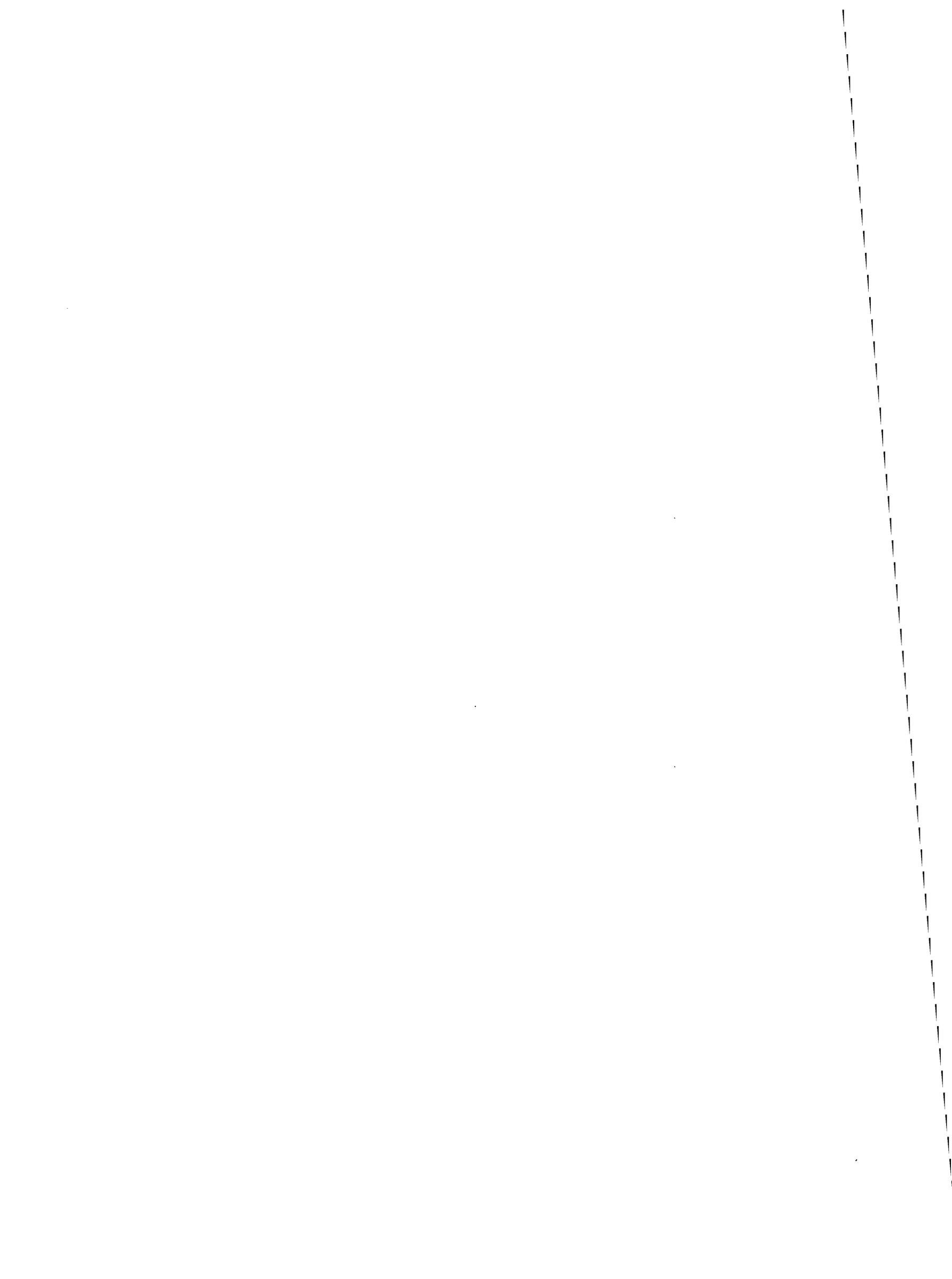


**Skew PBW monads and
representations**

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INTRODUCTION.

Let R be an associative ring, θ a map from some set \mathcal{G} to the group $Aut(R)$ of automorphisms of the ring R such that the image of θ is a semi-group in $Aut(R)$. A skew PBW (Poincaré-Birkhoff-Witt) ring related to the map θ is an associative ring $R\langle\theta, \xi\rangle$ which contains R as a subring and is a free right R -module with a basis $\{x_s | s \in \mathcal{G}\}$ such that $rx_s = \theta_s(r)x_s$ for any $s \in \mathcal{G}$ and all $r \in R$. The symbol ξ stands for the multiplication table: $x_s x_t = \sum_u \xi(s, t | u) x_u$. We assume that \mathcal{G} has a marked element $*$ which the map θ sends into id_R ; and $x_* = 1$ - the identity element of the ring R .

Important special cases of skew PBW rings are (quantized) enveloping algebras of an arbitrary Kac-Moody Lie algebra, enveloping algebras of reductive Lie algebras, Heisenberg and Weyl algebras (of an arbitrary rank) and their 'quantum' deformations, enveloping algebra of the Virasoro Lie algebra, crossed products.

The main result of this paper, Theorem 6.6.3, describes (gives a canonical realization of) the spectrum of the category $R\langle\theta, \xi\rangle\text{-mod}$ in terms of the spectrum of $R\text{-mod}$.

Recall that annihilators of modules from the spectrum are prime ideals, and if the ring is left noetherian, any prime ideal is the annihilator of a module of the spectrum (cf. [R2]).

Another fact is that the spectrum of a category contains isomorphy classes of all simple objects. And Theorem 6.6.3 allows to single out immediately a series of irreducible representations of $R\langle\theta, \xi\rangle$ which could be called *generalized Harish-Chandra modules*. A more involved application of Theorem 6.6.3 (and some other facts of the developed in [R4]-[R6] noncommutative spectral theory) allows to find some natural classes of non-diagonalizable irreducible representations of reductive Lie algebras and Kac-Moody Lie algebras.

Thus, Theorem 6.6.3 can be used (at least) in two ways: for classification of prime ideals of skew PBW rings; and for the study of their irreducible representations.

It happens that the language of rings and ideals is not convenient for the

study of the spectrum and simple objects which are of categorical nature. So, we give to the problem a more appropriate setting, and, as a result, investigate a more general object - modules over a *skew PBW monad*. Recall that a monad in a category \mathcal{A} is a pair (F, μ) , where F is a functor from \mathcal{A} to \mathcal{A} , and μ is a functor morphism from $F \circ F$ to F such that $\mu \circ F\mu = \mu \circ \mu F$ and $\mu \circ F\eta = \mu \circ \eta F$ for a (uniquely defined) morphism $\eta: Id \longrightarrow F$ (- the unity). An (F, μ) -module is a pair (M, m) , where M is an object of \mathcal{A} and m is an arrow from $F(M)$ to M such that $m \circ Fm = m \circ \mu(M)$ and $m \circ \eta(M) = id_M$.

We say that a monad (F, μ) is a *skew PBW monad* if $F = \bigoplus_{s \in \mathfrak{G}} \Theta_s$, where all Θ_s , $s \in \mathfrak{G}$, are auto-equivalences of the category \mathcal{A} , and the image of the set $\{\Theta_s \mid s \in \mathfrak{G}\}$ in the group $Aut \mathcal{A}$ of isomorphy classes of auto-equivalences is a semigroup. We assume that \mathfrak{G} has a marked object, $*$, and $\Theta_* \simeq Id_{\mathcal{A}}$.

A standart example: with any ring morphism $R \longrightarrow B$ (respecting the identity elements) one associates a monad by taking as F the functor $B \otimes_R$ and as μ the morphism $B \otimes_R (B \otimes_R) \longrightarrow B \otimes_R$ induced by the multiplication in B .

If B is a skew PBW ring over R , then $B \otimes_R$ defines a skew PBW monad in the category $R\text{-mod}$. But, of course, there are lots of skew PBW monads of different nature. For instance, \mathcal{A} might be the category of quasi-coherent modules on some scheme and Θ a map taking values in (tensoring by) invertible sheaves on this scheme.

The paper is organized as follows.

Section 1 contains some preliminaries about the spectrum of nonabelian categories we need in the sequel.

In Section 2, we introduce the principal characters of this work - skew PBW rings and skew PBW monads.

In Section 3, some auxiliary facts about the spectrum and associated points are proved.

Section 4 is concerned with monads and modules having gradings related to a triple (H, π, \mathbf{X}) , where H is a semigroup, and π an H -set surjection from H onto \mathbf{X} . The principal (for us) example is a special grading of this kind associated with a point of the spectrum.

In Section 5, we get a description of the spectrum of the category $gr_{\mathbf{X}} \mathbb{F}\text{-mod}$ of \mathbf{X} -graded \mathbb{F} -modules.

Section 6 contains the main theorem of this paper. We show that all points of the spectrum of $\mathbb{F}\text{-mod}$ which "grow up" over a given point of $\mathbf{Spec} \mathcal{A}$ can be represented by a graded module with the grading associated to this point. So that we can use the results of Section 5 to get a description of the spectrum.

Then follow 'Complementary facts and examples'. Most of them are motivated by representation theory.

In Section C1, we discuss shortly some of the functorial properties of our setting.

Section C2 is concerned with *quasi-holonomic* modules. Note that the conventional representation theory is restricted to holonomic modules. The (much larger) class of quasi-holonomic modules is a natural domain of definition of the formal character.

In Section C3 we show how to dualize the main results of the paper and to apply them to the study of comodules over skew PBW comonads.

In Section C4, we study the spectrum of the Weyl algebras and their quantized versions.

In Section C5, we sketch some of the lying on the surface consequences of our approach to the study of representations of reductive and Kac-Moody Lie algebras. Namely, given a reductive (or Kac-Moody) Lie algebra, we single out a natural class of representations which is, on a generic level at least, related with representations of certain hyperbolic ring. In the case of reductive Lie algebras over a field of zero characteristic, this class coincides (conjecturally) with the class of quasi-holonomic modules introduced in Section C.2. The significance of this fact is that the spectrum of hyperbolic rings is much easier to study.

In Appendix, we apply the main theorem to get the spectral picture of the two-parameter deformation of the coordinate algebras of $M(2)$ and $GL(2)$.

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1. THE SPECTRUM OF A QUASI-EXACT CATEGORY.

Here we give a sketch of an extension of the developed in [R3] spectral theory to nonabelian case (which is of independent interest) in the degree required by the main body of this work.

1.0. Quasi-exact categories. A *quasi-exact category* is a triple $(\mathcal{C}; \mathfrak{M}, \mathfrak{C})$, where \mathcal{C} is a category, and \mathfrak{M} and \mathfrak{C} are classes of arrows of \mathcal{C} which enjoy the following properties:

(a) Both \mathfrak{M} and \mathfrak{C} are closed with respect to compositions, and contain all isomorphisms of the category \mathcal{C} .

(b) If a morphism $f: X \longrightarrow Y$ has a kernel and $f \circ g \in \mathfrak{E}$ for some arrow $g: W \longrightarrow X$, then $f \in \mathfrak{E}$. Dually for morphisms in \mathfrak{M} .

(c) Every diagram $X \xrightarrow{e} V \xleftarrow{m} Y$ such that $e \in \mathfrak{E}$ and $m \in \mathfrak{M}$ can be extended to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{e} & V \\ m' \uparrow & & \uparrow m \\ X' & \xrightarrow{e'} & Y \end{array}$$

where $m' \in \mathfrak{M}$ and $e' \in \mathfrak{E}$.

(d) If $g: X \longrightarrow X'$ and $h: Y \longrightarrow Y'$ are morphisms from \mathfrak{M} (resp. from \mathfrak{E}), then their product, $g \amalg h: X \amalg Y \longrightarrow X' \amalg Y'$, belongs to \mathfrak{M} (resp. to \mathfrak{E}) too.

1.1. Example. Let \mathfrak{E} be a full subcategory of an abelian category \mathcal{A} which contains all subobjects of any of its objects. Take as \mathfrak{E} all morphisms in \mathfrak{E} which are epimorphisms in \mathcal{A} ; and set \mathfrak{M} to be all monoarrows in \mathfrak{E} . Clearly the conditions (a) - (d) above hold. Moreover, (c) follows from a much stronger property:

(c') Any arrow in \mathfrak{E} is represented as a composition $m \circ e$, where $m \in \mathfrak{M}$ and $e \in \mathfrak{E}$. ■

1.2. Example: the category of torsion free objects. Let \mathfrak{T} be a topologizing subcategory in an abelian category \mathcal{A} ; and let $\text{f}\mathfrak{T}$ denote the full subcategory of \mathcal{A} generated by \mathfrak{T} -torsion free objects. Clearly the subcategory $\text{f}\mathfrak{T}$ satisfies the conditions of Example 1.1: every subobject of a \mathfrak{T} -torsion free object is \mathfrak{T} -torsion free.

Note that $\text{f}\mathfrak{T}$ is also closed under extensions in \mathcal{A} : if in the exact sequence (in \mathcal{A})

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{e} M'' \longrightarrow 0 \quad (1)$$

M' and M'' are \mathfrak{T} -torsion free, then M is \mathfrak{T} -torsion free. In particular, the subcategory $\text{f}\mathfrak{T}$ is closed under finite products in \mathcal{A} . ■

1.3. Example. Let \mathfrak{T} be a topologizing subcategory of an abelian category \mathcal{A} . Take as \mathfrak{M} the family of all arrows f in \mathcal{A} such that $\text{Ker}(f) \in \text{Ob}\mathfrak{T}$. Dually, define \mathfrak{E} to be the class of arrows g such that $\text{Cok}(g) \in \mathfrak{T}$.

An easy way to see that all the conditions hold (even (c') in Example 1.1) is to notice that the family \mathfrak{M} (resp. \mathfrak{E}) is the preimage under the localiza-

tion $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ of the class of all monomorphisms (resp. epimorphisms) of the quotient category \mathcal{A}/\mathbb{T} . ■

A *quasi-exact functor* $F: (\mathcal{C}; \mathbb{M}, \mathcal{E}) \longrightarrow (\mathcal{C}'; \mathbb{M}', \mathcal{E}')$ between quasi-exact categories is a functor $F: \mathcal{C} \longrightarrow \mathcal{C}'$ which carries morphisms in \mathbb{M} (resp. \mathcal{E}) into morphisms in \mathbb{M}' (resp. morphisms of \mathcal{E}').

1.4. Examples. In Example 1.1, the natural embedding $\mathcal{C} \longrightarrow \mathcal{A}$ is a quasi-exact functor. In particular, the inclusion functor $f\mathbb{T} \longrightarrow \mathcal{A}$ of Example 1.2.

Clearly the localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ in Example 1.3 is a quasi-exact functor. ■

1.4. A preorder in quasi-exact categories. Fix a quasi-exact category $(\mathcal{A}; \mathbb{M}, \mathcal{E})$. When it does not create an ambiguity, we shall write \mathcal{A} instead of $(\mathcal{A}; \mathbb{M}, \mathcal{E})$.

For any two objects X and Y of the category \mathcal{A} , we shall write $X \succ Y$ if there is a diagram $(k)X \xleftarrow{m} U \xrightarrow{e} Y$, where $m \in \mathbb{M}$, $e \in \mathcal{E}$, and $(k)X$ denotes the product of k copies of the object X .

1.1.1. Lemma. *The relation \succ is a preorder in $Ob\mathcal{A}$.*

Proof. In fact, let $X \succ Y$, and $Y \succ Z$; i.e. there exist the diagrams

$$(k)X \xleftarrow{i} U \xrightarrow{e} Y \quad (1)$$

and

$$(n)Y \xleftarrow{j} V \xrightarrow{e'} Z \quad (2)$$

in which the arrows $i, j \in \mathbb{M}$, and $e, e' \in \mathcal{E}$. The product

$$(nk)X \xleftarrow{(n)i} (n)U \xrightarrow{(n)e} (n)Y$$

of n copies of the diagram (1) is of the same type (cf. 1.0.1).

There exists a diagram

$$\begin{array}{ccc} (n)U & \xrightarrow{(n)e} & (n)Y \\ \uparrow j' & & \uparrow j \\ W & \xrightarrow{\varepsilon} & V \end{array}$$

where $j' \in \mathbb{M}$ and $\varepsilon \in \mathcal{E}$. Hence $(n)i \circ j': W \longrightarrow (nk)X$ belongs to \mathbb{M} and the composition $e' \circ \varepsilon: W \longrightarrow Z$ is a morphism in \mathcal{E} ; i.e. $X \succ Z$. ■

1.5. The notation. Denote by $|\mathcal{A}|$, or by $|(\mathcal{A}; \mathfrak{M}, \mathfrak{E})|$, the ordered set of equivalence classes of objects of \mathcal{A} with respect to the relation \succ . We save the same symbol, \succ , for the induced order on $|\mathcal{A}|$.

Clearly any quasi-exact functor $F: (\mathfrak{E}; \mathfrak{M}, \mathfrak{E}) \longrightarrow (\mathfrak{E}'; \mathfrak{M}', \mathfrak{E}')$ defines a morphism $|F|: |\mathfrak{E}| \longrightarrow |\mathfrak{E}'|$ of the corresponding ordered sets.

1.6. The spectrum of a quasi-exact category. Let M be a nonzero object of the category \mathcal{A} . We write $M \in \text{Spec}(\mathcal{A}; \mathfrak{M}, \mathfrak{E})$ (or $\text{Spec}\mathcal{A}$ if classes $\mathfrak{M}, \mathfrak{E}$ are fixed), if the existence of a nonzero arrow $i: N \longrightarrow M$ in \mathfrak{M} implies that $N \succ M$.

Since $M \succ N$, we can say that $M \in \text{Spec}\mathcal{A}$ if and only if it is equivalent with respect to the preorder \succ to any of its nonzero \mathfrak{M} -subobjects.

Denote by $\text{Spec}\mathcal{A}$ the ordered set of equivalence classes (with respect to \succ) of elements of $\text{Spec}\mathcal{A}$. The set $\text{Spec}\mathcal{A}$ shall be called the *spectrum of the quasi-exact category* \mathcal{A} .

1.7. Spectrum and \mathfrak{M} -simple objects. Call an object M of $(\mathcal{A}; \mathfrak{M}, \mathfrak{E})$ *\mathfrak{M} -simple* if every nonzero morphism $i: L \longrightarrow M$ in \mathfrak{M} belongs also to \mathfrak{E} .

Clearly every \mathfrak{M} -simple object of the category \mathcal{A} belongs to the spectrum.

1.8. Example. If $(\mathfrak{E}; \mathfrak{M}, \mathfrak{E})$ is a quasi-exact category of Example 1.1, then the preorder \succ on $Ob\mathfrak{E}$ is induced by the preorder \succ in $Ob\mathcal{A}$, and $\text{Spec}(\mathfrak{E}; \mathfrak{M}, \mathfrak{E})$ coincides with $\text{Spec}\mathcal{A} \cap |\mathfrak{E}|$.

In particular, for any topologizing subcategory \mathfrak{T} of the category \mathcal{A} , we can write: $\text{Spec}\mathcal{A} = \text{Spec}\mathfrak{T} \cup \text{Spec}\mathfrak{f}\mathfrak{T}$.

In fact, if \mathfrak{T} is a topologizing category, then any object M from $\text{Spec}\mathcal{A}$ is either in \mathfrak{T} , or in $\mathfrak{f}\mathfrak{T}$. Since, if M' is a nonzero subobject of M such that $M' \in Ob\mathfrak{T}$, then $M' \succ M$ which implies that $M \in Ob\mathfrak{T}$. ■

1.9. Example. For the quasi-exact category $(\mathcal{A}; \mathfrak{M}, \mathfrak{E})$ of Example 1.3, we have:

$$\text{Spec}(\mathcal{A}; \mathfrak{M}, \mathfrak{E}) = \text{Spec}\mathcal{A}/\mathfrak{T}. \quad \blacksquare$$

2. FROM HYPERBOLIC RINGS TO SKEW PBW MONADS.

2.1. Hyperbolic rings. Fix an associative ring R and consider the following data:

a set $\mathbf{x} = (x_i | i \in J)$, $\mathbf{y} = (y_i | i \in J)$ of indeterminates;

a set $\Xi = \{\theta_i | i \in J\}$ of automorphisms of the ring R ;

a set $\xi = \{\xi_i | i \in J\}$ is of central elements of the ring R .

The corresponding to this data hyperbolic ring $R(\Xi, \xi) = R(x, y; \Xi, \xi)$ is defined by the following relations:

$$\theta_i(r)x_i = x_i r \quad \text{and} \quad ry_i = y_i \theta_i(r) \quad (1)$$

for any $r \in R$ and $i \in J$;

$$x_i y_i = \xi_i, \quad y_i x_i = \theta_i^{-1}(\xi_i) \quad \text{for all } i \in J. \quad (2)$$

Besides, we assume that $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ for all i, j , and

$$x_i y_j = y_j x_i \quad \text{if } i \neq j. \quad (3)$$

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i \quad \text{for all } i, j \in J. \quad (4)$$

2.2. Examples. (a) The n -th Weyl algebra A_n over a field k . Here R is the ring $k[\xi_1, \dots, \xi_n]$ of polynomials, and $\theta_i \xi_j = \xi_j + \delta_{ij}$ for all $1 \leq i, j \leq n$.

(b) Similarly with n -th Heisenberg algebra H_n . Only this time R is the ring of polynomials in $\xi_i, z_j, 1 \leq i, j \leq n$, and the automorphisms θ_i are given by $\theta_i \xi_j = \xi_j + \delta_{ij} z_i \xi_j, \theta_i z_j = z_j$.

(c) A (generalized) algebra of q -differential operators: R is a polynomial ring in ξ_i (as in (a)); $\theta_i \xi_j = q_{ij} \xi_j + \delta_{ij}$ for all i, j , where q_{ij} are nonzero elements of the field k .

(d) Most of 'small' algebras of mathematical physics to begin with the enveloping algebra $U(sl(2))$ of the Lie algebra $sl(2)$ and its quantized version $U_q(sl(2))$, algebra of functions of $SL_q(2)$, dispin algebra etc. (see [R10]). ■

2.3. Cross-products. Fix an associative ring R . Let G be a group, and θ a group morphism from G to $Aut(R)$. Let ζ be a map $G \times G \longrightarrow R$ such that

$$\zeta(s, tu) \cdot \zeta(t, u) = \zeta(st, u) \cdot \theta_u \zeta(s, t), \quad (1)$$

$$\zeta(s, 1) = 1 = \zeta(1, s) \quad \text{for any } s \in G. \quad (2)$$

Define a multiplication on the free right module $\bigoplus_{g \in G} x_g R$, where $x_e = 1$, (here e is the unity element of G) by the formula:

$$\left(\sum_{s \in G} x_s r_s \right) \left(\sum_{t \in G} x_t r'_t \right) = \sum_{g \in G} x_g \left(\sum_{st=g} \theta_t(r_s) r'_t \zeta(s, t) \right). \quad (3)$$

The equalities (1) and (2) imply (are equivalent to the fact) that the formula (3) defines an associative multiplication, ζ , on $\bigoplus_{g \in G} x_g R$ with the unity element $x_e = 1$. Denote thus defined ring by $R(\theta, \zeta)$. It is generated by its subring R and the elements $x_g, g \in G$, subject to the following relations:

$$x_g x_h = x_{gh} \zeta(g, h), \quad r x_g = x_g \theta_g(r) \quad (4)$$

for all $(g,h) \in G \times G$ and $r \in R$.

In the special case, $\zeta(g,h) \equiv 1$, $R(\theta,\zeta)$ is a skew group ring (cross-product). If, in addition, the action of G on R is (i.e. the map θ) trivial, then $R(\theta,\zeta)$ is the group ring of G with coefficients in R .

Clearly the hyperbolic ring defined in 2.1 is a very special case of the ring $R[G,\zeta]$. Namely, take as θ the group morphism of the product \mathbb{Z}^J of J copies of \mathbb{Z} to the group $\text{Aut}(R)$ which assigns to the i^{th} canonical generator of \mathbb{Z}^J the automorphism θ_i .

$$\zeta(s,t) = 1 \text{ if } st \neq 1, \text{ and } \zeta(s,s^{-1}) := \xi$$

for all s .

2.3. From the ring $R[G,\zeta]$ to a skew PBW monad. It is very inconvenient to study the spectrum in terms of rings and ideals. So, we need to switch to categorical notions. While realizing this transition, we shall considerably extend the set up.

Fix a category \mathcal{A} . Let $\text{Aut}\mathcal{A}$ denote the category objects of which are auto-equivalences of \mathcal{A} and arrows are functor isomorphisms. Denote by $(\mathfrak{G}, \Theta, \xi)$ the following data:

(a) \mathfrak{G} is a set with a marked element 1 and $\Theta: \mathfrak{G} \longrightarrow \text{ObAut}\mathcal{A}$ is a map such that $\Theta_1 = \text{Id}_{\mathcal{A}}$

(b) ξ is a function which assigns to any triple $s,t,u \in \mathfrak{G}$ a functor morphism $\xi(s,t|u): \Theta_s \circ \Theta_t \longrightarrow \Theta_u$ such that

(i) the family $\{\xi(s,t|u) \mid u \in \mathfrak{G}\}$ defines a morphism

$$\xi(s,t): \Theta_s \circ \Theta_t \longrightarrow \bigoplus_{u \in \mathfrak{G}} \Theta_u$$

(clearly the family $\{\xi(s,t|u) \mid u \in \mathfrak{G}\}$ defines a morphism

$$\Theta_s \circ \Theta_t \longrightarrow \prod_{u \in \mathfrak{G}} \Theta_u;$$

so, what we really require is that this morphism should factor through the natural arrow $\bigoplus_{u \in \mathfrak{G}} \Theta_u \longrightarrow \prod_{u \in \mathfrak{G}} \Theta_u$);

(ii) for any $s, t, t', u \in \mathfrak{G}$,

$$\sum_{w \in \mathfrak{G}} \xi(t',w|u) \circ \Theta_{t'} \xi(s,t|w) = \sum_{w \in \mathfrak{G}} \xi(w,t|u) \circ \xi(t',s|w) \Theta_{t'}$$

(iii) for any $s, t \in \mathfrak{G}$,

$$\xi(s,1|t) = \xi(1,s|t) = \text{id} \text{ if } t = s$$

and

$$\xi(s,1|t) = \xi(1,s|t) = 0 \text{ if } t \neq s.$$

(c) Denote by $\mathbf{SAut}\mathcal{A}$ the set of equivalence classes with respect to the following equivalence relation on $Ob\mathbf{Aut}\mathcal{A}$: $\theta \cong \vartheta$ if their actions on $\mathbf{Spec}\mathcal{A}$ coincide; i.e. $\langle \theta(P') \rangle = \langle \vartheta(P') \rangle$ for every $\langle P' \rangle \in \mathbf{Spec}\mathcal{A}$. Clearly $\mathbf{SAut}\mathcal{A}$ is a subgroup of the group of homeomorphisms of the topological space $(\mathbf{Spec}\mathcal{A}, \tau)$. Denote by $\mathbf{S}\Theta$ the image of the composition of

$$\Theta: \mathfrak{G} \longrightarrow Ob\mathbf{Aut}\mathcal{A} \text{ and } Ob\mathbf{Aut}\mathcal{A} \longrightarrow \mathbf{SAut}\mathcal{A}.$$

We assume that $\mathbf{S}\Theta$ is a subsemigroup of $\mathbf{SAut}\mathcal{A}$.

We relate to this data the monad $\mathbb{F} = (F, \xi)$, where $F = \bigoplus_{s \in \mathfrak{G}} \Theta(s)$ and the multiplication $\xi: F \circ F \longrightarrow F$ is defined by the morphisms $\xi(s, t|u)$.

We call a monad of this type a *skew PBW (Poincaré-Birkhoff-Witt) monad*.

Note that any \mathbb{F} -module (M, m) is described by the family of compositions

$$m_t: \Theta_t(M) \longrightarrow M, \quad t \in \mathfrak{G}, \quad (1)$$

of the embedding $\Theta_t(M) \longrightarrow F(M)$ and the action $m: F(M) \longrightarrow M$.

Conversely, a set of morphisms (1) defines an \mathbb{F} -module structure on M iff the following conditions hold:

$$\sum_{w \in \mathfrak{G}} m_w \circ \Theta_w \xi(s, t|w) = m_s \circ \Theta_s(m(t)), \quad m_1 = id.$$

The spectrum of the forgetting functor $\mathbb{F}\text{-mod} \longrightarrow \mathcal{A}$, where \mathbb{F} is a skew PBW monad, shall be the object of our study. To refer to the base category \mathcal{A} and to the data $\{\mathfrak{G}, \Theta, \xi\}$, we shall write sometimes $\mathcal{A}(\Theta, \xi)$ instead of $\mathbb{F}\text{-mod}$.

2.4. Skew PBW monads and skew PBW rings. The reflection of the notion of a skew PBW monad in **RINGS** provides a useful generalization of Example 2.2. The corresponding data is:

an associative ring R ; and a set \mathfrak{G} with a marked element 1 ;

a map $\vartheta: \mathfrak{G} \longrightarrow Aut(R)$ which sends the marked element into id_R ;

a map $\xi: \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \longrightarrow R$, $(s, t, u) \longmapsto \xi(s, t|u)$ which has the properties:

(i) For any $(s, t) \in \mathfrak{G} \times \mathfrak{G}$, the set $\{u \in \mathfrak{G} \mid \xi(s, t|u) \neq 0\}$ is finite.

(ii) For any $s, t, t', u \in \mathfrak{G}$,

$$\sum_{w \in \mathfrak{G}} \xi(t', w|u) \xi(s, t|w) = \sum_{w \in \mathfrak{G}} \xi(w, t|u) \cdot \Theta_t(\xi(t', s|w)).$$

(ii) For any $s, t \in \mathfrak{G}$,

$$\xi(s, 1|t) = \xi(1, s|t) = \delta_{st}$$

where, as usual, $\delta_{st} = 1$ if $t = s$, and 0 if $t \neq s$.

We define the ring $R[\vartheta, \xi]$ as a free right R -module $\bigoplus_{s \in \mathfrak{G}} x_s R$ with the multiplication given by

$$x_s x_t = \sum_{u \in \mathfrak{G}} x_u \xi(s,t|u), \quad rx_s = x_s \vartheta_s(r) \quad \text{for all } r \in R.$$

Set $\mathcal{A} := R\text{-mod}$, and denote by Θ the composition of ϑ with the natural map $\text{Aut}(R) \longrightarrow \text{Aut}\mathcal{A}$. The map ξ defines the multiplication table

$$\zeta(s,t|u): \Theta_s \circ \Theta_t \longrightarrow \Theta_u, \quad (s,t,u) \in \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}.$$

One can check that the equalities (i) - (iii) are equivalent to the corresponding equalities in 2.5. So that the data $(\mathfrak{G}, \Theta, \zeta)$ defines a skew PBW monad $\mathbb{F}\text{-mod}$ in the category $\mathcal{A} := R\text{-mod}$.

It is easy to see that the categories $\mathbb{F}\text{-mod}$ and $R\{\vartheta, \xi\}\text{-mod}$ are naturally equivalent.

2.5. Example: Kac-Moody Lie algebras. Fix a field k . Let J be a finite set (the index set of simple roots); and let \mathfrak{h} be a finite-dimensional k -vector space. Assume that elements

$$h_i \in \mathfrak{h} \quad \text{and} \quad \alpha_i \in \mathfrak{h}^*, \quad i \in J,$$

are given which satisfy the following conditions:

- (a) $a_{ij} := \langle h_i, \alpha_j \rangle$ form a generalized Cartan matrix; i.e.
 $a_{ii} = 2$; $a_{ji} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$; and $a_{ij} = 0$ iff $a_{ji} = 0$.
- (b) $\{\alpha_i\}$ and $\{h_i\}$ are linearly independent.

Recall that the Kac-Moody Lie k -algebra associated to this data contains \mathfrak{h} as a commutative subalgebra and is generated by \mathfrak{h} and elements $\{x_i\}$, $\{y_i\}$ subject to the following conditions:

$$[h, x_i] = \langle \alpha_i, h \rangle x_i, \quad [h, y_i] = - \langle \alpha_i, h \rangle y_i$$

$$[x_i, y_j] = \delta_{ij} h_i$$

for any $h \in \mathfrak{h}$ and $i, j \in J$;

$$(ad_{x_i})^{1-a_{ij}} x_j = 0, \quad (ad_{y_i})^{1-a_{ji}} y_j = 0 \quad (i \neq j)$$

For any $\alpha \in \mathfrak{h}^*$,

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x\},$$

and

$$\Delta := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\},$$

$$\Delta^+ := \Delta \cap \sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i, \quad \Delta^- := \Delta \cap \sum_{i \in J} \mathbb{Z}_{\leq 0} \alpha_i$$

Let \mathfrak{n} and \mathfrak{n}^- be the subalgebras generated resp. by $\{x_i\}$ and $\{y_i\}$. Then $\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- := \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$, and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.

The latter decomposition induces an isomorphism

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}).$$

Now we shall reformulate this setting a little bit.

First, denote the ring $U(\mathfrak{h}) = S(\mathfrak{h})$ by R ; and, for any $\alpha \in \Delta$, let ϑ_α be the automorphism of the k -algebra R defined by

$$h \longmapsto h - \langle \alpha, h \rangle_1 \quad \text{for all } h \in \mathfrak{h}.$$

Fix a linear order in Δ . Set $\mathfrak{G} := (\mathbb{Z}_{\geq 0})^\Delta := \{\text{maps from } \Delta \text{ to } \mathbb{Z}_{\geq 0}\}$. The enveloping algebra $U(\mathfrak{g})$ is a free right R -module with the basis $\{\mathbf{x}(\mathbf{i}) \mid \mathbf{i} \in \mathfrak{G}\}$. Define the map $\Theta: \mathfrak{G} \longrightarrow \text{Aut}(R)$ by $\mathbf{i} \longmapsto \prod_{\alpha \in \Delta} \vartheta_\alpha^{\mathbf{i}(\alpha)}$, $\mathbf{i} \in \mathfrak{G}$.

We shall identify any map $\mathbf{i}: \Delta \longrightarrow \mathbb{Z}_{\geq 0}$ with the word $\prod_{\alpha \in \Delta} \alpha^{\mathbf{i}(\alpha)}$. For instance, α denotes the map \mathbf{i} such that $\mathbf{i}(\beta) = \delta_{\alpha\beta}$, and $\alpha\beta$ (with $\alpha < \beta$) replaces $\mathbf{i}: \gamma \longmapsto \delta_{\alpha\gamma} + \delta_{\beta\gamma}$.

Define the multiplication table by

$$\xi(\alpha, \beta \mid \gamma) = \delta_{\alpha\beta, \gamma} \quad \text{if } \alpha \leq \beta;$$

if $\beta > \alpha$, then

$$\xi(\alpha, \beta \mid \gamma) = \begin{cases} 1 & \text{if } \gamma = \beta\alpha \\ a_{\alpha\beta} & \text{if } \gamma = \alpha + \beta \\ 0 & \text{for any other } \gamma \in \Delta \end{cases}$$

Thus the enveloping algebra $U(\mathfrak{g})$ of a Kac-Moody Lie algebra is a skew PBW ring over the polynomial ring $U(\mathfrak{h}) = S(\mathfrak{h})$.

2.6. Remark. Suppose we are given the following data:

a ring R ;

a set $\{\vartheta_i \mid i \in J\}$ of pairwise commuting automorphisms of the ring R ;

a set $\{h_i \mid i \in J\}$ of central elements of R ;

a set $\{\lambda_{ij} \mid i, j \in J\}$ of central invertible elements of R .

Denote the data $\{\vartheta_i, h_i, \lambda_{ij} \mid i, j \in J\}$ by Ξ . We relate to Ξ an associative ring $R(\Xi)$ which contains R as a subring and is generated by R and elements $\{x_i\}, \{y_i\}$ subject to the following relations:

$$x_i r = \vartheta_i(r) x_i, \quad r y_i = y_i \vartheta_i(r) \quad (1)$$

$$x_i y_j - \lambda_{ij} y_j x_i = \delta_{ij} h_i \quad (2)$$

for any $r \in R$ and $i, j \in J$;

Clearly $R(\Xi)$ is a skew PBW ring with $\mathfrak{G} = \mathfrak{G}^+ \amalg \mathfrak{G}^-$, where \mathfrak{G}^+ (resp. \mathfrak{G}^-) is the set of all monomials in $\{x_i \mid i \in J\}$ (resp. in $\{y_i \mid i \in J\}$). The map

$\vartheta: \mathfrak{G} \longrightarrow \text{Aut}(R)$ is defined by

$$x_i \longmapsto \vartheta_i, \quad y_i \longmapsto \vartheta_i^{-1} \quad \text{for any } i \in J$$

and by the requirement that the restrictions of ϑ to \mathfrak{G}^+ and \mathfrak{G}^- respect the multiplication. Finally, the function ξ is defined by

$$\xi(s,t|u) = \delta_{st,u} \quad \text{if } (s,t) \in (\mathfrak{G}^+ \amalg \mathfrak{G}^+) \amalg (\mathfrak{G}^- \amalg \mathfrak{G}^-) \amalg (\mathfrak{G}^- \amalg \mathfrak{G}^+),$$

and $\xi(s,t|u)$ is determined by the relations (1) and (2) when $(s,t) \in \mathfrak{G}^+ \amalg \mathfrak{G}^-$.

Clearly the image G of the map ϑ is an abelian subgroup in $\text{Aut}(R)$. The map ϑ defines a partition of \mathfrak{G} (two elements, s and t are in one class iff $\vartheta(s) = \vartheta(t)$) and, therefore, a G -grading of the free right R -module $\bigoplus_{s \in \mathfrak{G}} x(s)R$. One can see that it is a G -grading of the ring $R(\Xi)$.

Consider now a homogenous (with respect to the G -grading) two-sided ideal \mathfrak{J} of the G -graded ring $R(\Xi)$. The ring of factors $R(\Xi)/\mathfrak{J}$ is, therefore, also G -graded. If it happens to be a free R -module, then it is also a skew PBW.

One of the simplest occurrences of this kind is the two-sided ideal \mathfrak{J} generated by

$$x_i x_j - x_j x_i, \quad y_i y_j - y_j y_i, \quad (i,j) \in J \times J.$$

Then the quotient ring $R(\Xi)$ is hyperbolic. Weyl and Heisenberg algebras, and their quantum deformations are examples of this (cf. Examples 2.2).

A more sophisticated example is the enveloping algebra $U(\mathfrak{g})$ of a Kac-Moody Lie algebra \mathfrak{g} . Here $R = S(\eta)$, $\lambda_{ij} = 1$ for all i, j , and ϑ_i is the automorphism defined by the i^{th} simple root (cf. 2.5). The two-sided ideal \mathfrak{J} is generated by

$$(adx_i)^{1-a_{ij}} x_j, \quad (ady_i)^{1-a_{ij}} y_j \quad (i \neq j)$$

Clearly \mathfrak{J} is G -homogenous. And it follows from PBW theorem that $R(\Xi) \simeq U(\mathfrak{g})$ is a free R -module.

Another set of important examples of skew PBW rings of the form $R(\Xi)/\mathfrak{J}$ are quantum enveloping algebras. ■

2.7. Virasoro algebra. Recall that the Virasoro Lie algebra is the universal central extension of the Lie algebra of regular vector fields. It has a basis $\{d_n, c \mid n \in \mathbb{Z}\}$, where c is a central element, and the following commutation relations:

$$[d_m, d_n] = (n - m)d_{m+n} + (1/12)(n^3 - n)\delta_0(n-m)c, \quad (1)$$

for all $m, n \in \mathbb{Z}$.

Set $R := k[z, c]$; and let ϑ denote an automorphism of the ring R defined by

$$\vartheta f(z, c) := f(z^{-1}, c).$$

Let \mathfrak{G} denote the set of all functions $i: \mathbb{Z} \longrightarrow \mathbb{Z}_+$ with finite support. And define by Θ the composition of the map

$$\mathfrak{G} \longrightarrow \mathbb{Z}, \quad i \longmapsto \sum_{n \in \mathbb{Z}} ni(n),$$

and the group morphism $\mathbb{Z} \longrightarrow \text{Aut}(R)$ which sends i into ϑ .

The multiplication table is defined by

$$\xi(n, m | v) = \begin{cases} 1 & \text{if } v = \delta_n + \delta_m \\ a_{n, m} & \text{if } v = \delta_{n+m} \\ 0 & \text{for any other } v \in \mathbb{Z} \end{cases}$$

where $a_{n, m} := \max(n-m, 0)$ if $n+m \neq 0$, and $a_{n, m} = (1/12)(n^3 - n)c$ if $m+n$ is equal to zero.

Thus defined skew PBW ring coincides with the enveloping algebra of the Virasoro Lie algebra.

3. PREPARATION.

To get to the punch line, we need several auxiliary facts. First of them is the following Lemma:

3.1. Lemma. *Let G be a left exact functor from \mathcal{A} to \mathcal{A} ; and let λ be an arbitrary functor morphism from $\text{Id}_{\mathcal{A}}$ to G . Then, for any $P \in \text{Spec} \mathcal{A}$, either $\lambda(P) = 0$, or $\lambda(P)$ is a monomorphism.*

Proof. 1) Note that if $\lambda(M) = 0$, and $M \succ L$, then $\lambda(L) = 0$.

In fact, the relation $M \succ L$ means that there is a diagram

$$({}_l)M \xleftarrow{i} K \xrightarrow{e} L,$$

where i is a monoarrow and e is an epimorphism. Since the functor G is left exact, the morphism G_i in the commutative diagram

$$\begin{array}{ccccc} ({}_l)M & \xleftarrow{i} & K & \xrightarrow{e} & L \\ \downarrow ({}_l)\lambda(M)=0 & & \downarrow \lambda(K) & & \downarrow \lambda(L) \\ ({}_l)G(M) & \xleftarrow{G_i} & G(K) & \xrightarrow{G_e} & G(L) \end{array}$$

is a monoarrow. Therefore the equalities

$$G_i \circ \lambda(K) = ({}_l)\lambda(M) \circ i = 0 \circ i = 0$$

imply that $\lambda(K) = 0$. Since e is an epimorphism the equalities

$$\lambda(L) \circ e = Ge \circ \lambda(K) = Ge \circ 0 = 0$$

imply that $\lambda(L) = 0$.

2) Note now that, for any $M \in \text{Ob}\mathcal{A}$, $\lambda(\text{Ker}\lambda(M)) = 0$.

Indeed, let $k(M)$ denote the canonical monoarrow $\text{Ker}\lambda(M) \longrightarrow M$. Since $Gk(M)$ is a monomorphism, the equalities

$$Gk(M) \circ \lambda(\text{Ker}\lambda(M)) = \lambda \circ k(M) = 0$$

imply that $\lambda(\text{Ker}\lambda(M)) = 0$.

3) Let now $P \in \text{Spec}\mathcal{A}$. If $\text{Ker}\lambda(P) \neq 0$, then $\text{Ker}\lambda(P) \succ P$. This and the equality $\lambda(\text{Ker}\lambda(P)) = 0$ (cf. 2)), implies, according to the heading 1) of the argument, that $\lambda(P) = 0$. ■

3.2. Example. Let \mathbb{F} be a skew PBW monad defined by the data $(\mathfrak{G}, \Theta, \xi)$. By Lemma 3.1, for any $P \in \text{Spec}\mathcal{A}$ and $s, t, u \in \mathfrak{G}$, we have:

either $\xi(s, t|u)(P) = 0$, or $\xi(s, t|u)(P)$ is a monomorphism. ■

3.3. Lemma. Let \mathcal{A} be a local category with a quasi-final object P . Let θ and ϑ be auto-equivalences of \mathcal{A} and $\alpha: \theta \longrightarrow \vartheta$ a functor morphism. Then either $\alpha(P) = 0$, or α is an isomorphism.

Proof. Let $\alpha^\wedge: Id \longrightarrow \theta^\wedge \circ \vartheta$ be the adjoint to α morphism and M an arbitrary object of the category \mathcal{A} . If $\text{Ker}\alpha^\wedge(M) \neq 0$, then $\text{Ker}\alpha^\wedge(M) \succ P$. This relation implies, since $\alpha^\wedge(\text{Ker}\alpha^\wedge(M))$ equals to zero, that $\alpha^\wedge(P) = 0$. One can see that $\text{Ker}\alpha^\wedge(M) \neq 0 \Leftrightarrow \text{Ker}\alpha(M) \neq 0$, and $\alpha^\wedge(P) = 0 \Leftrightarrow \alpha(P) = 0$.

Similarly, consider the morphism $\alpha': \theta \circ \vartheta^\wedge \longrightarrow Id$ which is the composition of $\alpha\vartheta^\wedge: \theta \circ \vartheta^\wedge \longrightarrow \vartheta \circ \vartheta^\wedge$ and the adjunction arrow $\vartheta \circ \vartheta^\wedge \longrightarrow Id$. If $\text{Coker}\alpha(M) \neq 0$, then $\text{Coker}\alpha'(M) \neq 0$ which implies that $\text{Coker}\alpha'(M) \succ P$. Therefore $\alpha'(P) = 0$ (we use the equality $\alpha'(\text{Coker}\alpha'(M)) = 0$). Again, $\alpha'(P) = 0$ if and only if $\alpha(P) = 0$. ■

4. GRADED MONADS AND MODULES.

Fix a semigroup H and an H -set $X = (X, \cdot)$. Here \cdot denotes the action of H on X , $(h, x) \longmapsto h \cdot x$.

4.1. Graded monads. We call a monad $\mathbb{F} = (F, \mu)$ in the category \mathcal{A} X -graded if it is provided with decompositions

$$F = \bigoplus_{h \in H} F(h), \quad F = \bigoplus_{x \in X} F[x]$$

such that the restriction of the multiplication μ to $F(h) \circ F[x]$ takes values in $F[h \cdot x]$; i.e. the composition of the action $F(h) \circ F[x] \longrightarrow F$ and the projection $F \longrightarrow F[y]$ is zero if $y \neq h \cdot x$.

4.2. Graded \mathbb{F} -modules. Fix an X -graded monad $\mathbb{F} = (F, \mu)$. An X -graded \mathbb{F} -module as an (F, μ) -module (M, m) provided with a decomposition $M = \bigoplus_{x \in X} M[x]$ which is compatible with the action of $F = \bigoplus_{h \in H} F(h)$. This compatibility means that the composition

$$F(h)(M[x]) \longrightarrow F(M) \xrightarrow{m} M \longrightarrow M[y],$$

(where the first arrow is the canonical embedding, and the third one is the projection) equals to zero if $h \cdot x \neq y$.

A morphism from an X -graded \mathbb{F} -module (M, m) to an X -graded \mathbb{F} -module (M', m') is any (F, μ) -module morphism $g: (M, m) \longrightarrow (M', m')$ which has a diagonal matrix; i.e. the composition (entry) $g[x, y]$

$$M[x] \longrightarrow M \xrightarrow{g} M' \longrightarrow M'[y]$$

equals to zero if $x \neq y$.

The composition of arrows is inherited from \mathcal{A} .

We denote the category of X -graded \mathbb{F} -modules by $\text{gr}_X \mathbb{F}\text{-mod}$.

4.3. The grading associated with a point of the spectrum. Fix a data $(\mathcal{G}, \Theta, \xi)$ (cf. 2.3) and a point $\langle P \rangle \in \text{Spec } \mathcal{A}$. Take as H the image $S\Theta$ of the composition of $\Theta: \mathcal{G} \longrightarrow \text{Aut } \mathcal{A}$ and of the canonical map $\text{Aut } \mathcal{A} \longrightarrow \text{Aut}(\text{Spec } \mathcal{A})$ (cf. (c) in 2.3); and let X be the $S\Theta$ -orbit of the point $\langle P \rangle$.

We identify this orbit with the set $\mathcal{G}\langle P \rangle$ of equivalence classes with respect to the relation on \mathcal{G} : $s \approx t$ iff $\Theta_s(P) \approx \Theta_t(P)$.

Denote by π the projection $S\Theta \longrightarrow \mathcal{G}\langle P \rangle$. Consider the full subcategory \mathcal{B} of the category \mathcal{A} generated by all objects M of \mathcal{A} such that

$$(a) \text{Supp}(M) \subseteq \bigcup_{s \in \mathcal{G}} \text{Supp}(\Theta_s(P));$$

(b) M is the supremum of its subobjects from $\text{Spec } \mathcal{A}$.

Note that

- 1) The subcategory \mathcal{B} depends only on the orbit X of the point $\langle P \rangle$.
- 2) It follows from the property (sup) that the subcategory \mathcal{B} contains all subobjects of any of its objects and is closed under direct sums. But, in general, it is not closed under taking quotients. In particular, \mathcal{B} is not, usually, an abelian category.

An obvious exception is the case when the point $\langle P \rangle$ is closed.

3) Each auto-equivalence Θ_s , $s \in \mathfrak{G}$, induces an auto-equivalence, Θ'_s , of the category \mathfrak{B} . For any $h \in \mathbf{S}\Theta$ and any class $x \in \mathfrak{G}\langle P \rangle$, set

$$F(h) = \bigoplus_{\mathbf{S}\Theta(s)=h} \Theta'(s), \quad F[x] := \bigoplus_{t \in x} \Theta'(t).$$

Thus, we have two decompositions of the functor $F = \bigoplus_{s \in \mathfrak{G}} \Theta'(s)$ associated with the map Θ :

$$F = \bigoplus_{h \in \mathbf{S}\Theta} F(h) \quad \text{and} \quad F = \bigoplus_{x \in \mathfrak{G}\langle P \rangle} F[x]. \quad (1)$$

4.3.1. Lemma. *For any $\langle P \rangle \in \mathbf{Spec}\mathcal{A}$, the decompositions (1) turn the associated to the data $\{\mathfrak{G}, \Theta, \xi\}$ monad, $\mathbb{F} = (F, \xi)$, into an $\mathfrak{G}\langle P \rangle$ -graded monad.*

Proof. Fix an $x \in \mathfrak{G}\langle P \rangle$ and $h \in \mathbf{S}\Theta$. Take an $s \in x$; and let $t \in \mathfrak{G}$ be such an element that the action of Θ_t on $\mathbf{Spec}\mathcal{A}$ coincides with h .

For any $P' \in \mathbf{Spec}\mathcal{A}$ and $u \in \mathfrak{G}$, the inequality $\xi(t, s|u)(P) \neq 0$ implies that $\Theta_t \circ \Theta_s(P')$ is a nonzero subobject of $\Theta_u(P') \in \mathbf{Spec}\mathcal{A}$ (cf. Example 3.2). Hence $\Theta_u(P') \approx \Theta_t \circ \Theta_s(P')$; i.e. $u \in [h \cdot x]$. It follows from the property (b) of the category \mathfrak{B} that the composition of the action $F(h) \circ F[x] \longrightarrow F$ and the projection $F \longrightarrow F[y]$ is zero if $y \neq h \cdot x$. ■

Denote by i' the equivalence class of the element i in $\mathfrak{G}\langle P \rangle$; i.e. i' is the stabilizer of the point $\langle P \rangle$.

4.3.2. Lemma. *The subfunctor $F[i'] \longrightarrow F$ defines a submonad, $\mathfrak{S} = (F[i'], \nu)$, of the monad \mathbb{F} .*

Proof. In fact, for any $s, t \in i'$, and any $u \in \mathfrak{G}$ such that $\xi(s, t|u) \neq 0$, we have: $\Theta_s \circ \Theta_t(P) \approx \Theta_u(P)$. But, $\Theta_s \circ \Theta_t(P) \approx \Theta_s(P) \approx P$; hence $\Theta_u(P) \approx P$. ■

Note that the monad \mathfrak{S} is the skew PBW monad defined by the data $\{i', \Theta_{i'}, \xi_{i'}\}$, where

$\Theta_{i'}$ is the restriction of the map Θ to the subset i' ,
 $\xi_{i'}$ the restriction of the function ξ to $i' \times i' \times i'$.

4.4. Tensor products. Let $\mathbb{F} = (F, \mu)$ and $\mathbb{G} = (G, \nu)$ be monads in a category \mathcal{A} and φ a morphism $\mathbb{G} \longrightarrow \mathbb{F}$. The morphism φ induces the functor

$$\varphi_* : \mathbb{F}\text{-mod} \longrightarrow \mathbb{G}\text{-mod}, \quad (M, m) \longmapsto (M, m \circ \varphi(M)), \quad f \longmapsto f.$$

The functor φ_* has left adjoint, $\mathbb{F} \otimes_{\mathbb{G}}$, which sends a \mathbb{G} -module $\mathfrak{V} = (V, \nu)$ into the coequalizer, M , of the pair of arrows $F \circ \nu, \mu \circ F \circ \varphi(V) : F \circ G(V) \longrightarrow F(V)$

with the action $m : F(M) \longrightarrow M$ being the unique arrow which makes the diagram

$$\begin{array}{ccc} FFG(V) & \xrightarrow{Fc} & F(M) \\ \mu \downarrow & & \downarrow m \\ FG(V) & \xrightarrow{c} & M \end{array}$$

commutative. Here c is the universal morphism.

We need a graded version of these facts. Consider the following data:

an H -set epimorphism $\pi: H \longrightarrow \mathbf{X} = (X, \cdot)$;

a monad $\mathbb{F} = (F, \mu)$; and a decomposition

$$F = \bigoplus_{h \in H} F(h) \quad (1)$$

of the functor F . Set

$$F[x] := \bigoplus_{h \in \pi^{-1}(x)} F(h) \quad (2)$$

for every $x \in X$. The two decompositions, (1) and (2), define a structure of an \mathbf{X} -graded monad on (F, μ) if and only if, for any $s, t, u \in H$, the composition

$$F(s) \circ F(t) \longrightarrow F \circ F \longrightarrow F(u)$$

is zero when $\pi(st) \neq \pi(u)$.

4.4.1. Example. The associated with a point of the spectrum of the category \mathcal{A} graded monad (cf. Example 4.3) is exactly of this kind. ■

4.4.2. The functor \mathfrak{F} . Note that, for any $x \in X$, the component $F[x]$ stands the right action of $F[\iota']$, where $\iota' := \pi(\iota)$; i.e. the composition

$$F[x] \circ F[\iota'] \longrightarrow F \circ F \xrightarrow{\mu} F \longrightarrow F[y]$$

equals to zero if $x \neq y$. In particular, the multiplication μ defines a multiplication, μ' , on $F[\iota']$.

Denote the submonad $(F[\iota'], \mu')$ by \mathfrak{S} , $\mathfrak{S} = (S, \mu')$.

The map which assigns to any \mathbf{X} -graded \mathbb{F} -module (M, m) its component $M[\iota']$ with the induced by m action $m': S(M[\iota']) \longrightarrow M[\iota']$ and to any morphism f of \mathbf{X} -graded \mathbb{F} -modules the morphism $f[\iota']$ is a functor from $\text{gr}_{\mathbf{X}} \mathbb{F}\text{-mod}$ to $\mathfrak{S}\text{-mod}$ which we denote by \mathfrak{F} .

4.4.3. Lemma. For any \mathfrak{S} -module $\mathfrak{V} = (V, \nu)$, the \mathbb{F} -module $\mathbb{F} \otimes_{\mathfrak{S}} \mathfrak{V} = (V', \nu')$ has a

natural X -grading.

Proof. For every $x \in X$, denote by $V'[x]$ the coequalizer of the pair of arrows $\mu[x](V), F[x](\nu) : F[x] \circ S(V) \longrightarrow F[x](V)$.

Here $\mu[x]$ is the induced by μ action $F[x] \circ S \longrightarrow F[x]$.

The commutativity of the diagram (with exact rows)

$$\begin{array}{ccccccc}
 F[x] \circ S(V) & \xrightarrow{\mu[x](V) - F[x](\nu)} & F[x](V) & \longrightarrow & V'[x] & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow i[x] & & \\
 F \circ S(V) & \xrightarrow{\mu(V) - F(\nu)} & F(V) & \longrightarrow & V' & \longrightarrow & 0
 \end{array} \tag{1}$$

implies the existence of unique morphism $i[x]: V'[x] \longrightarrow V$ such that the adjoining of $i[x]$ to the diagram (1) does not disturb its commutativity.

The set of arrows $\{i[x] \mid x \in X\}$ defines a morphism $i: \bigoplus_{x \in X} V'[x] \longrightarrow V'$.

One can see that i is an isomorphism. We claim that i is a structure of an X -graded F -module. This is easily seen from the commutative diagram

$$\begin{array}{ccc}
 F \circ F(V) & \xrightarrow{\quad} & F(V) \\
 \downarrow & \swarrow & \downarrow \\
 & F(h) \circ F[x](V) \longrightarrow F[h \cdot x](V) & \\
 & \downarrow & \downarrow \\
 & F(h)(V'[x]) \longrightarrow V'[h \cdot x] & \\
 \downarrow & \swarrow & \downarrow \\
 F(V') & \xrightarrow{\nu'} & V'
 \end{array}$$

The remaining details are left to the reader. ■

Clearly the map which assigns to any S -module V the X -graded F -module $\text{gr}F \otimes_S V$ extends uniquely to a functor $\text{gr}F \otimes_S : S\text{-mod} \longrightarrow \text{gr}_X F\text{-mod}$.

4.4.4. Proposition. The functor $\text{gr}F \otimes_S$ is left adjoint to the functor

$$\mathfrak{F} : \text{gr}_X F\text{-mod} \longrightarrow S\text{-mod}$$

(cf. 4.4.2).

Proof is left to the reader. ■

4.5. The functor \mathfrak{L} . We continue to work in the setting of 4.4; i.e. we are given: an H -set $X = (X, \cdot)$, an H -set surjection $\pi: H \longrightarrow X$, an X -graded mo-

Let $\mathbb{F} = (F, \mu)$ with the grading $F = \bigoplus_{x \in X} F[x]$ defined by an 'H-decomposition' $F = \bigoplus_{h \in H} F(h)$; i.e. $F[x] := \bigoplus_{h \in \pi^{-1}(x)} F(h)$ for every $x \in X$.

For an \mathbf{X} -graded \mathbb{F} -module (M, m) , consider the family Ω of all \mathbf{X} -graded submodules $(M', m') \longrightarrow (M, m)$ such that $M'[1']$ is zero. Clearly $\text{sup}(\Omega)$ is the largest \mathbf{X} -graded submodule of (M, m) having this property. We denote this submodule by $\mathfrak{Y}(M, m)$. The map $(M, m) \longmapsto \mathfrak{Y}(M, m)$ extends naturally to a functor \mathfrak{Y} which is a subfunctor of the identical functor. Denote the quotient functor Id/\mathfrak{Y} by \mathfrak{L} and the canonical epimorphism $Id \longrightarrow \mathfrak{L}$ by ε .

Consider the full subcategory $\mathfrak{f}\mathfrak{Y}$ of the category $\text{gr}_{\mathbf{X}}\mathbb{F}\text{-mod}$ generated by all modules (M, m) such that $\mathfrak{Y}(M, m) = 0$. Clearly $\mathfrak{Y}(M, m) = 0$ iff $\varepsilon(M, m)$ is an isomorphism.

Another useful property: the restriction of the functor \mathfrak{Y} to the subcategory $\mathfrak{f}\mathfrak{Y}$ is faithful; i.e. if $\mathfrak{Y}(M, m) = 0$ and $\mathfrak{Y}(M, m) = 0$, then $M = 0$.

4.5.1. Proposition. (a) The functor $\mathfrak{L} : \text{gr}_{\mathbf{X}}\mathbb{F}\text{-mod} \longrightarrow \text{gr}_{\mathbf{X}}\mathbb{F}\text{-mod}$ takes values in the subcategory $\mathfrak{f}\mathfrak{Y}$, and its corestriction, \mathfrak{L}' , to $\mathfrak{f}\mathfrak{Y}$ is left adjoint to the embedding $J: \mathfrak{f}\mathfrak{Y} \longrightarrow \text{gr}_{\mathbf{X}}\mathbb{F}\text{-mod}$.

(b) The functor \mathfrak{L} is exact.

Proof. (a) It is clear that $\mathfrak{Y} \circ \mathfrak{L} = 0$ which means that \mathfrak{L} takes values in the subcategory $\mathfrak{f}\mathfrak{Y}$. Denote the corestriction of \mathfrak{L} to $\mathfrak{f}\mathfrak{Y}$ by \mathfrak{L}' . One can see that the canonical epimorphism $\varepsilon: Id_{\text{gr}_{\mathbf{X}}\mathbb{F}\text{-mod}} \longrightarrow \mathfrak{L} = J \circ \mathfrak{L}'$ and its 'inverse', $\varepsilon^{-1}: \mathfrak{L}' \circ J \longrightarrow Id_{\mathfrak{f}\mathfrak{Y}}$, are adjunction arrows.

(b) Given an exact sequence

$$0 \longrightarrow (M', m') \xrightarrow{i} (M, m) \xrightarrow{e} (M'', m'') \longrightarrow 0$$

of \mathbf{X} -graded \mathbb{F} -modules, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (M', m') & \xrightarrow{i} & (M, m) & \xrightarrow{e} & (M'', m'') & \longrightarrow & 0 \\ & & \downarrow l' & & \downarrow l & & \downarrow l'' & & \\ 0 & \longrightarrow & (W, w) & \xrightarrow{g} & \mathfrak{L}(M', m') & \xrightarrow{\mathfrak{L}i} & \mathfrak{L}(M, m) & \xrightarrow{\mathfrak{L}e} & \mathfrak{L}(M'', m'') \longrightarrow 0 \end{array}$$

Here the vertical arrows are the canonical epimorphisms.

Since $l'' \circ e$ is an epimorphism and $l'' \circ e = \mathfrak{L}e \circ l$, $\mathfrak{L}e$ is also an epimorphism.

It follows from the definition of the functor \mathfrak{L} that the canonical arrows $l'[1']$ and $l[1']$ are isomorphisms. Hence $\mathfrak{L}i[1']$ is a monomorphism which im-

plies that $g[1'] = 0$.

On the other hand, if the monoarrow g were nonzero, then, since $\tilde{\mathfrak{F}} \circ \mathfrak{L} = 0$, the morphism $g[1']$ should be nonzero. Therefore g equals to zero; i.e. $\mathfrak{L}i$ is a monoarrow. ■

Denote by \mathfrak{L}'' the composition $\mathfrak{L} \circ \text{grF} \otimes_{\mathfrak{S}} : \mathfrak{S}\text{-mod} \longrightarrow \mathfrak{f}\mathfrak{Y}$ and by $\tilde{\mathfrak{F}}''$ the composition of the functor $\tilde{\mathfrak{F}} : \text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}} \longrightarrow \mathfrak{S}\text{-mod}$ and the inclusion functor $J : \mathfrak{f}\mathfrak{Y} \longrightarrow \text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$.

4.5.2. Proposition. (a) The functor \mathfrak{L}'' is left adjoint to the functor $\tilde{\mathfrak{F}}''$.

(b) The adjunction arrow $\gamma : \text{Id}_{\mathfrak{S}\text{-mod}} \longrightarrow \tilde{\mathfrak{F}}'' \circ \mathfrak{L}''$ is an isomorphism; i.e. the functor \mathfrak{L}'' is fully faithful.

(c) The adjunction morphism $\phi : \mathfrak{L}'' \circ \tilde{\mathfrak{F}}'' \longrightarrow \text{Id}_{\mathfrak{f}\mathfrak{Y}}$ is a monomorphism; i.e. the functor $\tilde{\mathfrak{F}}''$ is faithful.

(d) The functor $\tilde{\mathfrak{F}}''$ is exact.

Proof. (a) By definition, $\mathfrak{L}'' = \mathfrak{L}' \circ \text{grF} \otimes_{\mathfrak{S}}$ and $\tilde{\mathfrak{F}}'' = \tilde{\mathfrak{F}} \circ J$. The functor \mathfrak{L}' is left adjoint to the functor J (cf. Proposition 4.5.1), and the functor $\text{grF} \otimes_{\mathfrak{S}}$ is left adjoint to the functor $\tilde{\mathfrak{F}}$ (cf. Proposition 4.4.4). Therefore \mathfrak{L}'' is left adjoint to the functor $\tilde{\mathfrak{F}}''$.

(b) We have: $\tilde{\mathfrak{F}}'' \circ \mathfrak{L}'' := \tilde{\mathfrak{F}} \circ J \circ \mathfrak{L}' \circ \text{grF} \otimes_{\mathfrak{S}} = \tilde{\mathfrak{F}} \circ \mathfrak{L} \circ \text{grF} \otimes_{\mathfrak{S}}$.

It follows from the definition of the functors $\tilde{\mathfrak{F}}$, $\text{grF} \otimes_{\mathfrak{S}}$ and \mathfrak{L} that the composition, γ , of the adjunction morphism $\eta : \text{Id}_{\mathfrak{S}\text{-mod}} \longrightarrow \tilde{\mathfrak{F}} \circ \text{grF} \otimes_{\mathfrak{S}}$ and the epimorphism $\tilde{\mathfrak{F}}(\varepsilon) : \tilde{\mathfrak{F}} \circ \text{grF} \otimes_{\mathfrak{S}} \longrightarrow \tilde{\mathfrak{F}} \circ \mathfrak{L} \circ \text{grF} \otimes_{\mathfrak{S}}$ is an isomorphism. One can see that γ is the adjunction arrow.

(c) The monomorphness of the second adjunction arrow,

$$\phi : \mathfrak{L}'' \circ \tilde{\mathfrak{F}}'' := \mathfrak{L}' \circ \text{grF} \otimes_{\mathfrak{S}} \circ \tilde{\mathfrak{F}} \circ J \longrightarrow \text{Id}_{\mathfrak{f}\mathfrak{Y}},$$

is equivalent to) the faithfulness of the functor $\tilde{\mathfrak{F}}'' := \tilde{\mathfrak{F}} \circ J$.

(d) The functor $\tilde{\mathfrak{F}}''$, being a right adjoint functor to \mathfrak{L}'' , is left exact. So, it remains to prove that it is right exact. Actually,

an arrow $f : \mathfrak{M} \longrightarrow \mathfrak{M}'$ of the category $\mathfrak{f}\mathfrak{Y}$ is an epimorphism if and only if $\tilde{\mathfrak{F}}f$ is an epimorphism.

Since the functor \mathfrak{L} is left adjoint to the embedding $\mathfrak{f}\mathfrak{Y} \longrightarrow \text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$, the object $\mathfrak{L}(\mathfrak{M}\mathfrak{M}')$ is a cokernel of the arrow f in $\mathfrak{f}\mathfrak{Y}$. Clearly $\tilde{\mathfrak{F}}f$ is an epimorphism if and only if $\mathfrak{L}(\mathfrak{M}\mathfrak{M}') = 0$. In other words, f is an epimorphism if and only if $\tilde{\mathfrak{F}}f$ is an epimorphism. ■

5. THE SPECTRUM OF THE CATEGORY OF X-GRADED MODULES.

First we need a couple of auxiliary facts.

5.1. Lemma. *Let $G: \mathcal{B} \longrightarrow \mathcal{E}$ be a fully faithful functor which has a faithful right adjoint functor, $G^\wedge: \mathcal{E} \longrightarrow \mathcal{B}$.*

Then the functor G respects monomorphisms.

Proof. Let $\iota: V \longrightarrow W$ be a monoarrow in \mathcal{B} ; and let $g, h: M \longrightarrow G(V)$ be arrows such that $G\iota \circ g = G\iota \circ h$. This implies that

$$G^\wedge G\iota \circ G^\wedge g = G^\wedge G\iota \circ G^\wedge h. \quad (1)$$

Since the adjunction arrow $\delta: Id_{\mathcal{B}} \longrightarrow G^\wedge \circ G$ is an isomorphism, the monomorphness of ι implies that of $G^\wedge G\iota$. Therefore it follows from the equality (1) that $G^\wedge h = G^\wedge g$. Since the functor G^\wedge is faithful, the equality $G^\wedge h = G^\wedge g$ means that $g = h$. This shows that $G\iota$ is a monoarrow. ■

5.2. Proposition. *Let $G: \mathcal{B} \longrightarrow \mathcal{E}$ be a fully faithful functor which has a faithful right adjoint functor, $G^\wedge: \mathcal{E} \longrightarrow \mathcal{B}$. Then the functor G induces a continuous injection $\mathbf{Sp}G: (\mathbf{Spec}\mathcal{B}, \tau) \longrightarrow (\mathbf{Spec}\mathcal{E}, \tau)$.*

If the functor G^\wedge is right exact, then the map $\mathbf{Sp}G$ is a homeomorphism.

Proof. Fix some $P \in \mathbf{Spec}\mathcal{B}$ and a nonzero monomorphism $\iota: M \longrightarrow G(P)$.

Consider the diagram

$$G^\wedge(M) \xrightarrow{G^\wedge \iota} G^\wedge \circ G(P) \xleftarrow{\delta(P)} P,$$

where $\delta: Id_{\mathcal{B}} \longrightarrow G^\wedge \circ G$ is the adjunction arrow.

Since the functor G is fully faithful, the adjunction arrow δ is an isomorphism. Since the functor G^\wedge is left exact (as any functor which has a left adjoint), the arrow $G^\wedge \iota$ is a monomorphism. Thanks to the faithfulness of G^\wedge , the monomorphism $G^\wedge \iota$ is nonzero.

Thus, we have a nonzero monoarrow $\delta(P)^{-1} \circ G^\wedge \iota: G^\wedge(M) \longrightarrow P$.

Since $P \in \mathbf{Spec}\mathcal{B}$, there is a diagram

$$({}_n)G^\wedge(M) \xleftarrow{i} K \xrightarrow{e} P \quad (1)$$

where i is a monoarrow and e is an epimorphism.

Consider the diagram

$$({}_n)M \xleftarrow{\varepsilon({}_n)M} G \circ G^\wedge({}_n)M \simeq G({}_n)G^\wedge(M) \xleftarrow{Gi} G(K) \xrightarrow{Ge} G(P),$$

where ε is the adjunction arrow. The faithfulness of the functor G^\wedge means exactly that ε is a monomorphism.

By Lemma 5.1, the functor G respects monoarrows, and it is right exact as

any functor which has a right adjoint functor. Therefore G_i is a monomorphism and G_e is an epimorphism.

All together shows that $G(P) \in \text{Spec}\mathfrak{C}$.

The exactness of the functor G implies that it respects the preorder \succ . Hence G induces a morphism $\mathbf{Sp}G: \mathbf{Spec}\mathfrak{B} \longrightarrow \mathbf{Spec}\mathfrak{C}$ of ordered sets. It follows from the definition of the topology τ that $\mathbf{Sp}G$ is a continuous map from $(\mathbf{Spec}\mathfrak{B}, \tau)$ to $(\mathbf{Spec}\mathfrak{C}, \tau)$.

Since $\delta(P): P \longrightarrow G^\wedge \circ G(P)$ is an isomorphism, it is clear that the functor G^\wedge induces a map $\text{Im}(\mathbf{Sp}G) \longrightarrow \mathbf{Spec}\mathfrak{B}$ which is inverse to $\mathbf{Sp}G$ (or, rather, to the corestriction of $\mathbf{Sp}G$ to its image). In particular, the map $\mathbf{Sp}G$ is injective.

(b) Suppose now that the functor G^\wedge is right exact.

Let $V \in \text{Spec}\mathfrak{C}$; and let $l: L \longrightarrow G^\wedge(V)$ be an arbitrary nonzero monomorphism. In the diagram

$$G(L) \xrightarrow{Gl} G \circ G^\wedge(V) \xrightarrow{\varepsilon(V)} V,$$

both $\varepsilon(V)$ and Gl are monoarrows; and $Gl \neq 0$, since G is a faithful functor. Therefore, since V belongs to the spectrum of \mathfrak{C} , there exists a diagram

$$({}_n)G(L) \xleftarrow{i} W \xrightarrow{e} V \quad (2)$$

where i is a monomorphism and e is an epimorphism. Consider the diagram

$$({}_n)L \xleftarrow{\delta^{-1}} G^\wedge \circ G({}_n)L \simeq G^\wedge({}_n)G(L) \xleftarrow{G^\wedge i} G^\wedge(W) \xrightarrow{G^\wedge e} G^\wedge(V).$$

Thanks to the right exactness of the functor G^\wedge , the arrow $G^\wedge e$ is an epimorphism, and the diagram above means that $L \succ G^\wedge(V)$. Thus, $\langle G^\wedge(V) \rangle$ is a point of $\mathbf{Spec}\mathfrak{B}$. The exactness of the functor G^\wedge implies that it respects the preorder \succ . In particular, G^\wedge induces a morphism, $\mathbf{Sp}G^\wedge: \mathbf{Spec}\mathfrak{C} \longrightarrow \mathbf{Spec}\mathfrak{B}$, of ordered sets. This implies that the map $\mathbf{Sp}G^\wedge$ is continuous with respect to the topology τ .

The map $\mathbf{Sp}G^\wedge$ is injective.

In fact, for any $V \in \text{Spec}\mathfrak{C}$, we have a nonzero monoarrow

$$\varepsilon(V): G \circ G^\wedge(V) \longrightarrow V$$

which means that $\langle G \circ G^\wedge(V) \rangle = \langle V \rangle$; i.e. the functor G induces the map from $\text{Im}(\mathbf{Sp}G^\wedge)$ to $\mathbf{Spec}\mathfrak{C}$ which is inverse to $\mathbf{Sp}G^\wedge$.

This shows that $\mathbf{Sp}G^\wedge$ is the inverse to $\mathbf{Sp}G$ map. ■

5.3. Remark. Note that if, under the conditions of Proposition 5.2, the category

\mathcal{E} is abelian (or, more generally, if any bimorphism (i.e. mono- and epimorphism) in \mathcal{E} is an isomorphism), then the functor G is an equivalence of categories (which implies immediately Proposition 5.2).

First note that, since G is fully faithful, the functor G^\wedge is a localization by Proposition 1.1.3 in [GZ].

If $s \in \text{Hom}\mathcal{E}$ is such that $G^\wedge s$ is invertible, then, thanks to the faithfulness of G^\wedge , the arrow s is a bimorphism. Therefore, by assumption, s is an isomorphism. By the universal property of localizations, this implies that G^\wedge (hence G) is an equivalence of categories. ■

5.4. The spectrum of the category $\text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$. Return now to the setting of 4.4. Note that

$$\text{Specgr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}} = \text{SpecKer}\mathfrak{F} \cup \text{Specf}\mathfrak{F}.$$

In fact, since the functor \mathfrak{F} is exact, $\text{Ker}\mathfrak{F}$ is a thick subcategory of $\text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$. Therefore, if a module \mathbb{M} belongs to $\text{Specgr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$, and \mathbb{M} has a nonzero subobject from $\text{Ker}\mathfrak{F}$, then \mathbb{M} is in $\text{Ker}\mathfrak{F}$.

This shows that an object from $\text{Specgr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$ is either $\text{Ker}\mathfrak{F}$ -torsion free, or belongs to $\text{Ker}\mathfrak{F}$. By definition, $f\mathfrak{F}$ is the subcategory of $\text{Ker}\mathfrak{F}$ -torsion free modules.

Thus, the study of the spectrum of $\text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$ splits in two parts, accordingly with the decomposition (1). By the reason which shall become clear later, we are interested in the description of $\text{Specf}\mathfrak{F}$ much more than in the description of $\text{SpecKer}\mathfrak{F}$.

5.4.1. Theorem. The functor $\mathfrak{L}'' := \mathfrak{L}' \circ \text{gr}\mathbb{F} \otimes_{\mathfrak{S}} : \mathfrak{S}\text{-mod} \longrightarrow f\mathfrak{F}$ induces a homeomorphism $(\text{Spec}\mathfrak{S}\text{-mod}, \tau) \longrightarrow (\text{Specf}\mathfrak{F}, \tau)$.

Proof. 1) According to Proposition 4.5.2, the functor

$$\mathfrak{L}'' := \mathfrak{L}' \circ \text{gr}\mathbb{F} \otimes_{\mathfrak{S}} : \mathfrak{S}\text{-mod} \longrightarrow f\mathfrak{F}$$

is fully faithful and has a faithful and exact right adjoint \mathfrak{F}'' which is equal to the composition of the embedding $J: f\mathfrak{F} \longrightarrow \text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}}$ and the functor

$$\mathfrak{F} : \text{gr}_{\mathbf{X}}^{\mathbb{F}\text{-mod}} \longrightarrow \mathfrak{S}\text{-mod}.$$

The assertion follows now from Proposition 5.2. ■

6. THE SPECTRUM OF THE CATEGORY OF MODULES OVER A PBW MONAD.

Now fix a data $(\mathfrak{G}, \Theta, \xi)$ and consider the corresponding to this data monad

$\mathbb{F} = (F, \xi)$ (cf. 2.5). Denote by $\mathcal{A}\{\Theta, \xi\}$ the category $\mathbb{F}\text{-mod}$.

For any element $\langle P \rangle \in \mathbf{Spec} \mathcal{A}$, denote by $\mathbf{Spec}_{\langle P \rangle} \mathcal{A}\{\Theta, \xi\}$ the subset of all $\langle (M, m) \rangle \in \mathbf{Spec}_{\langle P \rangle} \mathcal{A}\{\Theta, \xi\}$ such that $\langle P \rangle \in \text{Ass}(M)$.

Our goal is to describe $\mathbf{Spec}_{\langle P \rangle} \mathcal{A}\{\Theta, \xi\}$ for all $\langle P \rangle \in \mathbf{Spec} \mathcal{A}$.

6.1. The case of a local category. Let the category \mathcal{A} be local, and let P be a quasi-final object of \mathcal{A} . This implies that $\langle P \rangle$ is $\Theta(s)$ -stable, i.e.

$$\langle \Theta(s)P \rangle = \langle P \rangle, \text{ for any } s \in \mathfrak{G}.$$

It follows from Lemma 3.4 that, for any triple $s, t, u \in \mathfrak{G}$, either $\xi(s, t|u)(P) = 0$, or $\xi(s, t|u)$ is an isomorphism.

Thus, the data $(\mathfrak{G}, \Theta, \xi)$ induces the same kind of data on the thick subcategory $\mathcal{A}(\langle P \rangle)$ which allows us to replace the category \mathcal{A} by its subcategory $\mathcal{A}(\langle P \rangle)$. In other words, we assume that the spectrum of \mathcal{A} consists of a single point $\langle P \rangle$.

Now fix an object $\langle (M, m) \rangle \in \mathbf{Spec}_{\langle P \rangle} \mathcal{A}\{\Theta, \xi\}$.

Note that M is equivalent to its submodule (M', m') such that $M' = \sum_{s \in Y} P(s)$ where Y is a subset of \mathfrak{G} and $\langle P(s) \rangle = \langle P \rangle$ for any $s \in Y$.

In fact, by assumption, there is a monomorphism $\iota: P \longrightarrow M$. Take the adjoint \mathbb{F} -module morphism $\iota^\wedge: \mathbb{F}(P) := (\bigoplus_{t \in \mathfrak{G}} \Theta(t)(P), \xi(P)) \longrightarrow (M, m)$ (which is uniquely defined by the fact that its composition with $P \longrightarrow \bigoplus_{t \in \mathfrak{G}} \Theta(t)(P)$ coincides with ι).

Since $(M, m) \in \mathbf{Spec} \mathcal{A}\{\mathfrak{G}, \xi\}$, it is equivalent to the image of the morphism ι^\wedge which we denote by (M', m') . Clearly M' is the sum of images of $\Theta(t)(P)$. Since P is a quasi-final object, $\Theta(t)(P) \approx P$ for every t , and any nonzero image of $\Theta(t)(P)$ is equivalent to P .

Suppose that the category \mathcal{A} has objects of finite type. Then any \mathbb{F} -module from $\mathbf{Spec} \mathcal{A}\{\Theta, \xi\}$ is equivalent to an \mathbb{F} -module (M', m') , such that the object M' of the category \mathcal{A} is semisimple.

6.2. Stable case. Suppose that $P \approx \Theta(s)(P)$ for any $s \in \mathfrak{G}$; or, equivalently, the Serre subcategory $\langle P \rangle$ is $\Theta(s)$ -stable for all s . Hence each $\Theta(t)$, $t \in \mathfrak{G}$, induces an auto-equivalence, $\Theta'(t)$, of the quotient category $\mathcal{A}' = \mathcal{A}/\langle P \rangle$; i.e. the map Θ induces a map $\Theta': \mathfrak{G} \longrightarrow \mathbf{Pic} \mathcal{A}'$.

To the monad $\mathbb{F} = (F, \xi)$, $F = \bigoplus_{t \in \mathfrak{G}} \Theta(t)$, there corresponds a monad $\mathbb{F}' = (F', \xi')$ defined as follows: $F' := \bigoplus_{t' \in \mathfrak{G}'} \Theta'(s')$, and the multiplication is induced by ξ .

Denote by P' the image of P in \mathcal{A}' which is the unique up to equivalence quasi-final object of \mathcal{A}' . Clearly the localization at $\langle P \rangle$ provides an injective map $\text{Spec}_{\langle P \rangle} \mathcal{A}(\Theta, \xi) \longrightarrow \text{Spec}_{\langle P' \rangle} \mathcal{A}'(\Theta', \xi')$.

Thus, if \mathbf{P} is $\Theta(s)$ -stable for all $s \in \mathcal{G}$, the description of $\text{Spec}_{\mathbf{P}} \mathcal{A}(\Theta, \xi)$ is reduced to the case of a local 'base' category.

6.3. A general construction. Let $G: \mathcal{B} \longrightarrow \mathcal{E}$ be a functor which has an exact right adjoint functor, G^\wedge .

Suppose that \mathcal{B} and \mathcal{E} are Grothendieck categories; and let \mathbb{T} be a Serre subcategory of the category \mathcal{B} . Since the functor G^\wedge is exact, the pre-image $G^{\wedge^{-1}}(\mathbb{T})$ of \mathbb{T} is a Serre subcategory of the category \mathcal{E} .

Let $f\mathbb{T}$ denote the full subcategory of the category \mathcal{B} generated by \mathbb{T} -torsion free objects. The embedding

$$J = J_{f\mathbb{T}}: f\mathbb{T} \longrightarrow \mathcal{B}$$

has a left adjoint functor, ${}^\wedge J_{f\mathbb{T}}$, which assigns to any object M of \mathcal{B} the quotient of M by its \mathbb{T} -torsion.

Thus, we have the following commutative diagram of functors:

$$\begin{array}{ccccc} f\mathbb{T} & \xrightarrow{\tilde{\gamma}} & f\mathbb{T}' & \xrightarrow{\tilde{\gamma}'} & f\mathbb{T} \\ J_{f\mathbb{T}} \searrow & & {}^\wedge J_{f\mathbb{T}'} \nearrow & & \searrow J_{f\mathbb{T}'} \\ \mathcal{B} & \xrightarrow{G} & \mathcal{E} & \xrightarrow{G^\wedge} & \mathcal{B} \\ & & & & \nearrow {}^\wedge J_{f\mathbb{T}} \end{array}$$

Here \mathbb{T}' denotes the subcategory $G^{\wedge^{-1}}(\mathbb{T})$.

Note that the functor ${}^\wedge J_{f\mathbb{T}'} \circ G := G_{\mathbb{T}}$ is left adjoint to the functor $G^\wedge \circ J_{f\mathbb{T}'} := G_{\mathbb{T}}^\wedge$. So, we have a canonical arrow

$$\lambda: Id_{f\mathbb{T}} \longrightarrow \tilde{\gamma}' \circ \tilde{\gamma}$$

which is equal to the composition ${}^\wedge J_{f\mathbb{T}'} \circ J_{f\mathbb{T}} \circ \delta^{-1}$, where

$$\gamma: Id_{f\mathbb{T}'} \longrightarrow G_{\mathbb{T}}^\wedge \circ G_{\mathbb{T}} \quad \text{and} \quad \delta: {}^\wedge J_{f\mathbb{T}'} \circ J_{f\mathbb{T}} \longrightarrow Id_{f\mathbb{T}}$$

are adjunction morphisms.

As to the composition $\tilde{\gamma} \circ \tilde{\gamma}'$, we have the diagram:

$$\tilde{\gamma} \circ \tilde{\gamma}' = G_{\mathbb{T}} \circ (J_{f\mathbb{T}} \circ {}^\wedge J_{f\mathbb{T}'}) G_{\mathbb{T}}^\wedge \longleftarrow G_{\mathbb{T}} \circ G_{\mathbb{T}}^\wedge \longrightarrow Id_{f\mathbb{T}'}$$

Here the right arrow is the adjunction morphism, the left arrow is the morphism $G_{\mathbb{T}} \circ \varepsilon_{\mathbb{T}} \circ G_{\mathbb{T}}^\wedge$, where $\varepsilon_{\mathbb{T}}: Id_{\mathcal{B}} \longrightarrow J_{f\mathbb{T}} \circ {}^\wedge J_{f\mathbb{T}}$ is the adjunction arrow.

The functor $\tilde{\gamma}'$ is faithful.

In fact, let $f: M \longrightarrow L$ be a morphism in $f\mathbb{T}'$ such that $\tilde{\gamma}' f := {}^\wedge J_{f\mathbb{T}'} \circ G^\wedge f = 0$. By definition, this means that the image of $G^\wedge f$ is a subobject of the

\mathbb{T} -torsion of $G^\wedge(L)$; or, equivalently, $im(f)$ belongs to the subcategory \mathbb{T}' . Since L is assumed to be \mathbb{T}' -torsion free, $im(f) = 0$. ■

6.3. Canonical gradings. Fix an element $\langle P \rangle$ of $\mathbf{Spec} \mathcal{A}$. Let $\mathfrak{S} = \mathfrak{S}\langle P \rangle$ be the corresponding submonad of \mathbb{F} (cf. 4.3.2).

Denote by $\mathfrak{S}\langle P \rangle\text{-mod}\langle P \rangle$ the full subcategory of $\mathfrak{S}\langle P \rangle\text{-mod}$ formed by all $\mathfrak{S}\langle P \rangle$ -modules (V, υ) such that $Supp(V) \subseteq Supp(P)$. And let $\mathbb{F}\text{-mod}\langle \Theta | P \rangle$ be the full subcategory of the category $\mathbb{F}\text{-mod}$ formed by all \mathbb{F} -modules (M, m) such that

$$Supp(M) \subseteq \bigcup_{s \in \mathfrak{S}} Supp(\Theta(s)(P)).$$

Clearly the functor $\mathbb{F} \otimes_{\mathfrak{S}}$ induces a functor

$$\mathbb{F} \otimes_{\mathfrak{S}} |_{\langle P \rangle} : \mathfrak{S}\langle P \rangle\text{-mod}\langle P \rangle \longrightarrow \mathbb{F}\text{-mod}\langle \Theta | P \rangle.$$

6.3.1. Lemma. *The functor $\mathbb{F} \otimes_{\mathfrak{S}} |_{\langle P \rangle}$ has a right adjoint functor, and this right adjoint functor is exact.*

Proof. For any Serre subcategory \mathfrak{X} of the category \mathcal{A} , the embedding $J_{\mathfrak{X}} : \mathfrak{X} \longrightarrow \mathcal{A}$ has a right adjoint functor, $J_{\mathfrak{X}}^\wedge$, which assigns to any object of \mathcal{A} its \mathfrak{X} -torsion. In our case, when \mathfrak{X} coincides with $\mathfrak{S}\langle P \rangle\text{-mod}\langle P \rangle$,

Since the forgetting functor $|_{\mathfrak{S}} : \mathbb{F}\text{-mod} \longrightarrow \mathfrak{S}\langle P \rangle\text{-mod}$ is right adjoint to $\mathbb{F} \otimes_{\mathfrak{S}}$, the restriction to $\mathbb{F}\text{-mod}\langle \Theta | P \rangle$ of the composition $J_{\mathfrak{X}}^\wedge \circ |_{\mathfrak{S}}$, where \mathfrak{X} is $\mathfrak{S}\langle P \rangle\text{-mod}\langle P \rangle$, is a right adjoint functor to the functor $\mathbb{F} \otimes_{\mathfrak{S}} |_{\langle P \rangle}$. Thanks to the exactness of the functors $|_{\mathfrak{S}}$ and $J_{\mathfrak{X}}^\wedge$, the functor $\mathbb{F} \otimes_{\mathfrak{S}} |_{\langle P \rangle}^\wedge$ is exact. ■

Now we apply the construction of 6.3 to

$$\mathcal{B} := \mathfrak{S}\langle P \rangle\text{-mod}\langle P \rangle, \quad \mathcal{C} := \mathbb{F}\text{-mod}\langle \Theta | P \rangle, \quad G := \mathbb{F} \otimes_{\mathfrak{S}} |_{\langle P \rangle}, \quad \text{and} \quad \mathbb{T} := \langle P \rangle.$$

Let $\langle P \rangle\mathbb{F}$ (resp. $\langle P \rangle\mathfrak{S}$) denote the full subcategory of the category $\mathcal{C} := \mathbb{F}\text{-mod}\langle \Theta | P \rangle$ (resp. of the category $\mathcal{B} := \mathfrak{S}\text{-mod}\langle P \rangle$) generated by those modules (M, m) for which M is an object of the subcategory $\langle P \rangle$. We have:

$$\mathbf{Spec} \mathbb{F}\text{-mod} = \mathbf{Spec} \langle P \rangle\mathbb{F} \cup \mathbf{Spec} \mathfrak{f}\langle P \rangle\mathbb{F},$$

and

$$\mathbf{Spec} \mathfrak{S} = \mathbf{Spec} \langle P \rangle\mathfrak{S} \cup \mathbf{Spec} \mathfrak{f}\langle P \rangle\mathfrak{S},$$

where $\mathfrak{f}\langle P \rangle\mathbb{F}$ is the full subcategory of the category $\mathbb{F}\text{-mod}\langle \Theta | P \rangle$ generated by all $\langle P \rangle\mathbb{F}$ -torsion free modules, and $\mathfrak{f}\langle P \rangle\mathfrak{S}$ is the subcategory of $\langle P \rangle\mathfrak{S}$ -torsion free modules. Clearly

$$\mathbf{Spec}_{\langle P \rangle} \mathbb{F}\text{-mod} \subseteq \mathbf{Spec} \mathfrak{f}\langle P \rangle\mathbb{F} \quad \text{and} \quad \mathbf{Spec}_{\langle P \rangle} \mathfrak{S}\text{-mod} \subseteq \mathbf{Spec} \langle P \rangle\mathfrak{S}.$$

There is the following commutative diagram

$$\begin{array}{ccc}
 & \text{grf}\mathbb{F}\langle P \rangle & \xrightarrow{\Phi} & \mathbb{F}\langle P \rangle \\
 \mathfrak{g} \nearrow & & \searrow \mathfrak{g}^\wedge & \\
 \mathbb{F}\mathfrak{S}\langle P \rangle & & \mathbb{F}\mathfrak{S}\langle P \rangle & \xleftarrow{G^\wedge}
 \end{array}$$

where the functor \mathfrak{g} is fully faithful, and its right adjoint functor, \mathfrak{g}^\wedge , is faithful and exact.

6.3.2. Lemma. Suppose that $\langle P \rangle \in \text{Spec}\mathcal{A}$ is such that

$$\langle \Theta(s)(P) \rangle^- \cap \langle \Theta(t)(P) \rangle^- \neq \emptyset \text{ iff } s \approx t$$

(for instance, $\langle P \rangle$ is a closed point). Then the functor

$$\Phi: \text{grf}\mathbb{F}\langle P \rangle \longrightarrow \mathbb{F}\langle P \rangle$$

is an equivalence of categories.

Proof. This is a special case of the following fact:

Let $\text{Supp}(M) = \bigcup_{W \in \Omega} W$ be a disjoint union of closed (in the topology τ) subsets. Then $M = \bigoplus_{W \in \Omega} M(W)$, where $M(W)$ is the $\mathcal{A}(W)$ -torsion of the object M .

Indeed, under the conditions, $\text{Supp}(M(\bigoplus_{W \in \Omega} M(W))) = \emptyset$ which means that the quotient module $M(\bigoplus_{W \in \Omega} M(W))$ is zero. ■

6.5. The general case. Let now $\langle P \rangle$ be an arbitrary element of the spectrum of the category \mathcal{A} . We cannot maintain any more that all modules from $\mathbb{F}\text{-mod}\langle \Theta | P \rangle$ are (canonically) \mathbf{X} -graded. However, as we shall see, it is still true for elements of $\text{Spec}_{\langle P \rangle} \mathbb{F}\text{-mod}$.

6.5.1. Lemma. For every $\mathbb{V} \in \text{Spec}_{\langle P \rangle} \mathfrak{S}\text{-mod}$, the $\mathbb{F}\langle P \rangle$ -torsion of $\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V}$ is an \mathbf{X} -graded submodule.

Proof. (a) Fix an \mathfrak{S} -module $\mathbb{V} = (V, \nu)$ from $\text{Spec}_{\langle P \rangle} \mathfrak{S}\text{-mod}$; and take the \mathbb{F} -module $\mathbb{F}\mathbb{V} := (F(V), \mu(V))$ with the canonical $\mathfrak{G}(P)$ -grading: $F(V) = \bigoplus_{x \in X} F[x](V)$.

Consider $\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V} = (|\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V}|, \nu')$. One can see that $|\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V}| = \bigoplus_{x \in X} \mathbb{V}[x]$, where $\mathbb{V}[x] \approx F[x] \otimes_{\mathfrak{S}} \mathbb{V}$ for any $x \in X$. Thus, we have a canonical \mathbf{X} -grading on $\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V}$ such that the natural epimorphism

$$e: \mathbb{F}\mathbb{V} \longrightarrow \mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V} \quad (1)$$

is a morphism of \mathbf{X} -graded \mathbb{F} -modules.

(b) Note that the $\langle P \rangle$ -torsion of $F(V)$ coincides with $\bigoplus_{x \in X-[1]} F[x](V)$.

For any $t \in \mathfrak{G}$, denote by $V(t)$ the pullback of the pair of arrows

$$\langle P \rangle F(V) \longrightarrow \Theta(t)^\wedge F(V) \longleftarrow \Theta(t)^\wedge (\langle P \rangle F(V)),$$

where the left arrow is the adjoint to the action $\Theta(t)(\langle P \rangle F(V)) \longrightarrow \langle P \rangle F(V)$ morphism, and the right arrow is $\Theta(t)^\wedge$ of the embedding $\langle P \rangle F(V) \longrightarrow F(V)$.

Clearly $V(t)$ is an \mathbf{X} -graded subobject of $F(V)$; hence the intersection

$\bigcap_{t \in \mathfrak{G}} V(t)$ is an \mathbf{X} -graded subobject of $F(V)$. But, this intersection coincides with the $\mathbb{F}\langle P \rangle$ -torsion of $\mathbb{F}V$.

(c) The functor which assigns to an \mathbb{F} -module its $\mathbb{F}\langle P \rangle$ -torsion is (right) exact. In particular, it sends the epimorphism (1) into the epimorphism of $\mathbb{F}\langle P \rangle$ -torsions; i.e. the $\mathbb{F}\langle P \rangle$ -torsion of $\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V}$ is the image of the $\mathbb{F}\langle P \rangle$ -torsion of $\mathbb{F}V$; therefore it is an \mathbf{X} -graded submodule of $\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V}$. ■

6.5.2. Corollary. For any $\mathbb{V} \in \text{Spec}_{\langle P \rangle} \mathfrak{S}\text{-mod}$, the \mathbb{F} -torsion free quotient module $\mathbb{F}\langle P \rangle(\mathbb{F} \otimes_{\mathfrak{S}} \mathbb{V})$ has a canonical \mathbf{X} -grading.

6.6. The spectrum of $\mathbb{F}\text{-mod}/\mathcal{A}$. The following two theorems give a complete description of the spectrum of the forgetting functor $\mathbb{F}\text{-mod} \longrightarrow \mathcal{A}$.

6.6.1. Theorem. For every $\langle P \rangle \in \text{Spec} \mathcal{A}$ such that

$$\text{Supp}(\Theta(t)(P)) \cap \text{Supp}(\Theta(s)(P)) \neq \emptyset \text{ iff } \Theta(s)(P) \approx \Theta(t)(P),$$

the functor $\wedge_{\mathbb{F}\langle P \rangle}^{\circ} \mathbb{F} \otimes_{\mathfrak{S}}$ induces a homeomorphism

$$(\text{Spec} \mathfrak{S}\text{-mod}_{\langle P \rangle}, \tau) \longrightarrow (\text{Spec} \mathbb{F}\text{-mod}_{\langle \Theta | P \rangle}, \tau).$$

The functor $\wedge_{\mathbb{F}\langle P \rangle}^{\circ} \mathbb{F} \otimes_{\mathfrak{S}}$ induces a bijection between the sets of isomorphism classes of simple objects of the category $\mathfrak{S}\text{-mod}_{\langle P \rangle}$ and the category $\mathbb{F}\text{-mod}_{\langle \Theta | P \rangle}$.

6.6.2. Note. Clearly if an element $\langle P \rangle$ of $\text{Spec} \mathcal{A}$ satisfies the condition of Theorem 6.6.1, then this condition holds for all specializations of $\langle P \rangle$. ■

6.6.3. Theorem. For every $\langle P \rangle \in \text{Spec} \mathcal{A}$ the functor $\wedge_{\mathbb{F}\langle P \rangle}^{\circ} \mathbb{F} \otimes_{\mathfrak{S}}$ induces a homeomorphism $(\text{Spec}_{\langle P \rangle} \mathfrak{S}\text{-mod}, \tau) \longrightarrow (\text{Spec}_{\langle P \rangle} \mathbb{F}\text{-mod}, \tau)$.

The functor $\wedge_{\mathbb{F}\langle P \rangle}^{\circ} \mathbb{F} \otimes_{\mathfrak{S}}$ sends simple objects into simple objects. In particular, it induces a bijection

$$\text{Simple}_{\langle P \rangle} \mathfrak{S}\text{-mod} \longrightarrow \text{Simple}_{\langle P \rangle} \mathbb{F}\text{-mod},$$

where

$$\mathbf{Simple}_{\langle P \rangle}^{\mathbb{F}\text{-mod}} := \mathbf{Spec}_{\langle P \rangle}^{\mathbb{F}\text{-mod}} \cap \mathbf{Simple}^{\mathbb{F}\text{-mod}}$$

and

$$\mathbf{Simple}_{\langle P \rangle}^{\mathbb{S}\text{-mod}} := \mathbf{Spec}_{\langle P \rangle}^{\mathbb{S}\text{-mod}} \cap \mathbf{Simple}^{\mathbb{S}\text{-mod}}.$$

As usual, $\mathbf{Simple}^{\mathcal{B}}$ denotes the set of isomorphism classes of simple objects of the category \mathcal{B} .

Proofs. Theorem 6.6.1 follows from Theorem 5.4.1 and Lemma 6.4.2. Theorem 6.6.3. is a direct consequence of the same Theorem 5.4.1 and Corollary 6.5.2. ■

6.7. The case of PBW rings. For the readers' convenience, we will translate Theorem 6.6.3 to the language of rings and modules.

Fix a PBW ring $\mathcal{R} := R/\langle \vartheta, \xi \rangle = (\bigoplus_{s \in \mathcal{S}} x_s R, \xi)$, where ξ is the multiplication given by the table:

$$x_s x_t = \sum_{u \in \mathcal{S}} x_u \xi(s, t | u) \quad (1)$$

for all $s, t \in \mathcal{S}$, and

$$rx_s = \vartheta_s(r)x_s \quad (2)$$

for all $r \in R$ and $s \in \mathcal{S}$ (cf. 2.4).

Fix $\mathbf{P} \in \mathbf{Spec}(R\text{-mod})$. By abuse the language, we shall call *the stabilizer of \mathbf{P} in $R/\langle \vartheta, \xi \rangle$* the ring $\mathcal{S}_{\mathbf{P}} = \mathcal{S}_{\mathbf{P}}/\langle \vartheta, \xi \rangle := (\bigoplus_{s \in \mathcal{S}_{\mathbf{P}}} x_s R, \xi')$, where $\mathcal{S}_{\mathbf{P}}$ is the set $\{s \in \mathcal{S} \mid \vartheta_s \mathbf{P} = \mathbf{P}\}$, ξ' is induced by ξ ; i.e. the multiplication in $\mathcal{S}_{\mathbf{P}}$ is given by

$$x_s \cdot x_t = \sum_{u \in \mathcal{S}_{\mathbf{P}}} x_u \xi(s, t | u) \quad (3)$$

and (2) with \mathcal{S} replaced by $\mathcal{S}_{\mathbf{P}}$.

Note that $\mathcal{S}_{\mathbf{P}}$ is usually not a subring in $R/\langle \vartheta, \xi \rangle$. Denote the minimal subring in $R/\langle \vartheta, \xi \rangle$ which contains $\mathcal{S}_{\mathbf{P}}$ by $\mathbf{S}_{\mathbf{P}}$.

Consider the full subcategory $\sim \mathcal{K}_{\mathbf{P}}$ of the category $R\text{-mod}$ generated by modules M such that any nonzero cyclic submodule X in M belongs to the spectrum and $\langle X \rangle = \mathbf{P}$. And let $\mathcal{S}_{\mathbf{P}}\text{-mod}^{\mathbf{P}}$ is the full subcategory of the category $\mathcal{S}_{\mathbf{P}}\text{-mod}$ generated by modules (M, m) such that $M \in \sim \mathcal{K}_{\mathbf{P}}$. Note that the category $\mathcal{S}_{\mathbf{P}}\text{-mod}^{\mathbf{P}}$ is isomorphic to the category $\mathbf{S}_{\mathbf{P}}\text{-mod}^{\mathbf{P}}$ defined the same way.

Now we have functors:

$$\text{inclusion } \mathcal{S}_{\mathbf{P}}\text{-mod}^{\mathbf{P}} = \mathbf{S}_{\mathbf{P}}\text{-mod}^{\mathbf{P}} \longrightarrow \mathbf{S}_{\mathbf{P}}\text{-mod};$$

$$\text{tensoring } \mathcal{R} \otimes_{\mathbf{S}_{\mathbf{P}}} : \mathbf{S}_{\mathbf{P}}\text{-mod} \longrightarrow \mathcal{R}\text{-mod};$$

factorization by $\mathcal{F}^{-1}\mathbf{P}$ -torsion, where \mathcal{F} is the forgetting functor

$$\mathcal{R}\text{-mod} \longrightarrow R\text{-mod}.$$

The composition of these functors is the functor which establishes the isomorphism of spectra:

$$\mathbf{Spec}_{\mathbf{P}}(\mathcal{S}_{\mathbf{P}}\text{-mod}) \longrightarrow \mathbf{Spec}_{\mathbf{P}}(\mathcal{R}\text{-mod}).$$

Thus the problem of computing $\mathbf{Spec}_{\mathbf{P}}(\mathcal{R}\text{-mod})$ splits into

- a) finding $\mathbf{Spec}_{\mathbf{P}}(\mathcal{S}_{\mathbf{P}}\text{-mod})$;
- b) factorizing $\mathcal{R} \otimes_{\mathcal{S}_{\mathbf{P}}} M$ by $\mathcal{F}^{-1}\mathbf{P}$ -torsion.

If the ring R is noetherian, the first problem can be simplified by using the embedding $\mathbf{Spec}_{\mathbf{P}}(\mathcal{S}_{\mathbf{P}}\text{-mod}) \longrightarrow \mathbf{Spec}(\mathcal{S}_{\mathbf{P}}\text{-mod})$, where $\mathcal{S}_{\mathbf{P}} = \mathcal{S}_{\mathbf{P}} \otimes_R \mathcal{K}$, $\mathcal{K} = \mathcal{K}(\mathbf{P})$ is the residue (skew) field of \mathbf{P} , and $\mathcal{S}_{\mathbf{P}}\text{-mod}$ is the category of representations of $\mathcal{S}_{\mathbf{P}}$ in \mathcal{K} -vector spaces.

The situation is slightly more comfortable when the 'base' ring R is commutative. Since in this case the residue category $\mathcal{K}_{\mathbf{P}}$ is equivalent to the category $K_{\mathbf{p}}$ -vector spaces, where $\mathbf{p} = \text{Ann}(\mathbf{P})$ is the prime ideal corresponding to \mathbf{P} and $K_{\mathbf{p}} = R/\mathbf{p}$ is its residue (commutative) field in the usual sense.

The second problem involves combinatorics related to the multiplication in the ring \mathcal{R} . The solution is pretty simple in the case of the so called hyperbolic rings (cf. [R10]).

COMPLEMENTARY FACTS AND EXAMPLES.

C1. Morphisms of graded monads and the spectrum. The most natural question concerning our constructions of the spectrum is how they behave under a base (= monad) change?

Let \mathfrak{E} denote the category of triples (H, π, \mathbf{X}) , where H is a semigroup, $\mathbf{X} = (X, \cdot)$ an H -set, and π is an H -set epimorphism. A morphism from (H', π', \mathbf{X}') to (H, π, \mathbf{X}) is a pair (ϕ, ψ) , where $\phi : H' \longrightarrow H$ is a semigroup morphism, and ψ is a map from X' to X which is uniquely defined by the compatibility condition: $\psi \circ \pi' = \pi \circ \phi$.

Fix a morphism $\Phi = (\phi, \psi) : (H', \pi', \mathbf{X}') \longrightarrow (H, \pi, \mathbf{X})$.

The morphism Φ induces a map which assigns to any \mathbf{X} -graded monad $\mathbb{F} = (F, \mu)$ in a category \mathcal{A} , the \mathbf{X}' -graded monad $\Phi_{\#}\mathbb{F} = (\Phi_{\#}F, \mu')$ in \mathcal{A} which is defined by $\Phi_{\#}F(h') := F(\phi(h'))$, and the multiplication μ' is induced by μ .

Similarly, for any \mathbf{X} -graded \mathbb{F} -module $\mathbb{M} = (M, m)$, we denote by $\Phi_{\#}\mathbb{M} = (\Phi_{\#}M, m')$ the \mathbf{X}' -graded $\Phi_{\#}\mathbb{F}$ -module defined by $\Phi_{\#}M[x] := M[\psi(x)]$ for any $x \in X'$ with the obvious action m' . Clearly the map $\mathbb{M} \longrightarrow \Phi_{\#}\mathbb{M}$ extends naturally to a functor $\Phi_{\#} : \mathbb{F}\text{-mod} \longrightarrow \Phi_{\#}\mathbb{F}\text{-mod}$.

Now, we define the (meta)category of graded monads in \mathcal{A} in the most standard way. Namely, a morphism from an \mathbf{X}' -graded monad \mathbb{F}' to an \mathbf{X} -graded monad

\mathbb{F} is a pair (Φ, φ) , where $\Phi = (\phi, \psi)$ is a morphism from (H', π', \mathbf{X}') to (H, π, \mathbf{X}) , and $\varphi : \mathbb{F}' \longrightarrow \Phi_{\#}\mathbb{F}$ is an \mathbf{X}' -graded monad morphism. The definition of the composition is standart as well. We denote the category of graded monads in \mathcal{A} by the symbol $\text{gr}|\mathcal{A}$.

The map which assigns to any \mathbf{X} -graded monad $\mathbb{F} = (F, \mu)$ its submonad $\mathbb{F}[1] = (F[1], \mu')$ extends to a functor, $[1]$, from the category $\text{gr}|\mathcal{A}$ to the category $\text{Mon}\mathcal{A}$ of monads in \mathcal{A} .

Clearly any morphism $\Phi' = (\Phi, \varphi)$ from an \mathbf{X}' -graded monad \mathbb{F}' to an \mathbf{X} -graded monad \mathbb{F} induces a functor $\Phi'_{*}: \mathbb{F}\text{-mod} \longrightarrow \mathbb{F}'\text{-mod}$ which is the composition of $\Phi_{\#}$ and the functor $\varphi_{*}: \Phi_{\#}\mathbb{F}\text{-mod} \longrightarrow \mathbb{F}'\text{-mod}$.

One can see that there is a commutative diagram

$$\begin{array}{ccc}
 \mathbb{F}\text{-mod} & \xrightarrow{\Phi'_{*}} & \mathbb{F}'\text{-mod} \\
 \mathfrak{F} \downarrow & & \downarrow \mathfrak{F}' \\
 \mathbb{F}[1]\text{-mod} & \xrightarrow{\Phi'[1]_{*}} & \mathbb{F}'[1]\text{-mod}
 \end{array} \tag{1}$$

The diagram (1) induces the commutative diagram

$$\begin{array}{ccc}
 \mathfrak{f}Ker\mathfrak{F} & \xrightarrow{\mathfrak{f}\Phi'_{*}} & \mathfrak{f}Ker\mathfrak{F}' \\
 \mathfrak{F} \downarrow & & \downarrow \mathfrak{F}' \\
 \mathbb{F}[1]\text{-mod} & \xrightarrow{\Phi'[1]_{*}} & \mathbb{F}'[1]\text{-mod}
 \end{array} \tag{2}$$

The commutativity of (2) and (the proof of) Theorem 5.4.1 imply that given a module $\mathbb{M} \in \text{Spec}\mathfrak{f}Ker\mathfrak{F}$, the module $\mathfrak{f}\Phi'_{*}\mathbb{M}$ is in $\text{Spec}\mathfrak{f}Ker\mathfrak{F}'$ if and only if $\Phi'_{*}[1] \circ \mathfrak{F}(\mathbb{M}) \in \text{Spec}\mathbb{F}'[1]\text{-mod}$.

The assertion above is not particularly applicable. A really useful fact is the following lemma.

C1.1. Lemma. For any \mathbb{F} -module \mathbb{M} from $\mathfrak{f}Ker\mathfrak{F}$, the functor \mathfrak{F}' induces a bijection of $\text{Ass}(\mathfrak{f}\Phi'_{*}(\mathbb{M}))$ onto $\text{Ass}(\Phi'[1]_{*} \circ \mathfrak{F}(\mathbb{M}))$. In particular, for any $\mathbb{F}[1]$ -module \mathbb{V} , the functor \mathfrak{F}' induces a bijection

$$\text{Ass}(\mathfrak{f}\Phi'_{*} \circ \mathfrak{F}(\mathbb{V})) \longrightarrow \text{Ass}(\Phi'_{*}[1](\mathbb{V})).$$

Here \mathfrak{F} is, as usually, a left adjoint to \mathfrak{F} (cf. Theorem 5.4.1).

C2. QUASI-HOLONOMIC MODULES AND CHARACTERS.

C2.1. Essential length. Let \mathcal{B} be a local category. We say that an object M of \mathcal{B} is of finite length if either $M = 0$, or there is a finite filtration

$$0 := M_0 \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_n := M \quad (1)$$

such that every quotient M_i/M_{i-1} is a quasi-final object.

We define the essential length, $\text{el}(M)$, of the object M as the minimal n having this property.

If \mathcal{B} has simple objects, then M_i/M_{i-1} is quasi-final iff it is semi-simple of finite length. In particular, $\text{el}(M) = 1$ if and only if M is a semi-simple object of finite length.

C2.1.1. Lemma. *The full subcategory of objects of finite length of a local category \mathcal{B} is thick.*

Proof is a standart argument left to a reader. ■

C2.2. Locally finite objects. We call an object M of a category \mathcal{A} locally finite if,

- a) the support of any nonzero subobject of M is nonempty;
- b) for any $\mathbf{P} \in \text{Spec} \mathcal{A}$, $Q_{\mathbf{P}}(M) \in \text{Ob} \mathcal{A}_{\mathbf{P}}$ has a finite length.

Let $\mathcal{A}_{\mathcal{F}}$ denote the full subcategory of \mathcal{A} generated by locally finite objects in \mathcal{A} .

C2.2.1. Note. In 'real life' examples of categories \mathcal{A} , the only objects with empty support are zero objects. So, the condition a) holds automatically. ■

C2.2.2. Lemma. (a) *If \mathcal{A} is a local category, then $\text{Ob} \mathcal{A}_{\mathcal{F}}$ is the family of all objects of finite length in \mathcal{A} .*

(c) *For any $M \in \text{Ob} \mathcal{A}_{\mathcal{F}}$ the support of M consists only of closed points.*

(b) *The subcategory $\mathcal{A}_{\mathcal{F}}$ is thick.*

Proof. (a) Let \mathcal{A} be a local category. It follows from definition of $\mathcal{A}_{\mathcal{F}}$ that all its objects has a finite length (since the zero subcategory belongs to the support of any nonzero object of \mathcal{A}). Suppose now that $M \in \text{Ob} \mathcal{A}$ is of finite length; i.e. there is a filtration (1) such that each quotient M_i/M_{i-1} is a quasi-final object. This implies that $\text{Supp}(M) = \mathbf{0}$. Therefore $Q_{\mathbf{P}}(M) = 0$ for any $\mathbf{P} \in \text{Spec} \mathcal{A}$ which is not equal to $\mathbf{0}$.

(b) First note that if $\mathbf{P} \in \text{Supp}(X)$ and $\mathbf{P}' \in \text{Spec} \mathcal{A}$ is any specialization

of \mathbf{P} (i.e. $\mathbf{P}' \subseteq \mathbf{P}$), then $\mathbf{P}' \in \text{Supp}(X)$.

Fix an $M \in \text{Ob}\mathcal{A}_{\mathcal{P}}$. Let $\mathbf{P} \in \text{Supp}(M)$; and let \mathbf{P}' be a specialization of the point \mathbf{P} . Since $\mathbf{P}' \in \text{Supp}(M)$, $Q_{\mathbf{P}'}(M)$ is a nonzero object of finite length in the local category $\mathcal{A}_{\mathbf{P}'}$. It follows from the argument in (a) that the support of $Q_{\mathbf{P}'}(M)$ is $\mathbf{0}$. But $\mathbf{P}/\mathbf{P}' \in \text{Supp}(Q_{\mathbf{P}'}(M))$. Therefore $\mathbf{P} = \mathbf{P}'$.

(c) It follows from the definition of $\mathcal{A}_{\mathcal{P}}$ that

$$\mathcal{A}_{\mathcal{P}} = \bigcap_{\mathbf{P} \in \text{Spec}\mathcal{A}} Q_{\mathbf{P}}^{-1}((\mathcal{A}/\mathbf{P})_{\mathcal{P}}).$$

Since all localizations $Q_{\mathbf{P}}$ are exact, it suffices to check that $\mathcal{A}_{\mathcal{P}}$ is thick when the category \mathcal{A} is local. According to the assertion (a), in this case, $\mathcal{A}_{\mathcal{P}}$ is the category of objects of finite length. So the assertion follows from Lemma C2.1.1. ■

C2.3. Locally associated points. For any $M \in \text{Ob}\mathcal{A}$, set

$$L\text{Ass}(M) = \{\mathbf{P} \in \text{Spec}\mathcal{A} \mid Q_{\mathbf{P}}(\mathbf{P}) \in \text{Ass}(Q_{\mathbf{P}}(M))\} \quad (1)$$

Clearly $\text{Ass}(M) \subseteq L\text{Ass}(M)$ for all M .

The inverse inclusion is true if $\mathcal{A} = R\text{-mod}$, where R is a commutative noetherian ring ([B], Ch.IV, 1.2, Cor. of Prop. 5).

It is still true if $\mathcal{A} = R\text{-mod}$, where R is a left \leq -noetherian ring; i.e. a ring with maximality condition for left ideals with respect to the preorder \leq (cf. [R3]). Recall that left \leq -noetherian becomes just noetherian if the ring R is commutative.

In the non-affine situation, a sufficient condition is: \mathcal{A} has Gabriel-Krull dimension and any Serre subcategory \mathcal{S} of \mathcal{A} such that \mathcal{A}/\mathcal{S} is local belongs to $\text{Spec}\mathcal{A}$. The category $\mathcal{A} = R\text{-mod}$ has these properties if R is left \leq -noetherian.

We summarize the main properties of the map $L\text{Ass}$ in the following assertion.

C2.3.1. Proposition. (a) $L\text{Ass}(M) \subseteq \text{Supp}(M)$ for all M .

(b) For any short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$,

$$L\text{Ass}(M') \subseteq L\text{Ass}(M) \subseteq L\text{Ass}(M') \cup L\text{Ass}(M'').$$

(c) If Ω is a directed family of subobjects of M and $\sup\Omega = M$, then

$$L\text{Ass}(M) = \bigcup_{X \in \Omega} L\text{Ass}(X).$$

In particular, $L\text{Ass}(\bigoplus Y) = \bigcup_{Y \in \Xi} L\text{Ass}(Y)$ whenever $\bigoplus_{Y \in \Xi} Y$ exists.

Proof is analogous to the proof of similar statements about $\text{Ass}(\)$ (cf. [R5], Section 8). ■

C2.3.2. Lemma. For any object M of the subcategory \mathcal{A}_f $\text{Supp}(M) = \text{LAss}(M)$.

Proof follows from definitions. ■

C2.3.3. The category $\mathcal{C}_f(\mathcal{F})$. Fix an exact functor $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$ between abelian categories. Call an object M of the category \mathcal{B} *\mathcal{F} -locally finite* if $\mathcal{F}(M)$ is locally finite: $\mathcal{F}(M) \in \text{Ob}\mathcal{A}_f$. Denote the full subcategory of \mathcal{B} generated by \mathcal{F} -locally finite objects by $\mathcal{C}_f(\mathcal{F})$. When \mathcal{F} is the forgetting functor from $\mathbb{F}\text{-mod}$ to \mathcal{A} for some monad \mathbb{F} in \mathcal{A} , we could write $\mathcal{C}_f(\mathbb{F})$ instead.

Since the functor \mathcal{F} is exact, and \mathcal{A}_f is thick, the category $\mathcal{C}_f(\mathcal{F})$ is thick too.

An example to keep in mind: for a reductive Lie algebra \mathfrak{g} and its Cartan subalgebra \mathfrak{h} , let \mathcal{F} be the forgetting functor from $\mathfrak{g}\text{-mod}$ to $\mathfrak{h}\text{-mod}$. Note that, the category $\mathcal{C}_f(\mathcal{F})$ contains the category \mathcal{O} .

C2.4. Quasi-holonomic objects. We define the subcategory of *quasi-holonomic* objects in \mathcal{A} as the full subcategory $\text{Qh}\mathcal{A}$ of the category \mathcal{A} generated by all $M \in \text{Ob}\mathcal{A}$ such that

(a) for any nonzero subobject X of M , $\text{LAss}(X) \neq \emptyset$;

(b) for any $\langle P \rangle \in \text{LAss}(M)$, the localization of M at $\langle P \rangle$ is an object of finite length.

Clearly $\mathcal{A}_f \subseteq \text{Qh}\mathcal{A}$. The inverse inclusion does not hold if $\text{Spec}\mathcal{A}$ has non-closed points. In fact, any object $P \in \text{Spec}\mathcal{A}$ is quasi-holonomic. While P belongs to \mathcal{A}_f iff it is closed.

Given a functor $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$, denote by $\text{Qh}(\mathcal{F})$ the preimage of the subcategory $\mathcal{F}^{-1}(\text{Qh}\mathcal{A})$. We call the objects of $\text{Qh}(\mathcal{F})$ *\mathcal{F} -quasi-holonomic*. Again, we may write $\text{Qh}(\mathbb{F})$ if \mathcal{F} is the forgetting functor $\mathbb{F}\text{-mod} \longrightarrow \mathcal{A}$ for some monad \mathbb{F} in \mathcal{A} .

C2.4.1. Remarks. (a). If in the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the objects M' and M'' are quasi-holonomic, then such is M .

If $M \in \text{Qh}\mathcal{A}$, then, certainly, M' is quasi-holonomic; but, M'' might be not. Which means that the subcategory $\text{Qh}\mathcal{A}$ is not, in general, topologizing.

The same is true for \mathcal{F} -quasi-holonomic objects for any exact functor \mathcal{F} .

(b) One can show that \mathcal{A}_f is the largest thick subcategory of \mathcal{A} contained in $Qh\mathcal{A}$. ■

C2.5. Characters. The main reason for introducing the category $Qh(\mathcal{F})$ is that, for objects of $Qh(\mathcal{F})$, there is a well defined notion of a (formal) character.

Denote by $GSpec\mathcal{A}$ the subset (or subspace) of all points $\mathbf{P} \in Spec\mathcal{A}$ such that \mathcal{A}/\mathbf{P} has simple objects. For any $M \in Ob\mathcal{A}$, set

$$GAss(M) := GSpec\mathcal{A} \cap LAss(M), \quad GSupp(M) := GSpec\mathcal{A} \cap Supp(M).$$

Fix an exact functor $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$. The formal character of an object M of $Qh(\mathcal{F})$ is a function $ch_{\mathcal{F}, M} = ch_M$ which assigns to any $\mathbf{P} \in GAss(\mathcal{F}(M))$ the length of $Q_{\mathbf{P}}(\mathcal{F}(M))$. In other words, formal character as an element of the free abelian group generated by $GAss(\mathcal{F}(M))$:

$$ch_M = \sum_{\mathbf{P} \in GAss(M)} length(Q_{\mathbf{P}}(\mathcal{F}(M)))e^{\mathbf{P}}. \quad (1)$$

Note, that if the category \mathcal{A} has Gabriel-Krull dimension (e.g. \mathcal{A} is locally noetherian) which is the case of most of examples, and in many other cases, $GSpec\mathcal{A} = Spec\mathcal{A}$. (cf. [R5]).

C2.5.1. Lemma. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence.

(a) If $M \in \mathcal{E}_f(\mathcal{F})$, then $ch_M = ch_{M'} + ch_{M''}$

(b) If $M \in Qh(\mathcal{F})$ and, for all $\mathbf{P} \in GAss(M'')$,

$$Q_{\mathbf{P}}(\mathcal{F}(M)) \simeq Q_{\mathbf{P}}(\mathcal{F}(M')) \oplus Q_{\mathbf{P}}(\mathcal{F}(M'')), \quad (2)$$

then $ch_M = ch_{M'} + ch_{M''}$

Proof. (a) The assertion (a) follows from the equalities

$$GAss(M) = GSupp(M) = GSupp(M') \cup GSupp(M'') = GAss(M') \cup GAss(M'')$$

(cf. Lemma C2.3.2) and the equality $length(M) = length(M') + length(M'')$ in the case when M is of finite length.

(b) The splitting (2) implies that $GAss(\mathcal{F}(M)) = GAss(\mathcal{F}(M')) \cup GAss(\mathcal{F}(M''))$.

The rest of the argument is the same as in (a). ■

C2.5.2. Remarks. (a) A standart interpretation of the assertion (a) is that the function

$$ch|_{\mathcal{E}_f(\mathcal{F})} : Ob\mathcal{E}_f(\mathcal{F}) \longrightarrow \mathbf{Maps}(GSpec\mathcal{A}, \mathbb{Z}), \quad M \longmapsto ch_M$$

factorizes through the canonical map $\mathcal{E}(\mathcal{F}) \longrightarrow K_0(\mathcal{E}(\mathcal{F}))$. So that we have the uniquely defined \mathbb{Z} -module morphism $ch_0: K_0(\mathcal{E}(\mathcal{F})) \longrightarrow \mathbf{Maps}(GSpec\mathcal{A}, \mathbb{Z})$.

Clearly the morphism ch_0 is injective if $GSpec\mathcal{A} = Spec\mathcal{A}$.

(b) There is a similar interpretation of the whole map

$$\mathcal{Ch}: \text{Ob}Q\mathcal{h}(\mathcal{F}) \longrightarrow \mathbf{Maps}(\mathbf{G}\text{Spec}\mathcal{A}, \mathbf{Z}).$$

Only this time, one should replace K_0 be the *relative* Grothendieck group $K_{0, \mathcal{F}}$ corresponding to a class \mathbf{E} of short exact sequences defined as follows:

An exact sequence $E = (0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0)$ in \mathcal{B} belongs to $\mathbf{E} = \mathbf{E}_{\mathcal{F}}$ iff $M \in \text{Ob}Q\mathcal{h}(\mathcal{F})$, and $\text{GAss}(\mathcal{F}(M'')) \subseteq \text{GAss}(\mathcal{F}(M))$ (this implies that E belongs to $Q\mathcal{h}(\mathcal{F})$). According to Proposition C2.3.1, the class \mathbf{E} contains all exact sequences E such that $Q_{\mathbf{P}}(E)$ splits for any $\mathbf{P} \in \text{GAss}(M'')$ (cf. the assertion (b) of Lemma C2.5.1). ■

C2.6. Quasi-holonomic modules over a skew PBW monad. We begin with the following observation: for any exact functor $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$,

$$\text{Spec}\mathcal{B} \cap Q\mathcal{h}(\mathcal{F}) = \bigcup_{\mathbf{P} \in \text{Spec}\mathcal{A}} \text{Spec}_{\mathbf{P}}(\mathcal{F}) \cap Q\mathcal{h}(\mathcal{F}). \quad (1)$$

So that the description of the intersection $\text{Spec}\mathcal{B} \cap Q\mathcal{h}(\mathcal{F})$ is reduced to that of $\text{Spec}_{\mathbf{P}}(\mathcal{F}) \cap Q\mathcal{h}(\mathcal{F})$ for all $\mathbf{P} \in \text{Spec}\mathcal{A}$.

Similarly,

$$\text{Spec}\mathcal{B} \cap \mathcal{E}_{\mathbf{P}}(\mathcal{F}) = \bigcup_{\mathbf{P} \in \text{Spec}\mathcal{A}} \text{Spec}_{\mathbf{P}}(\mathcal{F}) \cap \mathcal{E}_{\mathbf{P}}(\mathcal{F}). \quad (2)$$

Note that, since $\mathcal{E}_{\mathbf{P}}(\mathcal{F})$ is a topologizing (actually, thick) subcategory in \mathcal{B} , we can identify $\text{Spec}\mathcal{B} \cap \mathcal{E}_{\mathbf{P}}(\mathcal{F})$ with $\text{Spec}\mathcal{E}_{\mathbf{P}}(\mathcal{F})$ and $\text{Spec}(\mathcal{F})_{\mathbf{P}} \cap Q\mathcal{h}(\mathcal{F})$ with $\text{Spec}_{\mathbf{P}}\mathcal{E}_{\mathbf{P}}(\mathcal{F})$.

Suppose now that \mathbb{F} is a skew PBW monad in \mathcal{A} , and \mathcal{F} is a forgetting functor $\mathbb{F}\text{-mod} \longrightarrow \mathcal{A}$. The bijection

$$\text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} \longrightarrow \text{Spec}_{\mathbf{P}}\mathbb{S}\mathbb{P}\text{-mod}$$

of Theorem 6.6.3 induces an injection of

$$Q\mathcal{h}\text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} := \{ \langle (M, m) \rangle \in \text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} \mid M \in \text{Ob}Q\mathcal{h}(\mathbb{F}) \}$$

into

$$Q\mathcal{h}\text{Spec}_{\mathbf{P}}\mathbb{S}\mathbb{P}\text{-mod} := \{ \langle (V, \omega) \rangle \in \text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} \mid \langle V \rangle = \mathbf{P} \}.$$

Under certain (pretty mild) finiteness conditions, this map

$$Q\mathcal{h}\text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} \longrightarrow Q\mathcal{h}\text{Spec}_{\mathbf{P}}\mathbb{S}\mathbb{P}\text{-mod}$$

is also surjective. In particular, it is surjective for all examples of skew PBW monads we consider here. ■

C2.6.1. Lemma. *An object $\mathbf{M} = (M, m)$ of $\text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} \cap \text{Ob}Q\mathcal{h}(\mathbb{F})$ has nonzero character iff $\mathbf{P} \in \mathbf{G}\text{Spec}\mathcal{A}$.*

Proof. Note that $\mathbf{G}\text{Spec}\mathcal{A}$ is $\mathbf{Aut}\mathcal{A}$ -stable.

In fact, $\mathbf{P} \in \mathbf{G}\text{Spec}\mathcal{A}$ if and only if there is an object V which is *simple*

modulo \mathbf{P} ; i.e. $V \notin \mathbf{P}$, and, for any monoarrow $g: V' \longrightarrow V$, either $V' \in \mathbf{P}$, or $\text{Cok}(g) \in \mathbf{P}$. Clearly simple modulo \mathbf{P} objects are exactly those objects which are made simple by the localization at \mathbf{P} .

For any auto-equivalence ϑ of \mathcal{A} and a simple modulo \mathbf{P} object V , the object $\vartheta(V)$ is simple modulo $\vartheta(\mathbf{P})$.

The $\text{Aut}\mathcal{A}$ -stability of $\text{GSpec}\mathcal{A}$ implies that $\text{Ass}(\mathcal{F}(M)) \subseteq \text{GSpec}\mathcal{A}$ iff one of the points of $\text{GAss}(\mathcal{F}(M))$ belongs to $\text{GSpec}\mathcal{A}$. ■

Actually, it is possible to get a more explicit picture.

Fix any point \mathbf{P} of $\text{Spec}\mathcal{A}$. And consider the associated with \mathbf{P} submonad $\mathcal{S}_{\mathbf{P}} := \left(\bigoplus_{t \in \mathcal{S}\mathbf{P}} \Theta(t), \xi' \right)$, where $\mathcal{S}\mathbf{P} := \{t \in \mathcal{S} \mid \Theta(t)\mathbf{P} = \mathbf{P}\}$.

For any $t \in \mathcal{S}\mathbf{P}$, the auto-equivalence $\Theta(t)$ induces an auto-equivalence, $\Theta(t)'$, of the thick subcategory $\mathcal{A}(\mathbf{P}^-) := \mathcal{A}(\text{Supp}(\mathbf{P}))$ which, in turn, defines an auto-equivalence, $\Theta(t)^-$, of the quotient category $\mathcal{K}^-\mathbf{P} := \mathcal{A}(\mathbf{P}^-)/\mathbf{P}$.

Recall that the category $\mathcal{K}^-\mathbf{P}$ is "zero-dimensional"; i.e. (since $\mathcal{K}^-\mathbf{P}$ is local) $\text{Spec}\mathcal{K}^-\mathbf{P}$ consists of only one point.

Consider the full subcategory $\mathcal{K}\mathbf{P}$ of the category $\mathcal{K}^-\mathbf{P}$ generated by all objects M of $\mathcal{K}^-\mathbf{P}$ which are supremums of its subobjects $V \longrightarrow M$ such that $\langle V \rangle = \mathbf{P}$. One can check that the subcategory $\mathcal{K}\mathbf{P}$ is topologizing which implies that $\mathcal{K}\mathbf{P}$ inherits the nice properties of the category $\mathcal{K}^-\mathbf{P}$: it is local and its spectrum consists of only one point.

Clearly the subcategory $\mathcal{K}\mathbf{P}$ is stable with respect to any auto-equivalence ϑ such that $\vartheta(\mathbf{P}) = \mathbf{P}$. Therefore the monad $\mathcal{S}_{\mathbf{P}}$ defines a monad

$$\mathcal{K}\mathcal{S}_{\mathbf{P}} = \left(\bigoplus_{t \in \mathcal{S}\mathbf{P}} \Theta(t), \zeta \right)$$

in the category $\mathcal{K}\mathbf{P}$. The localization at \mathbf{P} provides an embedding

$$i: \text{Spec}_{\mathbf{P}}\mathcal{S}_{\mathbf{P}}\text{-mod} \longrightarrow \text{Spec}\mathcal{K}\mathcal{S}_{\mathbf{P}}\text{-mod}.$$

C2.6.2. Lemma. *Suppose that \mathcal{A}/\mathbf{P} has simple objects. Then the monad $\mathcal{K}\mathcal{S}_{\mathbf{P}}$ is isomorphic to $R_{\mathbf{P}} \otimes_K$ for a certain ring $R_{\mathbf{P}}$ over the skew field $K = K_{\mathbf{P}}$.*

Proof. By Lemma 5.4.1 in [R3], the residue category $\mathcal{K}\mathbf{P}$ of the point \mathbf{P} is equivalent to the category $K_{\mathbf{P}}\text{-mod} = K_{\mathbf{P}}\text{-Vec}$ of $K_{\mathbf{P}}$ -vector spaces for a skew field $K = K_{\mathbf{P}}$ - the residue skew field of \mathbf{P} . The functor $\bigoplus_{t \in \mathcal{S}\mathbf{P}} \Theta(t)^-$ is isomorphic to $R_{\mathbf{P}} \otimes_K$ for a K -module $R_{\mathbf{P}}$. The multiplication ξ^- of $\mathcal{S}_{\mathbf{P}}$ defines (uniquely) an associative ring structure $\mu: R_{\mathbf{P}} \otimes_K R_{\mathbf{P}} \longrightarrow R_{\mathbf{P}}$. ■

Thanks to the Lemma C2.6.1, we can identify the category $\mathcal{K}\mathcal{S}_{\mathbf{P}}\text{-mod}$ with the category $R_{\mathbf{P}}\text{-mod}/K_{\mathbf{P}}$ of $R_{\mathbf{P}} \otimes_K$ -modules.

C2.7. Lemma. Let $\langle (M, m) \rangle$ be the image of an element $\langle \mathbb{V} \rangle = \langle (V, \mathfrak{v}) \rangle$ under the canonical bijection $\phi_{\mathbf{P}}: \text{Spec}_{\mathbf{P}} \mathcal{S}_{\mathbf{P}}\text{-mod} \longrightarrow \text{Spec}_{\mathbf{P}} \mathcal{F}\text{-mod}$ for some $\mathbf{P} \in \mathbf{G}\text{Spec} \mathcal{A}$.

If the module (M, m) is quasi-holonomic, then $i(\mathbb{V})$ is the coproduct of a finite number of copies of a simple $K_{\mathbf{P}}$ -finite dimensional $R_{\mathbf{P}}$ -module (cf. Lemma C2.6.1).

Proof. If (M, m) is quasi-holonomic, then the image, $i(\mathbb{V}) = (Q_{\mathbf{P}}V, \mathfrak{v}')$, of $\mathbb{V} = (V, \mathfrak{v})$ in $R_{\mathbf{P}}\text{-mod}$ is $K_{\mathbf{P}}$ -finite dimensional. Since $i(\mathbb{V})$ is K -finite dimensional, it is artinian; in particular, it contains a simple submodule, say \mathcal{P} . But, being an object of the spectrum, $i(\mathbb{V})$ is equivalent to \mathcal{P} . The latter means, since \mathcal{P} is simple, that $i(\mathbb{V})$ is isomorphic to the coproduct of a finite number of copies of \mathcal{P} . ■

C2.8. Example: skew PBW monads of rank one. 'Rank one' means that $\mathcal{G} = \mathbb{Z}$ and Θ is a map from \mathbb{Z} to $\mathbf{Aut}(\mathcal{A})$ which sends $n \geq 0$ into θ^n and $n \leq -1$ into $\theta^{\wedge -n}$, where θ is an auto-equivalence and θ^{\wedge} its right adjoint. The action is given by the data

$$\xi = \{ \xi(s, t | u): \theta^s \circ \theta^t \longrightarrow \theta^u \mid s, t, u \in \mathbb{Z} \},$$

where $\theta^s := \theta^{\wedge -s}$, if s is negative.

For any $\mathbf{P} \in \text{Spec} \mathcal{A}$, the stabilizer, $\mathcal{G}\mathbf{P} := \{ t \in \mathbb{Z} \mid \Theta(t)\mathbf{P} = \mathbf{P} \}$, coincides with $m\mathbb{Z}$ for some nonnegative integer m . Thus we have a partition:

$$\text{Spec} \mathcal{A} = \bigcup_{n \geq 0} \text{Spec}^n \mathcal{A},$$

where $\text{Spec}^n \mathcal{A} := \{ \mathbf{P} \in \text{Spec} \mathcal{A} \mid \mathcal{G}\mathbf{P} = n\mathbb{Z} \}$.

(0) Theorem 6.4.3 provides a map

$$\phi: \text{Spec}^0 \mathcal{A} \longrightarrow \text{Spec} \mathcal{A} / \langle \Theta, \xi \rangle.$$

One can easily find the preimage of the map ϕ . Note first that, for any $P \in \text{Spec} \mathcal{A}$, the morphism $\xi(s, t | u)(P): \theta^s \circ \theta^t(P) \longrightarrow \theta^u(P)$ equals to zero if $s + t \neq u$.

In fact, if $\xi(s, t | u)(P)$ is nonzero, then it is a monomorphism. The latter case implies that $\theta^s \circ \theta^t(P) \approx \theta^u(P)$, or, equivalently, $\langle \theta^{s+t-u}(P) \rangle = \mathbf{P}$ which, by hypothesis, means that $s + t = u$.

Let $\mathbf{P} = \langle P \rangle$. Set

$$\Xi_{-}\mathbf{P} := \{ s \in \mathbb{N} \mid \xi(1, -s | -s+1)(P) = 0 \} \cup \{ -\infty \}, \quad \Xi_{+}\mathbf{P} := \{ s \in \mathbb{Z} \mid \xi(-1, s | s-1)(P) = 0 \} \cup \{ \infty \},$$

and denote by $\ell_{+}\mathbf{P}$ the minimal element of $\Xi_{+}\mathbf{P}$ and by $\ell_{-}\mathbf{P}$ the maximal ele-

ment of $\Xi_{\mathbf{P}}$. Let $I_{\mathbf{P}}$ denote the interval $(\ell_{+\mathbf{P}}, \ell_{-\mathbf{P}})$ and $I_{\mathbf{P}}^{\perp} := \mathbf{z} - I_{\mathbf{P}}$. One can see that $\bigoplus_{n \in I_{\mathbf{P}}^{\perp}} \theta^n(P)$ is a maximal \mathbb{F} -submodule of $\mathbb{F}(P) := (\bigoplus_{n \in \mathbf{z}} \theta^n(P), \xi)$, and the map ϕ assigns to the point \mathbf{P} the equivalence class of the quotient \mathbb{F} -module $\Phi(P) := (\bigoplus_{n \in I_{\mathbf{P}}} \theta^n(P), \xi'(P))$.

In particular, if $\xi_{\{s, -s\} \mid 0}(P) \neq 0$ for every $s \in \mathbf{z}$. Then

$$\Phi(\mathbf{P}) = \langle \mathbb{F}(P) \rangle := \langle (\bigoplus_{n \in \mathbf{z}} \theta^n(P), \xi(P)) \rangle.$$

Clearly

$$\text{ch}_{\Phi(P)} = \text{length}(Q_{\mathbf{P}}(P)) \sum_{n \in I_{\mathbf{P}}} e^{\langle \theta^n(P) \rangle}.$$

Choosing P in such a way that $Q_{\mathbf{P}}(P)$ is a simple object, we obtain:

$$\text{ch}_{\Phi(P)} = \sum_{n \in I_{\mathbf{P}}} e^{\langle \theta^n(P) \rangle}.$$

(+) Let now $\mathbf{P} \in \text{Spec}^m \mathcal{A}$; i.e. the stabilizer $\mathfrak{S}_{\mathbf{P}}$ of the point \mathbf{P} is $m\mathbf{z}$ for some m . Thus $\mathfrak{S}_{\mathbf{P}} = \mathbb{F}_m := (\bigoplus_{n \in \mathbf{z}} \theta^{mn}(P), \xi_m(P))$, where ξ_m is the restriction of the multiplication table to $m\mathbf{z}$.

The localization at \mathbf{P} provides an embedding

$$\iota: \text{Spec}_{\mathbf{P}}^{\mathbb{F}_m} \text{-mod} \longrightarrow \text{Spec}^{\mathcal{K}\mathbb{F}_m} \text{-mod}, \quad (1)$$

where $\mathcal{K}\mathbb{F}_m = (\bigoplus_{n \in \mathbf{z}} \theta^n, \zeta)$ is the induced by \mathbb{F}_m monad in the (semisimple) residue category $\mathcal{K}\mathbf{P}$.

Suppose that $\mathbf{P} \in \text{GSpec} \mathcal{A}$. Then the category $\mathcal{K}\mathbf{P}$ is equivalent to $K_{\mathbf{P}} \text{-mod}$ for some skew field $K = K_{\mathbf{P}}$. And the monad $\mathcal{K}\mathbb{F}_m$ is isomorphic to the monad $R_{\mathbf{P}} \otimes_K$ for a certain ring $R_{\mathbf{P}}$ over a skew field K . According to Lemma C2.7, the image, (M, m) , of an element $\langle \mathbb{V} \rangle = \langle (V, \mathfrak{v}) \rangle$ under the canonical bijection

$$\phi_m: \text{Spec}_{\mathbf{P}}^{\mathbb{F}_m} \text{-mod} \longrightarrow \text{Spec}_{\mathbf{P}}^{\mathbb{F}} \text{-mod}$$

is quasi-holonomic if and only if $\iota(\mathbb{V})$ is the coproduct of a finite number of copies of a simple K -finite dimensional R -module.

In general, there might be lots of infinite dimensional (over $K_{\mathbf{P}}$) simple objects in $\mathbb{F}_m \text{-mod}$.

C2.9. Example: skew group monads (crossed products). Fix a group G . Consider a skew PBW monad \mathbb{F} in \mathcal{A} defined by a map $\Theta: G \longrightarrow \text{Aut}(\mathcal{A})$, and morphisms

$$\xi(s, t | u): \Theta_s \circ \Theta_t \longrightarrow \Theta_u \quad s, t, u \in G,$$

such that $\xi(s, t | u) = 0$ if $u \neq st$. So that we can set $\zeta(s, t) := \xi(s, t | u)$.

The fact that ζ defines an associative multiplication is given by:

$$\begin{aligned}\zeta(s,tu)\zeta(t,u) &= \zeta(st,u)\Theta_u\zeta(s,t), \\ \zeta(s,1) &= \zeta(1,s) = id.\end{aligned}$$

Suppose that $\mathbf{P} \in \text{Spec}\mathcal{A}$ is such that $G_{\mathbf{P}} := \{g \in G \mid \Theta_g(\mathbf{P}) = \mathbf{P}\}$ is a normal subgroup in G . Then the functor

$$\text{Spec}_{\mathbf{P}}\mathcal{S}_{\mathbf{P}}\text{-mod} \longrightarrow \text{Spec}_{\mathbf{P}}\mathbb{F}\text{-mod} \quad (1)$$

of Theorem 6.6.3 sends (quasi-)locally finite $\mathcal{S}_{\mathbf{P}}$ -modules into (quasi-)locally finite \mathbb{F} -modules. This follows from the fact that, under the assumptions, the functor $\mathbb{F} \otimes_{\mathcal{S}_{\mathbf{P}}}$ sends any $\mathcal{S}_{\mathbf{P}}$ -module $\mathbb{V} = (V, \alpha)$ such that $\langle V \rangle = \mathbf{P}$ into a quasi-holonomic \mathbb{F} -module which is locally finite iff \mathbf{P} is a closed point.

In particular, if the group G is commutative, this holds for all \mathbf{P} .

Note that hyperbolic monads of any rank are special cases of the latter situation. ■

C3. Dualization. Fix an abelian category \mathcal{A} . As usual, \mathcal{A}^{op} denotes its dual category.

C3.1. Lemma. (a) The preorder \succ coincides with its dual.

(b) $\text{Spec}\mathcal{A}^{\text{op}} \cap \text{Spec}\mathcal{A} = \{P \in \text{Ob}\mathcal{A} \mid P \succ X \neq 0 \Rightarrow X \succ P\}$.

In particular, $\text{Spec}\mathcal{A}^{\text{op}} \cap \text{Spec}\mathcal{A} \subseteq \text{Max}\mathcal{A} := \{P \in \text{Spec}\mathcal{A} \mid \langle P \rangle \text{ is closed}\}$.

(c) If any nonzero object in \mathcal{A} has a nonempty support, then

$$\text{Spec}\mathcal{A}^{\text{op}} \cap \text{Spec}\mathcal{A} = \text{Max}\mathcal{A}.$$

Proof. (a) By definition, $X \succ Y$ iff Y is a subquotient of a direct sum of a finite number of copies of X . Dualization saves finite direct sums; and if Y is a subquotient of X' in \mathcal{A} , then it is a subquotient of X' in \mathcal{A}^{op} . The latter is due to the fact that if in the fiber coproduct

$$\begin{array}{ccc} & \iota & \\ & \longrightarrow & \\ K & \longrightarrow & X' \\ e \downarrow & & \downarrow e' \\ & \iota' & \\ Y & \longrightarrow & Y \amalg X' \\ & & K \end{array}$$

the arrow e (resp. ι) is an epimorphism (resp. monomorphism), then e' is an epimorphism (resp. ι' is a monomorphism).

(b) Suppose that $P \in \text{Spec}\mathcal{A}^{\text{op}} \cap \text{Spec}\mathcal{A}$. Then any nonzero object X such that $P \succ X$ is equivalent to P . In particular, P is closed.

In fact, $P \succ X$ means that there exists a subobject K of $(n)P$ for some n and an epimorphism $K \longrightarrow X$. Thus $K \approx P$. And, since X is nonzero and P (hence K) belongs to $\text{Spec} \mathcal{A}^{\text{op}}$, $X \approx K$.

(c) Suppose now that $\text{Supp}(M) = \emptyset$ only if $M = 0$. And let $\langle P \rangle$ is a closed point of $\text{Spec} \mathcal{A}$. Let $P \succ X \neq 0$. Since X is nonzero, $X \succ P'$ for some element P' of $\text{Spec} \mathcal{A}$. Since $\langle P \rangle$ is closed, $P \approx P'$. Therefore $P \approx X$. ■

C3.2. Corollary. (a) A category \mathcal{A} is local iff \mathcal{A}^{op} is local.

(b) $\text{Spec}^{\wedge} \mathcal{A} := \{\text{thick subcategories } \mathbf{P} \subseteq \mathcal{A} \text{ such that } \mathcal{A}/\mathbf{P} \text{ is local}\}$ is self-dual; i.e. $\text{Spec}^{\wedge} \mathcal{A} = \text{Spec}^{\wedge} \mathcal{A}^{\text{op}}$.

We will indicate the dualization by $^{\circ}$. For instance, $\text{Spec}^{\circ} \mathcal{A}$ denotes $\text{Spec} \mathcal{A}^{\text{op}}$; $\text{Supp}^{\circ}(M)$ is the support of M in \mathcal{A}^{op} , etc..

C3.3. Residue and coresidue category. Let \mathcal{A} be a local category. Recall that its residue subcategory $\mathcal{K}(\mathcal{A})$ is a full subcategory of \mathcal{A} generated by objects M which are supremums of their subobjects X such that $\langle X \rangle = \mathbf{0}$. The coresidue category $\mathcal{K}^{\circ}(\mathcal{A})$ is defined dually: $\mathcal{K}^{\circ}(\mathcal{A}) := \mathcal{K}(\mathcal{A}^{\text{op}})^{\text{op}}$.

The subcategories $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}^{\circ}(\mathcal{A})$ are topologizing.

If \mathcal{A} has the property (sup), then the subcategory $\mathcal{K}(\mathcal{A})$ is coreflective. If \mathcal{A} has the dual property (sup $^{\circ}$):= [(sup) in \mathcal{A}^{op}], $\mathcal{K}^{\circ}(\mathcal{A})$ is reflective.

For any $\mathbf{P} \in \text{Spec}^{\wedge} \mathcal{A}$, we have the residue and coresidue categories at \mathbf{P} : $\mathcal{K}_{\mathbf{P}}(\mathcal{A}) := \mathcal{K}(\mathcal{A}/\mathbf{P})$ and $\mathcal{K}_{\mathbf{P}}^{\circ}(\mathcal{A}) := \mathcal{K}_{\mathbf{P}}^{\circ}(\mathcal{A}/\mathbf{P})$ respectively.

C3.4. Quasi-finite objects. We say that an object M of the category \mathcal{A} is quasi-finite if, for any nonzero subobject M' of M ,

(a) $\text{Supp}(M') = \text{LAss}(M') \neq \emptyset$,

(b) for any $\mathbf{P} \in \text{LAss}(M')$, $\mathcal{K}_{\mathbf{P}}^{(n)} M'$ is finite for all n .

Denote the full subcategory of quasi-finite objects in \mathcal{A} by \mathcal{A}_{qf} .

Clearly $\mathcal{A}_{\text{f}} \subseteq \mathcal{A}_{\text{qf}}$.

C3.4.1. Lemma. \mathcal{A}_{qf} is a thick subcategory in \mathcal{A} .

Proof. The condition (a) for all nonzero subobjects defines a Serre subcategory. The condition (b) defines a thick subcategory. Details are left to the reader. ■

Given an exact functor $\mathcal{F}: \mathcal{B} \longrightarrow \mathcal{A}$, set $\mathcal{E}_{\text{qf}}(\mathcal{F}) := \mathcal{F}^{-1}(\mathcal{A}_{\text{qf}})$, and call ob-

jects of the subcategory $\mathcal{E}_{\mathfrak{qf}}(\mathcal{F})$ \mathcal{F} -quasifinite. It follows from Lemma C.?.4.1 that $\mathcal{E}_{\mathfrak{qf}}(\mathcal{F})$ is a thick subcategory of \mathcal{B} .

If \mathcal{F} is a forgetting functor $\mathbb{F}\text{-mod} \longrightarrow \mathcal{A}$ for some monad \mathbb{F} in \mathcal{A} , we could write $\mathcal{E}_{\mathfrak{qf}}(\mathbb{F})$ instead of $\mathcal{E}_{\mathfrak{qf}}(\mathcal{F})$ and call objects of this category *quasi-finite \mathbb{F} -modules*.

C3.5. Quasi-cofinite objects. Actually, we are more interested in the subcategory $\mathcal{A}_{\mathfrak{qf}}^\circ$ of *quasi-cofinite* objects which is by definition $(\mathcal{A}^{\text{op}})_{\mathfrak{qf}}^{\text{op}}$, and the corresponding relative subcategories $\mathcal{E}_{\mathfrak{qf}}^\circ(\mathcal{F})$. One of the reasons of this interest is the following example.

C3.5.1. Example. Let \mathcal{A} be the category of quasi-coherent sheaves on a smooth variety X ; and let \mathcal{D} be the sheaf of differential operators on X which we identify with the corresponding monad. Then $\mathcal{E}_{\mathfrak{qf}}^\circ(\mathcal{D})$ is the category of holonomic \mathcal{D} -modules. ■

C3.6. Skew PBW comonads and their spectrum. A prototype (and a consequence) of Theorem 6.6.3 is the description of highest weight representations as unique irreducible quotients of Verma modules. Or a similar fact about Harish-Chandra modules. There is also a widely used (say, in representation theory of finite groups) dual way to get irreducible representations as unique subobjects of coinduced representations. This dualization can be obtained as a corollary of Theorem 6.6.3 as follows.

A comonad $\mathbb{G} = (G, \delta: G \longrightarrow G \circ G)$ in \mathcal{A} is a *skew PBW comonad* if the dual monad \mathbb{G}° in \mathcal{A}^{op} is PBW. Thus we can apply Theorem 6.6.3 to get a description of $\text{Spec}_{\mathbb{P}}^\circ \mathbb{G}\text{-comod}$ for any element \mathbb{P} of $\text{Spec}^\circ \mathcal{A}$ in terms of $\text{Spec}_{\mathbb{P}}^\circ \mathbb{G}_{\mathbb{P}}\text{-comod}$, where $\mathbb{G}_{\mathbb{P}}$ is the comonad generated by the stabilizer of \mathbb{P} . The corresponding ('dual') functor of Theorem 6.6.3 maps any $V \in \text{Spec}_{\mathbb{P}}^\circ \mathbb{G}_{\mathbb{P}}\text{-comod}$ into the uniquely defined subcomodule of the coinduced \mathbb{G} -comodule.

Suppose now that $\mathbb{P} \in \text{Spec}^\circ \mathcal{A} \cap \text{Spec} \mathcal{A}$ (for instance, $\mathbb{P} = \langle P \rangle$ for a simple object P ; or, every nonzero object of \mathcal{A} has a nonempty support, and \mathbb{P} is a closed point in $\text{Spec} \mathcal{A}$; cf. Lemma C.3.1). And let the $\mathbb{G}_{\mathbb{P}}$ -comodule V be a closed point (resp. a simple comodule), then this uniquely defined subcomodule is a closed point of $\text{Spec}_{\mathbb{P}}^\circ \mathbb{G}\text{-comod}$ (resp. a simple \mathbb{G} -comodule).

C4. WEYL ALGEBRAS.

Presently, (at least) three types of Weyl algebras are known:

the 'classical' Weyl algebra $A_f(k)$ over a field k ;

the algebra $\mathbb{D}_{q,J} = \mathbb{D}_{q,J}(k)$ of q -differential operators with polynomial coefficients which is a most straightforward one-parameter deformation of the classical Weyl algebra;

the introduced by Hayashi [Ha] quantum deformation, $W_{q,J} = W_{q,J}(k)$ of the Weyl algebra A_J which is called *the quantum Weyl algebra*.

Recall their definitions.

The algebra $\mathbb{D}_{q,J}(k)$ is generated by the elements $x_i, y_i, i \in J$, subject to the relations

$$x_i y_j - q y_j x_i = \delta_{ij}, \quad x_i y_i = y_i x_i \quad (1)$$

for every $i, j \in J$ such that $i \neq j$. Here q is an invertible element of the field k .

The classical Weyl algebra $A_f(k)$ coincides with $\mathbb{D}_{1,J}(k)$.

The quantum Weyl algebra $W_{q,J}(k)$ is generated by $x_i, y_i, z_i, i \in J$, which are related as follows:

$$\begin{aligned} x_i z_i &= q z_i x_i, & z_i y_i &= q y_i z_i \\ x_i y_j - q y_j x_i &= z_i, & x_i y_i - q^{-1} y_i x_i &= z_i^{-1}; \\ x_i y_j &= y_j x_i, & x_j z_i &= z_i x_j, & y_i z_j &= z_j y_i \end{aligned}$$

for any $i, j \in J$ such that $i \neq j$.

Denote by R the ring of polynomials in the indeterminates $\xi_J = (\xi_i | i \in J)$ over a field k : $R := k[\xi_J]$. Let ϑ_i denote the automorphism of R given by $\vartheta_i(\xi_i) = q\xi_i + 1$ and $\vartheta_i(\xi_j) = \xi_j$ if $i \neq j$.

The algebra $\mathbb{D}_{q,J}(k)$ is isomorphic to the constructed by the data $(R, \vartheta_i | i \in J)$ hyperbolic ring; i.e. $\mathbb{D}_{q,J}(k)$ is defined by the relations

$$x_i r = \vartheta_i(r) x_i, \quad y_i \vartheta_i(r) = r y_i \quad \text{for any } r \in R; \quad (1)$$

$$x_i y_i = \xi_i, \quad y_i x_i = \vartheta_i^{-1}(\xi_i) \quad (2)$$

$$x_i y_j = y_j x_i, \quad x_j z_i = z_i x_j, \quad y_i z_j = z_j y_i \quad (3)$$

where i, j run through J , and $i \neq j$.

Similarly, the algebra $W_{q,J}(k)$ is given by the relations (1), (2), (3) with different R and $\vartheta_i, i \in J$. Namely,

R is the ring of polynomials in $\xi_i, z_i, z_i^{-1}, i \in J$;

$$\vartheta_i(\xi_i) = q\xi_i + 1, \quad \vartheta_i(z_i) = qz_i$$

and ϑ_i acts identically on ξ_j, z_j for $j \in J - \{i\}$.

Denote by G the generated by the set $\{\vartheta_i \mid i \in J\}$ subgroup of $\text{Aut}(R)$. According to Theorem 6.6.3, we need to find, for any prime ideal in the ring R its stabilizer in G . We shall do it for each of the three rings - $A_f(k)$, $\mathbb{D}_{q,J}(k)$, and $W_{q,J}(k)$ - separately.

C4.1. The case of the Weyl algebra. For any map $\alpha: J \longrightarrow k$, denote by θ_α the translation by α ; i.e. $\theta_\alpha f(\xi_j) = f(\xi_j + \alpha)$ for any function $f \in R$.

The translations $\theta_\alpha, \alpha \in k^J$, form, obviously, an abelian subgroup, $\mathfrak{Tr}(R)$, in $\text{Aut}(R)$. For any subgroup G in $\mathfrak{Tr}(R)$, define $\text{rank}(G)$ - the rank of G - as the dimension of the k -vector space generated by all α such that $\theta_\alpha \in G$.

The following assertion was, probably, known a hundred years ago; but, it is easier to prove it than to find a reference.

C4.1.1. Lemma. *Let G be a finitely generated subgroup of the group $\mathfrak{Tr}(R)$. Then the subspace $\text{Spec}^G R$ of G -stable prime ideals in R is naturally embedded into the spectrum $\text{Spec} R^G$ of the subring R^G of G -invariants of the ring R .*

If $\text{char}(k) = 0$, then this embedding is an isomorphism, and $\text{Spec} R^G$ is an affine subspace in $\text{Spec} R$ of the codimension $r := \text{rank}(G)$.

Proof. a) Consider first the case when G is a subgroup of rank 1; i.e. G is generated by one non-identical translation, say θ_α ,

$$\theta_\alpha(\xi_j) = \xi_j + \alpha(j), \quad j \in J.$$

Choose i such that $\alpha(i) \neq 0$, and denote by $T\langle\alpha; i\rangle$ the linear map

$$\xi_j \longmapsto \alpha(i)\xi_j - \alpha(j)\xi_i \quad \text{for all } j \in J. \quad (1)$$

Set $\zeta := T\langle\alpha; i\rangle(\xi)$, i.e. $\zeta_j := T\langle\alpha; i\rangle(\xi_j)$ for all $j \in J$. Clearly

$$\theta_\alpha(\zeta_j) = \zeta_j \quad \text{for all } j \in J.$$

Since $\alpha(i) \neq 0$, the operator $T\langle\alpha; i\rangle$ is a surjection onto the subspace of codimension 1.

Denote by R_α the subring of polynomials in $\zeta_j, j \in J - \{i\}$. Let \mathfrak{p} be an arbitrary θ_α -stable prime ideal in R . Set $\mathfrak{p}_\alpha := \mathfrak{p} \cap R_\alpha$; and let K_α denote the field of fractions of the domain $R_\alpha/\mathfrak{p}_\alpha$.

The claim is that \mathfrak{p} is generated by \mathfrak{p}_α : $\mathfrak{p} = (\mathfrak{p}_\alpha)$.

Take the image, \mathfrak{p}' , of the ideal \mathfrak{p} in the quotient ring $R/(\mathfrak{p}_\alpha)$. The

localization at the multiplicative system $R_\alpha - \mathfrak{p}_\alpha$ transfers the ring $R/(\mathfrak{p}_\alpha)$ into the ring $K_\alpha[\xi_i, J]$ and the ideal \mathfrak{p}' into a prime ideal \mathfrak{p}'' in the ring $K_\alpha[\xi_i, J]$. Since $K_\alpha[\xi_i, J]$ is a principal ideal domain, the ideal \mathfrak{p}'' is generated by a polynomial, say f , and the θ_α -stability of the initial ideal \mathfrak{p} is equivalent to the equality

$$f(\xi_i + \alpha(i)) = f(\xi_i). \quad (2)$$

Since $\alpha(i) \neq 0$, the equality (2) is satisfied if and only if $f \in K_\alpha[\xi_i^p, \xi_i, J]$, where $p := \text{char}(k)$.

b) Now, pick another translation, θ_β . Note that

$$\theta_\beta(\zeta_j) = \zeta_j + \{\alpha, \beta\}(i, j) \quad (3)$$

for all $j \in J - \{i\}$. Here

$$\{\alpha, \beta\}(i, j) := \alpha(i)\beta(j) - \alpha(j)\beta(i).$$

The formula (2) shows that $\theta_\beta(\zeta_j) = \zeta_j$ for all $j \in J - \{i\}$ if and only if $\beta = \lambda\alpha$ for some $\lambda \in k$.

So, if α and β are linearly independent, then there is an index $j \in J$, $j \neq i$, such that $\{\alpha, \beta\}(i, j) \neq 0$. And we apply the transformation $T\langle \{\alpha, \beta\}(i, j); j \rangle$ (cf. (1)) to $\{\zeta_t \mid t \in J - \{i\}\}$. And then apply the obtained in a) result to the automorphism θ_β .

c) Etc.. ■

C4.1.2. Stabilizers. Fix a subgroup G of the group $\text{Tr}(R)$. For any finitely generated subgroup H in G , denote by $S(H, G)$ the subgroup of all $t \in G$ such that every H -stable prime ideal in R is t -stable. It is not difficult to find $S(H, G)$.

In fact, $S(H, G) = H^- \cap G$, where $H^- := S(H, \text{Tr}(R))$.

It follows from Lemma C4.1.1 that $H^- = \{\theta_\alpha \mid \alpha \in \text{Vec}(H)\}$, where $\text{Vec}(H)$ denotes the k -vector space spanned on $\{\beta \mid \theta_\beta \in H\}$.

Thus, $S(H, G) = \{\theta_\alpha \mid \alpha \in \text{Vec}(H) \cap G\}$.

C4.2. The case of the algebra of q -differential operators. Consider now a subgroup G of $\text{Aut}(R)$ generated by a set of automorphisms $\{\vartheta_i \mid i \in J\}$ which act as follows:

$$\vartheta_i(\xi_i) = q_i \xi_i + \alpha_i, \quad \vartheta_i(\xi_j) = \xi_j \quad \text{if } i \neq j.$$

Set $J' := \{i \in J \mid q_i \neq 1\}$. For every $i \in J'$, define a new element, ζ_i , by $\zeta_i := \xi_i - \gamma(i)$, where $\gamma(i) := \alpha_i / (1 - q_i)$.

Since $\vartheta_i(\zeta_i) = q_i \zeta_i$ and $\vartheta_j(\zeta_i) = \zeta_i$ if $i \neq j$, the prime ideal $(\zeta_i) := R\zeta_i$ is G -stable.

Denote by $J'(m)$ the set $\{i \in J' \mid q_i^m = 1, q_i^s \neq 1 \text{ for } 1 \leq s \leq m-1\}$, $m \geq 2$; and let $J(0) := \{i \in J' \mid q_i \text{ is not a root of one}\}$. Set $x(i) := \zeta_i^m$ for every $i \in J(m)$.

We assume that $\text{char}(k) = 0$.

Clearly the subalgebra B in R generated by $\{x(i) \mid i \in J\}$ consists of G -stable elements.

Now, fix a subgroup H of the group G . We want to describe the set $\text{Spec}^H R$ of H -stable prime ideals in R .

Clearly the subalgebra $R_{J''}$ generated by $\{\xi_i \mid i \in J'' := J - J'\}$ is G -stable. Let H'' denote the induced by H subgroup of $\text{Aut}(R_{J''})$.

Finally, R_H denotes the subalgebra in R generated by all ξ_i such that $i \in J'$ and ξ_i is H -stable.

According to Lemma C4.1.1, H'' -stable primes of $R_{J''}$ form an affine subspace, $\text{Spec} R_{J'', H''}$ of codimension $r = \text{rank}(H'')$ in $\text{Spec} R_{J''}$.

Thus, we have an embedding

$$\iota_H: \text{Spec} B \amalg \text{Spec} R_H \amalg \text{Spec} R_{J'', H''} \amalg \left(\prod_{i \in J'H} \{(\zeta_i)\} \right) \longrightarrow \text{Spec}^H R,$$

where $J'H := \{i \in J' \mid \xi_i \text{ is not } H\text{-stable}\}$.

C4.2.1. Lemma. *The embedding ι_H is a homeomorphism.*

Proof, similar to that of Lemma C4.1.1, is left to the reader. ■

C4.3. Example: the second Weyl algebra. Consider the second Weyl algebra $A_2(k)$ over a field k of characteristic zero. So that $A_2(k)$ is a hyperbolic ring, $R/(\Theta, \xi_j)$, where

$$\begin{aligned} R &= k[\xi_1, \xi_2]; \quad \theta_i(\xi_j) = \xi_j + \delta_{ij}; \\ x_i x_j &= x_j x_i; \quad x_j y_i = y_i x_j \quad \text{if } i \neq j. \\ x_i y_i &= \xi_i; \quad y_i x_i = \theta_i^{-1}(\xi_i) \quad \text{for } i = 1, 2. \end{aligned}$$

Let G be the subgroup in $\text{Aut}(R)$ generated by θ_1 and θ_2 . Fix a pair (m, n) of nonzero integers, and consider the subgroup $(\vartheta_{m, n})$ in G generated by $\vartheta_{m, n} := \theta_1^m \circ \theta_2^n$.

It follows from C4.1.3, that the following subgroups in G are stabilizers of nonzero prime ideals in $R = k[\xi_1, \xi_2]$:

$$G, (\theta_1), (\theta_2) \text{ and } (\vartheta_{m, n}), \text{ where } (m|n) = 1,$$

Here $(m|n)$ denotes the highest common factor of m and n .

And the stabilizer of any nonzero prime ideal generated by a prime ideal in the algebra $k[\gamma]$, where γ is resp. ξ_1, ξ_2 , or $n\xi_1 - m\xi_2$, coincides with

the corresponding subgroup from the list (1) (cf. Lemma C4.1).

Now, fix a stabilizer $(\theta_1^m \circ \theta_2^n)$, where $(m|n) = 1$. Let \mathfrak{p} be a nonzero prime (hence maximal) ideal in the corresponding ring $k[\gamma]$. Consider the generic case: $m \neq 0, n \neq 0$.

Set $\mathbf{x} := x_1^m x_2^n$ and $\mathbf{y} := y_1^n y_2^m$. The corresponding to the point $\mathfrak{p} = k[\lambda, \gamma]/\mathfrak{p}$ ring is the hyperbolic ring $R(\mathbf{x}, \mathbf{y}; \vartheta, \zeta)$.

Here R is the quotient ring $k[\lambda, \gamma]/(\mathfrak{p}) \simeq K[\lambda]$; (\mathfrak{p}) is the generated by \mathfrak{p} (prime) ideal in the ring $k[\lambda, \gamma]$; K is the quotient field $k[\gamma]/\mathfrak{p}$; $\zeta = \xi_1^m \xi_2^n \text{ mod } (\mathfrak{p})$; ϑ is the induced by $\theta_1^m \circ \theta_2^n$ automorphism of the ring $K[\lambda]$. Clearly $\vartheta f(\lambda) = f(\lambda + n + m)$.

The quotient ring over \mathfrak{p} is isomorphic to the ring $K'[x, x^{-1}; \vartheta]$ of skew Laurent polynomials over the field $K' := K(\lambda)$.

Now consider special cases.

(a) Let \mathfrak{p} be a nonzero prime ideal in $k[\xi_1, \xi_2]$ which is (θ_1) -stable; i.e. $\mathfrak{p} = k[\xi_1, \xi_2]/\mathfrak{p}$, where \mathfrak{p} is some maximal ideal in $k[\xi_2]$. The corresponding to the point \mathfrak{p} hyperbolic ring is $R(\vartheta_1, \xi_1) = R(x_1, y_1; \vartheta_1, \xi_1)$, where

$$R = K[\xi_1], \quad K = k[\xi_2]/\mathfrak{p}, \quad \vartheta_1 f(\xi_1) = f(\xi_1 + 1).$$

The quotient ring over \mathfrak{p} is isomorphic to the ring $K'[x_1, x_1^{-1}; \vartheta_1]$ of skew Laurent polynomials over the field $K' = K(\xi_1)$.

Similarly, any θ_2 -stable nonzero prime ideal in $k[\xi_1, \xi_2]$ is generated by a maximal ideal \mathfrak{p} in $k[\xi_1]$; and the corresponding to \mathfrak{p} hyperbolic ring is

$$R(\vartheta_2, \xi_2) = R(x_2, y_2; \vartheta_2, \xi_2),$$

where $R = K[\xi_2]$, $K = k[\xi_1]/\mathfrak{p}$, $\vartheta_2 f(\xi_2) = f(\xi_2 + 1)$. The quotient ring over \mathfrak{p} is isomorphic to the ring $K'[x_2, x_2^{-1}; \vartheta_2]$ of skew Laurent polynomials over the field $K' = K(\xi_2)$.

(b) The only remaining possibility is the stabilizer equal to the whole group G . Since $\text{char}(k) = 0$, the only G -stable proper ideal in $k[\xi_1, \xi_2]$ is the zero ideal. The localization at $k[\xi_1, \xi_2] - \{0\}$ provides an embedding of the set $\text{Spec}_0 R(\theta_1, \theta_2; \xi_1, \xi_2) \text{-mod}$ into $\text{Spec} K[x_1, x_2, x_1^{-1}, x_2^{-1}; \theta_1, \theta_2]$, where K is the field of rational functions $k(\xi_1, \xi_2)$. ■

C4.4. Example: the second algebra of q -differential operators. Consider now the k -algebra $\mathbb{D}_{q,2}$ of q -differential operators in two indeterminates with polynomial coefficients. We assume that both $q(1)$ and $q(2)$ are not roots of one.

Consider the subgroup of G generated by the automorphism $\vartheta := \theta_1^m \circ \theta_2^n$, where m and n are nonzero integers. Set $\zeta_i := (\xi_i - \gamma_i)$, where γ_i denotes

$1/(1-q(i)), i = 1, 2$. We have:

$$\vartheta(\zeta_1) = q(1)^m \zeta_1, \quad \vartheta(\zeta_2) = q(2)^n \zeta_2$$

which implies that the only nonzero primes in $k[\xi_1, \xi_2] = k[\zeta_1, \zeta_2]$ stabilized by ϑ are

$$(\zeta_1), (\zeta_2), \text{ and } (\zeta_1, \zeta_2). \quad (4)$$

Note that the stabilizer of each of the ideals (4) is the whole group G .

Thus, the list of stabilizers is pretty short this time:

the group G , and its cyclic subgroups (θ_1) and (θ_2) .

Consider each of these cases:

a) Fix a prime ideal \mathfrak{p} in $k[\xi_1, \xi_2]$ the stabilizer of which is the subgroup (θ_1) . According to Lemma C4.2.1, the ideal \mathfrak{p} is either generated by some maximal ideal p in $k[\xi_2]$ which is not equal to (ζ_2) (cf. the list (4)), or it is generated by a maximal ideal $p \neq (\zeta_2)$ in $k[\xi_2]$ and by (ζ_1) .

The corresponding to the point \mathfrak{p} ring is the hyperbolic ring $R'/(\vartheta, \xi)$. Here $R' = k[\xi_1, \xi_2]/\mathfrak{p}$ is isomorphic either to $K[\xi_1]$, where K is the quotient field $k[\xi_2]/p$, $p := k[\xi_2] \cap \mathfrak{p}$, or just to the field K (if $\mathfrak{p} \cap k[\xi_1] = (\zeta_1)$);

ϑ is the induced by θ_1 automorphism of R' ;

ξ is the image of ξ_1 .

If $\mathfrak{p} \cap k[\xi_1] = 0$, then the hyperbolic ring $R'/(\vartheta, \xi)$ is isomorphic to the ring \mathbb{D}_{q_1} of ordinary q_1 -differential operators with coefficients in $K[\xi_1]$.

And the quotient ring over \mathfrak{p} is isomorphic to the ring $K'[x_1, x_1^{-1}; \theta_1]$ of skew Laurent polynomials over the field $K' := K(\xi_1)$.

If $\mathfrak{p} \cap k[\xi_1] \neq 0$, i.e. $\mathfrak{p} = (p, \zeta_1)$ for some maximal ideal p in $k[\xi_2]$, then R' is the field $K := k[\xi_2]/p$, and the image of ξ_1 in K coincides with the invertible element γ_1 . Clearly ϑ is the identical map. All this implies that the quotient ring over the point \mathfrak{p} is isomorphic to the ring $K[x, x^{-1}]$ of Laurent polynomials over K .

b) Suppose now that the stabilizer of a prime ideal \mathfrak{p} coincides with the group G . There are four points in $\text{Spec} k[\xi_1, \xi_2]$ which have this property (cf. the argument above): 0 , (ζ_1) , (ζ_2) , and (ζ_1, ζ_2) .

Consider each of these possibilities in the inverse order.

(i) $\mathfrak{p} = (\zeta_1, \zeta_2)$. Then there is a natural isomorphism

$$k[\xi_1, \xi_2]/\mathfrak{p} \longrightarrow k, \quad \xi_1 \longmapsto \gamma_1, \quad \xi_2 \longmapsto \gamma_2.$$

The corresponding to the point \mathfrak{p} ring is isomorphic to the ring $k[x_1, x_2, x_1^{-1}, x_2^{-1}]$ of Laurent polynomials over k .

(ii) $\mathfrak{p} = (\zeta_1)$. Then $k[\xi_1, \xi_2]/\mathfrak{p} \cong k[\xi_2]$. The quotient ring over \mathfrak{p} is isomorphic to the ring $K[x_2, x_2^{-1}; \theta_2]$ of skew Laurent polynomials over the field $K := k(\xi_2)$.

Dually, if $\mathfrak{p} = (\zeta_2)$, then $k[\xi_1, \xi_2]/\mathfrak{p} \cong k[\xi_1]$; and the quotient ring over \mathfrak{p} is isomorphic to the ring $K[x_1, x_1^{-1}; \theta_1]$ of skew Laurent polynomials over the field $K := k(\xi_1)$.

(iii) There remains the case $\mathfrak{p} = 0$. So, we localize at the set $k[\xi_1, \xi_2] - \{0\}$. The corresponding ring is isomorphic to the ring $K[x_1, x_2, x_1^{-1}, x_2^{-1}; \theta_1, \theta_2]$ of skew Laurent polynomials over the field $K := k(\xi_1, \xi_2)$ of rational functions in two variables.

C4.5. Quantum Weyl algebras. Denote by $\mathfrak{Xc}(R)$ the set of morphisms

$$\langle \mathbf{q}, \alpha \rangle: R \longrightarrow R, f(\xi, z) \longmapsto f(\mathbf{q}\xi + \alpha z, \mathbf{q}z),$$

where \mathbf{q} is a function from J to k^* , and α is a function from J to k . Here $\xi = (\xi_j)$, $z = (z_j)$; so that

$$(\mathbf{q}\xi + \alpha z)_i = \mathbf{q}(i)\xi_i + \alpha(i)z_i.$$

Clearly $\mathfrak{Xc}(R)$ is a subgroup in $\text{Aut}(R)$.

C4.5.1. Lemma. Suppose that $\text{char}(k)=0$. Let G be a subgroup in $\mathfrak{Xc}(R)$ generated by a set of automorphisms

$$\langle \mathbf{q}, \alpha \rangle: f(\xi, z) \longmapsto f(\mathbf{q}\xi + \alpha z, \mathbf{q}z)$$

in G such that, for any $i \in J$, $\mathbf{q}(i) = 1$, if $\alpha(i) = 0$.

Then the natural map $\text{Spec}R \longrightarrow \text{Spec}R^G$ induces a homeomorphism of the subspace $\text{Spec}^G R$ of G -stable points of $\text{Spec}R$ onto $\text{Spec}R^G$.

Proof. It suffices to prove the assertion in the case when G is a cyclic subgroup generated by an automorphism Θ . Set

$$J(\Theta) := \{j \in J \mid \Theta(\xi_j) \neq \xi_j\}, \text{ and } J\perp(\Theta) := J - J(\Theta).$$

Fix a Θ -stable prime ideal \mathfrak{p} in R . Set $\mathfrak{p}' := R_{J(\Theta)} \cap \mathfrak{p}$ and $R' := R/\mathfrak{p}'$.

The localization at $R_{J(\Theta)}/\mathfrak{p}' - \{0\}$ transfers the ideal $\mathfrak{p}/R_{\mathfrak{p}'}$ of the ring R' into a prime ideal \mathfrak{p}'' of the ring $K''[(\xi_j \mid j \in J\perp(\Theta))]$, where K'' denotes the field of rational functions (over k) in variables ξ_t , $t \in J(\Theta)$. Clearly $\mathfrak{p} = R_{\mathfrak{p}'}$ iff $\mathfrak{p}'' = 0$.

Therefore we can (and will) assume that $J(\Theta) = \emptyset$, hence K'' coincides with k , \mathfrak{p}'' with \mathfrak{p} . And what we need to show is that \mathfrak{p} equals to zero.

(a) Note first that the intersection of \mathfrak{p} with the subring of polynomials in $(z_i \mid i \in J)$ is zero.

Indeed, $p := \mathfrak{p} \cap k[(z_i)]$ is G -stable, and the restriction of G to the subring $k[(z_i)]$ consists of contractions, $\theta_{\mathfrak{q}, 0}$, where \mathfrak{q} runs through a certain subgroup, H , of $(k^*)^J$. Since, for every $i \in J$, there is $\mathfrak{q} \in H$ such that $\mathfrak{q}(i)$ is not a root of one, it follows from Lemma C4.2.1 that p is either zero, or it is generated by $\{z_i \mid i \in J'\}$ for some subset J' in J . But, the second issue cannot happen, because all z_i are invertible.

(b) So, the localization at the set $k[(z_i)] - \{0\}$ transfers the prime ideal \mathfrak{p} into a prime ideal \mathfrak{p}' in the polynomial ring $K[(\xi_i)]$ over the field $K = k((z_i))$ of rational functions in the indeterminates (z_i) . And $\mathfrak{p} = 0$ if and only if $\mathfrak{p}' = 0$.

(c) Now, we shall show by induction in $|J|$ that $\mathfrak{p}' = 0$.

1) Let $|J| = 1$; i.e. $R = k(z)[\xi]$, and Θ acts by

$$\Theta f(\xi, z) = f(q\xi + \alpha z, qz) \text{ for any } f(\xi, z) \in R.$$

The ideal \mathfrak{p} is generated by a polynomial $f(\xi, z)$, and the Θ -stability of \mathfrak{p} is equivalent to the equality

$$f(q\xi + \alpha z, qz) = af(\xi, z). \quad (1)$$

for some rational function $a = a(z)$. Write f as $\sum_{s \geq 0} \xi^s f_s(z)$. The equation (1) is equivalent to

$$f_s(qz)(q\xi + \alpha z)^s = a(z)f_s(z)\xi^s$$

for all nonnegative integers s . Since $\alpha \neq 0$, and $\text{char}(k) = 0$, the only solution of (5) for $s \geq 1$ is $f_s \equiv 0$; i.e. f does not depend on ξ ; hence f is zero.

2) Let now J be a finite set. Let J'' be the union of all subsets I of J such that the ideal \mathfrak{p}' has zero intersection with the subring of polynomials in ξ_i , $i \in I$. Localization at $K[(\xi_i \mid i \in J'')]$ sends the ideal \mathfrak{p}' into a nonzero prime ideal \mathfrak{p}^- in the ring of polynomials in $(\xi_i \mid i \in J - J'')$ over the field $K' := K((\xi_i \mid i \in J''))$ of rational functions in $(\xi_i \mid i \in I)$.

Note that, since $\mathfrak{p}^- \neq 0$, the set $J - J''$ is nonempty. By definition of J'' , for any $i \in J$, the intersection $\mathfrak{p}^- \cap K'[\xi_i]$ is a nonzero prime (hence maximal) ideal in $K'[\xi_i]$ which is stable under the automorphism Θ .

$$\Theta: f(\xi_i) \longmapsto \vartheta f(q(i)\xi_i + \alpha(i)z_i).$$

The ideal $\mathfrak{p}^- \cap K'[\xi_i]$ is generated by an irreducible polynomial, say $f = f(\xi_i)$; and the Θ -stability of \mathfrak{p}^- means that $\Theta f = af$ for some $a \in K'$; i.e.

$$\vartheta f(q(i)\xi_i + \alpha(i)z_i) = af(\xi_i). \quad (2)$$

Let D denote the derivative with respect to ξ_i . Clearly the operator D

commutes with ∂ . So, applying D to both parts of the equation (2), one gets the system:

$$q(i)^s \partial D^s f(\alpha(i)z_i) = a D^s f(0), \quad s \geq 0. \quad (3)$$

By the induction assumption, $D^s f(0) \in k$ for every $s \geq 0$. Since $\text{char}(k) = 0$, this implies that $f \in k[\xi_i]$. In particular, $\partial f = f$; i.e. the equation (2) is

$$f(q(i)\xi_i + \alpha(i)z_i) = af(\xi_i). \quad (4)$$

But, this case was already considered in 1). ■

C4.5.2. Stabilizers. Now, given a subgroup G in $\text{Aut}(R)$ such that any element $\langle q, \alpha \rangle \in G$ has the property

$$\alpha(i) = 0 \Rightarrow q(i) = 1,$$

it is easy to describe stabilizers of prime ideals in G :

just single out a subset $I \subseteq J$, and take the subgroup $G(I)$ of all $\langle q, \alpha \rangle \in G$ such that $\alpha(i) = 0$ (hence $q(i) = 1$) for all $i \in I$.

C4.5.3. The second quantum Weyl algebra. The description of the spectrum of the algebra $W_{q,2}(k)$ follows the same pattern as in the case of the second Weyl algebra $A_2(k)$ (cf. C4.3). We leave details to the reader.

C5. A REMARK ABOUT RELATIONS BETWEEN THE SPECTRUM OF REDUCTIVE AND (QUANTIZED) KAC-MOODY LIE ALGEBRAS AND THE SPECTRUM OF CERTAIN HYPERBOLIC RINGS.

Theorem 6.6.3 shows that the complexity of representation theory (or local algebra in our approach) is concentrated in the stabilizers of points of the spectrum of the base category \mathcal{A} . To see what is going on, consider one of the most important examples: monads associated to the enveloping algebras of Kac-Moody Lie algebras and their quantized versions.

C5.1. A hyperbolic ring associated to a Kac-Moody Lie algebra. Fix a Kac-Moody (or a reductive) Lie algebra \mathfrak{g} with the Cartan subalgebra η .

We shall use the notations of 2.5. Our 'base category' \mathcal{A} is the category $U(\eta)\text{-mod}$ of modules over the Cartan subalgebra. Fix a point $\mathbf{P} \in \text{Spec } \mathcal{A}$. The stabilizer $\mathcal{S}_{\mathbf{P}}$ of \mathbf{P} is the submonad generated by all $\Theta(i)$, such that $\Theta(i)(\mathbf{P}) = \mathbf{P}$. Clearly $\mathcal{S}_{\mathbf{P}}$ contains the submonad

$$Z(\eta) := \bigoplus_{i \in \Gamma_0} \Theta(i),$$

where $\Gamma_0 := \{i \in \mathfrak{G} \mid i(\alpha) = i(-\alpha) \text{ for any root } \alpha\}$. In other words, $Z(\eta)$ is isomorphic to $\mathfrak{C}(\eta) \otimes U(\eta)$, where $\mathfrak{C}(\eta)$ is the centralizer of the Cartan subalgebra. And if $\text{rank}(\eta) \geq 2$, $\mathfrak{C}(\eta)$ is a pretty nasty noncommutative ring.

Theorem 6.6.3 shows that we cannot really avoid dealing with this ring, explicitly or implicitly. But, we can control the classes of representations (or points of the spectrum) of $U(\mathfrak{g})$ we want to study by choosing the class of 'admissible' representations of $\mathfrak{C}(\eta)$.

Consider the quotient $\mathfrak{C}'(\eta)$ of the ring $\mathfrak{C}(\eta)$ by its commutant. The ring $\mathfrak{C}'(\eta)$ is isomorphic to the ring of polynomials in ξ_α , $\alpha \in \Delta_+$, with coefficients in the enveloping algebra $S(\eta)$ of the Cartan subalgebra η ; and the isomorphism is given by $x_\alpha x_{-\alpha} \longmapsto \xi_\alpha$ for each positive root α .

Fix a prime ideal \mathfrak{p} in $S(\eta)$. And let ρ be any prime ideal in $\mathfrak{C}'(\eta)$ such that $\rho \cap S(\eta) = \mathfrak{p}$. Now we choose any $\mathbf{M} \in \text{Spec}_\rho(\mathcal{P}_\rho\text{-mod})$ and map it using the canonical functor to $\text{Spec}_\rho(U(\mathfrak{g})\text{-mod})$ (cf. Theorem 6.6.3).

Note that $R = \mathfrak{C}'(\eta)$ can be regarded as a coefficient ring of the hyperbolic ring $R/\langle \theta, \xi \rangle$, where θ is a homomorphism from $\mathbf{Maps}(\Delta_+, Z)$ to $\text{Aut}(R)$ given by:

$$\begin{aligned} \theta_\alpha \xi_\beta &= \xi_\beta + \delta_{\alpha, \beta} h_\alpha \text{ for any } \alpha, \beta \in \Delta_+. \\ x_\alpha x_{-\alpha} &= \xi_\alpha \text{ for all } \alpha \in \Delta_+. \end{aligned}$$

Denote this hyperbolic ring by $\mathcal{H}(\mathfrak{g})$.

Since the stabilizer $\mathcal{H}(\mathfrak{g})_\rho$ of the ideal ρ in $R/\langle \theta, \xi \rangle$ and \mathcal{P}_ρ have naturally isomorphic bases (of monomials), they are isomorphic as R -modules. But not as rings.

(a) **A generic point.** Denote by $X_0(\eta)$ the set of $\rho \in \text{Spec} S(\eta)$ such that $\mathcal{P}_\rho = \mathfrak{C}(\eta)$, hence, $\mathcal{H}(\mathfrak{g})_\rho = \mathfrak{C}'(\eta)$. Theorem 6.6.3 provides injective maps:

$$\text{Spec}_\rho(\mathcal{H}(\mathfrak{g})\text{-mod}) \ni (R/\rho) \longleftarrow X_0(\eta) \longrightarrow \text{Spec}_\rho(U(\mathfrak{g})\text{-mod}) \quad (1)$$

In other words, in a generic case, points of the spectrum of $U(\mathfrak{g})$ are in one-to-one correspondence with points of the spectrum of the associated hyperbolic ring.

(b) **Degenerate cases.** The relation with the spectrum of $R/\langle \theta, \xi \rangle\text{-mod}$ still might hold when the stabilizer of $\rho \in \text{Spec} R$ is nontrivial. This is clear for the stabilizer generated by θ_α , $\alpha \in \Delta_+$, since in this case the stabilizer in the hyperbolic ring is generated by R , x_α , and $x_{-\alpha}$. The stabilizer in $U(\mathfrak{g})$ corresponding to the pair (\mathfrak{p}, ρ) is generated by $\mathfrak{C}(\eta)$, x_α , and $x_{-\alpha}$. So that, being restricted to the corresponding subcategory, it acts as $R/\langle \theta_\alpha, \xi_\alpha \rangle$.

A higher rank case: let the stabilizer be generated by a subset $\{\theta_\alpha \mid \alpha \in X\}$ such that $\{x_\alpha \mid \alpha \in X\}$ pairwise commute. Then the stabilizer in $U(\mathfrak{g})$ of the

A.2. Special cases. If $q = p$, then $\theta = id$; so that t and u commute between themselves and with elements of R ; and D is a central element in the ring $M_{p,q}(2)$. In this case $M_{p,q}(2)$ is a hyperbolic ring over $k[t,u,D]$ corresponding to the automorphism

$$\vartheta': f(t,u,D) \longmapsto f(qt,qu,D)$$

and the element $D + qtu$; i.e. $xy = D + qtu$. One can see that $M_{q,q}(2)$ coincides with $M_q(2)$, and D is the q -determinant.

If $q = p^{-1}$, then ϑ is identical; so that x and y commute with elements of R and between themselves. And $M_{p,q}(2)$ is a hyperbolic ring over $k[x,y,D]$ with generators t, u corresponding to the automorphism

$$\theta': f(x,y,D) \longmapsto f(qx,qy,q^2D)$$

and the element $(xy - D)/p$: $tu = q(xy - D)$.

A.3. The spectrum. The ring $M_{p,q}(2)$ given by the relations (6) - (9) in A.1 is not hyperbolic if $q \neq p \neq q^{-1}$. But, of course, it is a skew PBW ring.

We assume that $q \neq p \neq q^{-1}$, and proceed following the general scenario imposed by Theorem 6.6.3.

A.3.1. Stable points. There are the following stable points (with respect to θ and ϑ) in $SpecR$, $R = k[\xi, D]$:

- (a) the generic point 0 ;
- (b) (D) ;
- (c) (η) ;
- (d) (η, D) .

Note that, for any stable point p of $SpecR$, the left ideal generated by p in $\mathcal{R} = M_{p,q}(2)$ is two-sided; i.e. $\mathcal{R}p$ defines a closed subscheme which deserves a special attention.

Consider each of these stable points in the inverse order.

(d) The corresponding ideal is maximal, and the generated by it left ideal in $M_{p,q}(2)$ is two-sided. The quotient algebra is given by relations:

$$xy = yx = 0; \quad tu = ut = 0.$$

(c) Here the relations are:

$$xy = D = yx.$$

The quotient algebra $R/(\eta)$ is naturally isomorphic to $k[D]$; and t, u generate a ring over $k[D]$ defined by the relations:

$$tu = o = ut. \tag{1}$$

pair (\mathfrak{p}, ρ) is generated by $\mathfrak{E}(\eta)$ and $\{x_\alpha, x_{-\alpha} \mid \alpha \in X\}$ and acts on modules of the corresponding subcategory as the hyperbolic ring $R(\Theta, \xi)$ of rank $|X|$ defined by the data $\{\theta_\alpha, \xi_\alpha \mid \alpha \in X\}$.

C5.2. Quantized enveloping algebras and hyperbolic rings. Consider now the quantized enveloping algebra $U_q(\mathfrak{g})$ of a Kac-Moody Lie algebra \mathfrak{g} . Of course, we have a situation similar to that of the non-quantized case. Only instead of the polynomial ring $U(\eta) = S(\eta)$, we should consider the quantized version of enveloping algebra of the Cartan subalgebra - Laurent polynomials; and the automorphisms θ_α are defined differently. The corresponding hyperbolic ring, denote it by $\mathcal{H}_q(\mathfrak{g})$, can be regarded as a quantized version of the introduced in C5.1 hyperbolic ring $\mathcal{H}(\mathfrak{g})$.

APPENDIX: TWO-PARAMETER DEFORMATIONS OF $M(2)$ AND $GL(2)$.

We begin with $M_{p,q}(2)$. By definition [SW], this is a k -algebra with generators x, y, t, u subject to the following relations:

$$xt = ptx, \quad xu = qux; \quad xy = yx + (p - 1/q)tu \quad (1)$$

$$tu = (q/p)ut, \quad ty = qyt, \quad uy = puy. \quad (2)$$

A.1. Rewriting the relations. Set $\eta := tu$, $D := xy - p\eta$. We have:

$$\eta t = (p/q)t\eta, \quad \eta u = (q/p)u\eta; \quad (1)$$

$$\eta x = (1/qp)x\eta, \quad \eta y = (qp)y\eta \quad \alpha; \quad (2)$$

One can check that

$$Dx = xD, \quad Dy = yD, \quad (3)$$

$$Dt = (p/q)tD, \quad Du = (q/p)uD. \quad (4)$$

Now take $R = k[\eta, D]$ ($= k[\xi, \eta]$). Define automorphisms ϑ and θ of the ring R by

$$\vartheta f(\xi, D) = f(a\eta, D), \quad \theta f(\eta, D) = f(b\eta, bD) \quad (5)$$

where $a = 1/qp$, $b = q/p$. Note that $\vartheta \circ \theta = \theta \circ \vartheta$.

Now the defining relations can be rewritten as follows:

$$xy = D + p\eta, \quad yx = \vartheta^{-1}(D + p\eta); \quad (6)$$

$$xr = \vartheta(r)x, \quad ry = y\vartheta(r); \quad tr = \theta(r)t, \quad ru = u\theta(r) \quad (7)$$

for all $r \in R$;

$$tu = \eta, \quad ut = \theta^{-1}\eta; \quad (8)$$

$$xt = ptx, \quad xu = qux; \quad ty = qyt, \quad uy = puy. \quad (9)$$

Note that each of the pairs (x, y) and (t, u) generates a hyperbolic ring over R .

$$tr = \theta'(r)t, \quad ru = u\theta'(r) \quad (2)$$

for all $r \in k[D]$, where $\theta'(r)(D) = r(bD)$. It follows from the relations (1), (2) that the closed subscheme defined by the generated by D ideal is homeomorphic to the fibered coproduct $\text{Spec}k_{b,t}[D] \amalg_{\text{Spec}k[D]} \text{Spec}k_{1/b}[u,D]$ over $\text{Spec}k[D]$ of b - and $1/b$ -quantum planes.

The fiber of \mathcal{R} over (η) is $\text{Spec}k_{b,t}(D)[t] \amalg_{\text{Spec}k(D)} \text{Spec}k_{1/b}(D)[u]$.

(b) We identify the quotient algebra $R/(D)$ with $k[\eta]$. The relations describing the closed subscheme defined by (D) are

$$xy = p\eta, \quad yx = p\vartheta^{-1}(\eta); \quad (3)$$

$$xr = \vartheta'(r)x, \quad ry = y\vartheta'(r); \quad tr = \theta'(r)t, \quad ru = u\theta'(r) \quad (4)$$

for all $r \in k[\xi]$; here $\vartheta'(\eta) = a\eta$, $\theta'\eta = b\eta$;

$$tu = \eta, \quad ut = \theta^{-1}\eta; \quad (5)$$

$$xt = ptx, \quad xu = qux; \quad ty = qyt, \quad uy = pyu. \quad (6)$$

Note that the relations (3)-(6) describe a 4-dimensional quantum space.

To get the fiber of $\text{Spec}\mathcal{R}\text{-mod}$ ($\simeq \text{Spec}_f\mathcal{R}$) over (D) , we should go to the residue field $k(\eta)$ of the point (D) . The relations are the same, but over $k(\eta)$. In particular, η becomes invertible which allows to get rid of half of the variables. We drop y and u . The remaining relations are:

$$xr = \vartheta'(r)x, \quad tr = \theta'(r)t \quad (7)$$

for all $r \in k(\eta)$, $\vartheta'(\eta) = a\eta$, $\theta'\eta = b\eta$; and

$$xt = ptx. \quad (8)$$

These are equations of an iterated skew polynomial ring.

(a) The fiber of $\text{Spec}\mathcal{R}\text{-mod}$ over the generic point (0) is the spectrum of (the category of left modules over) the ring described by the relations (6)-(9) in A.1, but with $R = k[\eta, D]$ replaced by its fraction field $k(\eta, D)$. Thus, we can drop variables y and u . The remaining relations are:

$$xr = \vartheta(r)x, \quad tr = \theta(r)t; \quad (9)$$

for all $r \in k(\eta, D)$;

$$xt = ptx. \quad (10)$$

Here $\vartheta f(\eta, D) = f(a\eta, D)$, $\theta f(\eta, D) = f(b\eta, bD)$ for all rational functions $f(\eta, D)$ (cf. (5) in A.1).

Again, we have an iterated skew polynomial ring (as in the case (b)), but, with $k(\eta)$ replaced by $k(\eta, D)$.

A.3.2. Partially stable points. Any irreducible polynomial f in D defines a ϑ -stable point in $\text{Spec}\mathcal{R}$. Suppose that $f(0) \neq 0$, and $f(b^m D) \neq f(D)$ for any

m (e.g. b is not a root of one), then the subgroup (ϑ) generated by ϑ is the stabilizer of the prime ideal (f) . The corresponding subring $\mathcal{R}_{(f)}$ is generated over $R/(f) = K[\eta]$ (here $K = k[D]/(f)$) by x, y subject to the relations:

$$xy = D + p\eta, \quad yx = \vartheta^{-1}(D + p\eta) = D + a^{-1}p\eta \quad (1)$$

$$xr = \vartheta'(r)x, \quad ry = y\vartheta'(r) \quad (2)$$

where $\vartheta'f(\eta) = f(a\eta)$ for any $f \in K[\eta]$.

Thus, $\mathcal{R}_{(f)}$ is a hyperbolic ring over $K[\eta]$, and its spectrum can be described explicitly using results of [R4]. Note that we need only the part of the spectrum which 'sits' over the generic point of $K[\eta]$. This part coincides with the spectrum of a skew polynomial ring given by the relations

$$xr = \vartheta''(r)x \quad \text{for all } r \in K(\eta) \quad (3)$$

where $\vartheta''r(\eta) = r(a\eta)$.

In the *generic* cases (i.e. when the equality $a^n b^m = 1$ implies that $m = n = 0$), the primes listed above are the only points of $\text{Spec}R$ having nontrivial stabilizer.

A.3.2. Some of the degenerate cases. For all nonnegative integers m, n and any $f \in R$, we have:

$$\theta^m \vartheta^n f(\eta, D) = f(a^n b^m \eta, b^m D). \quad (1)$$

where $a = 1/qp$, $b = q/p$.

θ^m -stable points. If $b = q/p$ is an m -th root of one, then any prime ideal in $R = k[\eta, D]$ generated by polynomials $f(\eta, D) = \sum \eta^i D^j a_{ij}$ such that $a_{ij} = 0$ if $i+j \not\equiv m$ is θ^m -stable. The 'stabilizing' subring $\mathcal{R}_{(f)}$ is generated over the quotient $R/(f) =$ by $z := t^m$ and $w := u^m$ satisfying the relations:

$$zw = b^{m-1}\eta, \quad ut = b^{1-m}\eta;$$

$$zr = \theta^m(r)t, \quad rw = w\theta^m(r)$$

for all $r \in R/(f)$.

If (θ^m) is the stabilizer of (f) , then, by Theorem 6.6.3, the fiber of the spectrum of \mathcal{R} -mod over (f) is in one-to-one correspondence with the points of the spectrum of the skew polynomial ring given by

$$zr = \theta^m(r)z \quad \text{for all } r \in \mathcal{K},$$

where \mathcal{K} is the field of fractions of $R/(f)$, θ' the automorphism of \mathcal{K} induced by θ .

ϑ^n -stable points. If $a = 1/qp$ is an n -th root of 1, then any prime ideal in R generated by a polynomial f in η^n and D is ϑ^n -stable. If (ϑ^n) is a stabilizer of (f) , then, as above, the (left) spectrum of the skew polynomial

ring in one variable over a skew field (residue field of (f)) describes the fiber of $\text{Spec}\mathcal{R}\text{-mod}$ over (f) .

$\vartheta^n\theta^m$ -stable points. If $a^n b^m = 1$, then any irreducible polynomial $f \in k[\eta]$ such that $f(0) \neq 0$ defines a $\vartheta^n\theta^m$ -stable prime ideal. We leave to a reader the defining of the skew polynomial ring responsible for the fiber over (f) .

Other degeneracies which might occur lead to the situations listed above, only coefficients change.

A.4. The ring $GL_{q,p}(2)$. It follows from the relations (3), (4) in A.1 that the multiplicative set $(D) := \{D^n \mid n \geq 0\}$ satisfies (right and left) Ore conditions. By definition, the algebra $GL_{q,p}(2)$ is the localization of $M_{q,p}(2)$ at the Ore set (D) . In other words, $GL_{q,p}(2)$ is the ring corresponding to the complement of the closed subset defined by the ideal $M_{q,p}(2)D$.

The defining relations are given by the equalities (6)-(9) in A.1, only the ring R should mean $k[\eta, D, D^{-1}]$.

References

- [AS] M. Artin, W. Shelter, Graded algebras of global dimension 3, *Advances in Mathematics* 66 (1987), 171-216.
- [ATvdB1] M. Artin, J. Tate, M. van den Berg, Some algebras associated to automorphisms of elliptic curves, *The Grothendieck Festschrift*, v.1, Birkhouser, Boston, 1990, pp. 33-85.
- [ATvdB2] M. Artin, J. Tate, M. van den Berg, Modules over regular algebras of dimension 3, *Invent. Math.* 106 (1991), 335-388.
- [AvdB] M. Artin, M. van den Berg, Twisted homogenous coordinate rings, *J. Algebra* 133 (1990), 249-271.
- [APK] H. H. Andersen, P. Polo, W. Kexin, Representations of quantum algebras, Preprint (1990).
- [AJS] H. H. Andersen, J. C. Jantzen, W. Soergel, Representations of quantum groups at a p -th root of unity and of semisimple groups in characteristic p : independence on p , Preprint (1993).
- [B1] R. Block, The irreducible representations of the Lie algebra $sl(2)$ and of the Weyl algebra, *Advances in Mathematics*, 39, 69-110 (1981).
- [B2] R. Block, The irreducible representations of the Weyl algebra A_1 , in "Sèminaire d'Algèbre Paul Dubreil" (M.P. Malliavin, Ed), LNM 740, pp. 69-79, Springer Verlag, 1979.
- [BD] I.Bucur, A.Deleanu, Introduction to the Theory of Categories and Functors, Pure and Appl. Math., V.XIX, John Wiley & Sons LTD, London-New York-Sydney, 1969.
- [B] N. Bourbaki, Algèbre commutative, Hermann, Paris, 1965.
- [CK] C.De Concini, V.Kac, Representations of quantum groups at root of 1, Preprint, 1990
- [D1] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France* 96 (1968), 209-242.
- [D2] J. Dixmier, Sur les algèbres de Weyl, *Bull. Soc. Math. France* 94 (1970), 289-301.
- [D3] J. Dixmier, Quotients simples de l'algèbre enveloppante de sl_2 , *J. Algebra* 24 (1973), 551-564.
- [D4] J. Dixmier, Algèbres Enveloppantes, Gauthier-Villars, Paris/Bruxelles/Montreal, 1974.
- [Dr] V.G. Drinfeld, Quantum Groups, Proc. Int. Cong. Math., Berkeley (1986), 798-820.
- [FRT] L. Faddeev, N. Reshetikhin, L. Takhtajan, Quantization of Lie Groups and

- Lie Algebras, preprint, LOMI -14-87; Algebra and Analysis, vol.1, no. 1 (1989).
- [Gab] P.Gabriel, Des catégories abéliennes, Bull. Soc. Math. France, 90 (1962), 323-449
- [GZ] P. Gabriel, M. Zisman, Calculus of Fractions and homotopy theory, Ergebnisse der Mathematik, Vol. 35, Berlin - Heidelberg - New York; Springer Verlag (1967)
- [J] M. Jimbo, A q -Difference Analog of $U(\mathfrak{g})$ and the Yang-Baxter Equation, Lett. Math. Phys. 10, (1985), 63-69.
- [KS] E.E. Kirkman, L.W. Small, q -Analog of harmonic oscillators and related rings, Preprint, UCSD, 1990.
- [M1] Yu.I. Manin, Quantum Groups and Non-commutative Geometry, CRM, Université de Montréal (1988).
- [M2] Yu. I. Manin, Topics in Noncommutative Geometry, Princeton University Press, Princeton New Jersey (1991).
- [M4] Yu. I. Manin, Multiparametric quantum deformation of the general linear supergroup, Comm. Math. Phys. 123 (1989), 123-135.
- [Ma] M. P. Malliavin, L'algèbre d'Heisenberg quantique, to appear in Bull., Sc. Math. France.
- [MNSU] T.Masuda, K.Mimachi, Yo.Nakagami, M.Noumi, Yu.Saburi, K.Ueno, Unitary Representations of the Quantum Group $SU_q(1,1)$ I, II, Letters in Math. Physics 19: 187-194, 195-204 (1990).
- [R1] A.L.Rosenberg, Noncommutative Affine Schemes and Schemes, 'Seminar on Supermanifolds', v.26, Reports of Dept. of Math. of Stockholm Univ., 1988, 1-317.
- [R2] A.L. Rosenberg, Noncommutative affine schemes, Preprint MPI/53, 1994
- [R3] A.L. Rosenberg, The spectrum of irreducible representations of the hyperbolic algebra and the skew enveloping algebra of $sl(2)$, Preprint, Harvard University, 1990
- [R4] A. L. Rosenberg, Noncommutative Local Algebra, Preprint MPI/8, 1994, (to appear in GAFA).
- [R5] A.L. Rosenberg, Six spectra and two dimensions of an abelian category, Preprint MPI/16, 1994
- [R6] A.L. Rosenberg, Noncommutative local algebra and representations of certain rings of mathematical physics, Preprint MPI, 1994
- [R7] A.L.Rosenberg, Left Spectrum, Levitzki Radical and Noncommutative Schemes, Proc. of Natnl. Acad., v.87, 1990
- [R8] A.L. Rosenberg, The Spectrum of the Algebra of Skew Differential Operators

- and the Irreducible Representations of the Quantum Heisenberg Algebra, *Commun. Math. Phys.* 142, 567-588, 1991
- [R9] A. L. Rosenberg, The unitary irreducible representations of the quantum Heisenberg algebra, *Commun. Math. Phys.* 144, 41-51, 1992.
- [R10] A. L. Rosenberg, The left spectrum and irreducible representations of 'small' quantized and classical rings, Preprint MPI/41, 1994
- [Ro] M. Rosso, Finite-dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra, *Comm. Math. Phys.* 117 (1988), 581-593.
- [S1] S.P. Smith, A class of algebras similar to the enveloping algebra of $sl(2)$, *Trans. Amer. Math. Soc.*, 322 (1990) 285-314.
- [S2] S.P. Smith, Quantum Groups: an Introduction and Survey for Ring Theorists, preprint. (1990)
- [SB] S.P. Smith, A.D. Bell, Some 3-dimensional skew polynomial rings, Preprint (1991).
- [Sk] E. K. Sklyanin, On algebraic structures connected with Yang-Baxter equations, *Func. Analiz* 16 (1982), 22-34.
- [SW] X.D. Son, S.K. Wang, Bicovariant differential calculus on the two-parameter quantum group $GL_{p,q}(2)$, *J. Math. Phys.* 33(10), 1992.
- [VS] L. L. Vaksman, Ya. S. Soibelman, The algebra of functions on quantized $SU(2)$, *Func. Anal. and Appl.* 22 (1988), 170-181.
- [V] J-L. Verdier, Groupes quantiques, *Seminaire Bourbaki* (1986-87) no 685.
- [W1] S.L. Woronowicz, Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *Publ. RIMS, Kyoto Univ.*, 23 (1987) 117-181.
- [W2] S.L. Woronowicz, Compact matrix pseudogroups, *Commun. Math. Phys.* 111 (1987), 613-615.

commutes with ϑ . So, applying D to both parts of the equation (2), one gets the system:

$$q(i)^s \vartheta D^s f(\alpha(i)z_i) = a D^s f(0), \quad s \geq 0. \quad (3)$$

By the induction assumption, $D^s f(0) \in k$ for every $s \geq 0$. Since $\text{char}(k) = 0$, this implies that $f \in k[\xi_i]$. In particular, $\vartheta f = f$; i.e. the equation (2) is

$$f(q(i)\xi_i + \alpha(i)z_i) = af(\xi_i). \quad (4)$$

But, this case was already considered in 1). ■

C4.5.2. Stabilizers. Now, given a subgroup G in $\mathfrak{Ic}(R)$ such that any element $\langle q, \alpha \rangle \in G$ has the property

$$\alpha(i) = 0 \Rightarrow q(i) = 1,$$

it is easy to describe stabilizers of prime ideals in G :

just single out a subset $I \subseteq J$, and take the subgroup $G(I)$ of all $\langle q, \alpha \rangle \in G$ such that $\alpha(i) = 0$ (hence $q(i) = 1$) for all $i \in I$.

C4.5.3. The second quantum Weyl algebra. The description of the spectrum of the algebra $W_{q,2}(k)$ follows the same pattern as in the case of the second Weyl algebra $A_2(k)$ (cf. C4.3). We leave details to the reader.

C5. A REMARK ABOUT RELATIONS BETWEEN THE SPECTRUM OF REDUCTIVE AND (QUANTIZED) KAC-MOODY LIE ALGEBRAS AND THE SPECTRUM OF CERTAIN HYPERBOLIC RINGS.

Theorem 6.6.3 shows that the complexity of representation theory (or local algebra in our approach) is concentrated in the stabilizers of points of the spectrum of the base category \mathcal{A} . To see what is going on, consider one of the most important examples: monads associated to the enveloping algebras of Kac-Moody Lie algebras and their quantized versions.

C5.1. A hyperbolic ring associated to a Kac-Moody Lie algebra. Fix a Kac-Moody (or a reductive) Lie algebra \mathfrak{g} with the Cartan subalgebra η .

We shall use the notations of 2.5. Our 'base category' \mathcal{A} is the category $U(\eta)\text{-mod}$ of modules over the Cartan subalgebra. Fix a point $\mathbf{P} \in \text{Spec } \mathcal{A}$. The stabilizer $\mathcal{S}_{\mathbf{P}}$ of \mathbf{P} is the submonad generated by all $\Theta(\mathbf{i})$, such that $\Theta(\mathbf{i})(\mathbf{P}) = \mathbf{P}$. Clearly $\mathcal{S}_{\mathbf{P}}$ contains the submonad

$$Z(\eta) := \bigoplus_{\mathbf{i} \in \Gamma_0} \Theta(\mathbf{i}),$$