}{2} Л(\frac{\pi}{2} - \alpha_4)\end{aligned} \tag{13.47}$$

is an antiderivative of Ω in G_0. Restricted to $\alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)$, $\widehat{V}$ represents the volume $\mathrm{vol}_3(R_0)$ of R_0, since:

(i) For $\alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)$, Leibniz' Rule yields together with (I),(II) and Proposition 13.2, **13.2.3.**:

$$\begin{aligned}\frac{\partial \widehat{V}}{\partial \alpha_k} &= \frac{\partial}{\partial \alpha_k} \int_0^{\theta} W_4(\alpha_1, \alpha_2, \alpha_3, t)\, dt \\ &= W_4(\alpha_1, \alpha_2, \alpha_3, \theta) \cdot \frac{\partial \theta}{\partial \alpha_k} + \int_0^{\theta} \frac{\partial W_4}{\partial \alpha_k}(\alpha_1, \alpha_2, \alpha_3, t)\, dt \\ &= W_k(\alpha_1, \alpha_2, \alpha_3, \theta) - W_k(\alpha_1, \alpha_2, \alpha_3, 0) = W_k(\alpha_1, \alpha_2, \alpha_3, \theta) \\ &= -\frac{1}{2} V_k(\alpha_1, \alpha_2, \alpha_3) = \frac{\partial \mathrm{vol}_3(R_0)}{\partial \alpha_k} \quad , \quad k = 1, 2, 3 \quad .\end{aligned}$$

(ii) For $\alpha_4 = \theta = 0$, both $\widehat{V}$ and $\mathrm{vol}_3(R_0)$ vanish according to (13.47) and Proposition 13.2, **13.2.3.**.

Furthermore, the antiderivative $\widehat{V}$ of Ω certainly extends to the cube

$$\{(\alpha_1, \ldots, \alpha_4) \in \mathbf{R}^4 \mid 0 \le \alpha_1, \ldots, \alpha_4 \le \frac{\pi}{2}\}$$

in $\mathbf{R}^4$, still satisfying

$$\frac{\partial \widehat{V}}{\partial \alpha_k} = W_k(\alpha_1, \alpha_2, \alpha_3, \theta) \quad \text{for} \quad \alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)\,, \; 1 \le k \le 3\,,$$

$$\widehat{V}(\alpha_1, \alpha_2, \alpha_3, 0) = 0\,.$$

Hence, the restriction of $\widehat{V}(\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3))$ to the domain of definition of R_d, $d > 0$, is identical to $\mathrm{vol}_3(R_d)$ (see **13.1.3.**, (13.13),(13.14)). Thus, we proved the following

THEOREM 13.5. *Let R_d, $0 \le d \le 2$, denote a d-truncated orthoscheme with scheme Σ_d of type A and with essential angles $0 \le \alpha_1, \alpha_2, \alpha_3 \le \frac{\pi}{2}$. If σ denotes the maximal subscheme of Σ_d, then*

$$\begin{aligned}\mathrm{vol}_3(R_d) = \frac{1}{4}\{ & Л(\alpha_1+\theta) - Л(\alpha_1-\theta) + Л(\frac{\pi}{2}+\alpha_2-\theta) + Л(\frac{\pi}{2}-\alpha_2-\theta) + \\ & + Л(\alpha_3+\theta) - Л(\alpha_3-\theta) + 2Л(\frac{\pi}{2}-\theta)\} \quad , \quad \textit{where} \end{aligned} \tag{13.48}$$

$$0 \le \theta = \arccos\left(\frac{\mathrm{per}\sigma + \det\sigma - 2}{\mathrm{per}\sigma - \det\sigma - 2}\right)^{\frac{1}{2}} = \arctan\left(\frac{\cos^2\alpha_2 - \sin^2\alpha_1\sin^2\alpha_3}{\cos^2\alpha_1\cos^2\alpha_3}\right)^{\frac{1}{2}} \le \frac{\pi}{2} \quad .$$

B. Let R be a Lambert cube with essential angles $0 < \alpha_k < \frac{\pi}{2}$ and corresponding apices of length V_k, $k = 1,2,3$. By **13.2.3.**, (13.25), the principal parameter θ of R satisfies

$$\tan^2\theta = \frac{\cosh^2 V_k - \sin^2\alpha_{k-1}\sin^2\alpha_{k+1}}{\cos^2\alpha_{k-1}\cos^2\alpha_{k+1}} \quad , \quad k \bmod 3 \quad ,$$

which implies that $\theta > \alpha_1, \alpha_2, \alpha_3$. Hence, consider the convex region

$$H := \{ (\alpha_1, \ldots, \alpha_4) \in \mathbf{R}^4 \mid 0 \le \alpha_k < \alpha_4 \le \frac{\pi}{2}, \, k = 1,2,3 \},$$

choose an arbitrary point $P = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in H$ and integrate the differential form Ω from $(\alpha_1, \alpha_2, \alpha_3, \frac{\pi}{2})$ to P. Since $W_k(\alpha_1, \alpha_2, \alpha_3, \frac{\pi}{2}) = 0$ for $k = 1,2,3$, one obtains

$$\begin{aligned}\widehat{V} :&= \int_{\frac{\pi}{2}}^{\alpha_4} W_4(\alpha_1,\alpha_2,\alpha_3,t)\,dt \\ &= \frac{1}{4}\int_{\frac{\pi}{2}}^{\alpha_4} \log\frac{\sin^2 t\cos^4 t}{(\cos^2\alpha_1 - \cos^2 t)(\cos^2\alpha_2 - \cos^2 t)(\cos^2\alpha_3 - \cos^2 t)}\,dt \\ &= \frac{1}{4}\sum_{k=1}^{3}\{Л(\alpha_k+\alpha_4) - Л(\alpha_k-\alpha_4)\} + Л(\frac{\pi}{2}-\alpha_4) - \frac{1}{2}Л(\alpha_4) \quad .\end{aligned} \tag{13.49}$$

Restricting to the hypersurface $\alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbf{R}^4$, we can identify $\widehat{V}$ with the volume $\mathrm{vol}_3(R)$ of the Lambert cube R, since again (cf. A.):

(i) For $\alpha_4 = \theta(\alpha_1, \alpha_2, \alpha_3)$:

$$\frac{\partial\widehat{V}}{\partial\alpha_k} = \frac{\partial\mathrm{vol}_3(R)}{\partial\alpha_k} \quad , \quad k = 1,2,3 \quad .$$

(ii) For $\alpha_4 = \theta = \frac{\pi}{2}$, $\widehat{V}$ and $\mathrm{vol}_3(R)$ vanish according to (13.49) and Proposition 13.3,

13.2.3.

Thus, we derived the following

THEOREM 13.6. *Let R denote a Lambert cube with essential angles $0 \le \alpha_k \le \frac{\pi}{2}$ and corresponding apices of length V_k, $k = 1,2,3$. Then*

$$\mathrm{vol}_3(R) = \frac{1}{4}\sum_{k=1}^{3} \{Л(\alpha_k + \theta) - Л(\alpha_k - \theta)\} - \frac{1}{4}Л(2\theta) + \frac{1}{2}Л(\frac{\pi}{2} - \theta)\,, \text{ where} \tag{13.50}$$

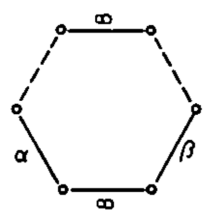
$$0 \le \theta = \arctan\left(\frac{\cosh^2 V_k - \sin^2\alpha_{k-1}\sin^2\alpha_{k+1}}{\cos^2\alpha_{k-1}\cos^2\alpha_{k+1}}\right)^{\frac{1}{2}} \le \frac{\pi}{2} \quad , \quad k \bmod 3 \quad .$$

Remarks.

(a) By means of hyperbolic trigonometry, the quantities $\cosh^2 V_k$ in Theorem 13.6 can be expressed as functions of the essential angles $\alpha_1, \alpha_2, \alpha_3$ according to

$$\cosh^2 V_k = 1 + \frac{1}{2}\left(\sqrt{A_k^2 + (2B_k\sin\alpha_k)^2} - A_k\right) \qquad \text{with} \tag{13.51}$$
$$A_k = \cos^2\alpha_{k-1} + \cos^2\alpha_{k+1} - B_k^2 \quad , \quad B_k = \frac{\cos\alpha_{k-1}\cos\alpha_{k+1}}{\cos\alpha_k} \quad , \quad k \bmod 3 \quad .$$

(b) In the limiting case of an asymptotic orthoscheme R_2 of degree 2 with scheme



the two formulae for $\mathrm{vol}_3(R_2)$ of Theorem 13.5 and Theorem 13.6 coincide. Apart from this exceptional case, these two formulae are conceptually different and cannot be related to each other by means of suitable functional equations for $Л(\omega)$. This can be seen by evaluating both expressions for the values $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{4}$ using (13.51).

13.3. Applications

13.3.1. Volumes of Coxeter polytopes

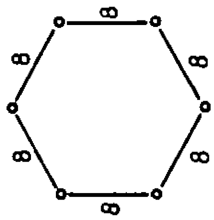
By means of Theorems 13.5, 13.6 and the Reduction formula 13.3, the volumes of all Coxeter orthoschemes of degree d, $0 \le d \le 2$, of dimension three and of even dimensions (existing only up to dimension eight) can be explicitly calculated. In dimension three, there are infinitely many of such Coxeter polyhedra. For $d = 0$, however, there are exactly 10 realizations with schemes and volumes according to the following list:

Scheme	Volume
$\circ - \circ - \circ \overset{6}{-} \circ$	$\frac{1}{8}\text{Л}(\frac{\pi}{3}) \simeq 0.0423$
$\circ - \circ = \circ = \circ$	$\frac{1}{6}\text{Л}(\frac{\pi}{4}) \simeq 0.0763$
$\circ - \circ \overset{5}{-} \circ - \circ$	$\simeq 0.0391$
$\circ - \circ \overset{6}{-} \circ - \circ$	$\frac{1}{2}\text{Л}(\frac{\pi}{3}) \simeq 0.1692$
$\circ = \circ - \circ \overset{5}{-} \circ$	$\simeq 0.0359$
$\circ = \circ - \circ \overset{6}{-} \circ$	$\frac{5}{24}\text{Л}(\frac{\pi}{6}) \simeq 0.1057$
$\circ = \circ = \circ = \circ$	$\frac{1}{2}\text{Л}(\frac{\pi}{4}) \simeq 0.2290$
$\circ \overset{5}{-} \circ - \circ \overset{5}{-} \circ$	$\simeq 0.0933$
$\circ \overset{5}{-} \circ - \circ \overset{6}{-} \circ$	$\simeq 0.1715$
$\circ \overset{6}{-} \circ - \circ \overset{6}{-} \circ$	$\frac{1}{2}\text{Л}(\frac{\pi}{6}) \simeq 0.2537$

For $d = 1$, the polyhedron with scheme

$$\circ \overset{\infty}{-} \circ = \circ = \circ \overset{\infty}{-} \circ$$

and with volume $\text{Л}(\frac{\pi}{4}) \simeq 0.4560$ is the simply truncated Coxeter orthoscheme of maximal volume. For $d = 2$, the volume is maximal and equal to $2\text{Л}(\frac{\pi}{4}) \simeq 0.9160$ for the totally asymptotic Lambert cube with scheme



The values for the even-dimensional cases may be found in [15, Appendix]. By dissection into orthoschemes, R. Meyerhoff determined the volumes of all Coxeter simplexes of dimension three (see [21, Appendix]).

13.3.2. Volumes of regular hyperbolic simplexes

Every simplex $S \subset X^n$ with acute dihedral angles can be dissected into orthoschemes in several ways. The dissection process $\chi_p := \chi_p(S)$ consists in taking a point $p \in S$ and in drawing successively the perpendiculars to the faces of lower dimensions. If p is an interior point of S, then S is subdivided into $(n+1)!$ orthoschemes. If p coincides with a vertex $v \in S$, then S is dissected into $n!$ orthoschemes.

Denote by S_{reg} a regular simplex (implying that all facets and vertex simplexes are regular), which is therefore parametrized by the dihedral angle 2α, say. Then, the dissecting orthoschemes with respect to χ_c, where c is the center of S, and with respect to χ_v are all congruent, and described by the characteristic scheme

$$\sigma_{n+1} = \sigma_{n+1}(\alpha) \quad : \quad \circ\overset{\alpha}{\text{---}}\circ\text{---}\circ\text{---} \quad \cdots \quad \text{---}\circ\text{---}\circ$$

of order $n+1$ for χ_c (cf. [4, Satz 1, p.271]), and by

$$\nu_{n+1} = \nu_{n+1}(\alpha) \quad : \quad \circ\overset{2\alpha}{\text{---}}\circ\overset{\alpha}{\text{---}}\circ\text{---}\circ\text{---} \quad \cdots \quad \text{---}\circ\text{---}\circ$$

of order $n+1$ for χ_v; this last result follows from the fact that $\chi_v(S)$ induces the process $\chi_{c'}(S(v))$ for the regular facet simplex $S(v)$ opposite to v, where c' is the center of $S(v)$. Then, $S(v)$ and, simultaneously, the vertex simplex S_v of $v \in S$ are dissected each into $n!$ congruent orthoschemes. Hence, the $(n+1)!$ congruent orthoschemes subdividing S_{reg} have principal vertices whose vertex orthoschemes are described by σ_n and ν_n, respectively.

In particular, if $S^{\infty}_{reg}(2\alpha) \subset \overline{H^n}$, $n \geq 2$, denotes a totally asymptotic regular simplex with scheme Σ^{∞}_{n+1}, then

$$F_n(\Sigma^{\infty}_{n+1}) = (n+1)!\, F_n(\sigma_{n+1}) = n!\, F_n(\nu_{n+1}) \quad . \tag{13.52}$$

Since the subschemes σ_n , ν_n are parabolic with $\det\sigma_n = \det\nu_n = 0$, it follows by Lemma 13.1, (13.5), using (13.18) that

$$\cos 2\alpha = \frac{1}{n-1} \quad . \tag{13.53}$$

Thus, up to motion, there is only one totally asymptotic regular simplex in $\overline{H^n}$, $n \geq 2$. Using Theorem 13.5 for the three-dimensional case, and for $n = 2m \geq 2$ even, the Reduction formula 13.3

$$F_{2m}(\Sigma^{\infty}_{2m+1}) = (2m+1)!\, F_{2m}(\sigma_{2m+1}) = (2m+1)! \sum_{k=0}^{m} \frac{(-1)^k}{k+1} \binom{2k}{k} \sum_{\sigma} f_{2m-(2k+1)}(\sigma)\,, \tag{13.54}$$

where σ runs through all elliptic subschemes of order $2(m-k)$ of σ_{2m+1} all of whose components are of even order, we obtain the following results:

(a) For $n = 2$, the area of a totally asymptotic triangle equals $\pi \simeq 3.1416$.

(b) For $n = 3$, the condition (13.53) implies that $\alpha = \frac{\pi}{6}$. If R_∞ denotes the doubly asymptotic orthoscheme given by

$$\nu_4(\frac{\pi}{6}) \quad : \quad \circ\text{---}\circ\overset{6}{\text{---}}\circ\text{---}\circ \quad ,$$

one gets by (13.52) (cf. [22, Corollary, p.20])

$$\mathrm{vol}_3(S^{\infty}_{reg}(\frac{\pi}{3})) = 3!\mathrm{vol}_3(R_\infty) = 3Л(\frac{\pi}{3}) \simeq 1.0149 \quad . \tag{13.55}$$

(c) For $n = 4$, we have $\cos 2\alpha = \frac{1}{3}$. Using (13.54) and results from **13.2.1.** we deduce

$$\begin{aligned}\frac{1}{5!}\, F_4(\Sigma^{\infty}_5) &= F_4(\sigma_5)\\ &= f_3(A_4) + f_1(\circ\text{---}\circ) f_1(\circ\overset{\alpha}{\text{---}}\circ) - \big(3f_1(\circ\text{---}\circ) - f_1(\circ\overset{\alpha}{\text{---}}\circ)\big) + 2\\ &= \frac{2}{3}(\frac{1}{5} - \frac{\alpha}{\pi}) \quad .\end{aligned}$$

Thus,

$$\mathrm{vol}_4(S^{\infty}_{reg}(2\alpha)) = \frac{\pi^2}{12}\, F_4(\Sigma^{\infty}_5) = \frac{4\pi}{3}(\pi - 5\alpha) \simeq 0.2689 \quad . \tag{13.56}$$

(d) For $n = 6$, we have $\cos 2\alpha = \frac{1}{5}$, and

$$\begin{aligned}\frac{1}{7!}\, F_6(\Sigma^{\infty}_7) &= F_6(\sigma_7)\\ &= f_5(A_6) + f_3(A_4) f_1(\sigma_2) + f_1(A_2) f_3(\sigma_4) -\\ &\quad - \big(3f_3(A_4) + f_3(\sigma_4) + 3f_1(A_2) f_1(\sigma_2) + 3(f_1(A_2))^2\big) +\\ &\quad + 2\big(5f_1(A_2) + f_1(\sigma_2)\big) - 5\\ &= -\frac{1}{3}\{f_3(\sigma_4) - \frac{4}{5\pi}\alpha + \frac{17}{105}\} \quad ,\end{aligned}$$

from which it follows that

$$\mathrm{vol}_6(S^{\infty}_{reg}(2\alpha)) = 2\pi^3\{7f_3(\sigma_4)+\frac{17}{15}\} - \frac{56\pi^2}{5}\alpha \quad . \tag{13.57}$$

By Schläfli's result for volumes of spherical orthoschemes (see **13.0.**, (13.4)), which can also be deduced from Theorem 13.5, (13.48), by means of analytic continuation using **13.2.4.**, (13.38), we obtain

$$\begin{aligned}\mathrm{vol}_6(S^{\infty}_{reg}(2\alpha)) = \frac{2}{\pi^2}\{&\mathrm{Li}_2(a,2\alpha)+\frac{1}{6}(\mathrm{Li}_2(a^3)-\mathrm{Li}_2(-a^3))-\frac{1}{2}(\mathrm{Li}_2(a)+\mathrm{Li}_2(-a))\}-\\ &-\frac{1}{2}(1-\frac{2}{\pi}\alpha)^2+\frac{1}{6}\} \quad ,\end{aligned} \tag{13.58}$$

where

$$\alpha = \arccos(\sqrt{\frac{3}{5}}) \quad , \quad a = \frac{\sqrt{3}-1}{\sqrt{3}+1} \quad , \quad \mathrm{Li}_2(a,2\alpha) = \sum_{r=1}^{\infty}\frac{a^r}{r^2}\cos(2r\alpha) \quad .$$

This yields

$$\mathrm{vol}_6(S^{\infty}_{reg}(2\alpha)) \simeq 0.0102 \quad . \tag{13.59}$$

By a result of Haagerup and Munkholm [11], a simplex in $\overline{H^n}$, $n \geq 2$, is of maximal volume if and only if it is totally asymptotic and regular. Hence, for $n = 2,3,4,6$, (a)-(d) provide an upper bound for the volume of an arbitrary simplex in $\overline{H^n}$, and we see that $\mathrm{vol}_n(S^{\infty}_{reg})$ is a decreasing function with respect to n verifying the inequality

$$\frac{n-1}{n^2} \leq \frac{\mathrm{vol}_{n+1}(S^{\infty}_{reg})}{\mathrm{vol}_n(S^{\infty}_{reg})} \leq \frac{1}{n} \tag{13.60}$$

due to Haagerup and Munkholm (see [11, Prop.2, p.4]).

13.3.3. Geometrical functional equations for $Л(\omega)$

Using different methods (e.g., cutting and pasting, limiting processes) for calculating the volume of a given convex polyhedron, one simultaneously obtains functional equations for the characteristic volume function $Л(\omega)$. In the following paragraph, we will present some examples which are derived by dissection into orthoschemes of degree $d = 0,1,2$.

Examples.

(a) Every orthoscheme $R_d \subset H^3$ of degree $d > 0$ admits a dissection into exactly three orthoschemes R^i_{d-1}, $i = 1,2,3$, of degree $d-1$. In the simplest case of a simply truncated asymptotic orthoscheme given by the graph

$$\circ\cdots\cdot\circ \stackrel{\frac{\pi}{2}-\alpha}{\text{———}} \circ \stackrel{\alpha}{\text{—}} \circ \stackrel{\beta}{\text{—}} \circ \quad , \quad \text{where} \quad 0 < \alpha + \beta < \frac{\pi}{2} \quad ,$$

this dissection (see Figure 13.2) implies the identity

$$\begin{aligned} 2Л(\alpha) + Л(\frac{\pi}{2} - \alpha + \beta) - Л(\frac{\pi}{2} + \alpha + \beta) &= 2\{\, Л(\alpha + \psi) - Л(\frac{\pi}{2} + \alpha) - Л(\frac{\pi}{2} + \psi)\} + \\ &+ Л(\psi + \phi) + Л(\psi - \phi) + Л(\frac{\pi}{2} - \phi + \alpha) + Л(\frac{\pi}{2} + \phi + \alpha) + \\ &+ Л(\frac{\pi}{2} + \beta - \alpha - \psi) - Л(\frac{\pi}{2} + \beta + \alpha + \psi) \quad , \end{aligned} \tag{13.61}$$

where $\phi, \psi \in (0, \frac{\pi}{2})$ with $\phi = \arccos\left(\dfrac{\sin\alpha}{\cos\beta}\right)$, $\psi = \arctan\left(\dfrac{\cos^2\beta - \sin^2\alpha}{\sin\alpha\cos\alpha}\right)$.

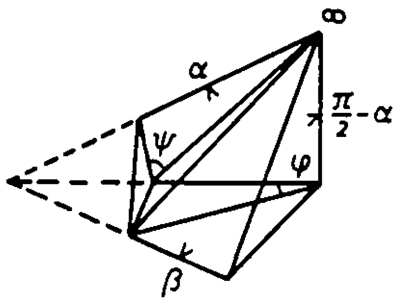


Figure 13.2

Lewin (private communication) showed that the second part of (13.61) can be reduced to the first part of (13.61) by two applications of Kummer's equation for $Л(\omega) = \frac{1}{2}\,\mathrm{Cl}(2\omega)$ (see [L, (4.68) and (4.69)]).

(b) Let R denote the totally asymptotic Lambert cube with dihedral angles $\frac{\pi}{4}$. Since $\mathrm{vol}_3(R) = 2 \cdot Л(\frac{\pi}{4}) = 4 \cdot \frac{1}{2} Л(\frac{\pi}{4})$ (cf. **13.3.1.**), one could expect a dissection of R into 2 congruent simply truncated orthoschemes with graph

$$\circ \stackrel{\infty}{\text{——}} \circ = \circ = \circ \stackrel{\infty}{\text{——}} \circ \quad ,$$

and into 4 congruent orthoschemes with graph

$$\circ = \circ = \circ = \circ \quad .$$

In fact, Figure 13.3 shows such a dissection.

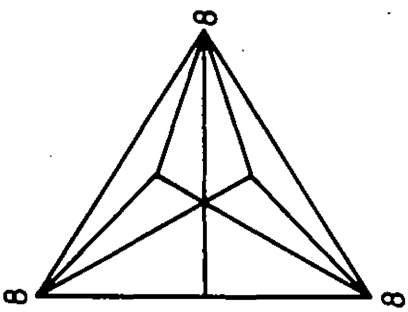
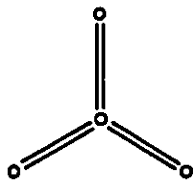


Figure 13.3

More obvious is the subdivision of R into 2 congruent simplexes S with scheme



Hence, $\mathrm{vol}_3(S) = Л(\frac{\pi}{4})$. (13.62)

(c) Denote by H^{∞}_{reg} the totally asymptotic regular hexahedron (or "cube") in $\overline{H^3}$ all of whose dihedral angles have to be equal to $\frac{\pi}{3}$. H^{∞}_{reg} may be viewed as totally asymptotic regular tetrahedron $S^{\infty}_{reg}(\frac{\pi}{3})$ to each of whose facets has been adjoined another tetrahedron. On the other hand, the dissection $\chi_c(H^{\infty}_{reg})$ (cf. **13.3.2.**), where c denotes the center of H^{∞}_{reg}, yields a subdivision into 48 orthoschemes with graph

Thus, by **13.3.1.** and (13.55), one gets

$$9Л(\frac{\pi}{3}) = 12(Л(\frac{\pi}{12}) - Л(\frac{5\pi}{12})) \quad .$$

By **13.2.4.**, (a), this equation is equivalent to the distribution law

$$Л(4 \cdot \frac{\pi}{12}) = 4 \cdot \sum_{1 \le k \le 3} Л(\frac{\pi}{12} + \frac{k\pi}{4}) \quad .$$

By another geometrical construction, Thurston proved the distribution law for $Л(3\alpha)$, $0 < \alpha < \frac{\pi}{3}$, (cf. [25, Prop. 4.12, p.201]).

(d) Denote by $g_m = g(P_1, \ldots, P_m)$, $m \geq 3$, a plane regular hyperbolic m-gon with vertices $P_1, \ldots, P_m$ and consider the pyramid $C := C_m(\alpha, 2\beta) \subset H^3$ over g_m with apex A such that the dihedral angles formed by g_m and the laterals equal α and the angles between two intersecting laterals equal 2β. With respect to the dissection $\chi_A(C)$, C admits a simplicial subdivision into $4m$ congruent orthoschemes given by

$$\overset{\alpha}{\circ\!\!-\!\!-\!\!-\!\!\circ}\!\!\overset{\beta}{-\!\!-\!\!-\!\!\circ}\!\!\overset{\frac{\pi}{m}}{-\!\!-\!\!-\!\!\circ} \quad .$$

If all $m+1$ vertices of C are points at infinity, then $\alpha = \frac{\pi}{2} - \beta = \frac{\pi}{m}$, and for the volume of $C_m^\infty := C_m(\frac{\pi}{m}, \frac{m-2}{m}\pi)$, Theorem 13.5 yields

$$\mathrm{vol}_3(C_m^\infty) = m Л(\frac{\pi}{m}) \quad . \tag{13.63}$$

In particular, for $m = 3$, we obtain again the formula (13.55) for the volume of S_{reg}^∞. Now, assume that the vertices $P_1, \ldots, P_m$ of the basis g_m are ideal in such a way that all edges P_iP_{i+1}, $1 \leq i \leq m-1$, *intersect* the absolute quadric $Q_{3,1}$. Furthermore, let the apex $A \in AQ_{3,1}$ be an ideal point; A can be chosen "near" the quadric $Q_{3,1}$ such that the edges AP_i, $i = 1, \ldots, m$, are *secants* with respect to $Q_{3,1}$ *of equal hyperbolic length.* Thus, polar truncation of each vertex of $C = C_m(\alpha, 2\beta)$ yields a $2m$-gonal prism $P_{2m}(\alpha, \alpha')$ whose mutually orthogonal laterals form by turns the angles $\alpha, \frac{\pi}{2}$ resp. $\alpha', \frac{\pi}{2}$ with one resp. the other totally rectangular $2m$-gon (see Figure 13.4 for the case $m = 3$). The dissection $\chi_A(C)$ of the underlying pyramid C in P^3 into orthoschemes induces a subdivision of $P_{2m}(\alpha, \alpha')$ into $2m$ congruent Lambert cubes with essential angles $\alpha, \alpha', \frac{\pi}{m}$.

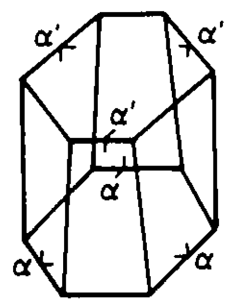


Figure 13.4

In the limiting case $\alpha = \alpha' = 0$, Theorem 13.6 yields for the volume of $P_{2m}^\infty := P_{2m}(0, 0)$

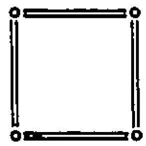
$$\mathrm{vol}_3(P_{2m}^\infty) = 2m Л(\theta_m) + \tfrac{m}{2}\{Л(\tfrac{\pi}{m} + \theta_m) - Л(\tfrac{\pi}{m} - \theta_m) - Л(2\theta_m) + 2Л(\tfrac{\pi}{2} - \theta_m)\}, \tag{13.64}$$

$$0 \leq \theta_m := \mathrm{arccot}(\cos\frac{\pi}{m}) \leq \frac{\pi}{2} \quad .$$

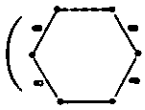
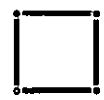
In particular, if $m = 3$, the prism P_6^∞ is the totally asymptotic regular octahedron O_{reg}^∞ with dihedral angles $\frac{\pi}{4}$, which can be dissected by means of the "central" dissection $\chi_c(O_{reg}^\infty)$ into 16 congruent orthoschemes given by

o═══o═══o═══o

as well as into 4 congruent simplexes S with scheme



Therefore, we obtain the relations

$6F_3\Big($  $\Big) = 16F_3($o═══o═══o═══o$) = 4F_3\Big($  $\Big)$,

which, by **13.3.1.** and (13.64), imply that $\mathrm{vol}_3(O_{reg}^\infty) = 8Л(\frac{\pi}{4})$, $\mathrm{vol}_3(S) = 2Л(\frac{\pi}{4})$, and

$$6Л(\theta_3) + \frac{3}{2}\{Л(\frac{\pi}{3} + \theta_3) - Л(\frac{\pi}{3} - \theta_3) - Л(2\theta_3) + 2Л(\frac{\pi}{2} - \theta_3)\} = 8Л(\frac{\pi}{4}) \simeq 3.6639\,, \quad (13.65)$$

where $\theta_3 = \arctan 2$.

(e) Finally, consider the remaining totally asymptotic regular polyhedron, the dodecahedron D_{reg}^∞ with dihedral angles $\frac{\pi}{3}$. The dissection $\chi_c(D_{reg}^\infty)$ starting from the center $c \in D_{reg}^\infty$ yields a subdivision into 120 congruent orthoschemes

$$\circ\overset{5}{\text{——}}\circ\text{——}\circ\overset{6}{\text{——}}\circ \quad ,$$

which leads to the equality

$$\mathrm{vol}_3(D_{reg}^\infty) = 30\{20Л(\frac{\pi}{3}) + Л(\frac{\pi}{5} + \frac{\pi}{6}) - Л(\frac{\pi}{5} - \frac{\pi}{6})\} \quad . \qquad (13.66)$$

13.4. Further Aspects

13.4.1. Volumes of hyperbolic 3-folds and Dedekind zeta functions
Denote by M^n, $n \geq 2$, an n-dimensional complete hyperbolic space form H^n/Γ of finite volume (orientable or non-orientable), where Γ is a discrete group of isometries of H^n. If Γ acts without fixpoints on H^n, then M^n is a manifold; otherwise it is an orbifold locally

modelled on $\mathbf{R}^n$ modulo a finite linear group action. One of the most important (topological) invariants of M^n is its volume. By a result of Wang [30], the volumes $\mathrm{vol}_n(M^n)$, where M^n runs through all n-manifolds, form a discrete subset of $\mathbf{R}_+$ if $n \neq 3$ (for n even, this is a consequence of the Gauss-Bonnet formula). For $n = 3$, however, Jørgensen and Thurston [27] proved that the volume spectrum

$$\mathbf{Vol} := \{\, \mathrm{vol}_3(M) \mid M \text{ hyperbolic 3-fold} \,\}$$

is a closed, non-discrete subset of $\mathbf{R}_+$, which is well-ordered and of order type ω^ω. In particular, there is a manifold (resp. orbifold) of minimum volume v_1 (resp. v_1'), and a 1-cusped manifold (resp. orbifold) of minimum volume v_ω (resp. v_c'). However, in contrast to the manifold case, there are non-compact 3-orbifolds whose volumes are isolated (cf. [20, p.278]). Up to now, very little is known about **Vol** and its smallest elements. In the sequel, we shall collect what is known to us (the lower volume bounds are due mostly to Meyerhoff [20]):

$$(v_1) \qquad 0.00115 < v_1 \leq \mathrm{vol}_3(W) \simeq 0.9427 \quad , \tag{13.67}$$

where W denotes the orientable Weeks manifold which is obtained by (5,1),(5,2) Dehn surgery on the complement of the Whitehead link in S^3 (cf. [7]). Here, the lower bound is due F. Gehring and G. Martin (cf. [F. Gehring, G. Martin, Inequalities for Möbius transformations and discrete groups, Preprint]) and improves the earlier result of 0.00082 found by Meyerhoff.

$$(v_1') \qquad 0.0000013 < v_1' \leq \mathrm{vol}_3(Q) \simeq 0.0391 \quad , \tag{13.68}$$

where Q is the following orientable tetrahedral orbifold (cf. [21]): Consider the reflection group Γ associated to the Coxeter orthoscheme

$$\circ\!-\!\!-\!\!-\!\circ\overset{5}{-\!\!-\!\!-}\circ\!-\!\!-\!\!-\!\circ \quad .$$

This simplex admits an inner twofold symmetry induced by a rotation of π. Denote by Γ' the isometry group generated by Γ and this rotation, and by Γ'_+ its subgroup of orientation-preserving isometries. Then, Q is the quotient H^n/Γ'_+ of volume $\simeq 0.0391$ (see **13.3.1.**).

$$(v_\omega) \qquad 0.5074 = \tfrac{v}{2} \leq v_\omega = \mathrm{vol}_3(G) = v \simeq 1.0149 \quad , \tag{13.69}$$

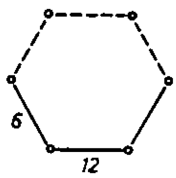
where $v = \mathrm{vol}_3(S^\infty_{reg})$, and where G denotes the Gieseking manifold; this is the unique non-compact (non-orientable) 3-manifold of minimal volume (cf. [1]). G is obtained from $S^\infty_{reg}(\frac{\pi}{3})$ by identifying two faces by means of a rotation of $\frac{2\pi}{3}$ about a common vertex and by identifying the opposite two faces by a rotation about a common vertex.

$$(v_c') \qquad \tfrac{\sqrt{3}}{24} \le v_c' \le \mathrm{vol}_3(Q_c) \simeq 0.0846 \quad , \tag{13.70}$$

where Q_c denotes the 1-cusped orientable double-cover of the (non-orientable) tetrahedral orbifold with Coxeter scheme

$$\circ\!-\!\!-\!\!-\!\circ\!-\!\!-\!\!-\!\circ\overset{6}{-\!\!-\!\!-}\circ \quad .$$

(v_∂) Kojima and Miyamoto [17] considered compact 3-manifolds with non-empty geodesic boundary and showed that the one of minimum volume v_∂, which is necessarily orientable but not unique, admits a subdivision into two regular truncated tetrahedra with dihedral angles $\frac{\pi}{6}$ or, equivalently, into 6 doubly truncated Coxeter orthoschemes with graph



Hence, by Theorem 13.5, $v_\partial \simeq 5.8735$.

(v'_{c^N}) Adams [2] investigated N-cusped 3-manifolds M_{c^N} and proved that $\mathrm{vol}_3(M_{c^N}) \ge N \cdot v$, where $v := \mathrm{vol}_3(S^\infty_{reg}) \simeq 1.0149$. These lower bounds are - in a unique way - realizable for $N = 1$ (by the Gieseking manifold) and for $N = 2$, whereas for $N > 2$, $\mathrm{vol}_3(M_{c^N}) > N \cdot v$.

For arithmetically definable hyperbolic space forms, the volumes can be related to values of Dedekind zeta functions at 2 (cf. [5] and [22]):

Denote by $F = \mathbf{Q}(\sqrt{-d})$, $d \ge 1$ square-free, an imaginary quadratic number field with discriminant $d > 0$, and by $\mathcal{O}_d$ its ring of integers. Then, the group $\Gamma = PSL(2, \mathcal{O}_d)$ is a discrete subgroup of $PSL(2, \mathbf{C})$ and acts therefore on H^3 by isometries.

Let ζ_F be the Dedekind zeta function associated to F, which can be written in the form

$$\zeta_F(s) = \zeta(s) \cdot \sum_{r=1}^{\infty} \left(\frac{-d}{r}\right) r^{-s} \quad , \tag{13.71}$$

where $(\frac{-d}{r})$ is the Kronecker symbol with values 0 or ± 1 associated to F. By a result of Humbert, the volume of a fundamental domain D of Γ is given by (cf. [22, p.20])

$$\mathrm{vol}_3(D) = d^{\frac{3}{2}} \zeta_F(2)/4\pi^2 \quad . \tag{13.72}$$

Borel [5] generalized Humbert's formula (13.72) to arbitrary number fields F having exactly b complex places and at least a real places, where a, b are non-negative integers with $a + b \ge 1$ (see Example (b) for $a = 2, b = 1$.)

Consider Humbert's formula (13.72) and use (see [22, p.21])

$$\left(\frac{-d}{r}\right)\sqrt{d} = \sum_{0<k<d} \left(\frac{-d}{k}\right) \sin\left(\frac{2\pi i k r}{d}\right) \quad .$$

Then, one obtains the following expression using **13.2.4.**, (13.40),

$$\mathrm{vol}_3(D) = \frac{d}{12} \sum_{0<k<d} \left(\frac{-d}{k}\right) Л\!\left(\frac{k\pi}{d}\right) \quad , \tag{13.73}$$

which together with (13.72) implies

$$\zeta_{\mathbf{Q}(\sqrt{-d})}(2) = \frac{\pi^2}{3\sqrt{d}} \sum_{0<k<d} \left(\frac{-d}{k}\right) Л\!\left(\frac{k\pi}{d}\right) \quad . \tag{13.74}$$

In [31], Zagier extended this result for $\zeta_F(2)$ to arbitrary number fields F first in terms of $A(x) = 2Л(\mathrm{arccot} x)$. This motivated his Conjecture about the representation of $\zeta_F(m)$, $m > 1$, in terms of (modified) polylogarithms (cf. [32]).

Examples.
(a) Consider the field $F = \mathbf{Q}(\sqrt{-3})$. Then, by a result of Meyerhoff (cf. [21]),

$$H^3/PSL(2, \mathcal{O}_3) = Q_c \quad .$$

Hence, by (13.72) and (v_c),

$$3^{\frac{3}{2}}\zeta_{\mathbf{Q}(\sqrt{-3})}(2)/4\pi^2 = \mathrm{vol}_3(Q_c) = \frac{1}{4}Л\!\left(\frac{\pi}{3}\right) \simeq 0.0846 \quad , \quad \text{and}$$

$$\zeta_{\mathbf{Q}(\sqrt{-3})}(2) = \frac{\pi^2}{3\sqrt{3}}Л\!\left(\frac{\pi}{3}\right) \simeq 0.6426 \quad .$$

(b) Consider the field $K = \mathbf{Q}(\sqrt{3+2\sqrt{5}})$ of discriminant -275 with $a = 2, b = 1$. Choose a maximal order $\mathcal{D}$ (all are mutually conjugate) of the Hamilton quaternion algebra over K. Then, one can associate to it a discrete subgroup $\Gamma_{\mathcal{D}}$ of $SL(2,\mathbf{C})$ such that $H^3/\Gamma_{\mathcal{D}} = Q$ (cf. [5, p.30] and [21, Remark (2), p.186]). Thus, the formula of Borel (cf. [6, **2.**]) and (v_1') yield

$$\frac{275^{\frac{3}{2}}}{2^7\pi^6}\zeta_K(2) = \mathrm{vol}_3(Q) \simeq 0.0391 \quad .$$

Therefore, we obtain

$$\zeta_K(2) = \frac{2^5\pi^6}{275^{\frac{3}{2}}}\left\{2Л\!\left(\frac{\pi}{3}+\theta\right) - 2Л\!\left(\frac{\pi}{3}-\theta\right) - Л\!\left(\frac{\pi}{6}+\theta\right) + Л\!\left(\frac{\pi}{6}-\theta\right) + 2Л\!\left(\frac{\pi}{2}-\theta\right)\right\} \simeq 1.0537 \quad ,$$

where $\theta = \arctan(\sqrt{2\sqrt{5}-3})$.

13.4.2. Scissors congruence and Dehn invariants

By cutting and pasting of polytopes in a space X^n of constant curvature, one immediately gets in touch with the notion of scissors congruence groups $\mathcal{P}(X^n)$ and Hilbert's Third Problem concerning scissors congruence (equidecomposability of polytopes):

Let G denote the group of isometries of X^n. Then, the scissors congruence group $\mathcal{P}(X^n)$ is defined to be the free abelian group generated by symbols $[P]$, one for each polytope $P \subset X^n$, modulo the relations

(i) $[P] = [P_1] + [P_2]$, if $P = P_1 + P_2$ in the sense of elementary geometry.

(ii) $[gP] = [P]$, where $g \in G$.

The scissors congruence problem consists in finding a complete system of invariants for the classes of polytopes in $\mathcal{P}(X^n)$. By means of the Dehn invariants (suitably defined and including volume) the scissors congruence problem was solved for $n \leq 2$ (this is a classical result), in E^3 by Sydler and in E^4 by Jessen (for references, see [9, p.159]). We are mainly interested in $\mathcal{P}(H^n)$ and, in particular, in the case $n = 3$. For the group $\mathcal{P}(H^3)$, Milnor conjectures that the scissors congruence class of a hyperbolic polyhedron is determined by its volume and its Dehn invariant; recall that the classical Dehn invariant is given by

$$\Psi \; : \; \mathcal{P}(H^3) \longrightarrow \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}/2\pi\mathbf{Z}$$

associating to a class of $\mathcal{P}(H^3)$, represented by a polyhedron $P \subset H^3$ with dihedral angles α_A along edges A of length $l(A)$, the expression

$$\Psi(P) = \sum_A l(A) \otimes \alpha_A \quad . \tag{13.75}$$

Remark.

The Dehn invariant looks very similar to Schläfli's volume differential (cf. **13.2.1.**, or [15, **2.2**])

$$d\mathrm{vol}_3(P) = -\frac{1}{2} \sum_A l(A) d\alpha_A \quad .$$

Like this Schläfli differential, the Dehn invariant can be extended to the set of asymptotic polyhedra representing the elements of $\mathcal{P}(\overline{H^3})$ (cf. [9, p.168]).

Beside $\mathcal{P}(H^n)$ and $\mathcal{P}(\overline{H^n})$ there are other notions of hyperbolic scissors congruence groups, e.g., the group $\mathcal{P}(\partial H^3)$ generated by classes of totally asymptotic polytopes. Some of these groups are identical; by a result of Dupont (cf. [9, Theorem 2.1, p.162]), the groups $\mathcal{P}(H^n)$ and $\mathcal{P}(\overline{H^n})$ are isomorphic for $n \geq 2$. If $\mathcal{P}(\overline{H^n})_\infty$ denotes the group associated to all totally asymptotic simplexes in $\overline{H^n}$, then $\mathcal{P}(\overline{H^{2k+1}})$ equals $\mathcal{P}(\overline{H^{2k+1}})_\infty$ for $k > 1$ (cf. [25, Prop. 3.7, p.195]). Moreover, one can show that $\mathcal{P}(H^n)$ (resp. $\mathcal{P}(\overline{H^n})$) can be generated by classes of orthoschemes (resp. doubly asymptotic orthoschemes) (cf. [25, 4.]), which implies that $\mathcal{P}(H^n)$, $n > 0$, (resp. $\mathcal{P}(\overline{H^n})$ for $n > 1$) is 2-divisible. In particular, $\mathcal{P}(\overline{H^3}) = \mathcal{P}(\overline{H^3})_\infty \cong \mathcal{P}(H^3)$ is 2-divisible.

Divisibility questions about $\mathcal{P}(\overline{H^3})$ can be reduced to corresponding problems for doubly asymptotic orthoschemes $R_\infty = R_\infty(\alpha)$ with graph

$$\circ \overset{\alpha}{\text{——}} \circ \overset{\frac{\pi}{2}-\alpha}{\text{————}} \circ \overset{\alpha}{\text{——}} \circ \quad ,$$

whose volumes equal $\frac{1}{2}\text{Л}(\alpha)$ (cf. Theorem 13.5): Consider the map

$$L \,:\, \mathbf{R} \longrightarrow \mathcal{P}(\overline{H^3})$$

given by

$$L(\alpha) := [R_\infty(\alpha)] \quad , \tag{13.76}$$

$$L(\alpha) = -L(-\alpha) \quad , \quad L(\alpha + \pi) = L(\alpha) \quad .$$

Then, one can show that L satisfies a distribution law analogous to **13.2.4.**, (a), for the Lobachevsky function $\text{Л}(\alpha)$ (cf. [25, p.200-202]), which therefore admits a geometrical interpretation in terms of cutting and pasting of totally asymptotic simplexes (see also **13.3.3.**, (c)), and from which the divisibility of $\mathcal{P}(\overline{H^3})$ follows.

All these results can be brought into a more general context which allows a very elegant description of volume and scissors congruence problems in $\overline{H^3}$. Consider the group $\mathcal{P}(\partial H^3)$ homologically defined by Dupont as follows (cf. [9, p.165-166]): Let $\mathcal{P}(\partial H^n)$ be the abelian group generated by $(a_0, \ldots, a_n)$, $a_i \in \partial H^n$, satisfying

(i) $(a_0, \ldots, a_n) = 0$, if $a_0, \ldots, a_n$ lie in a subspace of dimension $< n$.

(ii) $\sum_{0 \leq i \leq n+1} (-1)^i (a_0, \ldots, \widehat{a_i}, \ldots, a_n) = 0$ for $a_i \in \partial H^n$ arbitrary.

(iii) $(ga_0, \ldots, ga_n) = \det g \cdot (a_0, \ldots, a_n)$ for $a_i \in \partial H^n$ and g an isometry of H^n.

This group is closely related to Thurston's group $\mathcal{P}'(\partial H^n)$ which is obtained from $\mathcal{P}(\partial H^n)$ by replacing (i) and (iii) by

(i)' $(a_0, \ldots, a_n) = 0$, if $a_i = a_j$ for $i \neq j$.

(iii)' $(ga_0, \ldots, ga_n) = (a_0, \ldots, a_n)$ for $a_i \in \partial H^n$ and g an orientation preserving isometry of H^n.

For $n = 3$, the group $\mathcal{P}'(\partial H^3)$ can be viewed as a special case of a group $\mathcal{P}_F$ associated to an arbitrary field F, studied independently by Bloch, Wigner and Thurston: $\mathcal{P}_F$ is defined to be the abelian group generated by 4-tuples (x_0, x_1, x_2, x_3), $x_i \in P^1(F) = F \cup \{\infty\}$ and $x_i \neq x_j$ for $i \neq j$, subject to the relations

(i) $(gx_0, gx_1, gx_2, gx_3) = (x_0, x_1, x_2, x_3)$ for $g \in PGL(2, F)$.

(ii) $\sum_{0 \leq i \leq 4} (-1)^i (x_0, \ldots, \widehat{x}_i, \ldots, x_4) = 0$ for distinct $x_i \in P^1(F)$.

Consider the case $F = \mathbf{C}$, and use for H^3 the upper half space model $\mathbf{C} \times \mathbf{R}_+$ bounded by the Riemann sphere $\partial H^3 = \mathbf{C} \cup \{\infty\} = P^1(\mathbf{C})$. Then, $PGL(2, \mathbf{C}) = PSL(2, \mathbf{C})$ is the group of orientation preserving isometries of H^3 acting on ∂H^3 by

$$g(z) = \frac{az + b}{cz + d} \quad \text{for} \quad z \in \mathbf{C} \cup \{\infty\} \quad , \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathbf{C}) \quad . \tag{13.77}$$

Recall that the cross-ratio $\{a_0 : a_1 : a_2 : a_3\}$ of four distinct points $a_0, a_1, a_2, a_3 \in P^1(\mathbf{C})$ is defined by

$$\{a_0 : a_1 : a_2 : a_3\} = (a_0 - a_2)(a_1 - a_3)/(a_0 - a_3)(a_1 - a_2) \in \mathbf{C}\backslash\{0, 1\} \quad , \tag{13.78}$$

and that one has

(a) $\{\infty : 0 : 1 : z\} = z$.

(b) $PSL(2, \mathbf{C})$ acts 3-transitively on $P^1(\mathbf{C})$.

(c) For two 4-tuples (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) of distinct points in $P^1(\mathbf{C})$, one has

$$\begin{aligned} &\{a_0 : a_1 : a_2 : a_3\} = \{b_0 : b_1 : b_2 : b_3\} \iff \\ &\exists\, g \in PSL(2, \mathbf{C}) \quad \text{with} \quad (b_0, b_1, b_2, b_3) = (ga_0, ga_1, ga_2, ga_3) \quad . \end{aligned}$$

Using this, one observes that $\mathcal{P}'(\partial H^3)$ is the abelian group generated by $[z] := [(\infty, 0, 1, z)]$ for $z \in \mathbf{C}\backslash\{0, 1\}$, satisfying

$$\sum_{0 \leq i \leq 4} (-1)^i \left[\{a_0 : \cdots : \widehat{a}_i : \cdots : a_4\}\right] = 0 \quad \text{for arbitrary} \quad a_i \in P^1(\mathbf{C}) \quad . \tag{13.79}$$

If all components a_i in (13.79) are distinct, one can use (a)-(c) to express (13.79) in the form

$$[z_1] - [z_2] + [z_1/z_2] - [(1 - z_1)/(1 - z_2)] + [(1 - z_2)z_1/(1 - z_1)z_2] = 0 \quad , \tag{13.80}$$

where $z_1, z_2 \in \mathbf{C}\backslash\{0,1\}$, $z_1 \neq z_2$. In fact, Dupont and Sah showed that $\mathcal{P}'(\partial H^3)$ is completely characterized by this relation, which implies that $\mathcal{P}'(\partial H^n) = \mathcal{P}_{\mathbf{C}}$ (see [9, Lemma 5.11, p.176]).

Denote by $S_\infty \subset \overline{H^3}$ a totally asymptotic simplex. By the above, its vertices can be brought into the form $\infty, 0, 1, z$, where $z \in P^1(\mathbf{C})$, and the volume of $S_\infty(z) = (\infty, 0, 1, z)$ becomes an expression in z:

Denote by

$$D(z) := \mathrm{Im}\big(\mathrm{Li}_2(z)\big) + \arg(1-z)\log|z| \quad , \quad z \in \mathbf{C}\backslash\{0,1\} \quad , \quad -\pi < \arg(1-z) < \pi \quad ,$$

the Bloch-Wigner Dilogarithm which has the following properties (cf. [9, p.172]):

$$D(e^{2i\alpha}) = 2Л(\alpha) \quad ;$$

$$-D(z) = D(\overline{z}) = D(\frac{1}{z}) = D(1-z) \quad ;$$

$$D(z^n) = n \cdot \sum_{\mu^n=1} D(\mu z) \quad ;$$

$$D(z_1) - D(z_2) + D(z_1/z_2) - D((1-z_1)/(1-z_2)) + D((1-z_2)z_1/(1-z_1)z_2) = 0$$

$$\text{for} \quad z_1, z_2 \in \mathbf{C}\backslash\{0,1\}\, , \, z_1 \neq z_2 \quad .$$

Then, by a result of Bloch and Wigner (cf. [9, (4.13), p.173]),

$$D(z) = \mathrm{vol}_3(S_\infty(z)) \quad . \tag{13.81}$$

Bloch considered D as ("imaginary") part of a more general function, which, slightly modified by Dupont and Sah (cf. [9, (4.14), p.173]), is of the following form:

$$\varrho \; : \; \mathbf{C}\backslash\{0,1\} \longrightarrow \wedge^2_{\mathbf{Z}}(\mathbf{C}) \quad ,$$

$$\varrho(z) := \frac{1}{2\pi i}\log(z) \wedge \frac{1}{2\pi i}\log(1-z) + 1 \wedge \frac{1}{4\pi^2}\mathrm{Li}_2(z) - 1 \wedge \frac{1}{4\pi^2}\mathrm{Li}_2(1-z) \quad , \tag{13.82}$$

is well-defined for $0 < \mathrm{Re}(z) < 1$ satisfying $\varrho(z) + \varrho(1-z) = 0$ and can be analytically continued to $\mathbf{C}\backslash\{0,1\}$. Here, $\wedge^2_{\mathbf{Z}}(\mathbf{C})$ is the second exterior power written additively (i.e., it consists of formal sums of symbols $a \wedge b$, $a, b \in \mathbf{C}$, which are bimultiplicative and satisfy $a \wedge a = 0$). Consider the map

$$\lambda \; : \; \mathcal{P}_{\mathbf{C}} \longrightarrow \wedge^2_{\mathbf{Z}}(\mathbf{C}^\times)$$

defined by $\lambda[z] := z \wedge (1-z)$. Then by a result of Dupont and Sah, the following diagram commutes (cf. [9, p.171-172]):

$$\begin{array}{ccc} \mathcal{P}_{\mathbf{C}} & \xrightarrow{2\lambda} & \wedge^2_{\mathbf{Z}}(\mathbf{C}^\times) \\ \big\downarrow & & \big\downarrow{\scriptstyle -\log^-} \\ \mathcal{P}(\partial H^3) & \xrightarrow{\Psi_e} & \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}/2\pi\mathbf{Z} \end{array},$$

where the left vertical arrow is given by the surjective map sending $[z]$ onto $(\infty, 0, 1, z)$; on $\mathcal{P}(\partial H^3)$, mapped canonically onto $\mathcal{P}(\overline{H^3})$, the extended Dehn invariant Ψ_e is given by

$$\Psi_e(S_\infty) = 2 \sum_{k=1}^{3} \log(2 \sin \alpha_k) \otimes \alpha_k$$

for a totally asymptotic simplex S_∞ with angles $\alpha_1, \alpha_2, \alpha_3$ along edges intersecting in a vertex (cf. [9, p.168-169]). The function $-\log^-$ sends $r \wedge e^{2\pi i \alpha}$ onto $-\log|r| \otimes \alpha$.

Now, the function $\lambda[z]$ satisfies the 5-term-equation

$$\lambda[z_1] - \lambda[z_2] + \lambda[z_1/z_2] - \lambda[(1-z_1)/(1-z_2)] + \lambda[(1-z_2)z_1/(1-z_1)z_2] = 0 \tag{13.83}$$

Thus, one could expect that $\varrho(z)$ has an analogous property. In fact, the induced additive homomorphism

$$\varrho : \mathcal{P}_{\mathbf{C}} \longrightarrow \wedge^2_{\mathbf{Z}}(\mathbf{C})$$

even satisfies (cf. [9, (4.18), p.174])

$$\mathrm{Im}\varrho[z] = \frac{D(z)}{2\pi^2} \quad ! \tag{13.84}$$

References

[1] C.C. Adams, The Non-Compact Hyperbolic 3-Manifold of Minimal Volume, Proc. Amer. Math. Soc. **100** (1987), 601–606.

[2] C.C. Adams, Volumes of N-Cusped Hyperbolic 3-Manifolds, J. London Math. Soc. (2) **38** (1988), 507–565.

[3] E.M. Andreev, On Convex Polyhedra in Lobacevskii Spaces. Math. USSR Sbornik **10** No. 3 (1970), 413–440.

[4] J. Böhm, E. Hertel, Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung. Birkhäuser, Basel, 1981.

[5] A. Borel, Commensurability Classes and Volumes of Hyperbolic 3-Manifolds, Ann. Scuola Norm. Sup. Pisa, Serie IV, Vol. VIII, n.1 (1981), 1–33.

[6] T. Chinburg, E. Friedman, The Smallest Arithmetic Hyperbolic Three-Orbifold, Invent. math. **86** (1986), 507–527.

[7] T. Chinburg, A Small Arithmetic Hyperbolic Three-Manifold, Proc. Amer. Soc. **100** (1987), 140–144.

[8] H.S.M. Coxeter, The Functions of Lobatschefsky and Schläfli, Quart. J. Math. (Oxford) **6** (1935), 13–29.

[9] J.L. Dupont, C.H. Sah, Scissors Congruences, II, J. Pure Appl. Algebra **25** (1982), 159–195.

[10] O. Giering, Vorlesungen über höhere Geometrie, Vieweg, Braunschweig, Wiesbaden, 1982.

[11] U. Haagerup, H.J. Munkholm, Simplices of Maximal Volume in Hyperbolic n-Space, Acta Math. **147** (1981), 1–12.

[12] H. Hopf, Die Curvatura integra Clifford-Kleinscher Raumformen, Nachr. Ges. d. Wiss. Göttingen, Math.-Phys. Kl. (1925), 131–141.

[13] H.-C. Im Hof, Napier Cycles and Hyperbolic Coxeter Groups, to appear in Bull. Soc. Math. Belg.

[14] R. Kellerhals, On the Volume of Hyperbolic Polyhedra, Math. Ann. **285** (1989), 541–569.

[15] R. Kellerhals, On Schläfli's Reduction Formula, to appear in Math. Zeitschrift.

[16] H. Kneser, Der Simplexinhalt in der nichteuklidischen Geometrie, Deutsche Math. **1** (1936), 337–340.

[17] S. Kojima, Y. Miyamoto, The Smallest Hyperbolic 3-Manifolds With Totally Geodesic Boundary, Preprint Tokyo Inst. Tech. (1990).

[18] L. Lewin, Polylogarithms and Associated Functions, North Holland, N.Y., Oxford, 1981.

[19] W. Maier, Inhaltsmessung im R_3 fester Krümmung, Arch. Math. **5** (1954), 266–273.

[20] R. Meyerhoff, Sphere-Packing and Volume in Hyperbolic 3-Space, Comment. Math. Helv. **61** (1986), 271–278.

[21] R. Meyerhoff, A Lower Bound for the Volume of Hyperbolic 3-Orbifolds, Duke Math. J. **57** (1988), 185–203.

[22] J. Milnor, Hyperbolic Geometry: The First 150 Years, Bull. Amer. Math. Soc. **6** (1982), 9–24.

[23] H. Minc, Permanents, Encyclopedia of Mathematics and its Applications, Addison-Wesley, Reading, Massachusetts, 1978.

[24] E. Roeser, Die nicht-euklidischen Geometrien und ihre Beziehungen untereinander, Oldenburg, München, 1957.

[25] C.H. Sah, Scissors Congruences, I, Gauss-Bonnet Map, Math. Scand. **49** (1981), 181–210.

[26] L. Schläfli, Theorie der vielfachen Kontinuität, in: Gesammelte Mathematische Abhandlungen **1**, Birkhäuser, Basel, 1950.

[27] W.P. Thurston, Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry, Bull. Amer. Math. Soc. **6** (1982), 357–381.

[28] È. B.Vinberg, Hyperbolic Reflection Groups. Russian Math. Surveys **40** (1985), 31–75.

[29] È. B.Vinberg, The Absence of Crystallographic Groups of Reflections in Lobachevsky Spaces of Large Dimension. Trans. Moscow Math. Soc. **47** (1985), 75–112.

[30] H.C. Wang, Topics in Totally Discontinuous Groups, in: Symmetric Spaces, Ed. Boothby-Weiss, N.Y., 1972.

[31] D.B. Zagier, Hyperbolic Manifolds and Special Values of Dedekind Zeta-Functions, Invent. math. **83** (1986), 285–301.

[32] D.B. Zagier, Polylogarithms, Dedekind Zeta Functions, and the Algebraic K-Theory of Fields, Preprint MPI 90-44 (1990), to appear in Proc. Texel Conf.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3, FRG

Mathematisches Institut
der Universität
Wegelerstr. 10
D-5300 Bonn 1, FRG.