Asymptotic Solutions to Differential Equations on Manifolds with Cusps

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Contents

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Introduction

The topic of the present paper is the investigation of general elliptic differential operators on manifolds with singularities. The first work in this direction seems to be the well-known paper by V. Kondrat'ev [1], where elliptic operators on manifolds with conical points were investigated.

The further development of this theme is connected with the works of a lot of mathematicians (see, for example, [2] — [10]). In paritcular, in works [4] — [5] the Mellin pseudodifferential calculus on manifolds with singular points of *conical* type was constructed, the finiteness theorems were proved, and the asymptotic expansions of solutions near points of singularities were found. On the basis of the resurgent representation the *resurgent analysis* [11] of the problem can be carried out and in particular, exact asymptotic expansions of solutions near singular points of the manifold can be obtained. We emphasize that the main and adequate technical tool for the investigation of problems on the manifolds with singularities of the cone type is the *Mellin transform.*

The present paper is aimed at the investigation of the quite new situation in the elliptic theory on manifolds with singularities, that is of the situation when the underlying manifold has singularities of the *cusp* type. In this situation the Mellin transform apparatus occurs to be inapplicable. Hence, the first and the most important question is to create an adequate apparatus for investigating equations on manifolds with singularities of this new type. Fortunately, at present such an apparatus (*resurgent representation*) was already created by the authors [12] — [11]. We remark that the resurgent analysis[1] being rather modern but very powerful tool of asymptotic theory of differential equations includes, in particular, the *summation procedure* for divergent series (the asymptotic expansions are, as a rule, the series of this type). As we shall see below, the divergent series typically arise in the theory of partial differential equations on manifolds with cusps, and, hence, the application of the resurgent analysis in this situation is inavoidable.

[1]The above mentioned book [11] by the authors is an elementary introduction to the resurgent analysis and, at the same time, contains sufficiently full list of definitions, notions and theorems needed for applications of resurgent analysis in the asymptotic theory of differential equations. Besides, it contains sufficiently complete bibliography on the topic and is not too expensive. Below, we allow ourselves to cite this book often not pretending in all cases that all affirmations in this book belong to the authors. The book itself contains all the priority information.

We remark also that the application of the resurgent analysis for obtaining asymptotic expansions for solutions was useful in the situation of conical points as well [17] — [19].

The present paper is the first from the sery of papers on elliptic theory on manifolds with singularities of the cusp type. Here we construct asymptotics of solutions near points of singularity of the underlying manifold. We remark also that, except for the fact that it is interesting by itself, the results obtained here are the basis for the construction of the *solvability* theory (Fredholm property) of elliptic problems in the spaces with the weight determined by the asymptotics as well as for some other questions of the theory.

A few words of the structure of the paper. To begin with, we consider simple examples for Beltrami-Laplace operators accosiated with metrices induced by the geometry of the problem. These examples are sufficiently representative in the sence that in the process of their considerations all main effects (but, unfortunately, not the apparatus!) arising during the consideration of this problem will be shown.

The ideas and the technical tools used in the paper, that is, semiclassical approach and resurgent analysis, are presented in Sections 3 and 4. One have to keep in mind that the semi-classical approach gives just formal asymptotics[2], and the resurgent analysis resummates these series thus supplying one with the real (exact) asymptotics.

1 Examples

In this section, we consider two concrete examples. These are the case when the manifold has cusps of the first and, consequently, second order. By these two principally different situations we illustrate all the qualitative features of the theory. The presentation is carried out for Beltrami-Laplace equations associated with Riemannian metrics induced from the three-dimensional space by cusps of the first and the second order.

It is well-known that the simplest model for a singular point of the *conical* type is a vertex of the circular cone. The form of partial differential operators near such points can be obtained from the consideration of the Beltrami-Laplace operator corresponding to the metrics induced on the surface of the cone from the three-

[2]That is, formal series asymptotics.

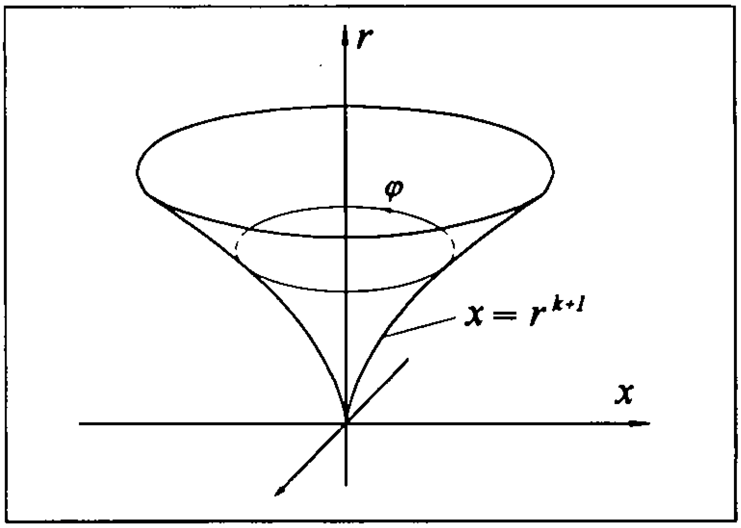

Figure 1. Singularity of the cusp type.

dimensional space. This operator reads

$$\Delta_g = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{c^2}{r^2}\frac{\partial^2}{\partial\varphi^2} = \frac{1}{r^2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + c^2\frac{\partial^2}{\partial\varphi^2}\right],$$

where c^2 is a constant determined by the angle of the cone.

Similar, in the constructing the general form of partial differential operator with point singularities *of the cusp type* one can use as a model the Beltrami-Laplace operator on the surface of the "thinning cone" in $\mathbf{R}^3$ obtained by rotation of the parabola of the $(k+1)$th order

$$x = r^{k+1} \tag{1}$$

around the axis Or (see Figure 1). The coordinates on such a surface are (r, φ), where φ corresponds to the angle of rotation around the axis Or. The number k will be called the *order of degeneracy*, and the point with $k = 1$ will be called a *simple* cusp point.

The Riemannian metrics induced on surface (1) from the metrics of $\mathbf{R}^3$ is given by the formula

$$ds^2 = \left(1 + (k+1)^2 r^{2k}\right) dr^2 + r^{2k+2}\, d\varphi^2.$$

Now, equating to zero the coefficients of powers of r in the latter relation, we arrive at the following recurrent system for the coefficients $u_j(\varphi)$ of expansion (4):

$$\begin{aligned}
\left[S^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_0(\varphi) &= 0, \\
\left[S^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_1(\varphi) &= -\left[2S\gamma + a\frac{\partial^2}{\partial\varphi^2}\right] u_0(\varphi), \\
\left[S^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_j(\varphi) &= -\left[2S(\gamma + j - 1) + a\frac{\partial^2}{\partial\varphi^2}\right] u_{j-1}(\varphi) \\
&\quad + \left[(\gamma + 2j - 4)(\gamma + 1) + (j-2)(j-3)\right] u_{j-2}(\varphi)
\end{aligned} \tag{5}$$

for $j = 2, 3, \ldots$.

Since we are constructing a nontrivial solution to (3), the function $u_0(\varphi)$ must be nonvanishing. This means that the operator

$$\left[S^2 + \frac{\partial^2}{\partial\varphi^2}\right]$$

is degenerate, or, in other words, that the number $z = S$ belongs to the spectrum of the analytic family

$$\hat{H}(z) = \left[z^2 + \frac{\partial^2}{\partial\varphi^2}\right] \tag{6}$$

of differential operators on the circle S^1. The spectrum of analytic family can be defined as the set of singular points of the inverse operator $\hat{H}^{-1}(z)$. In the considered case, this latter operator is a meromorphic one. This can be proved from the general theory, but here we prefer to verify this fact by a direct computation. To construct the operator $\hat{H}^{-1}(z)$ one needs to solve the equation

$$\hat{H}(z)\,u(\varphi) = \left[z^2 + \frac{\partial^2}{\partial\varphi^2}\right] u(\varphi) = f(\varphi).$$

Expanding the functions $u(\varphi)$ and $f(\varphi)$ into the Fourier series:

$$\begin{aligned}
u(\varphi) &= \sum_{k=-\infty}^{+\infty} e^{ik\varphi} u_k, \\
f(\varphi) &= \sum_{k=-\infty}^{+\infty} e^{ik\varphi} f_k,
\end{aligned}$$

So, the Beltrami-Laplace operator on the surface of the thinning cone has the form

$$\begin{aligned}\Delta_g &= r^{-2k-2}\left\{\frac{1}{1+(k+1)^2 r^{2k}}\left(r^{k+1}\frac{\partial}{\partial r}\right)^2\right.\\ &\qquad \left.-\frac{k(k+1)^2 r^{3k}}{\left(1+(k+1)^2 r^{2k}\right)^2}\left(r^{k+1}\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial\varphi^2}\right\}. \qquad (2)\end{aligned}$$

For the case of a simple cusp formula becomes

$$\Delta_g = r^{-4}\left\{\frac{1}{1+4r^2}\left(r^2\frac{\partial}{\partial r}\right)^2-\frac{4r^3}{(1+4r^2)^2}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial\varphi^2}\right\}.$$

In this section, we consider two examples of equations, which are simplified forms of equation (2) for $k=1$ and $k=2$, respectively. First can be considered as an example of a differential equation near a point of the cusp type of order 1, and the second as an example of a differential equation near a point of the cusp of order 2.

1.1 Cusp of order 1

Let us consider the equation

$$\hat{H}u \stackrel{\mathrm{def}}{=} r^{-4}\left\{\left(r^2\frac{\partial}{\partial r}\right)^2+(1+ar)\frac{\partial^2}{\partial\varphi^2}\right\}u(r,\varphi)=0, \qquad (3)$$

where a is some constant. First, we shall construct a formal solution to this equation. We search for a solution in the form[3]

$$u(r,\varphi)=e^{-\frac{S}{r}}r^{\gamma}\sum_{j=0}^{\infty}r^j u_j(\varphi) \qquad (4)$$

(the role of constants S and γ will be clear during the computations).

Cancelling out the inessential factor r^{-4} and substituting (4) into (3), we obtain the equation

$$\begin{aligned}e^{-\frac{S}{r}}r^{\gamma}&\left\{\sum_{j=0}^{\infty}\left[S^2+\frac{\partial^2}{\partial\varphi^2}\right]u_j(\varphi)r^j+\sum_{j=0}^{\infty}\left[2S\gamma+2Sj+a\frac{\partial^2}{\partial\varphi^2}\right]u_j(\varphi)r^{j+1}\right.\\ &\left.+\sum_{j=0}^{\infty}\left[\gamma(\gamma+1)+j(2\gamma+2)+j(j-1)\right]u_j(\varphi)r^{j+2}\right\} = 0.\end{aligned}$$

[3]The form of the anzatz (4) will be derived from the "semiclassical reasons" in Subsection 3.1 below.

we rewrite equation (6) in the form

$$\left[z^2 - k^2\right] u_k = f_k, \ k \in \mathbf{Z},$$

and, hence, the operator $\hat{H}^{-1}(z)$ is given by

$$u(\varphi) = \hat{H}^{-1}(z) f(\varphi) = \sum_{k=-\infty}^{+\infty} e^{ik\varphi} \frac{f_k}{z^2 - k^2}.$$

The latter formula shows that this operator is holomorphic at any point except for $z = k \in \mathbf{Z}$. One can see that the kernel of the operator $\hat{H}(z)$ is one and the same for $z = \pm k$. So, to construct different solutions to system (5) (and, hence, for equation (3)) it is sufficient to consider only the values $S \in \mathbf{Z}_+$.

The operator (6) has for every $k \in \mathbf{Z}_+$ two-dimensional kernel and cokernel with one and the same generators

$$U_k^{\pm}(\varphi) = e^{\pm ik\varphi}$$

(except for $k = 0$ where the kernel and cokernel are one-dimensional spaces with generator 1). So, the first equation from (5) shows that

- The number S must belong to $\mathbf{Z}_+$ ($S = k$).
- The function $u_0(\varphi)$ must be a linear combination of the generators of the kernel of operator (6) for $k = S$:

$$u_0^{(k)}(\varphi) = c_{0k}^+ e^{ik\varphi} + c_{0k}^- e^{-ik\varphi}, \tag{7}$$

  where the constants c_{0k}^+ and c_{0k}^- are, up to the moment, undefined.

We shall show that for each $k \in \mathbf{Z}$ there exists exactly one solution with the leading term (7). So, we fix some k (to be definite, we consider the case $k \neq 0$) and omit the corresponding indices.

Let us now proceed with the investigation of the second equation of (5). Substituting expression (7) into it, we arrive at the following equation for the function $u_1(\varphi)$ (we recall that $S = k$):

$$\left[k^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_1(\varphi) = -\left[2k\gamma + a\frac{\partial^2}{\partial\varphi^2}\right]\left[c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right]. \tag{8}$$

Since the operator on the left in the latter formula is not invertible, for this equation to be solvable, its right-hand part must satisfy the following compatibility conditions[4]:

$$\left\langle e^{ik\varphi}, \left[2k\gamma + a\frac{\partial^2}{\partial\varphi^2}\right]\left[c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right]\right\rangle = 0,$$
$$\left\langle e^{-ik\varphi}, \left[2k\gamma + a\frac{\partial^2}{\partial\varphi^2}\right]\left[c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right]\right\rangle = 0.$$

The two latter equations can be rewritten as

$$\begin{pmatrix} k(2\gamma - ak) & 0 \\ 0 & k(2\gamma - ak) \end{pmatrix} \begin{pmatrix} c_0^+ \\ c_0^- \end{pmatrix} = 0.$$

Since the vector $\left(c_0^+, c_0^-\right) \neq 0$, we obtain

$$\gamma = \frac{ak}{2},$$

and the components c_0^+ and c_0^- of the vector $\left(c_0^+, c_0^-\right)$ may be chosen arbitrarily. So, in this case, the right-hand side of equation (8) vanishes, and, hence, the general solution to equation (8) is

$$u_1(\varphi) = c_1^+ e^{ik\varphi} + c_1^- e^{-ik\varphi}. \tag{9}$$

Consider now the third equation from system (5). Substituting expressions (7) and (9) for functions $u_0(\varphi)$ and $u_1(\varphi)$ into this equation, for $u_2(\varphi)$ we obtain:

$$\begin{aligned}\left[k^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_2(\varphi) &= -\left[k(ak+2) + a\frac{\partial^2}{\partial\varphi^2}\right]\left(c_1^+ e^{ik\varphi} + c_1^- e^{-ik\varphi}\right) \\ &\quad + \frac{1}{4}ak(ak+2)\left(c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right).\end{aligned}$$

The compatibility conditions for this equation give

$$c_1^+ = \frac{ak(ak+2)}{8}c_0^+, \; c_1^- = \frac{ak(ak+2)}{8}c_0^-.$$

One can easily see that all the consequent equations can be solved in the same manner and that the obtained solution depends only on two arbitrary constants c_0^+

[4]Below, we use the notation

$$(f, g) = \int f(\varphi)\,\bar{g}(\varphi)\,d\varphi.$$

and c_0^-. So, the formal solution to equation (3) corresponding to any $k \in \mathbf{Z}_+$ (except for $k=0$) is given by

$$\begin{aligned} u_k(r,\varphi) &= c_0^+ e^{-\frac{k}{r}} r^{\frac{ak}{2}} \left[1 + \frac{ak(ak+2)}{8} r + \ldots\right] e^{ik\varphi} \\ &\quad + c_0^- e^{-\frac{k}{r}} r^{\frac{ak}{2}} \left[1 + \frac{ak(ak+2)}{8} r + \ldots\right] e^{-ik\varphi}. \end{aligned}$$

It can be shown that the series involved into the latter expansion diverge, and, hence, to obtain the exact asymptotics one should use the *resummation procedure*. It will be shown below, that in the case of a simple cusp such resummation can be carried out with means of the usual Borel resummation procedure.

1.2 Cusp of order 2

Our second example concerns a differential equation in a neighborhood of the cusp point of order 2. To simplify the computations, we slightly modify equation (4) with $k=2$ and consider the following equation

$$\hat{H}u \stackrel{\text{def}}{=} r^{-6}\left\{\left(r^3\frac{\partial}{\partial r}\right)^2 + \left(1 + ar + br^2\right)\frac{\partial^2}{\partial\varphi^2}\right\} u(r,\varphi) = 0. \tag{10}$$

We search for a solution to this equation in the form[5]

$$u(r,\varphi) = e^{-\frac{S_2}{2r^2} - \frac{S_1}{r}} r^{\gamma} \sum_{j=0}^{\infty} r^j u_j(\varphi),$$

where the constants S_1, S_2, and γ will be determined in the process of computations. Similar to the above subsection, we obtain the recurrent system of equations for the coefficients $u_j(\varphi)$ involved into the latter expansion. The first four of these equations are

$$\left[S_2^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_0(\varphi) = 0, \tag{11}$$

$$\left[S_2^2 + \frac{\partial^2}{\partial\varphi^2}\right] u_1(\varphi) = -\left[2S_1S_2 + a\frac{\partial^2}{\partial\varphi^2}\right] u_0(\varphi), \tag{12}$$

[5]See the footnote on page 5

$$
\begin{aligned}
\left[S_2^2+\frac{\partial^2}{\partial\varphi^2}\right]u_2(\varphi) &= -\left[2S_1S_2+a\frac{\partial^2}{\partial\varphi^2}\right]u_1(\varphi) \\
&\quad -\left[S_1^2+2S_2\gamma+b\frac{\partial^2}{\partial\varphi^2}\right]u_0(\varphi), \qquad (13) \\
\left[S_2^2+\frac{\partial^2}{\partial\varphi^2}\right]u_3(\varphi) &= -\left[2S_1S_2+a\frac{\partial^2}{\partial\varphi^2}\right]u_2(\varphi) \\
&\quad -\left[S_1^2+2S_2(\gamma+1)+b\frac{\partial^2}{\partial\varphi^2}\right]u_1(\varphi) \\
&\quad -S_1(2\gamma+1)u_0(\varphi), \qquad (14)
\end{aligned}
$$

and all the subsequent equations have the form

$$
\begin{aligned}
\left[S_2^2+\frac{\partial^2}{\partial\varphi^2}\right]u_j(\varphi) &= -\left[2S_1S_2+a\frac{\partial^2}{\partial\varphi^2}\right]u_{j-1}(\varphi) \\
&\quad -\left[S_1^2+2S_2(\gamma+j-2)+b\frac{\partial^2}{\partial\varphi^2}\right]u_{j-2}(\varphi) \\
&\quad -S_1(2\gamma+2j-5)u_{j-3}(\varphi) \\
&\quad -[\gamma(\gamma+2)+(2\gamma-3)(j-4)+(j-4)(j-5)]u_{j-4}(\varphi)
\end{aligned}
$$

for $j=4,5,\ldots$.

The solution to equation (11) goes quite similar to the previous subsection. The result is:

- The number S_2 must belong to $\mathbf{Z}_+$, $S_2=k$.
- The function $u_0(\varphi)$ must be a linear combination[6] of the generators of the kernel of operator $k^2+\frac{\partial^2}{\partial\varphi^2}$:

$$u_0(\varphi)=c_0^+e^{ik\varphi}+c_0^-e^{-ik\varphi},$$

  where the constants c_0^+ and c_0^- are, up to the moment, undefined (to be definite we consider here the case $k\neq 0$).

The compatibility conditions for equation (12) (we recall that $S_2=k$) are

$$\left\langle e^{\pm ik\varphi},\left[2S_1k+a\frac{\partial^2}{\partial\varphi^2}\right]\left(c_0^+e^{ik\varphi}+c_0^-e^{-ik\varphi}\right)\right\rangle=0,$$

[6]In what follows we, again, fix the value of k and omit the corresponding index.

which implies

$$2S_1 k - ak^2 = 0, \text{ or } S_1 = \frac{ak}{2}.$$

The general solution to (12) under such a choice of S_1 is

$$u_1(\varphi) = c_1^+ e^{ik\varphi} + c_1^- e^{-ik\varphi}.$$

Up to this moment the computations do not much differ from those of the preceding subsection. However, the consideration of equation (13) shows the difference between the cases of a simple cusp and a cusp of order 2. Namely, if we write down the compatibility conditions for this equation, we obtain

$$\begin{aligned} &\left\langle e^{\pm ik\varphi}, a\left(k^2 + \frac{\partial^2}{\partial\varphi^2}\right)\left(c_1^+ e^{ik\varphi} + c_1^- e^{-ik\varphi}\right)\right. \\ &\left.+\left[\frac{a^2k^2}{4} + 2k\gamma + b\frac{\partial^2}{\partial\varphi^2}\right]\left(c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right)\right\rangle \;=\; 0. \end{aligned}$$

Since the function $u_1(\varphi)$ belongs to the kernel of the operator $k^2 + \frac{\partial^2}{\partial\varphi^2}$, the latter relation can be rewritten in the form

$$\left\langle e^{\pm ik\varphi}, \left[\frac{a^2k^2}{4} + 2k\gamma + b\frac{\partial^2}{\partial\varphi^2}\right]\left(c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right)\right\rangle$$

which *does not contain* the constants c_1^+ and c_1^- at all. It is clear that for the latter relation to be valid with nonvanishing constants c_0^+ and c_0^-, one have to choose the constant γ from the relation

$$\frac{a^2k^2}{4} + 2k\gamma - bk^2 = 0,$$

that is,

$$\gamma = \frac{bk}{2} - \frac{a^2k}{8}.$$

Under such choice of γ, the general solution to equation (13) is

$$u_2(\varphi) = c_2^+ e^{ik\varphi} + c_2^- e^{-ik\varphi}$$

Up to the moment, the constants c_j^+ and c_j^-, $j = 0, 1, 2$ remain undefined. To determine c_1^+ and c_1^-(the pair c_0^+ and c_0^- are, in fact, the two arbitrary constants which determine a solution to the homogeneous equation in question), let us consider the compatibility conditions for equation (14). These conditions read

$$\left\langle e^{\pm ik\varphi}, 2k\left(c_1^+ e^{ik\varphi} + c_1^- e^{-ik\varphi}\right) + \frac{ak}{2}\left(bk - \frac{a^2k}{4} + 1\right)\left(c_0^+ e^{ik\varphi} + c_0^- e^{-ik\varphi}\right)\right\rangle = 0.$$

The values of the constants c_1^+ and c_1^- computed from the latter relation are

$$c_1^{\pm} = \frac{a}{4}\left(\frac{a^2k}{4} - bk - 1\right)c_0^{\pm}.$$

So, the asymptotic solution to equation (10) is

$$\begin{aligned} &c_0^+ \exp\left(-\frac{k}{2r^2} - \frac{ak}{2r}\right) r^{bk/2-a^2k/8}\left[1 + \frac{a}{4}\left(\frac{a^2k}{4} - bk - 1\right) + \dots\right] e^{ik\varphi} \\ &+c_0^- \exp\left(-\frac{k}{2r^2} - \frac{ak}{2r}\right) r^{bk/2-a^2k/8}\left[1 + \frac{a}{4}\left(\frac{a^2k}{4} - bk - 1\right) + \dots\right] e^{-ik\varphi}. \end{aligned} \tag{15}$$

Similar to the previous case, the resummation of the obtained series is nesessary for obtaining exact asymptotics of solutions to equation (10). However, in contrast to the case of the simple cusp, the resummation with the help of the 2-Borel transform *does not* lead to a resurgent function with simple singularities, so that the asymptotic information is lost during such a resummation. This fact is due to inhomogeneity of the phase function

$$-\frac{k}{2r^2} - \frac{ak}{2r}$$

of expansion (15) in the variable r. Below, we shall show that solutions obtained in this subsection can be resummated with the help of *resurgent representation*, introduced by the authors in [11].

2 Geometry of the problem

Let us desribe the form of general partial differential operators in a neighborhood of the singular points of the above considered type. We write down these forms having as models the Beltrami-Laplace operators near such points written down in the preceding section.

We begin with the standard case of conical point ([4]). The conical point can be described as the vertex of the cone K_Ω over the smooth manifold Ω:

$$K_\Omega = (\Omega \times [0,1]) \,/\, (\Omega \times \{0\}), \tag{16}$$

and differential operators in a neighborhood of such a point has the form

$$\hat{H} = r^{-m} H\left(r, \omega, r\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right),$$

where $m = \operatorname{ord}\hat{H}$ is an order of the operator $\hat{H}$. In the latter expression, the coordinates (r, ω) corresponding to the decomposition (16) were used. Namely, r is a coordinate along the directrice of the cone, $r \in [0,1]$, and $\omega = \left(\omega^1, \ldots, \omega^k\right)$ are local coordinates along Ω.

Similar, having in mind expression (2) for the Beltrami-Laplace operator, we see that the point m is a point of a cusp type of order k if the manifold near this point has (topologically) the cone structure (16), and differential operators in a neighborhood of such a point have the form

$$\hat{H} = r^{-(k+1)m} H\left(r, \omega, r^{k+1}\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right). \tag{17}$$

For the simple cusp this formula becomes

$$\hat{H} = r^{-2m} H\left(r, \omega, r^2\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right)$$

Similar considerations for the edges of the cusp type are as follows.

Chose as a model the direct product of the circular cusp K of order k by the straight line $\mathbf{R}^1$:

$$M = K \times \mathbf{R}^1.$$

The natural Riemannian metrics on the manifold M is given by

$$ds^2 = \left(1 + (k+1)^2 r^{2k}\right) dr^2 + r^{2k+2}\, d\varphi^2 + dx^2.$$

So, the Beltrami-Laplace operator on such a manifold has the form

$$\begin{aligned}\Delta_g \;=\;& r^{-2k-2}\left\{\frac{1}{1+(k+1)^2 r^{2k}}\left(r^{k+1}\frac{\partial}{\partial r}\right)^2 - \frac{k(k+1)^2 r^{3k}}{\left(1+(k+1)^2 r^{2k}\right)^2}\left(r^{k+1}\frac{\partial}{\partial r}\right)\right.\\ &\left. + \frac{\partial^2}{\partial \varphi^2} + \left(r^{k+1}\frac{\partial}{\partial x}\right)^2\right\},\end{aligned}$$

where x is a coordinate along $\mathbf{R}^1$. To write down the general form of a partial differential operator in the general case, let us represent the manifold in question near the considered edge in the form

$$M = K_\Omega \times X = (\Omega \times [0,1]) / (\Omega \times \{0\}) \times X,$$

and introduce the corresponding coordinates (r, ω, x). Then the general form of a differential operator near the edge of order k is

$$\hat{H} = r^{-(k+1)m} H\left(r, \omega, x, r^{k+1}\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}, r^{k+1}\frac{\partial}{\partial x}\right),$$

and in the case of simple cusp edge

$$\hat{H} = r^{-2m} H\left(r, \omega, x, r^2\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}, r^2\frac{\partial}{\partial x}\right).$$

The aim of the present paper is to investigate the asymptotic behavior of solutions to the differential equation

$$\hat{H}u = 0 \tag{18}$$

near singular points of the manifold M.

One has to keep in mind that, as it was already mentioned in Section 2 during the consideration of the examples in the preceding section, the cases of a simple cusp and of a cusp of higher order (both in point and edge type) are essentially different from the viewpoint of the asymptotic behavior of solutions.

3 Formal theory

Our main goal is the construction and investigation of asymptotic expansions of solutions to homogeneous differential equations near singular points of the underlying manifold having the cusp type. As it was shown in Section 2, the equation has the form

$$r^{-(k+1)m} H\left(r, \omega, r^{k+1}\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right) u = 0 \tag{19}$$

near such points. Here m is an order of the operator $\hat{H}$ and (r, ω) are coordinates on M associated with the following representation of this manifold:

$$M = ([0,1] \times \Omega) \,/\, (\{0\} \times \Omega)$$

near the cusp of order k. So, ω are coordinates on a smooth manifold Ω, and $r \in [0,1]$.

Equation (19) can be rewritten in the operator form

$$\hat{H}\left(r, r^{k+1}\frac{d}{dr}\right) u = 0, \tag{20}$$

where $\hat{H}$ is a differential operator whose coefficients are, in turn, differential operators on the smooth manifold Ω of corresponding orders.

From purely technical resons, we introduce a parameter h into equation (20), thus rewriting it in the form

$$\hat{H}\left(r, hr^{k+1}\frac{d}{dr}\right) u = 0; \tag{21}$$

initial equation (20) can be obtained from (21) with the help of the substitution $h = 1$.

Consequently, the exposition in the present section goes as follows. In the first subsection, we shall obtain general formulas valid for arbitrary values of k. The second subsection is aimed at concretization of these formulas for fixed values of the parameter. Finally, in the third subsection we present a procedure of the explicit computation of coefficients of asymptotic expansion (starting from the equation itself).

3.1 General asymptotic expansion

Let us search for solutions to equation (21) in the form

$$u(r) = e^{\frac{1}{h}S(r)}\varphi(r,h) = e^{\frac{1}{h}S(r)}\sum_{j=0}^{\infty} h^j \varphi_j(r), \tag{22}$$

where

$$\varphi_j : [0,1] \to E$$

is a function with values in some functional space[7] on the manifold Ω.

Expansion (18) has the form of a WKB-expansion in the small parameter h. At the same time, the parameter h is clearly not small, and the verification of such (convenient enough) expansion is in the fact that, as it will be shown *a posteriori*, the order of functions $\varphi_j(r)$ increase unboundedly with j, and, hence, (18) is in fact an asymptotic expansion as $r \to 0$.

So, substituting expansion (22) into equation (21) and using the evident relation

$$hr^{k+1}\frac{d}{dr}\left\{e^{\frac{1}{h}S(r)}\varphi(r,h)\right\} = e^{\frac{1}{h}S(r)}\left\{r^{k+1}S'(r) + hr^{k+1}\frac{d}{dr}\right\}\varphi(r,h),$$

we obtain the following relation for the series $\varphi(r,h)$:

$$\hat{H}\left(r, r^{k+1}S'(r) + hr^{k+1}\frac{d}{dr}\right)\sum_{j=0}^{\infty} h^j\varphi_j(r) = 0.$$

Expanding the operator involved into the latter relation in powers of h and equating the corresponding coefficients to zero, we arrive at the following recurrent system of

[7]One can use Sobolev spaces $H^s(\Omega)$ as the space E. Since the choice of these spaces is not essential for us in the sequel, we shall not concretize them here.

equations for the coefficients $\varphi_j(r)$ of the series $\varphi(r,h)$:

$$
\begin{aligned}
\hat{H}(r,P(r))\varphi_0(r) &= 0, \\
\hat{H}(r,P(r))\varphi_1(r) &= -\left[\frac{\partial \hat{H}}{\partial p}(r,P(r))\left(r^{k+1}\frac{d}{dr}\right)\right. \\
&\quad \left. +\frac{1}{2}r^{k+1}\frac{\partial^2 \hat{H}}{\partial p^2}(r,P(r))\frac{dP(r)}{dr}\right]\varphi_0(r), \qquad (23) \\
\hat{H}(r,P(r))\varphi_1(r) &= -\sum_{i=1}^{j}\hat{\mathcal{P}}_i\varphi_{j-i}(r),\ j=2,3,\ldots,
\end{aligned}
$$

where $P(r)=r^{k+1}S'(r)$, and the operator $\hat{\mathcal{P}}_1$ equals

$$\hat{\mathcal{P}}_1=\frac{\partial \hat{H}}{\partial p}(r,P(r))\left(r^{k+1}\frac{d}{dr}\right)+\frac{1}{2}r^{k+1}\frac{\partial^2 \hat{H}}{\partial p^2}(r,P(r))\frac{dP(r)}{dr}. \qquad (24)$$

To begin with, let us consider the first relation from (23). Since we are interested to construct *nontrivial* asymptotic solutions of the form (22), we have $\varphi_0(r)\neq 0$. Hence, the operator

$$\hat{H}\left(r,r^{k+1}S'(r)\right) \qquad (25)$$

must have a nontrivial kernel.

To analyse the condition of degeneracy of operator (25), let us consider the analytic family

$$\hat{H}(r,p) \qquad (26)$$

of operators on the manifold Ω as an analytic family in p for each fixed r. Since the initial operator is an elliptic one, the conditions of the finite-meromorphic invertibility of this operator are fulfilled. This means that:

1. The operators $\hat{H}(r,p)$ are Fredholm ones for each values of r,p.

2. For each $r>0$ there exists a p_0 such that the operator $\hat{H}(r,p_0)$ is invertible.

It is known (see, for example, [4]) that under the above conditions there exists an inverse $\hat{H}^{-1}(r,p)$ for operator (26), meromorphically dependent on p. Denote by

$$p=p(r)$$

the equation of the set of singularities of the operator $\hat{H}^{-1}(r,p)$ (the spectrum of operator family (26)). In general, the function $p(r)$ is a multivalued one.

Suppose, for simplicity, that all poles of the family $\hat{H}^{-1}(r,p)$ are simple up to the point $r=0$. Then it is evident that the function $p(r)$ splits to univalent branches

$$p = p_i(r)\,,\ i = 1,2,\ldots,$$

such that the functions $p_i(r)$ are regular functions in r (they are analytic for analytic $\hat{H}(r,p)$, and smooth for smooth $\hat{H}(r,p)$).

Remark 1 The latter assumption means that we are working in the situation of nondegenerate Lagrangian manifolds. The condition of the degeneracy of the operator $\hat{H}(r,p)$ determines a germ at $r=0$ of the Lagrangian manifold (possibly, with singularities) $p=p(r)$ in the phase space (r,p). The assumption that the function $p=p(r)$ splits into regular branches means exactly that this germ is split into the union of nondegenerate Lagrangian manifolds.

Now we have

$$S(r) = S_i(r)\,, \varphi_0(r) = \varphi_{0i}(r)\,,$$

where

$$r^{k+1}S_i'(r) = p_i(r)\,,\ \varphi_{0i}(r) \in \operatorname{Ker}\hat{H}(r,p_i(r))\,. \tag{27}$$

In what follows we fix some branch $p_i(r)$ omitting the subscript i.

Later on, we remark that, under the above assumptions, the operator

$$\hat{H}(r,p(r))$$

has zero index, so that the dimensions of the kernel and the cokernel of this operator coincide with each other. To simplify the presentation below, we suppose that

$$\dim\ \operatorname{Ker}\hat{H}(r,p(r)) = \dim\ \operatorname{Coker}\ \hat{H}(r,p(r)) = 1,$$

and denote by $U(r)$ and $V(r)$ the generators in the spaces $\operatorname{Ker}\hat{H}(r,p(r))$ and $\operatorname{Coker}\ \hat{H}(r,p(r))$, respectively.

So, we have

$$\varphi_0(r) = a_0(r)\,U(r)\,, \tag{28}$$

where the function $a_0(r)$ is, up to the moment, unknown.

Let us proceed with the investigation of the subsequent equations in (23). As we shall see below, the functions $\varphi_j(r)$ can be represented as series over powers of r:

$$\varphi_j(r) = r^{s_{j0}}\sum_{l=0}^{\infty} a_{jl}r^l.$$

The number s_{j0} will be called *the order* of the function $\varphi_j(r)$.

An operator $\hat{P}$ will be called an operator of (power) order l if the order of the function $\hat{P}\varphi(r)$ is not less than $s_0 + l$ for any function $\varphi(r)$ of order[8] s_0. The power order of an operator $\hat{P}$ will be denoted by $\mathrm{ord}_p \hat{P}$ (subscript p stands for "power").

The following affirmation takes place:

Lemma 1 *The operators $\hat{\mathcal{P}}_i$ involved into relations* (23) *are operators of power order ik. Besides, all these operators beginning from $i = 1$ can be represented in the form $r^{k+1}\hat{P}_i$.*

Proof. It is clear that the operators $\hat{\mathcal{P}}_i$ are determined from the relation

$$\hat{H}\left(\overset{2}{r}, \overline{r^{k+1}S'(r) + hr^{k+1}\frac{d}{dr}}^{\,1}\right) = \sum_{i=0}^{m} \hat{\mathcal{P}}_i,$$

where the numbers over the operators denote the order of their action (see, e. g. [20], [21]). Besides,

$$\hat{\mathcal{P}}_0 = \hat{H}(r, p(r))$$

and the operator $\hat{\mathcal{P}}_1$ is defined with relation (24). Clearly, we have

$$\begin{aligned} \mathrm{ord}\,\hat{\mathcal{P}}_0 &= 0, \\ \mathrm{ord}\,\hat{\mathcal{P}}_1 &= k. \end{aligned}$$

Let us carry out the proof by induction over the order m of the operator H. Clearly, our assertion is valid for operators of order 1. Later on, it suffices to prove the assertion for operators of order $m+1$ with the symbol of the form $H_1(r,p)\,p$ if that it is already proved for operators of order m. If

$$\hat{H}_1\left(\overset{2}{r}, \overline{r^{k+1}S'(r) + hr^{k+1}\frac{d}{dr}}^{\,1}\right) = \sum_{i=0}^{m} \hat{\mathcal{P}}_i^{(1)},$$

then, due to the inductive hypothesis, we have $\mathrm{ord}_p \hat{\mathcal{P}}_i^{(1)} = ik$. Then we have

$$\hat{H}_1\left(\overset{2}{r}, \overline{r^{k+1}S'(r) + hr^{k+1}\frac{d}{dr}}^{\,1}\right)\left(r^{k+1}S'(r) + hr^{k+1}\frac{d}{dr}\right) = \sum_{i=0}^{m+1} h^i \hat{\mathcal{P}}_i,$$

[8]More precisely, we shall use the real power order of the operator, that is, the maximal of all orders described above.

where

$$\hat{\mathcal{P}}_i = \hat{\mathcal{P}}_i^{(1)}\left(r^{k+1}S'(r)\right) + \hat{\mathcal{P}}_{i-1}^{(1)}\left(r^{k+1}\frac{d}{dr}\right). \tag{29}$$

Since

$$\operatorname{ord}_p r^{k+1}S'(r) = \operatorname{ord}_p p(r) = 0, \operatorname{ord}_p \left(r^{k+1}\frac{d}{dr}\right) = k,$$

then, clearly,

$$\operatorname{ord}_p \hat{\mathcal{P}}_i = \min\left\{\operatorname{ord}_p \hat{\mathcal{P}}_i^{(1)},\ \operatorname{ord}_p \hat{\mathcal{P}}_{i-1}^{(1)} + k\right\} = ik$$

as required. The last affirmation of the Lemma is a direct consequence of formula (29).

Let us consider now the second equation of system (23). Since under the above choice of the function $S(r)$ the operator

$$\hat{H}\left(r, r^{k+1}S'(r)\right) = \hat{H}\left(r, p(r)\right)$$

is not invertible, for this equation to be solvable its right-hand part must satisfy the following compatibility conditions:

$$\left\langle V(r), \left[\frac{\partial \hat{H}}{\partial p}(r, p(r))\left(r^{k+1}\frac{d}{dr}\right) + \frac{1}{2}r^{k+1}\frac{\partial^2 \hat{H}}{\partial p^2}(r, p(r))\frac{dp(r)}{dr}\right]\varphi_0(r)\right\rangle = 0.$$

Substituting expression (28) for the function $\varphi_0(r)$ into the last relation, we obtain the transport equation for the unknown $a_0(r)$:

$$\begin{aligned}&\left[\left\langle V(r), \frac{\partial \hat{H}}{\partial p}(r, p(r))\,U(r)\right\rangle \frac{d}{dr} + \left\langle V(r), \frac{\partial \hat{H}}{\partial p}(r, p(r))\frac{dU(r)}{dr}\right\rangle\right.\\ &\left. + \frac{1}{2}\left\langle V(r), \frac{\partial^2 \hat{H}}{\partial p^2}(r, p(r))\frac{dp(r)}{dr}U(r)\right\rangle\right] a_0(r) = 0,\end{aligned} \tag{30}$$

which is solvable in regular functions under the assumption that

$$\left\langle V(r), \frac{\partial \hat{H}}{\partial p}(r, p(r))\,U(r)\right\rangle \neq 0. \tag{31}$$

The value $a_0(0)$ remains undefined and determines an (arbitrary) multiplicative constant which is naturally involved into a solution to homogeneous equation (21). The power order of the function $a_0(r)$ is, clearly, equal to zero.

Let us proceed with the consideration of the rest transport equations. To be definite, we consider this process on the example of the third equation from system

(23) for $j = 2$. Clearly, if the compatibility condition (30) holds, the general solution to the second equation from (23) is

$$\varphi_1(r) = a_1(r)\,U(r) + \varphi_1^*(r)\,, \tag{32}$$

where $\varphi_1^*(r)$ is some particular solution to this equation (having, evidently, the power order $k+1$), and $a_1(r)$ is an arbitrary (up to the moment) function. Substituting expression (32) to the equation

$$\hat{H}(r, p(r))\,\varphi_2(r) = -\left[\hat{\mathcal{P}}_1\varphi_1(r) + \hat{\mathcal{P}}_2\varphi_0(r)\right],$$

we obtain an equation for the function $\varphi_2(r)$ in the form

$$\hat{H}(r, p(r))\,\varphi_2(r) = \hat{\mathcal{P}}_1\left[a_1(r)\,U(r)\right] + \hat{\mathcal{P}}_1\left[\varphi_1^*(r)\right] + \hat{\mathcal{P}}_2\left[\varphi_0(r)\right]. \tag{33}$$

The compatibility condition for the last equation reads

$$\left\langle V(r)\,, \hat{\mathcal{P}}_1\left[a_1(r)\,U(r)\right] + \hat{\mathcal{P}}_1\left[\varphi_1^*(r)\right] + \hat{\mathcal{P}}_2\left[\varphi_0(r)\right]\right\rangle = 0.$$

The latter equation can be rewritten in the form

$$\left[\left\langle V(r)\,, \frac{\partial \hat{H}}{\partial p}(r, p(r))\,U(r)\right\rangle \frac{d}{dr} + \left\langle V(r)\,, \frac{\partial \hat{H}}{\partial p}(r, p(r))\,\frac{dU(r)}{dr}\right\rangle \right.$$
$$\left. + \frac{1}{2}\left\langle V(r)\,, \frac{\partial^2 \hat{H}}{\partial p^2}(r, p(r))\,U(r)\right\rangle\right] a_1(r) = -\hat{P}_1\left[\varphi_1^*(r)\right] - \hat{P}_2\left[\varphi_0(r)\right], \tag{34}$$

where $\hat{P}_j$ are operators introduced in Lemma 1. Clearly, the last equation is solvable with respect to $a_1(r)$ in regular functions. Later on, since the power order of operators $\hat{P}_1$ and $\hat{P}_2$ equals -1 and $k-1$, respectively, the function on the right in the last equation has power order $k-1$. Hence, there exists a unique solution to (34) with power order k; this solution is determined by the Cauchy data $a_1(0) = 0$. Now the general solution to equation (33) is written down in the form

$$\varphi_2(r) = a_2(r)\,U(r) + \varphi_2^*(r)\,,$$

the function $\varphi_2^*(r)$ being of power order $2k$. We remark also that the power order of the function $\varphi_1(r)$ given by (32) equals k.

The continuation of the above described procedure leads us to the following affirmation:

Proposition 1 *There exists a formal solution $u(r)$ to equation (21) of the form (22) such that the functions $\varphi_j(r)$ have power order jk in the variable r. This solution is defined uniquely up to a multiplicative constant and is an entire function of the variable k.*

From this proposition it follows that expansion (22) (for $h = 1$) is an asymptotic expansion in r near $r = 0$ only for Re k > 0, since only in this case the terms of series (22) are graduated by descent as $r \to 0$. Moreover,

$$\varphi_j(r) = r^{jk}\psi_j(r)$$

with $\psi_j(r)$ possessing a standard Taylor expansion in powers of r.

3.2 The analysis of the asymptotic expansion

The aim of this subsection is to analyse the above asymptotic expansion[9]

$$u(r) = e^{S(r)}\sum_{j=0}^{\infty}\varphi_j(r) = e^{S(r)}\sum_{j=0}^{\infty} r^{jk}\psi_j(r) \tag{35}$$

for different values of k and to reduce this expansion to a more standard form. The matter is that the functions $S(r)$ and $\varphi_j(r)$ cannot be uniquely determined by expansion (35). Actually, if the function S can be represented in the form

$$S(r) = S_1(r) + S_2(r),$$

where S_2 is a regular function of the first power order near $r = 0$, then one can expand $\exp(S_2(r))$ in powers of r thus rewriting expansion (35) in the same form but with another action $S_1(r)$. To standartize an asymptotic expansion one should extract the singular part of the action $S(r)$ in some standard form and then consider asymptotic expansions with this standard form of the action.

To realize this program, we begin with the investigation of the function $S(r)$. In accordance to formula (27) above, we have

$$r^{k+1}S'(r) = p(r),$$

where the function $p(r)$ is regular at $r = 0$. The latter equation evidently has a unique solution up to an additive constant, and we fix this constant by putting

$$S(r) = S(r) = \int_{r_0}^{r} r^{-k-1}p(r)\,dr$$

[9]From now on we put $h = 1$.

with some r_0 from the interval $(0,1)$. One can see that $S(r)$ is evidently an entire function in k for any fixed value of r.

For more detailed investigation of the function $S(r)$ we expand the function $p(r)$ under the integral sign in the above expression for $S(r)$ into the Taylor series in r:

$$p(r) = \sum_{j=0}^{\infty} p_j r^j$$

and integrate the obtained integral term by term (this procedure is correct for sufficiently small values of r_0 and r, namely, both r_0 and r must belong to the domain of convergence of the above power series.) The result is

$$\begin{aligned} S(r) &= \int_{r_0}^{r} r^{-k-1} \sum_{j=0}^{\infty} p_j r^j \, dr \\ &= -\sum_{j=0}^{k-1} p_j \frac{r^{j-k}}{k-j} + p_k \ln r + A + S_1(r), \end{aligned} \tag{36}$$

where A is some constant and $S_1(r)$ is a regular function at $r=0$ of power order at least one.

Substituting the latter relation into expansion (35) and expanding $\exp[S_1(r)]$ and all amplitude functions in powers of r, we arrive at the expansion

$$u(r) = \exp\left\{ -\sum_{j=1}^{k} \frac{r^{-j}}{j} p_{j-k} \right\} r^{p_k} \sum_{l=0}^{\infty} r^l u_l. \tag{37}$$

This is exactly the formal asymptotic solution to equation (20).

3.3 Explicit computation of the coefficients

In this subsection, we present the method of computation of the coefficients of expansion (37) directly from equation (20). Besides, to list the effects which may occur during the consideration of asymptotic expansions in the case of an edge of the cusp type, we have included the corresponding computations into the second subsubsection of this subsection.

3.3.1 Isolated cusp point

1. Consider first the case of a simple cusp

$$\hat{H} = H\left(r, \omega, r^2 \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right) u = 0. \tag{38}$$

We suppose that the following condition is valid:
The operator (38), *included into the considered equation is an elliptic one.*
In particular, this yields that the family

$$H\left(0,\omega,\tau,\frac{\partial}{\partial\omega}\right), \ \tau\in\mathbf{C}$$

is meromorphically invertible.

In accordance to formula (37) above, we search for the solution to equation (38) in the form

$$u(r,\omega)=e^{-S/r}r^{\gamma}\sum_{j=0}^{\infty}r^{j}u_{j}(\omega)=e^{-S/r}\sum_{j=0}^{\infty}r^{j+\gamma}u_{j}(\omega). \tag{39}$$

Substituting the latter expression into (38), one obtains

$$\begin{aligned}H\left(r,\omega,r^{2}\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)\left[e^{-S/r}\sum_{j=0}^{\infty}r^{j+\gamma}u_{j}(\omega)\right] &= \\ e^{-S/r}H\left(r,\omega,S+r^{2}\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)\sum_{j=0}^{\infty}r^{j+\gamma}u_{j}(\omega) &= 0.\end{aligned}$$

Since both multiplication by r and the operator $r^{2}\partial/\partial r$ enlarge by 1 the power order of terms of the series $\sum_{j=0}^{\infty}r^{j+\gamma}u_{j}(\omega)$, it is natural to expand the operator $H\left(r,\omega,S+r^{2}\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)$ into the Taylor series in r and $r^{2}\partial/\partial r$:

$$H\left(r,\omega,S+r^{2}\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)=\sum_{i',i''=0}^{\infty}H_{i'i''}\left(\omega,S,\frac{\partial}{\partial\omega}\right)r^{i'}\left(r^{2}\frac{\partial}{\partial r}\right)^{i''},$$

where

$$H_{i'i''}\left(\omega,S,\frac{\partial}{\partial\omega}\right)=\frac{1}{i'!i''!}\frac{\partial^{i'+i''}H}{\partial r^{i'}\partial\tau^{i''}}\left(0,\omega,S,\frac{\partial}{\partial\omega}\right).$$

In particular,

$$H_{00}\left(\omega,S,\frac{\partial}{\partial\omega}\right)=H\left(0,\omega,S,\frac{\partial}{\partial\omega}\right). \tag{40}$$

As a result, equation (38) will be transformed to the form

$$\sum_{j,i',i''=0}^{\infty}\frac{\Gamma(j+\gamma+i'')}{\Gamma(j+\gamma)}r^{j+\gamma+i'+i''}H_{i'i''}\left(\omega,S,\frac{\partial}{\partial\omega}\right)u_{j}(\omega)=0.$$

Equating the coefficients of powers of r to zero, we arrive at the following recurrent system for computing the unknowns $u_j(\omega)$:

$$\sum_{j'+i'+i''=j}^{\infty} \frac{\Gamma(j'+\gamma+i'')}{\Gamma(j'+\gamma)} H_{i'i''}\left(\omega, S, \frac{\partial}{\partial\omega}\right) u_{j'}(\omega) = 0,\ j = 0, 1, \ldots, \tag{41}$$

where the sum is taken over nonnegative values of indices j', i', i''.

Let us write down several equations from (41) in the explicit form:

$$\begin{aligned} \hat{H}_{00} u_0(\omega) &= 0, && (42)\\ \hat{H}_{00} u_1(\omega) &= -\left[\hat{H}_{10} + \gamma \hat{H}_{01}\right] u_0(\omega), && (43)\\ \hat{H}_{00} u_2(\omega) &= -\left[\hat{H}_{10} + (\gamma+1)\hat{H}_{01}\right] u_1(\omega) \\ &\quad -\left[\hat{H}_{20} + \gamma\hat{H}_{11} + \frac{\gamma(\gamma+1)}{2}\hat{H}_{02}\right] u_2(\omega), && (44) \end{aligned}$$

where $\hat{H}_{ij} = H_{ij}(\omega, S, \partial/\partial\omega)$.

Equation (42) shows that to construct a nontrivial solution to equation (38) it is nesessary that:

- The number S is chosen in such a way that the operator (40) has an untrivial kernel (we suppose that S is chosen in this way in the sequel).
- The function $u_0(\omega)$ belongs to the kernel of the above mentioned operator.

Due to the condition posed in the beginning of this subsection, under the above conditions the operator (40) has finite-dimensional kernel and cokernel. Moreover, the dimensions of these two spaces coincide with each other. Denote by

$$\{U_1, \ldots, U_N\}$$

the base in Ker $\hat{H}_{00}$, and by

$$\{V_1, \ldots, V_N\} \tag{45}$$

the base in Coker $\hat{H}_{00}$. Then, due to relation (42), one evidently has

$$u_0(\omega) = \sum_{j=1}^{N} c_j^0 U_j(\omega). \tag{46}$$

Let us now turn our mind to the consideration of equation (43). For this equation to be solvable, its right-hand part

$$-\left[\hat{H}_{10}+\gamma\hat{H}_{01}\right]u_0(\omega)$$

is to be orthogonal to the functions (45), which form a base of the cokernel of the operator $\hat{H}_{00}$. Taking into account formula (46), we arrive at the following system of homogeneous equations for constants c_j^0 involved into the expansion (46):

$$\sum_{j=1}^{N}(A_{kj}+\gamma B_{kj})\,c_j^0=0,\ k=1,\dots N; \tag{47}$$

Here the matrices A_{kj} and B_{kj} are defined by relations

$$\begin{aligned} A_{kj} &= \left\langle V_k(\omega),\hat{H}_{10}\left(\omega,S,\frac{\partial}{\partial\omega}\right)U_j\right\rangle, \\ B_{kj} &= \left\langle V_k(\omega),\hat{H}_{01}\left(\omega,S,\frac{\partial}{\partial\omega}\right)U_j\right\rangle. \end{aligned}$$

So, for a nontrivial solution of the initial equation to exist, the number γ must be chosen from the relation

$$\det\|A_{kj}+\gamma B_{kj}\|_{k,j=1}^{N}=0, \tag{48}$$

and the vector $\vec{c^0}=(c_1^0,\dots,c_N^0)$ must satisfy (47). Then the general solution to equation (43) is given by

$$u_1(\omega)=u_1^*(\omega)+\sum_{j=1}^{N}c_j^1U_j(\omega), \tag{49}$$

where $u_1^*(\omega)$ is some particular solution of (43).

As above, arbitrary constants c_j^1 in the relation (49) must be chosen from the compatibility conditions of *the next* equation (44). It is not hard to show that this condition has the form

$$\sum_{j=1}^{N}(A_{kj}+(\gamma+1)\,B_{kj})\,c_j^1=b_k,\ k=1,\dots,N, \tag{50}$$

and that the numbers b_k are uniqely determined by the vector $\vec{c^0}$ and the function $u_1^*(\omega)$ computed on the earlier stage of the process. Under the assumption that the number $(\gamma+1)$ is not a root of equation (48), equation (50) is uniquely solvable with respect to the numbers c_j^1, $j=1,\dots N$.

The continuation of the described procedure leads us to the following result.

Theorem 1 *Let for each S from the spectrum of the operator $\hat{H}_{00}$, equation* (48) *has N different solutions $(\gamma_1, \dots, \gamma_N)$, and there exists no pair of these solutions which differ by an integer. Then each pair (S, γ_j) of the described type determines a nontrivial solution to equation* (38) *of the form* (39)[10]

2. Let us consider now a cusp of higher multiplicity. Since all effects can be seen on the case $k = 2$, we shall consider this case:

$$H\left(r, \omega, r^3\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right) u = 0. \tag{51}$$

As above, we shall search for asymptotic solutions to equation (51) in the form (which is exactly expansion (37) for $k = 2$)

$$u(r,\omega) = e^{-\frac{S_2}{2r^2}-\frac{S_1}{r}} r^{\gamma} \sum_{j=0}^{\infty} r^j u_j(\omega) = e^{-\frac{S_2}{2r^2}-\frac{S_1}{r}} \sum_{j=0}^{\infty} r^{j+\gamma} u_j(\omega).$$

The procedure similar to that described above, leads us to a recurrent system of equations; first four of these equations are

$$\begin{aligned}
\hat{H}_{00} u_0(\omega) &= 0, & (52)\\
\hat{H}_{00} u_1(\omega) &= -\left[\hat{H}_{10} + S_1 \hat{H}_{01}\right] u_0(\omega), & (53)\\
\hat{H}_{00} u_2(\omega) &= -\left[\hat{H}_{10} + S_1 \hat{H}_{01}\right] u_1(\omega) \\
&\quad + \left[\frac{1}{2}\hat{H}_{20} + S_1 \hat{H}_{11} + \frac{1}{2}\hat{H}_{02} + \gamma \hat{H}_{01}\right] u_0(\omega), & (54)\\
\hat{H}_{00} u_3(\omega) &= -\left[\hat{H}_{10} + S_1 \hat{H}_{01}\right] u_2(\omega) \\
&\quad + \left[\frac{1}{2}\hat{H}_{20} + S_1 \hat{H}_{11} + \frac{1}{2}S_1^2 \hat{H}_{02} + (\gamma+1) \hat{H}_{01}\right] u_1(\omega) \\
&\quad + \hat{A} u_0(\omega), & (55)
\end{aligned}$$

where the operators $\hat{H}_{ij} = H_{ij}(\omega, S_2, \partial/\partial\omega)$ are defined as above, and $\hat{A}$ is some operator (the exact form of this operator is not essential for what follows).

Let us begin our analysis with equation (52). As above, from this equation follows that the point S_2 is a spectral point of the operator family

$$\hat{H}_{00} = H_{00}(\omega, \tau, \partial/\partial\omega), \tag{56}$$

[10]The reader easily finds out that the assumptions of this theorem can be significantly weekend. For example, one can consider the case when the poles of $\hat{H}_{00}^{-1}$ are of more that first order.

and the function $u_0(\omega)$ is an element of the kernel of the corresponding operator of this family. Similar to the above described case, this family is meromorphically one invertible in τ. Denoting by

$$\{U_1, \ldots, U_N\}$$

the base in the kernel of operator (56) for $\tau = S_2$, and by

$$\{V_1, \ldots, V_N\} \tag{57}$$

the base in the cokernel of this operator, we obtain

$$u_0(\omega) = \sum_{i=0}^{N} c_j^0 U_j(\omega) \tag{58}$$

with some constants c_j^0, $j = 1, \ldots, N$.

Later on, the compatibility condition for equation (53), that is, the orthogonality of its right-hand part to the function system (57) has the form

$$\left\langle V_k, \left[\hat{H}_{10} + S_1 \hat{H}_{01}\right] \sum_{i=0}^{N} c_j^0 U_j(\omega) \right\rangle = 0, \; k = 1, \ldots, N,$$

or

$$\sum_{i=0}^{N} (A_{kj} + S_1 B_{kj}) c_j^0 = 0, \; k = 1, \ldots N. \tag{59}$$

The latter relation shows that the number S_1 have to be a root of the equation

$$\det \|A_{kj} + S_1 B_{kj}\|_{j,k=1}^{N} = 0, \tag{60}$$

and the vector $\vec{c^0} = (c_1^0, \ldots, c_N^0)$ have to be a solution to the corresponding equation. Denoting by $\left(S_1^{(1)}, \ldots, S_1^{(N)}\right)$ the roots of (60) and by $\vec{c^{0i}} = (c_1^{0i}, \ldots, c_N^{0i})$, $i = 1, \ldots, N$ the corresponding solutions[11] to equation (59) (for simplicity we suppose that the roots of (60) are simple) we arrive at the set of N solutions to equation (53)

$$u_1^k(\omega) = u_1^{k*}(\omega) + \sum_{i=0}^{N} c_j^{1k} U_j(\omega)$$

[11] Clearly, these solutions are determined up to a multiplicative constant.

Each of these solutions correspond to its own value of $S_1^{(k)}$. We remark that the particular solution $u_1^{k*}(\omega)$ to equation (53) with the right-hand part

$$-\sum_{i=0}^{N}\left[\hat{H}_{10}+S_1^{(k)}\hat{H}_{01}\right]c_j^{0k}U_j(\omega)$$

can be expressed as

$$u_1^{k*}(\omega)=\sum_{i=0}^{N}c_j^{0k}W_j^k(\omega),$$

where $W_j^k(\omega)$ are some fixed solutions to the equation

$$\hat{H}_{00}W_j^k(\omega)=-\left[\hat{H}_{10}+S_1^{(k)}\hat{H}_{01}\right]U_j(\omega).$$

So, we have

$$u_1^k(\omega)=\sum_{i=0}^{N}c_j^{0k}W_j^k(\omega)+\sum_{i=0}^{N}c_j^{1k}U_j(\omega). \tag{61}$$

Let us consider now equation (54). The compatibility condition for this equation is

$$\begin{aligned}&\Big\langle V_l(\omega),-\left[\hat{H}_{10}+S_1^{(k)}\hat{H}_{01}\right]u_1(\omega)\\&+\left[\frac{1}{2}\hat{H}_{20}+S_1^{(k)}\hat{H}_{11}+\frac{1}{2}\left(S_1^{(k)}\right)^2\hat{H}_{02}+\gamma\hat{H}_{01}\right]u_0(\omega)\Big\rangle=0.\end{aligned}$$

Substituting expressions (58) for $u_0(\omega)$ (here $\vec{c^0}=\vec{c^{0k}}$) and (61) for $u_1(\omega)$ in the latter equation, we rewrite the compatibility conditions in the form

$$\sum_{i=0}^{N}\left(A_{lj}+S_1^{(k)}B_{lj}\right)c_j^{1k}=-\sum_{i=0}^{N}\left(C_{lj}+\gamma D_{lj}\right)c_j^{0k},\ j=1,\dots,N, \tag{62}$$

where the matrices A_{lj} and B_{lj} are defined as above, and C_{lj} and D_{lj} are given by the formulas

$$\begin{aligned}C_{lj}&=\Big\langle V_l(\omega),\left[\hat{H}_{10}+S_1^{(k)}\hat{H}_{01}\right]W_j^k(\omega)\\&\quad+\left[\frac{1}{2}\hat{H}_{20}+S_1^{(k)}\hat{H}_{11}+\frac{1}{2}\left(S_1^{(k)}\right)^2\hat{H}_{02}\right]U_j(\omega)\Big\rangle,\\D_{lj}&=\Big\langle V_l(\omega),\hat{H}_{01}U_j\Big\rangle.\end{aligned}$$

We remark that the matrix on the left in (62) is degenerated. Its (one-dimensional) kernel is determined by the vector $\vec{c^{0k}}$. Clearly, this matrix has also a one-dimensional cokernel (the fact that both kernel and cokernel of $A_{lj} + S_1^{(k)} B_{lj}$ have dimension one follows from the above assumption that the roots of equation (60) are simple). Denote by $\vec{d^{0k}}$ the row-vector determining the cokernel of the latter matrix. Now the compatibility condition for equation (62) with respect to c_j^{1k} can be rewritten in the form

$$\sum_{j,l=0}^{N} d_l^{0k} \left(C_{lj} + \gamma D_{lj} \right) c_j^{0k} = 0. \tag{63}$$

Besides, the general solution to this equation is

$$\vec{c^{1k}} = \vec{c^{1k*}} + C\vec{c^{0k}} \tag{64}$$

with some (up to the moment undefined) constant C. Equation (63) allows one to determine the constant γ under the assumption that

$$\sum_{i,l=0}^{N} d_l^{0k} D_{lj} c_j^{0k} \neq 0. \tag{65}$$

It is not hard to show that the constant C involved into expression (64) for $\vec{c^{1k}}$ is determined from the compatibility condition of (55) *in the unique way.* Moreover, all consequent equations of the recursive scheme for the functions $u_j(\omega)$ are also uniquely solvable. The verification of this fact is left to the reader.

Therefore, the following statement is valid.

Theorem 2 *Let equation* (60) *has* N *different solutions* $\left(S_1^{(1)}, \ldots, S_1^{(N)}\right)$ *for any* S_2 *from the spectrum of this family. Later on, suppose that inequality* (65) *holds for each* k. *Then for each pair* $\left(S_2, S_1^k\right)$ *of the above described type there exists a nontrivial solution to equation* (51) *of the form*

$$u_k(r,\omega) = e^{-\frac{S_2}{2r^2} - \frac{S_1^k}{r}} r^{\gamma_k} \sum_{j=0}^{\infty} r^j u_j(\omega),$$

where γ_k *is a solution to equation* (63).

The contents of this section shows the *significant difference* between the case of a simple cusp and the case of a cusp of higher multiplicity. Namely, *in the case of higher multiplicity the phase* $-\frac{S_2}{2r^2} - \frac{S_1^k}{r}$ *of asymptotic expansion is not a homogeneous function of* r. As we shall see below, this difference essentially affects the construction of the real asymptotic expansions.

3.3.2 Edge of the cusp type

In this subsubsection, we shall make some remarks on the investigation of partial differential equations on manifolds with singularities of the cusp edge type. As we have already told in Section 2, the topology structure of such manifolds in a neighborhood of the edge is

$$M = K_\Omega \times X = (\Omega \times [0,1]) / (\Omega \times \{0\}) \times X,$$

where X and Ω are smooth manifolds and differential operators have the form

$$\hat{H} = r^{-(k+1)m} H \left(r, \omega, x, r^{k+1} \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}, r^{k+1} \frac{\partial}{\partial x} \right).$$

Considerations similar to that of the previous subsubsection using the asymptotic expansion of the form

$$u(r,\omega,x) = e^{-S/r} r^\gamma \sum_{j=0}^{\infty} r^j u_j(\omega,x) \tag{66}$$

(in the case of a simple cusp), or

$$u(r,\omega,x) = e^{-\frac{S}{2r^2} - \frac{S_1}{r}} r^\gamma \sum_{j=0}^{\infty} r^j u_j(\omega,x) \tag{67}$$

(for a cusp of the multiplicity 2) leads us to the following equation for the main coefficient of the asymptotics $u_0(\omega,x)$:

$$H_{00}(\omega,x,S,\partial/\partial\omega)\, u_0(\omega,x) = 0.$$

It is not hard to notice that in this case the analytic family

$$H_{00}(\omega,x,\tau,\partial/\partial\omega) = H\left(0,\omega,x,\tau,\frac{\partial}{\partial\omega},0\right) \tag{68}$$

of elliptic operators depends in addition on the coordinates x along the edge. As a consequence, the spectrum of the operator family in question depends also on these coordinates:

$$S = S(x),$$

the function $S(x)$ being, in general, multivalued. The ramification points (in x) of the spectrum, called *focal points of the family*, supply us with additional difficulties in the process of consideration of formal asymptotics. Actually, in a neighborhood of a focal point the asymptotic expansion of solutions to the equation

$$H\left(r,\omega,x,r^{l}\frac{\partial}{\partial r},\frac{\partial}{\partial \omega},r^{l}\frac{\partial}{\partial x}\right)u\left(r,\omega,x\right)=0$$

does not have the form (66), (67). However, in the case when there is no focal points in the edge X, the above presented theory can be worked out with evident modifications.

On the opposite, if focal points of the family (68) do exist on the edge X, then to construct the asymptotics one should use asymptotic expansions of the type of analytic functionals or of the type of Maslov's canonical operator (see [22], [23]). We shall not consider these questions in the present paper.

4 Construction of the resurgent solutions

In the previous sections we have constructed formal asymptotic expansions for solutions to homogeneous differential equations on manifolds having cusp-type singularities. However, as it is well-known, the series involved into such expansions are as a rule divergent. Therefore, to construct asymptotic solutions to corresponding problems, it is nesessary to resummate these series. In this section we shall prove that formal series of the form (37) obtained above, are summable with the appropriate choice of the resummation method. This means that these functions occurs to be *resurgent* ones (see, e. g. [11]).

We consider here the resummation procedure in the two cases[12] $k=1$ and $k=2$. It occurs that in the case $k=1$ the standard resummation procedure based on the Borel-Laplace transform supplies us with the representation of solutions which gives the full picture of asymptotic behavior of this solution near the considered cusp point. At the same time, for cusps of higher order (in fact, even for $k=2$) the application of the Borel-Laplace transform occurs to be inappropriate. The matter is that under the action, say, 2-Borel transform the corresponding function in the dual space is not more a function with simple singularities, and the information about the asymptotic behavior of solutions occurs to be "hidden" in such a representation. In this case one should use the *resurgent representation* introduced by the authors in [11].

[12]We consider here only the case of an isolated point of the cusp type.

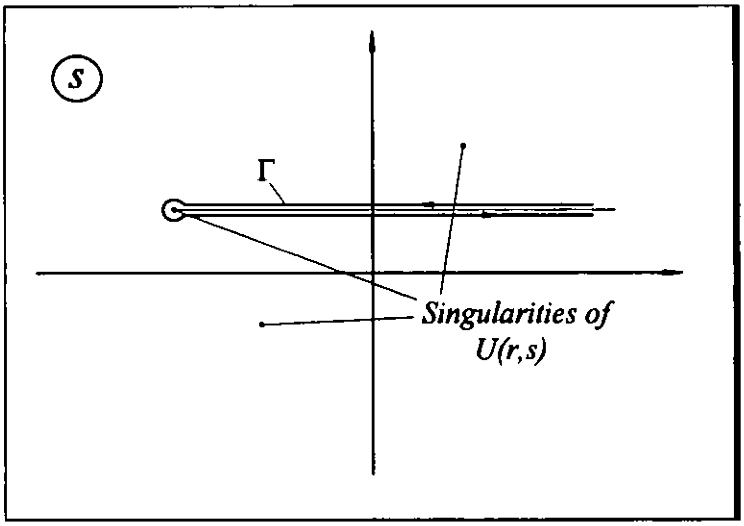

Figure 2. The standard integration contour.

Remark 2 We recall that the mentioned resurgent representation has the form

$$u(r) = \int_{\Gamma} e^{-s} U(r,s)\, ds,$$

where $U(r,s)$ is an endlessly-continuable function (that is, having the discrete set of singularities on its Riemannian surface), and Γ is a contour encircling some point of singularity of $U(r,s)$ and coming to infinity along the direction of the positive real axis (see Figure 2). We say that the function $U(r,s)$ has *simple singularities* near its singular point $s = S(r)$ if it can be represented in the form

$$U(r,s) = \frac{a_0(r)}{s - S(r)} + \ln(s - S) \sum_{j=0}^{\infty} \frac{(s - S(r))^j}{j!} a_{j+1}(r)$$

near this point. In this case, the asymptotic expansion of the function $u(r)$ is given by

$$u(r) \simeq e^{S(r)} \sum_{j=0}^{\infty} a_j(r),$$

so that the behavior of the function $U(r,s)$ near a point of singularity determines the asymptotic expansion of the function $u(r)$ as $r \to 0$. The details about resurgent representation and the functions with simple singularities the reader can find in the above cited book.

4.1 Case of a simple cusp

Consider the formal Borel transform $U(p,\omega)$ of the formal series (39):

$$U(p,\omega) = B\left[e^{-S/r}\sum_{j=0}^{\infty} r^{j+\gamma}u_j(\omega)\right].$$

The latter function satisfies the equation

$$\left[\hat{H}\left(0,\omega,p,\frac{\partial}{\partial\omega}\right) + \left(\frac{\partial}{\partial p}\right)^{-1}\frac{\partial\hat{H}}{\partial r}\left(0,\omega,p,\frac{\partial}{\partial\omega}\right)\right.$$
$$\left. + \left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_1\left(\left(\frac{\partial}{\partial p}\right)^{-1},\omega,p,\frac{\partial}{\partial\omega}\right)\right]U(p,\omega) = 0 \qquad (69)$$

which can be obtained from equation (38)

$$\hat{H}\left(r,\omega,r^2\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)u = 0 \qquad (70)$$

by application of the Borel transform (we have used the expansion of the latter equation into the Taylor series in r up to the second order:

$$\left[\hat{H}\left(0,\omega,r^2\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right) + r\frac{\partial\hat{H}}{\partial r}\left(0,\omega,r^2\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right) + r^2\hat{H}_1\left(r,\omega,r^2\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)\right]u = 0.$$

It is convenient to rewrite equation (69) in the operator form[13]

$$\left[\hat{H}(0,p) + \left(\frac{\partial}{\partial p}\right)^{-1}\frac{\partial\hat{H}}{\partial r}(0,p) + \left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_1\left(\left(\frac{\partial}{\partial p}\right)^{-1},p\right)\right]U(p) = 0, \qquad (71)$$

where all the operators considered are acting on functions determined on the manifold Ω. To investigate the asymptotic behavior of solutions to equation (70) one should:

[13]More precisely, we consider the operator

$$\hat{H}\left(r,\omega,r^2\frac{\partial}{\partial r},\frac{\partial}{\partial\omega}\right)$$

as the operator $\hat{H}\left(r,r^2\frac{d}{dr}\right)$, acting on functions with values in the Sobolev space scale on the manifold Ω. The same concernes all other operators involved into the considered equation.

- Prove the existence of an endlessly-continuable solution to equation (71) and investigate the position of singularities of this solution.

- To investigate the asymptotics in smoothness of the constructed solution at any point of its singularity.

For simplicity, we suppose that

The inverse $\hat{R}(z)$ of the operator $\hat{H}(0,z)$ has a simple poles only and that the kernel (and, hence, the cokernel) of the operator $\hat{H}(0,0)$ is one-dimensional.

Later on, we consider the case when the point $z = 0$ is a spectral point of the family $\hat{H}(0,z)$. The latter condition does not anyway lead to the loss of generality since in the opposite case it is sufficient to multiply the solution to the initial equation by the function $\exp(-S/r)$ with appropriate value of S.

The following affirmation takes place.

Theorem 3 *Under the above formulated conditions there exist endlessly-continuable solutions to equation* (71)*, with the singularity of the form*

$$U(p) = p^{\gamma} \sum_{j=0}^{\infty} c_j p^j$$

at the origin. The number γ in the latter relation is determined from

$$\left\langle V, \frac{\partial \hat{H}}{\partial r}(0,0)\, U \right\rangle + \gamma \left\langle V, \frac{\partial \hat{H}}{\partial z}(0,0)\, U \right\rangle = 0,$$

where U and V are generators of the kernel and cokernel of the operator $\hat{H}(0,0)$, respectively.

Now the resurgent analysis method shows that the above constructed formal asymptotic expansions are summable.

The rest part of this subsection is aimed at the proof of Theorem 3.

Proof. It is convenient to write down the equation for the Borel transform of the function $v(r,\omega)$ which is connected with the solution $u(r,\omega)$ of the initial problem by the relation

$$u(r,\omega) = r^{\gamma} v(r,\omega).$$

(Below we shall show that the value of γ must be determined from the last relation of the theorem, but temporarily we rest this value undefined.) Similar to the previous section, we obtain the following equation

$$\left\{\hat{H}(0,p) + \left(\frac{\partial}{\partial p}\right)^{-1}\left[\frac{\partial \hat{H}}{\partial r}(0,p) + \gamma\frac{\partial \hat{H}}{\partial p}(0,p)\right] + \left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_2\left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right)\right\} V(p) = 0.$$

for the function $V(p,\omega) = \mathcal{B}[v(r,\omega)]$. Let us perform one more modification of the latter equation putting

$$v(r,\omega) = \left(1 + r\hat{B}\right) u(r,\omega),$$

or

$$V(p) = \left(1 + \left(\frac{\partial}{\partial p}\right)^{-1}\hat{B}\right) W(p),$$

where the operator $\hat{B}$ will be determined later. The resulting equation for W is

$$\left\{\hat{H}(0,p) + \left(\frac{\partial}{\partial p}\right)^{-1}\left[\frac{\partial \hat{H}}{\partial r}(0,p) + \gamma\frac{\partial \hat{H}}{\partial p}(0,p) + \hat{H}(0,p)\hat{B}\right] + \left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_3\left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right)\right\} W(p) = 0. \tag{72}$$

We notice that due to the above assumptions the following relation takes place:

$$\left[\hat{H}(0,p)\right]^{-1} = \frac{R_0}{p} + R_1(p), \tag{73}$$

where R_0 is a finite-dimensional operator, and $R_1(p)$ is regular at $p = 0$. Let us search for the solution of equation (72) in the form

$$W(p) = \left[\hat{H}(0,p)\right]^{-1}\tilde{W}(p) + \left[\hat{H}(0,p)\right]^{-1} 0.$$

(We remark that the last summand does not vanish since equation (72) is considered as an equation in hyperfunctions. This summand equals $p^{-1}U_0(p) + U_1(p)$, where $U_1(p)$ is a regular function, and $U_0(p)$ is an element of the kernel of the operator $\hat{H}(0,p)$). We arrive at the following equation for $\tilde{W}(p)$:

$$\left\{1 + \left(\frac{\partial}{\partial p}\right)^{-1}\left[\frac{\partial \hat{H}}{\partial r}(0,p) + \gamma\frac{\partial \hat{H}}{\partial p}(0,p) + \hat{H}(0,p)\hat{B}\right]\left[\hat{H}(0,p)\right]^{-1} + \left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_3\left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right)\left[\hat{H}(0,p)\right]^{-1}\right\}\tilde{W}(p) = F(p), \tag{74}$$

where the function $F(p)$ has simple singularities at $p=0$:

$$F(p) = \ln p \sum_{j=0}^{\infty} F_j p^j .$$

To investigate the form of singularities of the function $U(p)$, it is sufficient to notice that the operator

$$\left(\frac{\partial}{\partial p}\right)^{-2} \hat{H}_3 \left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right)$$

involved into equation (72) has the form

$$\left(\frac{\partial}{\partial p}\right)^{-2} \hat{H}_3 \left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right) U(p) = \left(\frac{\partial}{\partial p}\right)^{-1} \hat{H}_4(p,p') * U(p)\Big|_{p=p'},$$

where the convolution is taken over the variable p, and the function $\hat{H}_4(p,p')$ equals

$$\hat{H}_4(p,p') = \sum_{i=0}^{\infty} \frac{p^j}{j!} \hat{h}_j(p'), \tag{75}$$

if the expansion of the function $\hat{H}_3(r,p)$ in powers of r is

$$\hat{H}_3(r,p') = \sum_{i=0}^{\infty} r^j \hat{h}_j(p').$$

We note that function (75) is an entire function in p for any fixed p.

Let us try now to define the number γ and the operator $\hat{B}$ in such a way that the operator

$$\left[\frac{\partial \hat{H}}{\partial r}(0,p) + \gamma \frac{\partial \hat{H}}{\partial p}(0,p) + \hat{H}(0,p)\hat{B}\right]\left[\hat{H}(0,p)\right]^{-1}, \tag{76}$$

involved in the left-hand part of (74), is regular at $p=0$. Due to (73), the singular part of this operator is

$$p^{-1}\left[\frac{\partial \hat{H}}{\partial r}(0,p) + \gamma \frac{\partial \hat{H}}{\partial p}(0,p) + \hat{H}(0,p)B\right] R_0.$$

Let P be a projector to the image of the operator $\hat{H}(0,0)$. The operator

$$\frac{\partial \hat{H}}{\partial r}(0,p)+\gamma\frac{\partial \hat{H}}{\partial p}(0,p)$$

can be rewritten in the following form:

$$\begin{aligned}\frac{\partial \hat{H}}{\partial r}(0,p)+\gamma\frac{\partial \hat{H}}{\partial p}(0,p) &= P\left[\frac{\partial \hat{H}}{\partial r}(0,p)+\gamma\frac{\partial \hat{H}}{\partial p}(0,p)\right] \\ &+(1-P)\left[\frac{\partial \hat{H}}{\partial r}(0,p)+\gamma\frac{\partial \hat{H}}{\partial p}(0,p)\right].\end{aligned}$$

Choose the number γ from the condition

$$(1-P)\left[\frac{\partial \hat{H}}{\partial r}(0,p)+\gamma\frac{\partial \hat{H}}{\partial p}(0,p)\right]R_0=0.$$

For this choice to be possible, it is nesessary that the inequality

$$\left\langle V,\frac{\partial \hat{H}}{\partial p}(0,0)\,U\right\rangle=0,$$

holds (see the previous section).Here, as above, U and V are generators of the kernel and cokernel of the operator $\hat{H}(0,0)$, consequently. Now we put[14]

$$\hat{B}=\left[\hat{H}(0,0)\right]^{-1}P\left[\frac{\partial \hat{H}}{\partial r}(0,0)+\gamma\frac{\partial \hat{H}}{\partial p}(0,0)\right].$$

The reader will easily verify that with such choice of γ and B operator (76) is regular.

So, the equation (74) for $\tilde{W}$ takes the form of a Volterra equation

$$\left[1+\left(\frac{\partial}{\partial p}\right)^{-1}\hat{C}_1(p)+\left(\frac{\partial}{\partial p}\right)^{-2}\hat{C}_2(p)\right]\tilde{W}(p)=F(p),$$

where the operator $\hat{C}_1(p)$ is regular for $p=0$, and the operator $\hat{C}_2(p)$ has a simple polar singularity at this point. The end of the proof of the theorem can be carried out with the help of a standard method of successive approximations.

[14]The operator $\hat{H}(0,0)$ is understood here as an operator from the subspace complementary to the image R_0 to the image of $\hat{H}(0,0)$. Then, the composition $\left[\hat{H}(0,0)\right]^{-1}P$ (and, hence, the operator $\hat{B}$) is well-defined.

4.2 Case of a cusp of higher multiplicity

As we have already mentioned, the case $k = 2$ is, in essence, a general one, and we restrict ourselves by the consideration of this case.

We recall (see formula (37) above) that formal solutions to equation (20) for $k = 2$ have the form

$$u(r) = \exp\left(-\frac{p_0}{2r^2} - \frac{p_1}{r}\right) r^{p_2} \sum_{l=0}^{\infty} r^l u_l. \tag{77}$$

The main difference between this case and the case considered in the previous subsection is that if we represent the solution $u(r)$ in the form of the 2-Borel transform

$$u(r) = \int_{\Gamma} \exp\left\{-\frac{p}{2r^2}\right\} U(p)\, dp$$

where the contour Γ is chosen similar to the contour for resurgent representation (see Figure 2 above), then the function $U(p)$ fails to have simple singularities. The reason for this phenomenon is that, as we have mentioned above, that the phase function (action) of asymptotic expansion (77) *is not a homogeneous function in* r. So, we use the change of the unknown which reduces our function to the case of homogeneous action, then investigate this function with the help of the 2-Laplace transform[15], and, finally, show that the initial function can be written down as a resurgent representation (see Remark 2 above) with the function $U(r, s)$ having simple singularities.

So, to have the possibility to use 2-Laplace transform for the construction of a function with simple singularities, one should perform the change of the unknown

$$u(r) = \exp\left(-\frac{p_1}{r}\right) r^{p_2} v(r). \tag{78}$$

Similar to the formal theory, we shall investigate a solution corresponding to some fixed value of p_0, which is chosen as

$$p = p(0),$$

where $p = p(r)$ is some branch of the spectrum of family (26).

Let us represent equation (20) in the form

$$\left[\hat{H}_0\left(r^3\frac{d}{dr}\right) + r\hat{H}_1\left(r^3\frac{d}{dr}\right) + r^2\hat{H}_2\left(r^3\frac{d}{dr}\right) + r^3\hat{H}_3\left(r, r^3\frac{d}{dr}\right)\right] u(r) = 0$$

[15] On this stage, a lot of computations are similar to that of Subsection 4.1, and we shall omit the corresponding details.

and substitute the function $u(r)$ in the form (78) in it. We have

$$\left(r^3\frac{d}{dr}\right)u(r)=e^{-\frac{p_1}{r}}r^{p_2}\left\{rp_1+r^2p_2+\left(r^3\frac{d}{dr}\right)\right\}v(r).$$

Therefore, after the substitution the equation is reduced to the form

$$\begin{aligned}\hat{H}_0\left(rp_1+r^2p_2+r^3\frac{d}{dr}\right)+r\hat{H}_1\left(rp_1+r^2p_2+r^3\frac{d}{dr}\right)\\ +r^2\hat{H}_2\left(rp_1+r^2p_2+r^3\frac{d}{dr}\right)+r^3\hat{H}_3\left(r,rp_1+r^2p_2+r^3\frac{d}{dr}\right)v(r) &= 0.\end{aligned}$$

Expanding operators

$$\hat{H}_j\left(rp_1+r^2p_2+r^3\frac{d}{dr}\right),\quad j=0,1,2$$

into the Taylor series in r up to the third power, we arrive at the equation

$$\begin{aligned}&\left\{\hat{H}_0\left(r^3\frac{d}{dr}\right)+r\left[\hat{H}_1\left(r^3\frac{d}{dr}\right)+p_1\frac{\partial\hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right)\right]\right.\\ &+r^2\left[\hat{H}_2'\left(p_1,r^3\frac{d}{dr}\right)+p_2\frac{\partial\hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right)\right]\\ &\left.+r^3\hat{H}_3'\left(r,r^3\frac{d}{dr}\right)\right\}v(r)=0,\end{aligned}\tag{79}$$

where

$$\hat{H}_2'\left(p_1,r^3\frac{d}{dr}\right)=\hat{H}_2\left(r^3\frac{d}{dr}\right)+p_1\frac{\partial\hat{H}_1}{\partial p}\left(r^3\frac{d}{dr}\right)+\frac{1}{2}\frac{\partial^2\hat{H}_0}{\partial p^2}\left(r^3\frac{d}{dr}\right).$$

We choose the numbers p_1 and p_2 in such a way that the operators

$$\left[\hat{H}_1(p)+p_1\frac{\partial\hat{H}_0}{\partial p}(p)\right]\hat{H}_0^{-1}(p)$$

and

$$\left[\hat{H}_2'(p_1,p)+p_2\frac{\partial\hat{H}_0}{\partial p}(p)\right]\hat{H}_0^{-1}(p)$$

have no singularity at the point considered. As we have already seen below, to do this one have to use one more change of the unknown. For $k = 2$ such a change has the form

$$v(r) = \left(1 + r\hat{B}_1 + r^2\hat{B}_2\right) w(r), \tag{80}$$

where $\hat{B}_1$ and $\hat{B}_2$ are some (unknown, up to the moment) operators in the functional space E. Substituting (80) into (79), we derive the equation for $w(r)$ in the form

$$\left\{\hat{H}_0\left(r^3\frac{d}{dr}\right) + r\left[\hat{H}_1\left(r^3\frac{d}{dr}\right) + p_0\frac{\partial \hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right) + \hat{H}_0\left(r^3\frac{d}{dr}\right)\hat{B}_1\right]\right.$$
$$+r^2\left[\hat{H}_2''\left(p_1, B_1, r^3\frac{d}{dr}\right) + p_2\frac{\partial \hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right) + \hat{H}_0\left(r^3\frac{d}{dr}\right)\hat{B}_2\right]$$
$$\left.+r^3\hat{H}_3''\left(r, r^3\frac{d}{dr}\right)\right\} w(r) = 0,$$

where

$$\hat{H}_2''\left(p_1, \hat{B}_1, r^3\frac{d}{dr}\right) = \hat{H}_2'\left(p_1, r^3\frac{d}{dr}\right) + \left[\hat{H}_1\left(r^3\frac{d}{dr}\right) + p_1\frac{\partial \hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right)\right]\hat{B}_1.$$

The last equation is alredy prepared for the application of the 2-Borel transform. Applying this transform, we arrive at the following equation for the Borel image $W(p)$ of the function $w(r)$:

$$\left\{\hat{H}_0(p) + \left(\frac{\partial}{\partial p}\right)^{-1/2}\left[\hat{H}_1(p) + p_1\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_1\right]\right.$$
$$+\left(\frac{\partial}{\partial p}\right)^{-1}\left[\hat{H}_2''\left(p_1, \hat{B}_1, p\right) + p_2\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_2\right]$$
$$\left.+\left(\frac{\partial}{\partial p}\right)^{-3/2}\hat{H}_3''\left(\left(\frac{\partial}{\partial p}\right)^{-1/2}, p\right)\right\} W(p) = 0. \tag{81}$$

We reduce the latter equation to an equation of the Volterra type by putting

$$W(p) = \hat{H}_0^{-1}(p)\, W_1(p) + \hat{H}_0^{-1}(p)\, 0$$

(the last summand on the right does not vanish since the equation is considered in hyperfunctions). For $W_1(p)$ we obtain

$$\left\{1 + \left(\frac{\partial}{\partial p}\right)^{-1/2}\left[\hat{H}_1(p) + p_1\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_1\right]\hat{H}_0^{-1}(p)\right.$$

$$+\left(\frac{\partial}{\partial p}\right)^{-1}\left[\hat{H}_2''(p_1,B_1,p)+p_2\frac{\partial\hat{H}_0}{\partial p}(p)+\hat{H}_0(p)\hat{B}_2\right]\hat{H}_0^{-1}(p)$$

$$+\left(\frac{\partial}{\partial p}\right)^{-3/2}\hat{H}_3''\left(\left(\frac{\partial}{\partial p}\right)^{-1/2},p\right)\hat{H}_0^{-1}(p)\Bigg\}W_1(p)=F(p) \qquad (82)$$

with some right-hand part $F(p)$ having a polar singularity of the first order at $p = p_0$. Equation (82) is an equation of the Volterra type. The coefficients of this equation have polar singularities of the first order at $p = p_0$. As above, for the solution to this equation obtained by the successive approximation method to have simple singularities, it is nesessary to require that the operator coefficients of $(\partial/\partial p)^{-1/2}$ and $(\partial/\partial p)^{-1}$ are regular at $p = p_0$. Since, under the above assumptions,

$$\hat{H}_0^{-1}(p)=\frac{R_0}{p-p_0}+R_1(p)$$

with $R_1(p)$ regular at $p = p_0$, the coefficient of $(\partial/\partial p)^{-1/2}$ can be written down in the form

$$\frac{1}{p-p_0}\left[\hat{H}_1(p_0)+p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)+\hat{H}_0(p_0)\hat{B}_1\right]R_0+\{\text{regular operator}\}.$$

We have to choose the number p_1 and the operator $\hat{B}_1$ in such a way that the operator in the square brackets on the left in the latter relation vanishes.

Since the image of the operator R_0 coincides with the kernel of the operator $\hat{H}_0(p)$, this image is a one-dimensional subspace generated by the vector $U = U(0)$. Hence, if we choose p_1 from the relation

$$\left\langle V,\left[\hat{H}_1(p_0)+p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)\right]U\right\rangle=0$$

(which is possible since $\left\langle V,\frac{\partial\hat{H}_0}{\partial p}(p_0)U\right\rangle\neq 0$, see relation (31) for $r = 0$), then the image of the operator

$$\left[\hat{H}_1(p_0)+p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)\right]R_0$$

will be contained in the image of the operator $\hat{H}_0(p_0)$ and, hence, we can determine the operator $\hat{B}_1$ by the relation

$$\hat{B}_1=-\hat{H}_0^{-1}(p_0)\left[\hat{H}_1(p_0)+p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)\right]R_0.$$

It is clear that for such choice of the operator $\hat{B}_1$ the coefficient of $(\partial/\partial p)^{-1/2}$ in equation (82) is regular at $p = p_0$.

Let us consider now the coefficient of the operator $(\partial/\partial p)^{-1}$ in equation (82). It equals

$$\left[\hat{H}_2''\left(p_1, \hat{B}_1, p\right) + p_2 \frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_2\right] \hat{H}_0^{-1}(p), \tag{83}$$

where p_1 and $\hat{B}_1$ are already fixed. The procedure of the choice of the number p_2 and the operator $\hat{B}_2$ from the condition of regularity of operator (83) is literally the same as the procedure of the choicze of p_1 and $\hat{B}_1$ above. We leave the description of this procedure to the reader.

Now equation (82) can be solved with the help of the successive approximation method. As a result, we obtain the following affirmation:

Proposition 2 *There exist a unique solution to equation* (82) *which is analytic everywhere except for poles of the operator family* $\hat{H}_0(p)$ *and which has simple singularities at the point* p_0.

We emphasize that the latter proposition *does not imply* that the corresponding solution $u(r)$ to equation (20) occurs to be a resurgent function with simple singularities in the sense of 2-Laplace transform. From the other hand, the following statement takes place:

Theorem 4 *For each pole* p_0 *of the operator family* $\hat{H}_0(p)$ *there exists a solution* $u(r)$ *to equation* (20) *of the form* (77), *which is a resurgent function with simple singularities in the sense of the resurgent representation*[16]

$$u(r) = \int_\Gamma e^{-s} V_1(s, r)\, ds. \tag{84}$$

Proof. As it follows from Proposition 2, the function $w(r)$, which is 2-Laplace transform of the solution $W(p)$ to equation (81) is a resurgent function with simple singularities in the sense of 2-Laplace transform. Clearly, the same is true for the function $v(r)$ defined via $w(r)$ by relation (80). This means that the function $v(r)$ admits a representation of the form

$$v(r) = \int_\Gamma e^{-\frac{p}{r^2}} V(p)\, dp, \tag{85}$$

[16]See Remark 2. The definition of the resurgent representation and its main properties the reader can find in the book [11].

where the function $V(p)$ is a function with simple singularities. Representation (85) can be easily rewritten in the form of resurgent representation (84). To complete the proof of the theorem, it remains to note that the operators of multiplication by $\exp(-p_1/r)$ and r^{p_2} involved into representation (78) of the function $u(r)$ preserve the class of resurgent functions with simple singularities (the first of these operators realises the shift in the s-plane, and the second just changes the powers of r involved into the considered expansion).

To conclude this section, we remark that the constructed resurgent solutions clearly coincide with results of resummation of formal silutions obtained in subsection 3.3. This follows from the fact that the computational procedure for coefficients of formal expansion is unique.

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