# ORBIFOLD-UNIFORMIZING DIFFERENTIAL EQUATIONS

## II Absense of acessory parameters

## III Arrangements defined by 3-dimensional primitive unitary reflection groups

by

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# Orbifold-uniformizing differential equations II

## - Absence of accessory parameters -

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§0 Introduction In this note, we show the absence of accessory parameters for some orbifold-uniformizing differential equations (OUDE). Accessory parameters for ordinary differential equations have a long history. For Fuchsian systems in several variables however, there are few studies about these. Only known result is their absence for Appell's hypergeometric differential equations ([Pi], [T]) and for some OUDE's ([Y1], [Y2]). Since the notion of accessory parameters is not so familiar, we shall explain it for some type of differential equations in several vaniables. And shall show their absence, in a weak form, for some OUDE's, which is proved by using a generalized version of Weil-Mostow's rigidity theorem (c.f. [R], [M]). To make the story more understandable and to see the contrast between OUDE's in several rariables and those in single variable,

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we shall briefly recall the situation in ordinary differential equations.

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## §1 Ordinary differential equations

Consider the Fuchsian equation

$$* \qquad \frac{d^2w}{dx^2} + p(x)w = 0$$

defined on $M = \mathbb{C}\cup\{\infty\}$ with singular points on $x = x_1, \ldots, x_m = \infty$, where $p(x)$ is a rational function. Let $w_0$ and $W_1$ be two linearly independent solution of * . The multi-valued mapping $\psi : M \to \mathbb{C}P^1$ defined by $x \to w_0(x) : w_1(x)$ is called the projective solution of * , which is determined up to projective transformations. Local property of * at $x = x_j$ is described by a complex number $\alpha_j$, called the exponent of * at $x_j$, which is the square of the difference of the indicial equation of * at $x_j$.

On the contrary, for given $m$ ($\geqq 3$) complex numbers $\alpha_1, \ldots, \alpha_m$, there exists an differential equation * whose exponent at $x_j$ is equal to $\alpha_j$. If $m = 3$ it is uniquely determined, in which case the equation * is equivalent to the hypergeometric equation. If $m \geqq 4$, some coefficients remain undetermined, which are called by a curious name "accessory parameters". For more genaral or modern treatment see [Oh] and [Ok].

§2 Fuchsian systems

Let $U$ be a domain in $\mathbb{C}^n$ with cordinate $x = (x_1, \ldots, x_n)$. Consider a completely integrable Fuchsian system

$$(E)_U : \frac{\partial^2 w}{\partial x_i \partial x_j} = \sum_{k=1}^{n} P_{ij}^k(x) \frac{\partial w}{x_k} + P_{ij}^0(x) w$$

satisfying $\sum_{\ell=1}^{n} P_i{}^{\ell}{}_{\ell} = 0$ $(i=1,\ldots, n)$ with reqular singularity along $A_U$, where $P_{ij}^k$ $(i,j=1,\ldots, n,\ k=0,1,\ldots, n)$ are meromorphic on $U$. Note that this equation reduces to * in §1 if $n = 1$. For $n+1$ linearly independent solutions $w_0, w_1, \ldots, w_n$ of $(E)_U$, we define the projective solution $\psi_U : U \to \mathbb{C}P^n$ by $x \to (w_0(x), w_1(x), \ldots, w_n(x))$. Two systems $(E)_U$ and $(E')_{U'}$ are said to be equivalent at $p \in U \cap U'$ if there is a germ of biholomorphic mapping $g$ from $\psi_U(p)$ to $\psi_{U'}(p)$ such that $\psi_{U'} = g \circ \psi_U$ as a germ at $p$. An equivalence class is called the local behavior at $p$.

Let $M$ be a projective algebraic manifold of complex dimension $n$. A Fuchsian system $(E)$ on $M$ is a collection of equations $(E)_U$ for all open sets $U$ in $M$ such that $\psi_U = \psi_{U'}$ on $U \cap U'$ up to projective transformations. The singular locus $A$ of $(E)$, which is by definition the union of $A_U$ for all $U$, is a hyhypersurface of $M$.
For a given system $(E)$, we consider the set $AP(E)$ of systems which have the same local behavior to that of $(E)$ at all points on $A$, so on $M$. The set $AP(E)$ forms an algebraic variety (c.f. [Y1 ; §4]) and will be called the space of accessory parameters of $(E)$.

## §3 Lattices in $PU(n,1)$ and OUDE

Let $\Gamma \subset PU(n,1)$ be a lattice acting on the unit ball $B_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{j=1}^{n} |z_j|^2 < 1\}$. Put

$M = \overline{B_n/\Gamma}$ a compactification of the orbit space,

$\pi : B_n \to M$ projection,

$A$ : union of critical points of $\pi$ and $M-(B_n/\Gamma)$ .

Assume that $M$ is smooth then there exists a unique system (E) on $M$, called OUDE, such that the projective solution gives an inverse of $\pi$. For more details see [Y1 ; §4], which can easily be generalized to $n$ variables.

We are interested in the variety AP(E). If $n = 1$, as we recalled in §1, AP(E) is a linear space of positive dimension unless $\Gamma$ is a triangle group. (Some classical ploblems are proposed in [Po].) Unlike the case $n = 1$, we shall propose for $n \geq 2$, that {E} is isolated in AP(E). Before stating the theorem we want to note that, for OUDE, at a regular point $p$ on $A$, the ramification index of $\psi$ determines the local behavior (cf. [Y1 ; §3]). Let $\tilde{A}P(E)$ be the set of systems which have the same local behaυior to that of (E) at all regular points on A. The set $\tilde{A}P(E)$ is an algebraic set containing AP(E).

<u>Theorem.</u> Let (E) be an OUDE on $M$ as above. If $n \geq 2$, then the component of $\tilde{A}P(E)$ including {E} is a point.

Remark. One can conjecture that $\tilde{A}P(E)$ itself is equal to {E}.

Some examples ([T], [Y1], [Y2]) support the conjecture.

§4 Proof.

Let $H$ be the fundamental group of $M-A$ with base point $P$, $A = \bigcup_j A_j$ be the decomposition into irreducible components, $\mu_j \in H$ be a normal loop around $A_j$ and $b_j$ be the ramification index of $\psi$ along $A_j$. The group $\Gamma$ is isomorphic to the quotient of $H$ by the minimal normal subgroup $H[\mu^b]$ of $H$ including all $\mu_j^{b_j}$'s ([ K ]).

Let $\{E(t)\}$ be an analytic family in $\widetilde{AP}(E)$ such that $E(0) = E$, and $W(t) = (w_0(t), \ldots, w_n(t))$ be a system of linearly independent solutions of $E(t)$ at $P$, depending holomorphically on $t$, such that $W(0)$ gives an inverse of $\pi$. The system $W(t)$ defines a representation

$$r(t) : H = \pi_1(M-A) \to PGL(n+1, \mathbb{C}),$$

and, for $t = 0$,

$$r(0) : H \to \Gamma(\cong H/_{H[\mu^b]}) \subset PU(n,1) \subset PGL(n+1, \mathbb{C}).$$

On account of the fixed local behavior of $W(t)$, along $A_j$'s, $r(t)$ is trivial on $H[\mu^b]$. Thus $r(t)$ induces a representation $\Gamma \to PGL(n+1, \mathbb{C})$. On the other hand we have the following

Proposition ([G], [JM]) If $\Gamma \subset PU(n,1)$ is a lattice, then every deformation $\psi_t : \Gamma \to PGL(n+1, \mathbb{C})$ (where $\psi_0$ is the inclusion $\Gamma \subset PU(n,1)$) is a trivial deformation (i.e. of the form $\psi_t(x) = g_t \psi_0(x) g_t^{-1}$, where $g_t$ is a path in $PGL(n+1, \mathbb{C})$).

In view of the $PGL(n+1)$-invariance of Schwarzian derivatives (c.f. [Y1]). this implies $E(t) = E(0)$. Q.E.D.

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# Orbifold-uniformizing differential equations III
## - Arrangements defined by 3-dimensional primitive unitary reflection groups -

by Masaaki YOSHIDA

Let $X$ be the complex projective plane and $A$ be a curve in $X$. One can ask the following. "Are there any system of linear differential equations of given rank $r$( = the dimension of the solution space) with singularity only on $A$ ?" Till now there are no theory to answer this question in general. It involves a non-linear problem : To solve a non-linear partial differential equation, which is called the integrability condition. Or from topological point of view, one must study the existence of non-trivial representations of the fundamental group $\pi_1(X-A)$ into $GL(r,\mathbb{C})$.

In this paper I shall construct several systems on $X$ of rank 3, which is the most interesting case, with regular singularity on some line arrangements, which has high symmetricity. To be more precise, let $G$ be a 3-dimensional primitive unitary reflection group acting on $X$ and $A$ be the line arrangement on $X$ defined by the group $G$. I shall construct, for each $G$, $G$-invariant system (E) of rank 3 with ramifying singularity along $A$. Each system

(E) involves one or two parameters. For special values of parameters, the systems (E) give orbifold-uniformizing differential equations ([Y1]) of the orbifolds obtained in (Hö] (cf. [$H_i$]). That is, the mapping defined by the ratio of three linearly independent solutions of the system (E) gives an equivalence between a covering of X, branching on A, and the unit ball in $\mathbb{C}^2$.

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## §1 Basic facts for differential equations ([Y1 ; §1,3,4], [Y3])

We consider the following completely integrable system

$$\text{(E)} \qquad \frac{\partial^2 w}{\partial x^i \partial x^j} = \sum_{k=1}^{2} P_{ij}^k(x)\frac{\partial w}{\partial x^k} + P_{ij}^0(x)w \qquad i,j = 1,2$$

with independent variables $x = (x^1,x^2) \in \mathbb{C}^2$ and an unknown $w$. Let $w_0$, $w_1$ and $w_2$ be linearly independent solutions of (E). The projective solution $\psi$ of (E) is defined by the mapping $x \to (w_1(x)/w_0(x),\ w_2(x)/w_0(x))$.

Definition. A system (E) satisfying the condition

$$\text{(1.1)} \qquad \sum_{i=1}^{2} P_{i\ell}^\ell(x) = 0 \qquad i = 1,2$$

will be said to be of canonical form.

Any system (E) can be transformed, without changing the projective solution, that is by a suitable change of unknown

$$w \to a(x)w,$$

uniquely into a system of canonical form. The coefficients change as follows :

$$\text{(1.2)} \qquad \begin{aligned} P_{ij}^k &\to P_{ij}^k + \delta_i^k\, P_j/3 - \delta_j^k\, P_i/3, \\ & \hspace{16em} i,j,k = 1,2 \\ P_{ij}^0 &\to P_{ij}^0 - \frac{\partial}{\partial x^i} P_j/3 + \sum_{k=1}^{2} P_k\, P_{ij}^k/3 - P_iP_j/9, \end{aligned}$$

where $\delta$ is the kronecker symbol and $P_k = \sum_{k=1}^{2} P_{k\ell}^\ell$. Here after we treat systems of canonical form only, unless otherwise stated.

Let us change the independent variable $x = (x^1,x^2)$ into $y = (y^1,y^2)$. One obtains a system of non-canonical form. Transform this system into the canonical form and let $Q^k_{ij}(y)$ be its coefficients. If $y$ is projectively related to $x$ then one can show, by using (1.2), that

$$(1.3)\qquad \begin{aligned} Q^k_{ij}(y) &= \sum_{\ell,m,n=1}^{2} P^\ell_{mn}(x)\,\frac{\partial x^m}{\partial y^i}\,\frac{\partial x^n}{\partial y^j}\,\frac{\partial y^k}{\partial x^\ell}, \qquad i,j,k = 1,2 \\ Q^0_{ij}(y) &= \sum_{m,n=1}^{2} P^0_{mn}(x)\,\frac{\partial x^m}{\partial y^i}\,\frac{\partial x^n}{\partial y^j}. \end{aligned}$$

Definition. A system (E) is said to have a ramifying singularity of exponent $\alpha(\neq 1)$ along $x^1 = 0$ at $(0,0)$ if a projective solution has the following expression

$$\psi = ((x^1)^\alpha v_1, v_2), \qquad \det\ (\partial\psi/\partial x) = (x_1)^{\alpha-1}u,$$

where $v_1, v_2$ and $u$ are holomorphic functions at $(0,0)$ which are not divisible by $x^1$ (if $\alpha=0$, then put $\log x^1$ instead of $(x^1)^\alpha$).

Proposition 1.1. ([Y1 ; Proposition 3]) If a system (E) has a ramifying singularity of exponent $\alpha$ along $X^1 = 0$ at $(0,0)$ then $x^1P^2_{11}(x)$, $P^2_{22}(x)$ and $P^1_{22}(x)/x^1$ are holomorphic at $(0,0)$ and

$$x^1\,P^1_{11}(x)\,\Big|_{x^1=0} = \frac{1}{3}\,(\alpha-1).$$

Proposition 1.2. (Y1 ; Proposition 4]) If the coefficients of a system (E) are rational functions and if (E) has a ramifying singularity at infinity then the total degree of $P^k_{ij}(x)$ is negative for $i,j,k = 1,2$.

## §2 Integrability condition and tensor form $\omega$

We shall study the integrability condition of the system (E). Here after we shall use Einstein's convention and the following abriviation

$$(P)_k = \frac{\partial}{\partial x^k} P(x).$$

<u>Lemma</u> 2.1. The system (E) is integrable if and only if

$$(2.1)\qquad \begin{aligned} P^k_{ij} &= P^k_{ji} \qquad i,j = 1,2, \qquad k = 0,1,2, \\ P^0_{ij} &= -(P^k_{ij})_k + P^k_{i\ell}\, P^\ell_{jk}, \qquad i,j = 1,2 \end{aligned}$$

and if the expressions

$$(P^0_{ij})_k + P^\ell_{ij}\, P^0_{k\ell} \qquad i,j,k = 1,2$$

are symmetric with respect to $(i,j,k)$.

<u>Proof.</u> Differentiating the system (E), we have

$$\frac{\partial^3 w}{\partial x^k \partial x^i \partial x^j} = (P^\ell_{ij})_k\, (w)_\ell + P^\ell_{ij}\, (w)_{k\ell} + (P^0_{ij})_k w + P^0_{ij}(w)_k$$

$$= \{(P^m_{ij})_k + P^\ell_{ij}\, P^m_{k\ell} + \delta^m_k\, P^0_{ij}\}(w)_m + \{(P^0_{ij})_k + P^\ell_{ij}\, P^0_{k\ell}\}w.$$

Since the coefficient of $(w)_m$ should be symmetric with respect to $(i,j,k)$, we have

$$P^0_{11} = (P^2_{21})_1 - (P^2_{11})_2 + P^\ell_{21}\, P^2_{1\ell} - P^\ell_{11}\, P^2_{2\ell},$$

$$P^0_{12} = (P^2_{22})_1 - (P^2_{12})_2 + P^\ell_{22}\, P^2_{1\ell} - P^\ell_{12}\, P^2_{2\ell},$$

$$P^0_{22} = (P^1_{12})_2 - (P^1_{22})_1 + P^\ell_{12}\, P^1_{2\ell} - P^\ell_{22}\, P^1_{1\ell}.$$

By the relation (1.1), one can check that the above expressions reduce to (2.1). Q.E.D.

This implies that the system (E) is determined only by the coefficients

$$P^k_{ij}(x) \qquad i,j,k = 1,2.$$

Since we know the transformation rule (1.3) of $\{P^k_{ij}\}$, we can say that the tensor

$$P^k_{ij}(x)\ dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

determines the system (E). It is more converient, for later use, to put

$$(2.2) \qquad P_{ij1} = -P^2_{ij},\ P_{ij2} = P^1_{ij} \qquad i,j = 1,2$$

and to consider a tensor form

$$(2.3) \qquad \omega = P_{ijk}\ dx^i \otimes dx^j \otimes dx^k (dx^1 \wedge dx^2)^{-1}.$$

Since we assumed that (E) is of canonical form, one can easily check that $\{P_{ijk}\}$ is symmetric.

Let $d$ denote the differentiation and $A$ the anti-symmetrizer of two components. We make the following calculation.

$$d\omega = (P_{ijk})_\ell\ dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell (dx^1 \wedge dx^2)^{-1},$$

$$A\ d\omega = (P_{ijk})_\ell\ dx^i \otimes dx^j\ \frac{dx^k \wedge dx^\ell}{dx^1 \wedge dx^2},$$

$$\omega \otimes \omega = P_{ijk}\ P_{abc}\ dx^i \otimes dx^j \otimes dx^k \otimes dx^a \otimes dx^b \otimes dx^c (dx^1 \wedge dx^2)^{-2},$$

$$A^2(\omega \otimes \omega) = P_{ijk}\ P_{abc}\ dx^i \otimes dx^a\ \frac{dx^j \wedge dx^b}{dx^1 \wedge dx^2}\ \frac{dx^k \wedge dx^c}{dx^1 \wedge dx^2}.$$

Put

$$(2.4) \qquad \eta = Ad\omega - A^2(\omega \otimes \omega)$$

then we have

$$\eta = \{(P_{ijk})_\ell \frac{dx^k \wedge dx^\ell}{dx^1 \wedge dx^2} - P_{ik\ell}\, P_{jbc} \frac{dx^k \wedge dx^b}{dx^1 \wedge dx^2} \frac{dx^\ell \wedge dx^c}{dx^1 \wedge dx^2}) dx^i \otimes dx^j$$

$$= -\{(P^k_{ij})_k + P^k_{i\ell}\, P^\ell_{jk}\}\ dx^i \times dx^j$$

$$= P^0_{ij}\ dx^i \times dx^j.$$

Since we have

$$d\eta = (P^0_{ij})_k \times dx^i \otimes dx^j \otimes dx^k,$$

$$\omega \otimes \eta = P_{ijk}\, P^0_{ab}\ dx^i \otimes dx^j \otimes dx^k \otimes dx^a \otimes dx^b (dx^1 \wedge dx^2)^{-1},$$

$$A(\omega \otimes \eta) = P_{ija}\, P^0_{kb} \times dx^i \otimes dx^j \otimes dx^k \frac{dx^a \wedge dx^b}{dx^1 \wedge dx^2}$$

$$= -\, P^b_{ij}\, P^0_{kb} \times dx^i \otimes dx^j \otimes dx^k,$$

one concludes, by Lemma 2.1, that the system (E) is integrable if and only if the 3-tensor

$$d\eta - A(\omega \otimes \eta)$$

is symmetric. Thus we have expressed the integrability condition of (E) free of coordinates.

<u>Proposition</u> 2.2. Let $\omega$ be the form corresponding to the system (E) by (2.2) and (2.3). The system (E) is integrable if and only if the 3-tensor

$$(2,5) \qquad dAd\omega - d\, A^2(\omega \otimes \omega) - A(\omega \otimes Ad\omega)$$

is symmetric.

<u>Proof.</u> By the expression (2.4) we have only to check that $A(\omega \otimes A^2(\omega \otimes \omega))$ is symmetric, which is easy to show.

Q.E.D.

## §3 Primitive unitary reflection groups ([S-T])

Let $V$ be a 3-dimensional complex linear space with positive definite Hermitian inner product, $G \subset GL(V)$ be a primitive unitary reflection group and $\bar{G} \subset PGL(V)$ be the group of homologies of $G$. Let $A_1, \ldots, A_k$ be the hyperplanes of $V$ which occur as reflecting hyperplanes of elements of $G$.

Following is a table of such $G$'s with Shefard-Todd number, the name so called, the order $|G|$ of $G$, the order $|\bar{G}|$ of $\bar{G}$, the degrees $d_1, d_2, d_3$ of fundamental invariants, the numbers $r_i$ of i-fold reflections and the number $k$ of reflecting hyperplanes.

| No | Name | $\|G\|$ | $\|\bar{G}\|$ | $d_1, d_2, d_3$ | $r_2$ | $r_3$ | k |
|---|---|---|---|---|---|---|---|
| 23 | Icosa. | 120 | 60 | 2,6,10 | 15 | 0 | 15 |
| 24 | Klein | 336 | 168 | 4,6,14 | 21 | 0 | 21 |
| 25 | Hesse | 648 | 216 | 6,9,12 | 0 | $2\cdot 12$ | 12 |
| 26 | ext.-Hesse | 1296 | 216 | 6,12,18 | 9 | $2\cdot 12$ | 21 |
| 27 | Valentiner | 2160 | 360 | 6,12,30 | 45 | 0 | 45 |

Let $X$ be the complex projective plane $V-\{0\}/\mathbb{C}^{\times}$,

$$\pi : V-\{0\} \to X$$

be the projection and put

$$\bar{A}_j = \pi A_j, \qquad j=1,\ldots,k$$

$$A = \bigcup_{j=1}^{k} A_j, \qquad \bar{A} = \bigcup_{j=1}^{k} \bar{A}_j .$$

$\bar{A}$ is called the arrangement of lines in $X$ defined by $G$.

## §4 Tensor form $\Omega$ on $V$

Let $\bar{A} = \bigcup_{j=1}^{k} \bar{A}_j$ be the arrangement of lines in $X$ defined by a primitive unitary reflection group $G \subset GL(V)$. Let (E) be a system on $X$ with at most ramifying singularity along $\bar{A}$. We moreover assume that the system (E) is invariant under the action of $\bar{G}$. That is , by (1.3), equivalent to assume the following identities

$$P^k_{ij}(z) = P^c_{ab}(x)\,\frac{\partial x^a}{\partial z^i}\,\frac{\partial x^b}{\partial z^j}\,\frac{\partial z^k}{\partial x^c} \qquad i,j,k = 1,2$$

for all $\bar{\sigma} \in \bar{G}$, where $z = \bar{\sigma}\,x$.

In §2, we introduced the tensor form $\omega$ on $X$ corresponding to the system (E). It is convenient to consider a tensor form $\Omega$ on $V$ which is the lift of $\omega$ by the map $\pi : V-\{0\} \to X$. Let $y = (y^0, y^1, y^2)$ be a coordinate of $V$. Then $\Omega$ is expressed as

$$(4.1) \quad \Omega = Q_{mn\ell}(y)\,dy^m \otimes dy^n \otimes dy^\ell \gamma(y)^{-1},$$

where

$$\gamma(y) = y^0 dy^1 \wedge dy^2 + y^1 dy^2 \wedge dy^0 + y^2 dy^0 \wedge dy^1,$$

and the indices $m,n$ and $\ell$ run through 0,1,2. Note that $\{Q_{mn\ell}\}$ is again symmetric. If $x = (x^1, x^2)$ is an inhomogeneous coordinate of $x$ related to $y$ by

$$x^1 = y^1/y^0; \quad x^2 = y^2/y^0,$$

and $\omega$ is expressed by $x$ as (2.3), then we have

$$P_{ijk}(x^1,x^2) = Q_{ijk}(y^0,y^1,y^2) \qquad i,j,k = 1,2.$$

Since $\Omega$ is a pull back by $\pi$, we have

$$(4.2) \quad y^\ell Q_{mn\ell} = 0 \qquad m,n = 0,1,2.$$

Let $\ell_1(y),\ldots,\ell_k(y)$ be the linear forms which define the planes $A_1,\ldots,A_k$. We fix the coordinate $y = (y^0,y^1,y^2)$ and the linear forms $\ell_1(y),\ldots,\ell_k(y)$ once for all and put

$$K(y) = \ell_1(y)\ldots\ell_k(y),$$

$$R_{\ell mn}(y) = K(y)\, Q_{\ell mn}(y), \quad \ell,m,n = 0,1,2.$$

In view of the argument in §1, one sees that the form $\Omega$ satisfies the following conditions.

(4.3) The form $\Omega$ is invariant under the action of $G$.

(4.4) Each $R_{\ell mn}(y)$ is a homogeneous polynomial of degree $k$.

(4.5) For each $j$ $(j=1,\ldots,k)$, choose a coordinate $\bar{y} = (\bar{y}^0,\bar{y}^1,\bar{y}^2)$ such that $\{\bar{y}^1 = 0\} = A_j$ and let $\bar{Q}_{mn\ell}(\bar{y})$ be the coefficients of $\Omega$ in the expression (4.1) with respect to the cordinate $\bar{y}$ and put

$$\bar{R}_{mn\ell}(\bar{y}) = K(y(\bar{y})\bar{Q}_{mn\ell}(\bar{y}) \qquad m,n,\ell = 0,1,2.$$

Then there is a constant $\alpha_j$ such that the polynomials

$$\bar{R}_{122}(\bar{y}), \quad \bar{R}_{222}(\bar{y})/\bar{y}^1$$

and

$$\bar{R}_{112}(\bar{y}) - \frac{1}{3}(\alpha_j-1)K(y(\bar{y}))\bar{y}^0/\bar{y}^1$$

are divisible by $\bar{y}^1$.

## §5 Result for Icosahedral, Klein and Valentiner group $G$

Let $G$ be the group with Shephard-Todd number 23,24 or 27, and $\Omega$ be a form on $V$ satisfying the conditions (4.2),...,(4.5). Let $R_{mn\ell}(y)$ be the polynomial defined in §4 and put

$$\Omega' = R_{mn\ell}(y)dy^m \otimes dy^n \otimes dy^\ell.$$

Lemma 5.1. $\Omega'$ is a $G$-invariant symmetric polynomial form on $V$.

Proof. In the table in §2, one reads that the group $G$ is generated by 2-fold reflections. Any 2-fold reflection in $G$ changes the signs of the polinomial $K(y)$ and the form $\ell(y)$. Therefore $K(y)\gamma(y)$ is $G$-invariant and so is $\Omega' = \Omega/K(y)\gamma(y)$.

Q.E.D.

The group $G$ acts transitively on the set of planes $A_1,\ldots,A_k$. Thus we can put

$$\alpha := \alpha_1 = \ldots = \alpha_k \ (\neq 1).$$

Let $\odot$ denote the symmetric tensor product. That is, for example,

$$dy^n \odot dy^m \odot dy^\ell = \frac{1}{3!}(dy^n\otimes dy^m\otimes dy^\ell + dy^n\otimes dy^\ell\otimes dy^m + dy^m\otimes dy^n\otimes dy^\ell + \ldots \ ).$$

Lemma 5.2. Let $A_a$, $A_b$ and $A_c$ be three mutually orthogonal planes in $A$. Then we have

$$\Omega' \equiv (\alpha-1)\ J(a,b,c)^{-1}\frac{K(y)}{\ell_b}(\ell_a d\ell_c - \ell_c d\ell_a)\odot(d\ell_b)^{\odot 2}$$

modulo $\ell_b$, where $J(a,b,c) = \det\left(\partial(\ell_a,\ell_b,\ell_c)/\partial(y^0,y^1,y^2)\right)$.

Proof. Choose a coordinate $\bar{y} = (\bar{y}^0,\bar{y}^1,\bar{y}^2)$ so that $\bar{y}^0 = \ell_a$, $\bar{y}^1 = \ell_b$, $\bar{y}^2 = \ell_c$. Let $\bar{R}_{mn\ell}(\bar{y})$ be as in (4.5). Since

$\eta(\bar{y})\eta(y)^{-1}$ is a constant which is equal to det $(\partial(\ell_a,\ell_b,\ell_c)/\partial (y^0,y^1,y^2))$, we have

$$\Omega' = J(a,b,c)^{-1}\ \bar{R}_{mn\ell}(\bar{y})\ d\bar{y}^m \otimes d\bar{y}^n \otimes d\bar{y}^\ell .$$

The invariance of $\Omega'$ under $G$ implies in particular that it is invariant under the reflection with reflecting hyperplane $A_b$. This reflection is represented, under the coordinate $\bar{y}$, by

$$(\bar{y}^0,\bar{y}^1,\bar{y}^2) \to (\bar{y}^0,-\bar{y}^1,\bar{y}^2).$$

Therefore the above expression of $\Omega'$ implies that

$$\bar{R}_{mn\ell}(\bar{y}) \equiv 0 \quad \text{modulo} \quad \bar{y}^1$$

if $\{m,n,\ell\} = \{1,p,q\}$ $p,q \neq 1$ or $\{1,1,1\}$. On the other hand (4.5) tells us $\bar{R}_{222}\ (y) \equiv 0$ modulo $\bar{y}^1$. Moreover, identities (4.2) with respect to the cordinate $\bar{y}$ implies, for $m,n = 0,1,2,$

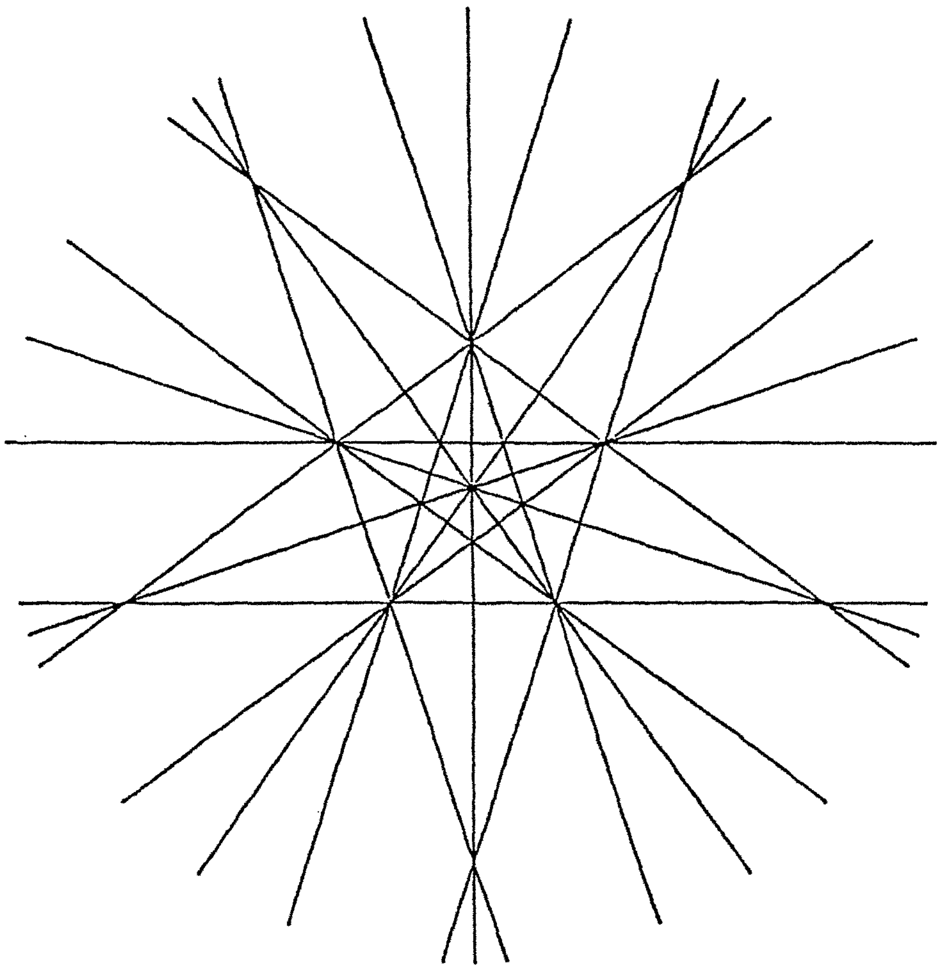
$$\bar{y}^0\bar{R}_{mn0}(\bar{y}) + \bar{y}^2\ \bar{R}_{mn2}(\bar{y}) \qquad 0 \quad \text{modulo} \quad \bar{y}^1.$$

By these equalites, one knows that all $\bar{R}_{mn}\ (\bar{y})$'s but $\bar{R}_{110}(\bar{y})$ and $\bar{R}_{112}(\bar{y})$ are zero modulo $\bar{y}^1$. Since the remaining two are related by the identity above, we have

$$\bar{R}_{110}(\bar{y}) \equiv -\ \frac{1}{3}\ (\alpha-1)\ K(y(\bar{y}))\bar{y}^2/\bar{y}^1 \quad \text{modulo} \quad \bar{y}^1.$$

Q.E.D.

<u>Lemma</u> 5.3. For each plane $A_b$ in $A$, there exist two planes $A_b'$ and $A_b''$ in $A$ such that the three planes $A_b', A_b, A_b''$ are orthogonal each other and that the set of triples $\{A_{b'}, A_b, A_{b''}\}$ $(b=1,\dots,k)$ is $G$-invariant.



Icosahedral arrangement

<u>Proof</u>. Let $G$ be the icosahedral group. The picture above tells us that, for a given plane $A_b$, there exist uniquely two planes in $A$ such that these three planes are orthogonal each other. Let $G$ be Klein or Valentiner group and $S$ be the set of triples of planes in $A$ which are mutually perpendicular. For a given plane $A_b$, there are four planes $A_a, A_c, A_{a'}, A_{c'}$ in $A$ such that $\{A_a, A_b, A_c\}$, $\{A_{a'}, A_b, A_{c'}\}$ $S$ (c,f. [S-T]). In particular we have $|S| = 2k/3$.

The group $G$ acts on $S$. The isotropy subgroup of $G$ at $\{A_a, A_b, A_c\}$ has the subgroup $H$ generated by the reflection with the reflecting plane $A_a$ and the symmetric group $\mathfrak{S}_3$ which permutes the three planes. So we have $|H| = 2^3\, 3!$. Since we have

$$
\begin{array}{lll}
G : \text{Klein} & |S| = 14 & |G/H| = 7 \\
G : \text{Valentiner} & |S| = 30 & |G/H| = 45,
\end{array}
$$

we can conclude that

$$S = G\{A_a, A_b, A_c\} \coprod G\,\{A_{a'}, A_b, A_{c'}\}.$$

Q.E.D.

Consider the form

$$\Omega'' = (\alpha - 1) \sum_{b=1}^{k} J(b', b, b'')^{-1} \frac{K(y)}{\ell_b} (\ell_{b'} d\ell_{b''} - \ell_{b''} d\ell_{b'}) \otimes (d\ell_b)^{\otimes 2}.$$

Since the group $G$ is generated by reflections with reflecting hyperplanes $A_1, \ldots, A_k$, one can easily check that $\Omega''$ is G-invariant. On account of Lemma 5.1 and 5.2, one knows that $\Omega' - \Omega''$ is a G-invariant form such that

$$\Omega' - \Omega'' \equiv 0 \text{ modulo } \ell_b, \qquad b=1,\ldots,k.$$

This implies that $(\Omega' - \Omega'')/K(y)$ is an anti-invariant 3-form with respect to $dy^0$, $dy^1$ and $dy^2$. Since there are no invariant of degree 3, as one can see in the table in §3, we can conclude that $\Omega' = \Omega''$. We have almost proved the following theorem.

__Theorem__ 1 Let $V$ be a 3-dimensional unitary space with cordinate $y = (y^0, y^1, y^2)$, $G \subset GL(V)$ be a unitary reflection group with Shephard-Todd number 23,24 or 27, $\{A_j\}_{j=1,\dots,k}$ be the set of reflecting hyperplanes, and $\ell_j(y)$ be a linear form which defines $A_j$. Let $X$, $\overline{G} \subset PGL(V)$ and $\overline{A}_j$ $(X)$ be the projectification of $V$, $G$ and $A_j$, respectively. For a given complex number $\alpha(\neq 1)$, there exists uniquely a completely integrable $\overline{G}$-invariant Fuchsian system $E(\alpha)$ of canonical form on $X$ only with ramifying singulality along $\overline{A}_1,\dots,\overline{A}_k$ of exponent $\alpha$. The corresponding form $\Omega(\alpha)$ on $V$ is given by

$$\Omega(a) = (\alpha-1)\sum_{b=1}^{k} J(b',b,b'')^{-1}\,\frac{\ell_{b'}d\ell_{b''}-\ell_{b''}d\ell_{b'}}{\ell_b}\otimes(d\ell_b)^{\otimes 2}\quad \gamma(y)^{-1}$$

where $J(b',b,b'') = \det\ (\partial(\ell_{b'},\ell_b,\ell_{b''})/\partial(y^0,y^1,y^2))$, $\gamma(y) = y^0dy^1\wedge dy^2+y^1dy^2\wedge dy^0+y^2dy^0\wedge dy^1$, and $A_{b'},A_{b''}$ are two planes such that three planes $A_{b'},A_b,A_{b''}$ are mutually perpendicular and that the set of triples $\{A_{b'},A_b,A_{b''}\}$ $(b=1,\dots,k)$ is G-invariant. The coefficients $P^k_{ij}(x)$ of $E(\alpha)$ with respect to an inhomogeneous coordinate $x = (x^1,x^2) = (y_1/y_0,\ y_2/y_0)$ are given by

$$\sum_{m,n,\ell=0,1,2} Q_{mn\ell}(y)\,dy^m\otimes dy^n\otimes dy^\ell$$

$$= (\alpha-1)\sum_{b=1}^{k} J(b',b,b'')^{-1}\frac{\ell_{b'}d\ell_{b''}-\ell_{b''}d\ell_{b'}}{\ell_b}\otimes(d\ell_b)^{\otimes 2},$$

$$P^1_{ij}(x) = Q_{ij2}(1,x^1,x^2),\ P^2_{ij}(x) = -\,Q_{ij1}(1,x^1,x^2)\qquad i,j = 1,2,$$

and (2.1).

<u>Proof</u>. First we show that the form $\Omega(\alpha)$ on $V$ is a pull back by $\pi : V-\{0\} \to X$. Let $v = y^i \frac{\partial}{\partial y^i}$ be a vector field on $V$, and $i(v)$ be the interior product operator with respect to $v$. Then the condition (4.2) is equivalent to

$$i(v)\Omega(\alpha) = 0.$$

If we notice that

$$v = \ell_{b'} \frac{\partial}{\partial \ell_{b'}} + \ell_b \frac{\partial}{\partial \ell_b} + \ell_{b''} \frac{\partial}{\partial \ell_{b''}},$$

we have

$$i(v)\Omega(\alpha) = (\alpha-1) \sum_{b=1}^{k} J(b',b,b'') \frac{1}{3\ell_b} \{(\ell_{b'} d\ell_{b''} - \ell_{b''} d\ell_{b'}) \times d\ell_b \cdot \ell_b$$

$$+ d\ell_b \otimes (\ell_{b'} d\ell_{b''} - \ell_{b''} d\ell_{b'}) \ell_b + d\ell_b \otimes d\ell_b (\ell_{b'} \ell_{b''} - \ell_{b''} \ell_{b'})\}$$

$$+ \frac{2}{3}(\alpha-1) \sum_{b=1}^{k} J(b',b,b'') (\ell_{b'} d\ell_{b''} - \ell_{b''} d\ell_{b'}) \odot d\ell_b.$$

Cyclic change of indices $(b',b,b'')$ tells us that this expression is zero.

Now we have only to show that the system $E(\alpha)$ is integrable. Let $\omega(\alpha)$ be the form on $X$ corresponding to the form $\Omega(\alpha)$, obtained by putting $y^0=1$, $y^1=x^1$, $y^2=x^2$. For notational simplicity, we use the same notation $\ell_a$ for $\ell_a(1,x^1,x^2)$, and

$$J(a,b) = \det \ (\partial(\ell_a,\ell_b)/\partial(x^1,x^2)).$$

By the caluculation below, one knows that the tensor $dAd\omega(\alpha)$ is symmetric.

$$d\omega = (\alpha-1)\sum_{b=1}^{k} J(b',b,b'')^{-1}\{\frac{d\ell_{b''}\odot d\ell_b\odot d\ell_b\otimes d\ell_{b'} - d\ell_{b'}\odot d\ell_b\odot d\ell_b\otimes d\ell_{b''}}{\ell_b}$$

$$- \frac{\ell_{b'}d\ell_{b''}-\ell_{b''}d\ell_{b'}}{(\ell_b)^2}\odot d\ell_b\odot d\ell_b\otimes d\ell_b\}\ (dx^1\wedge dx^2)^{-1},$$

and since we have in general

$$A(df_1\odot\cdots\odot df_n\otimes dg) = \frac{1}{n}\sum_{j=1}^{n} df_1\odot\cdots\overset{\vee}{j}\cdots\odot df_n(df_j\wedge dg),$$

we have

$$Ad\omega = \frac{\alpha-1}{3}\sum_{b=1}^{k} J(b',b,b'')^{-1}\{\frac{2d\ell_{b''}\odot d\ell_b J(b,b')+d\ell_b\odot d\ell_b J(b'',b')}{\ell_b}$$

$$- \frac{2d\ell_{b'}\odot d\ell_b J(b,b'') + d\ell_b\odot d\ell_b J(b',b'')}{\ell_b}$$

$$- \frac{\ell_{b'}d\ell_b\odot d\ell_b J(b'',b) - \ell_{b''}d\ell_b\odot d\ell_b J(b',b)}{(\ell_b)^2}\}$$

$$dAd\omega = \frac{\alpha-1}{3}\sum_{b=1}^{k} J(b',b,b'')^{-1}\{-\frac{2d\ell_{b''}\odot d\ell_b\otimes d\ell_b J(b,b')+d\ell_b\odot d\ell_b\otimes d\ell_b J(b'',b')}{(\ell_b)^2}$$

$$+ \frac{2d\ell_{b'}\odot d\ell_b\otimes d\ell_b J(b,b'') + d\ell_b\odot d\ell_b\otimes d\ell_b J(b',b'')}{(\ell_b)^2}$$

$$- \frac{d\ell_b\odot d\ell_b\otimes d\ell_{b'} J(b'',b) - d\ell_b\odot d\ell_b\otimes d\ell_{b''} J(b',b)}{(\ell_b)^2}$$

$$+ 2\,\frac{\ell_{b'}d\ell_b\odot d\ell_b\otimes d\ell_b J(b'',b) - \ell_{b''}d\ell_b\odot d\ell_b\otimes d\ell_b J(b',b)}{(\ell_b)^3}\}$$

$$= (\alpha-1) \sum_{b=1}^{k} J(b',b,b'')^{-1} \left\{ \frac{d\ell_{b'} \odot d\ell_b \odot d\ell_b}{(\ell_b)^2} J(b',b) \right.$$

$$\left. + \frac{d\ell_{b''} \odot d\ell_b \odot d\ell_b}{(\ell_b)^2} J(b\ ,b'') + \frac{2}{3} \frac{\ell_{b'} J(b'',b) - \ell_{b''} J(b',b)}{(\ell_b)^3} \cdot (d\ell_b)^{\odot 3} \right\}.$$

Instead of calculating directely the tensor

$$I(\alpha) := d\, A^2(\omega(\alpha) \otimes \omega(\alpha)) + A(\omega(\alpha) \otimes Ad\omega(\alpha)),$$

we note that

$$I(\alpha) = (\alpha-1)^2\, I(0)$$

and use the following fact due to Th. Höfer ([Hö], summary of the results is also in [Hi]).

" There exists a discrete reflection group $\Gamma \subset PU(2,1)$ acting on the unit ball

$$B_2 = \{(z^1, z^2) \in \mathbb{C}^2 \mid |z^1|^2 + |z^2|^2 < 1\}$$

such that

$$(B_2 - \tilde{A}) / \Gamma \cong X - \overline{A},$$

where $\tilde{A}$ is the union of reflecting lines of reflections in $\Gamma$. Let $n$ be the ramification index along $\overline{A}_j$, which is independent of $j$, of the projection

$$\rho : B_2 - \tilde{A} \to X - \overline{A}.$$

Then the complete list of such $\Gamma$ is given as follows".

| S - T number of $G$ | ramification index $n$ |
| --- | --- |
| 23 | 2,5 |
| 24 | 2,3,4 |
| 27 | 2 |

As is explained in [Y1], if such $\Gamma$ exists, there is a unique system (E) on $X$ with ramifying singularity of index $\alpha = 1/n$ along $\overline{A}$. The projective solution of (E) gives an inverse of $\rho$. By the uniqueness, this system must be our system $E(1/n)$. This implies that $E(1/n)$ is integrable and so that the tensor $I(1/n)$ is symmetric. Thus $I(\alpha)$ is symmetric for all $\alpha$.

Q.E.D.

## §6 Result for Hesse and extended Hesse group $G$

Let $G$ be the group with Shephard-Todd number 25 or 26, and $\overline{G}$ be the projectified group, which is common for both groups. We want to construct $\overline{G}$-invariant systems with ramifying singularity at most along $\overline{A}$. Although the method in §5 does not work in these cases, we can give the explicit form of them. We moreover show that they are obtained, by the change of independent variables, from the classical systems known as Appell's hypergeometric differential equations $F_1$.

Let $x = (x^0, x^1, x^2)$ be a homogeneous coordinate of $X$. Following [S-T], we put

$$I_6(x) = ((x^0)^3+(x^1)^3+(x^2)^3)^2-12((x^0x^1)^3+(x^1x^2)^3+(x^2x^0)^3),$$

$$I_9(x) = ((x^0)^3-(x^1)^3)((x^1)^3-(x^2)^3)((x^2)^3-(x^0)^3),$$

$$I'_{12}(x) = ((x^0)^3+(x^1)^3+(x^2)^3)(((x^0)^3+(x^1)^3+(x^2)^3)^3+216(x^0x^1x^2)^3),$$

and

$$I_{12}(x) = x^0x^1x^2 \prod_{a,b=0}^{2} (\omega^a x^0+\omega^b x^1+x^2),$$

where $\omega = \exp 2\pi i/3$. We have the relation

$$4\cdot 1728\ I_{12}^3 = (432\ I_9^2 - I_6^3 + 3\ I_6\ I'_{12})^2 - 4(I'_{12})^3.$$

We define, two kinds of line arrangements on $A$ as follows.

$$A'_x : \ I'_{12} = 0,$$

$$A''_x : \ I_q = 0.$$

The former is the image of the set of reflecting hyperplanes of Hesse group (25) and is called Hesse arrangement. The image of those of extented Hesse group (26) is $A_X^{'} \cup A_X^{''}$ and is called extended Hesse arrangement.

Let $S$ be the quotient variety of $X$ by the group $\overline{G}$ and

$$P : X \to S$$

be the projection. The variety $S$ is a weighted projective space of type $(2,3,4)$, that is,

$$S = \{(s_2,s_3,s_4) \in \mathbb{C}^3-\{0\}\}/ \sim$$

where

$$(s_2,s_3,s_4) \sim (\lambda^2 s_2,\lambda^3 s_3,\lambda^4 s_4) \qquad \lambda \in \mathbb{C} - \{0\},$$

and the map $p$ is given by

$$\text{(p)} \qquad s_2 = I_6(x), \ s_3 = I_9(x), \ s_4 = I_{12}^{'}(x).$$

The map $p$ is ramifying along

$$A_S^{'} := p\ (A_X^{'}) \quad \text{and} \quad A''_S := p(A_X^{''})$$

on $S$ with indices $3$ and $2$, respectively. We have

$$A_S^{'} : (432\ s_3^2 - s_3^2 + 3s_2s_4)^2 - 4s_4^3 = 0,$$

$$A_S^{''} : s_3 = 0.$$

Let $Y$ be another projective plane with a homogeneous coordinate $y = (y^0,y^1,y^2)$. Putting

$$y^3 = - y^0-y^1-y^2,$$

one finds that the symmetric group $\mathfrak{S}_4$ acts on $Y$ as permutations of four letters $y^0, y^1, y^2$ and $y^3$. Let $T$ be the quotient variety of $Y$ by the group $\mathfrak{S}_4$ and

$$q : Y \to T$$

be the projection. The variety $T$ is again a weighted projective space of type (2,3,4) and, by the homogeneous coordinate $t = (t_2, t_3, t_4)$, the map $q$ is given by

$$\text{(q)} \qquad \begin{aligned} t_2 &= \sum_{0\leqq i<j\leqq 3} y^i y^j, \\ t_3 &= -\sum_{0\leqq i<j<k\leqq 3} y^i y^j y^k, \\ t_4 &= y^0 y^1 y^2 y^3. \end{aligned}$$

The group $\mathfrak{S}_4$ has fixed points on $Y$ along

$$A'_Y := \bigcup_{a\leqq i<j\leqq 2} \{y^i - y^j = 0\} \bigcup_{\{i,j,k\}=\{0,1,2\}} \{2y^i + y^j + y^k = 0\},$$

$$A''_Y := \bigcup_{0\leqq i<j\leqq 2} \{y^i + y^j = 0\}$$

of multiplicity 2. The branch locus of $q$ on $T$ is given by

$$A'_T := q(A'_Y) \quad \text{and} \quad A''_T := q(A''_Y).$$

By compting the discriminant of the polynomial of degree 4

$$x^4 + t_2 x^2 + t_3 x + t_4,$$

one knows the defining equation of $A'_T$.

$$A'_T : (t_3^2 - (-\tfrac{2}{3} t_2)^3 + 3(-\tfrac{2}{3} t_2)(\frac{4t_4+t_2^2/3}{3})^2 - 4(\frac{4t_4+t_2^2/3}{3})^3 = 0.$$

Putting the expression $y^3 = -y^0-y^1-y^2$ into (q), one has

$$t_3 = (y^0+y^1)(y^1+y^2)(y^2+y^0).$$

Thus we have the equation of $A''_T$.

$$A''_T : t_3 = 0.$$

Let $\psi$ be the map $S \to T$ given by

$$(\psi) \quad \begin{cases} s_2 = -\tfrac{2}{3} t_2, \; s_3 = t_3/\sqrt{432} \\ s_4 = 4t_4/3 + t_2^2/9. \end{cases}$$

Then $\psi$ gives an isomorphism $S \to T$ which induces isomorphisms

$$A'_S \rightleftarrows A'_T \quad \text{and} \quad A''_S \rightleftarrows A''_T.$$

Let $Z$ be another projective plane with homogeneous coordinate $z = (z^0, z^1, z^2)$, Put

$$A'_Z : (z^0-z^1)(z^1-z^2)(z^2-z^1) = 0,$$

$$A''_Z : z^0z^1z^2 = 0$$

and let $r$ be the map $Y \to Z$ given by

$$(r) \qquad z^0 = (y^1+y^2)^2, \; z^1 = (y^2+y^0)^2, \; z^2 = (y^0+y^1)^2.$$

The map $r$, which is the quotient map by $(\mathbb{Z}/2\mathbb{Z})^2$, ramifies along $A''_Z$ with index 2. We have

$$r(A'_Y) = A'_Z, \qquad r(A''_Y) = A''_Z.$$

Summing up, we have had the following diagram.

$$\begin{tikzcd}
 & (Y,A_Y',A_Y'') \arrow[dl, "r"'] \arrow[dr, "q"] & & & (X,A_X',A_X'') \arrow[dl, "p"'] \\
(Z,A_Z',A_Z'') & & (T,A_T',A_T'') & (S,A_S'',A''_S) \arrow[l, "\psi"', "\sim"] & \\
1 \quad 2 & & 2 \quad 2 & 3 \quad 2 &
\end{tikzcd}$$

Here numerals below A's denote the braching indices of the corresponding coverings.

It is classically well known that for any system (E) on Z only with ramifying singularity along the complete quadrilateral

$$A_Z' \cup A_Z'' \ : \ z^0 z^1 z^2 (z^0 - z^1)(z^1 - z^2)(z^2 - z^0) = 0,$$

there exists a quadruple $(a,b,b',c) \in \mathbb{C}^4$ such that (E) is equivalent to the Appell's hypergeometric equation $(F_1(a,b,b',c))$, which is satisfied by the hypergeometric series

$$F_1(a,b,b',c;z^1,z^2) = \sum_{n,m=0}^{\infty} \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)(1,m)(1,n)} (z^1)^n (z^2)^m,$$

where $(a,m) = a(a+1)\dots(a+m-1)$ and $z^0 = 1$. We have almost proved the following theorem.

<u>Theorem</u> 2 Let (E) be a system on X only with ramifying singularity along the extended Hesse arrangement $A_X' \cup A_X''$. If the system (E) is invariant under the Hesse group $\overline{G}$, and $\alpha$ and $\beta$ denote the exponents along $A_X'$ and $A_X''$ respectively, then the system is transformed by the change of independent variables

$$z = r \circ q^{-1} \circ \psi \circ p(x)$$

into the Appell's hypergeometric equation $(F_1(a,b,b',c))$ with the special values of parameters

$$a = 1 - 2\alpha/3 - \beta/2,\ b = b' = 1/2 - \beta/4,$$
$$c = 3/2 - 2\alpha/3 - \beta/4.$$

Explicit form of the system is given, by inhomogeneous coordinates $x = z^1/z^0$, $y = z^2/z^0$, as follows.

$$P^1_{11} = 3\alpha'/x + 81\alpha' x^2y^3(2-x^3-y^3)/W + 3\beta' x^2(y^3-1)(1+x^3-2y^3)/2H,$$

$$P^2_{11} = 81\alpha' xy(1+x^3-y^6-x^3y^3)/W - 9\beta' xy(y^3-1)^2/2H,$$

$$P^2_{22} = 3\alpha'/y + 81\alpha' x^3y^2(2-x^3-y^3)/W - 3\beta' y^2(x^3-1)(1+y^3-2x^3)/2H,$$

$$P^1_{22} = 81\alpha' xy(1+y^3-x^6-x^3y^3)/W + 9\beta' xy(x^3-1)^2/2H,$$

where

$$W = \prod_{a,b=0,1,2}(\omega^a x+\omega^b y+1) = (x^3+y^3+1)^3 - 27x^3y^3,$$

$$H = (x^3-1)(y^3-1)(x^3-y^3),\ \alpha' = (\alpha-1)/3,\ \beta' = (\beta-1)/3.$$

Remaining coefficients are obtained by (1,1) and (2,1).

Remark 6.1. If one specializes the system (E), by putting $\beta=1$, it has singularity only on Hesse arrangement $A'_X$. This is the equation we constructed in [Y1].

Proof. Since, by assumption, (E) is $\bar{G}$-invariant, the system (E) is defined on $S$, so on $T$. Lifting the equation on $T$ by $q$, we have a system on $Y$ with ramifying singularity along $A'_Y$ and $A''_Y$ with respective exponents $2\alpha/3$ and $\beta$. On the other hand we have

Lemma 6.2. ([Y5]) Any system on $Y$ only with ramifying singularity along

$$y^0y^1y^2((y^0)^n-(y^1)^n)((y^1)^n-(y^2)^n)((y^2)^n-(y^0)^n) = 0$$

is invariant under the group $(\mathbb{Z}/n\mathbb{Z})^2$, which acts on $Y$ as

$$(y^0,y^1,y^2) \to (y^0,\varepsilon^i y^1,\varepsilon^j y^2) \qquad i,j = 1,\dots,n$$

where $\varepsilon = \exp 2\pi i/n$.

Applying this lemma for $n=2$, to our system on $Y$, we find that the system is defined on $Z$ and it has ramifying singularity along $A_2'$ and $A_2''$ with respective exponents $2\alpha/3$ and $\beta/2$. It is well known that the hypergeometric equation $(F(a,b,b',c))$ has exponents $1+b'-c$, $1+b-c$, $b+b'-a$, $1-b-b'$, $c-a-b'$, $c-a-b$ along $\{z^1=0\}$, $\{z^2=0\}$, $\{z^0=0\}$, $\{z^1=z^2\}$, $\{z^2=z^0\}$, $\{z^1=z^0\}$ respectively. Thus we have proved the first assertion of the theorem. To obtain an explicit form, one can transform the system $(F(a,b,b',c))$, with the special values of parameters as are in the theorem, by the change of independent variables

$$x = p^{-1}\circ\psi^{-1}\circ q\circ r^{-1}(z).$$

There is a more clever way. Show firstly that the coefficients $P_{ij}^k(x)$ $(i,j,k=1,2)$ are homogeneous linear forms of $\alpha-1$ and $\beta-1$. Next, put $\alpha=1$ then it reduces to a system with singularity only on

$$((x^0)^3-(x^1)^3)((x')^3-(x^2)^3)((x^2)^3-(x^0)^3) = 0.$$

Apply Lemma 6.2 for $n=3$ then one knows that it is again a lift of some system $(F(a,b,b',c))$. As is mentioned in Remark 6.1, we know the system for $\beta=1$. This completes the proof.

Q.E.D.

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