ALGEBRAIC VERSIONS OF A FINITE-DIMENSIONAL QUANTUM GROUPOID

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Abstract

We establish the equivalence of three versions of a finite dimensional quantum groupoid: a generalized Kac algebra introduced by T. Yamanouchi, a weak C^*-Hopf algebra introduced by G. Böhm–F. Nill–K. Szlachányi (with an involutive antipode), and a Kac bimodule – an algebraic version of a Hopf bimodule, the notion introduced by J.-M. Vallin. We also study the structure and construct examples of finite dimensional quantum groupoids.

INTRODUCTION

0.1. Our initial point is the following theorem conjectured by A. Ocneanu (see [19], the postface in [7] and remark 4.7a in [9]) and proved by W. Szymanski [22], R. Longo [12] and M.-C. David [4]:

if $M_0 \subset M_1 \subset M_2 \subset M_3$ is a *Jones's tower* of type II_1 von Neumann factors with finite index [9], which is *irreducible* (i.e., $M_0' \cap M_1 = C$) and *of depth 2* (i.e., $M_0' \cap M_3$ is a factor), then $M_1' \cap M_3$ has a natural finite-dimensional (f-d) Kac algebra [11], [7] structure which acts *outerly* on M_2 in such a way that the resulting cross-product is isomorphic to M_3. This gives an intrinsic characterization of inclusions of the mentioned type.

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A similar result for infinite index inclusions was conjectured in [10] and proved for arbitrary type factors in [6], [5] (see also [17]).

It is natural to extend the above theorem to *reducible* (i.e., $\mathbb{C} \subset M_0' \cap M_1$) inclusions of depth 2 – for example, an inclusion given by a cross-product $M_0 \subset M_0 \times_\alpha G$ (G is a finite group) is always of depth 2 and irreducible iff an action α is outer [6], [8], [17]. But for this one should replace a structure of a Kac algebra on $M_1' \cap M_3$ by some more general structure.

0.2. One of the suitable structures is a f-d *generalized (gen). Kac algebra* [27] – a C^*-bialgebra with a (generally) non-unital coproduct, an involutive antipode and a *Haar trace*. Exactly as for Kac algebras [11], [7], the corresponding duality theory is based on the usage of the *fundamental operator* W. W^* satisfies the *pentagonal relation* [1], it is a partial isometry and not necessarily a unitary. However, one can construct a gen. Kac algebra from a given f-d multiplicative partial isometry [26]. Since a gen. Kac algebra is commutative iff it is isomorphic to the algebra of complex-valued functions on a finite groupoid (see [27], Theorem 7.32), one can consider two above notions as "measurable versions" of a f-d *quantum groupoid* – "measurable", because their definitions contain explicitly (a gen. Kac algebra) or implicitly (a multiplicative partial isometry) a Haar measure.

The corresponding infinite-dimensional structures are *Hopf bimodules* [24] and *pseudo-multiplicative unitaries* [25] introduced using a fibered product of von Neumann algebras [21]. In their terms a structure of von Neumann algebra inclusions of depth 2 and any index with an operator-valued weight verifying some regularity conditions was studied [8]. On the other hand, a f-d *weak C^*-Hopf algebra* [2], [3] generalizing a Kac algebra and formulated in purely algebraic terms – a C^*-bialgebra with an antipode and a counit instead of a Haar measure – was used in [18] in order to show that any inclusion given by a cross-product with such an algebra is of depth 2.

Let us mention also some other points of view on quantum groupoids - [13], [15], [23].

In what follows all C^*-algebras and linear spaces over $\mathbb{C}$ are f-d.

0.3. In this paper we establish the equivalence of three versions of a f-d quantum groupoid: gen. and weak Kac algebra (a weak C^*-Hopf algebra with an involutive antipode), and a Kac bimodule – an "algebraization" of a Hopf bimodule. We also study the structure and construct examples of f-d quantum groupoids (some other examples are given in [8], [3], [2]).

The paper is organized as follows. In Section 1 (Preliminaries) we discuss, following [21], [8], [26], a relative tensor product of Hilbert modules and a fibered product of f-d C^*-algebras. Next, important notions of *Cartan subalgebras* (this term is borrowed from [20], [26]) and *counital maps* in C^*-bialgebras are introduced. In their terms we define two auxiliary structures – counital C^*-bialgebras and Hopf bimodules, study their elementary properties and prove their equivalence.

Section 2 is central. First we discuss weak Kac algebras. The theory of general weak C^*-Hopf algebras ([2], [3]) is much more complicated than the one of weak Kac algebras, that is why we give the proofs of all the statements. The most important results are existence and uniqueness of a Haar projection, normalized Haar trace and a Haar conditional expectation on any weak Kac algebra. This material can be read independently, one can look at the section of preliminaries in the case of necessity.

Then the equivalence of the categories of gen. and weak Kac algebras is established. For having a possibility to extend this theory to infinite-dimensional case, we formulate it also using the equivalent language of Kac bimodules – Hopf bimodules [8], [24], [26] equipped with a counit.

It is natural to consider a f-d quantum groupoid as non-trivial if it is non-commutative and non-cocommutative (i.e., does not come from groupoids) and is not a usual Kac algebra. Such examples are presented in Section 3. In particular, a finite group acting on a weak Kac algebra gives, via cross-product construction, a new weak Kac algebra. In such a way, we construct a series of concrete non-trivial quantum groupoids of dimensions n^3 $(n \geq 2)$.

Then we show that for any f-d C^*-algebra A there exists a unique weak Kac algebra structure (non-trivial when A is non-abelian) on the full matrix algebra $M_n(\mathbb{C})$ (where $n = \dim A$) with Cartan subalgebras isomorphic to A. This gives a classification of weak Kac algebras simple as C^*-algebras.

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1 Preliminaries

1.1 Relative tensor product of Hilbert modules

1.1.1. Let θ be a faithful trace on a C^*-algebra N, $L_2(N)$ a Hilbert space given by the GNS-construction for (N,θ), λ (resp., ρ) a faithful unital $*$-representation (resp., anti-$*$-representation) of N on $L_2(N)$ by operators of left (resp., right) multiplication. Clearly, the commutant $\lambda(N)' = \rho(N)$.

Let α be a faithful unital $*$-representation of N in some other Hilbert space H. Then H can be viewed as a left N-module ${}_\alpha H : n \cdot_\alpha \zeta = \alpha(n)\zeta$ $(\forall n \in N, \zeta \in H)$. An operator $R(\zeta) : L_2(N) \to H$ $(n \to \alpha(n)\zeta)$ is such that $R(\zeta)\lambda(n') = \alpha(n')R(\zeta)$ $(\forall n' \in N)$, so, for any $\zeta, \eta \in H$, operator $R(\zeta)R(\eta)^*$ belongs to the commutant $\alpha(N)'$ and $< \zeta, \eta >_N = R(\eta)^* R(\zeta)$ belongs to the commutant $\lambda(N)'$ which is isomorphic to the opposite C^*-algebra N^o.

In a similar way, if β be a faithful unital anti-$*$-representation of N in H, then H can be viewed simultaneously as a right N-module H_β or as a left N^o-module. An operator $R^o(\zeta) : L_2(N) \to H$ $(n \to \beta(n^*)\zeta)$ is such that $R^o(\zeta)\rho(n') = \beta(n'^*)R^o(\zeta)$ $(\forall n' \in N)$, so, for any $\zeta, \eta \in H$, operator $R^o(\zeta)R^o(\eta)^* \in \beta(N)'$ and $< \zeta, \eta >_{N^o} = R^o(\eta)^* R^o(\zeta) \in \lambda(N)$.

1.1.2. If K is another Hilbert space carrying a faithful unital $*$-representation γ of N, we obtain a relative tensor product $H \underset{N}{{}_\beta\otimes_\gamma} K$ from $H \otimes K$ introducing the following inner product (and factorizing, if necessary):

$$(\zeta_1 \otimes \eta_1 | \zeta_2 \otimes \eta_2) = (\gamma(< \zeta_1, \zeta_2 >_{N^o})\eta_1 | \eta_2) \ (\forall \zeta_1, \zeta_2 \in H, \eta_1, \eta_2 \in K).$$

Then there exists a linear operator $A_{\beta,\gamma} : H \otimes K \to H \underset{N}{{}_\beta\otimes_\gamma} K$, the image of $\zeta \otimes \eta$ being denoted by $\zeta \underset{N}{{}_\beta\otimes_\gamma} \eta$ $(\forall \zeta \in H, \eta \in K)$. One can also define $H \underset{N}{{}_\beta\otimes_\gamma} K$ by introducing the following inner product:

$$(\zeta_1 \otimes \eta_1 | \zeta_2 \otimes \eta_2) = (\beta(< \eta_1, \eta_2 >_N)\zeta_1 | \zeta_2) \ (\forall \zeta_1, \zeta_2 \in H, \eta_1, \eta_2 \in K),$$

this leads to a definition of a flip $\sigma_{\beta,\gamma} : H \underset{N}{{}_\beta\otimes_\gamma} K \to K \underset{N^o}{{}_\gamma\otimes_\beta} H$ by $\sigma_{\beta,\gamma} : \zeta \underset{N}{{}_\beta\otimes_\gamma} \eta \to \eta \underset{N^o}{{}_\gamma\otimes_\beta} \zeta$ $(\forall \zeta \in H, \eta \in K)$. This allows to define a flip $\Sigma_{\beta,\gamma} : L(H \underset{N}{{}_\beta\otimes_\gamma} K) \to L(K \underset{N^o}{{}_\gamma\otimes_\beta} H)$ by $\Sigma_{\beta,\gamma}(X) = \sigma^*_{\beta,\gamma} X \sigma_{\beta,\gamma}$ $(\forall X \in L(H \underset{N}{{}_\beta\otimes_\gamma} K))$. Recall also that $\beta(n)\zeta \underset{N}{{}_\beta\otimes_\gamma} \eta = \zeta \underset{N}{{}_\beta\otimes_\gamma} \gamma(n)\eta$ $(\forall \zeta \in H, \eta \in K, n \in N)$.

1.1.3. Finally, if ${}_\alpha H_\beta$ (resp., ${}_\gamma K_\delta$) is an (N,N)-bimodule with respect to a $*$-representation α (resp., γ) and an anti-$*$-representation β (resp., δ), then $H {}_\beta\!\underset{N}{\otimes}_\gamma K$ is also (N,N)-bimodule: $n \cdot_\alpha [\zeta {}_\beta\!\underset{N}{\otimes}_\gamma \eta] \cdot_\delta n' = [\alpha(n)\zeta] {}_\beta\!\underset{N}{\otimes}_\gamma [\delta(n')\eta]$ $(\forall \zeta \in H, \eta \in K, n, n' \in N)$.

The described construction does not depend on the choice of a faithful trace on N (up to an isomorphism of Hilbert spaces).

1.2 Fibered product of C^*-algebras

Given two C^*-algebras M_1, M_2 such that $\beta(N) \subset M_1, \gamma(N) \subset M_2$, we construct their *fibered product* - a canonical algebra $(M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2)$ isomorphic to the reduction of $M_1 \otimes M_2$ by some projection $e_{\beta,\gamma}$ depending on β, γ.

1.2.1. First, we define $(M_1') {}_\beta\!\underset{N}{\otimes}_\gamma (M_2')$ as a C^*-algebra generated by $x {}_\beta\!\underset{N}{\otimes}_\gamma y$ $(\forall x \in M_1', y \in M_2')$ and then $(M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2) := [(M_1') {}_\beta\!\underset{N}{\otimes}_\gamma (M_2')]'$. One can verify that:

$$\begin{aligned}
[(M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2)]^o &= (M^o{}_1) {}_\beta\!\underset{N^o}{*}_\gamma (M^o{}_2), \\
\Sigma_{\beta,\gamma}((M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2)) &= (M_2) {}_\gamma\!\underset{N^o}{*}_\beta (M_1), \\
(M_1 \cap \beta(N)') {}_\beta\!\underset{N}{\otimes}_\gamma (M_2 \cap \gamma(N)') &\subset (M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2), \\
(M_1) {}_\beta\!\underset{N}{*}_\gamma (N) = (M_1 \cap \beta(N)') {}_\beta\!\underset{N}{\otimes}_\gamma 1_K &= (1_H) {}_\beta\!\underset{N}{\otimes}_\gamma (M_2 \cap \gamma(N)').
\end{aligned}$$

If β' (resp., γ') is an injective unital anti-$*$-homomorphism (resp., $*$-homomorphism) of N into another C^*-algebra L_1 (resp., L_2), and $\Phi : M_1 \to L_1$ (resp., $\Psi : M_2 \to L_2$) is a unital $*$-homomorphism such that $\Phi \circ \beta = \beta'$ (resp., $\Psi \circ \gamma = \gamma'$), then one can define a unital $*$-homomorphism $\Phi {}_\beta\!\underset{N}{*}_\gamma \Psi : (M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2) \to (L_1) {}_{\beta'}\!\underset{N}{\otimes}_{\gamma'} (L_2)$. This means, in particular, that if β (resp.,γ) is an injective unital anti-$*$-homomorphism (resp., $*$-homomorphism) of N into M_1 (resp., M_2), then one can define $(M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2)$ (to be a C^*-algebra) without any references to a specific Hilbert space.

Obviously, if $\tau : P \to N$ is an isomorphism of C^*-algebras, then one can identify $H {}_\beta\!\underset{N}{\otimes}_\gamma K$ with $H {}_{\beta\circ\tau}\!\underset{P}{\otimes}_{\gamma\circ\tau} K$, $(M_1') {}_\beta\!\underset{N}{\otimes}_\gamma (M_2')$ with $(M_1') {}_{\beta\circ\tau}\!\underset{P}{\otimes}_{\gamma\circ\tau} (M_2')$, and $(M_1) {}_\beta\!\underset{N}{*}_\gamma (M_2)$ with $(M_1) {}_{\beta\circ\tau}\!\underset{P}{*}_{\gamma\circ\tau} (M_2)$.

1.2.2. Let now $H = K$ and $e_{\beta,\gamma}$ be the support of the above operator $A_{\beta,\gamma}$. Then one can show that $e_{\beta,\gamma} \in \beta(N) \otimes \gamma(N)$ and $e_{\beta,\gamma}(\beta(n) \otimes 1_H) = e_{\beta,\gamma}(1_H \otimes \gamma(n))$ $(\forall n \in N)$. Vice versa, let M_1 and M_2 be two C^*-algebras acting in H such that $\beta(N) \subset M_1$ and $\gamma(N) \subset M_2$, and let $e_{\beta,\gamma} \in \beta(N) \otimes \gamma(N)$ be a projection satisfying the above relation. Then automatically $e = (id \otimes \theta)(e_{\beta,\gamma}) \in Z(\beta(N))$ and

$$||(e_{\beta,\gamma})^{\frac{1}{2}}(\zeta \otimes \eta)||^2 = (\beta(< \eta, \eta >_N) e^{\frac{1}{2}}\zeta, e^{\frac{1}{2}}\zeta) \ (\forall \zeta, \eta \in H),$$

which equals to $||e^{\frac{1}{2}}\zeta \, {}_\beta\underset{N}{\otimes}{}_\gamma \, \eta||^2$. So $e^{\frac{1}{2}}$ is invertible and there exists an isometry $V_{\beta,\gamma} : H \, {}_\beta\underset{N}{\otimes}{}_\gamma \, H \to H \otimes H$ with final support $e_{\beta,\gamma}$, defined by: $V_{\beta,\gamma}(\zeta \, {}_\beta\underset{N}{\otimes}{}_\gamma \, \eta) = e_{\beta,\gamma}(e^{-\frac{1}{2}}\zeta \otimes \eta)$ $(\forall \zeta, \eta \in H)$.

This isometry satisfies the following relations:

$$\begin{aligned} V_{\beta,\gamma}(x \, {}_\beta\underset{N}{\otimes}{}_\gamma \, y) &= (x \otimes y)V_{\beta,\gamma} \ (\forall x \in \beta(N)', y \in \gamma(N)'); \\ (\beta(n) \otimes 1_H)V_{\beta,\gamma} &= (1_H \otimes \gamma(n))V_{\beta,\gamma} \ (\forall n \in N); \\ (M_1') \, {}_\beta\underset{N}{\otimes}{}_\gamma (M_2') &= V_{\beta,\gamma}^*(M_1' \otimes M_2')V_{\beta,\gamma}; \\ (M_1) \, {}_\beta\underset{N}{*}{}_\gamma (M_2) &= V_{\beta,\gamma}^*(M_1 \otimes M_2)V_{\beta,\gamma}, \end{aligned}$$

and $(M_1) \, {}_\beta\underset{N}{*}{}_\gamma (M_2)$ is isomorphic to the reduced C^*-algebra $(M_1 \otimes M_2)_{e_{\beta,\gamma}}$. Let $i_{\beta,\gamma}$ be a non-unital injection of $(M_1) \, {}_\beta\underset{N}{*}{}_\gamma (M_2)$ into $M_1 \otimes M_2$.

1.3 Counital C^*-bialgebras

Definition 1.3.1 a) A pair (M, Δ) is said to be a gen. C^*-bialgebra if M is a f-d C^*-algebra and a *coproduct* $\Delta : M \to M \otimes M$ is an injective $*$-homomorphism (not necessarily unital) such that $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.

b) A triplet $\mathcal{A} = (M, \Delta, S)$ is said to be a gen. coinvolutive C^*-bialgebra if (M, Δ) is a gen. C^*-bialgebra and a *co-involution* $S : M \to M$ is a unital anti-$*$-automorphism such that $S^2 = id$ and $(S \otimes S) \circ \Delta = \varsigma \circ \Delta \circ S$, where $\varsigma : M \otimes M \to M \otimes M$ is a usual flip.

c) A homomorphism of gen. coinvolutive C^*-bialgebras (M_1, Δ_1, S_1) and (M_2, Δ_2, S_2) is a unital $*$-homomorphism of C^*-algebras $\pi : M_1 \to M_2$ such that $(\pi \otimes \pi) \circ \Delta_1 = \Delta_2 \circ \pi$, $\pi \circ S_1 = S_2 \circ \pi$. Clearly, gen. coinvolutive C^*-bialgebras form a category.

d) [27] A quadruplet $GK = (M, \Delta, S, \phi)$ is said to be a gen. Kac algebra if $\mathcal{A} = (M, \Delta, S)$ is a gen. coinvolutive C^*-bialgebra and a *Haar trace* ϕ is a faithful trace on M such that $\phi \circ S = \phi$ and

$$(id \otimes \phi)[(1 \otimes y)\Delta(x)] = S[(id \otimes \phi)(\Delta(y)(1 \otimes x))] \ (\forall x, y \in M).$$

Gen. Kac algebra is *abelian*, i.e., M is abelian (resp., *symmetric*, i.e., $\varsigma \circ \Delta = \Delta$), iff it is isomorphic to the algebra of functions on a finite groupoid (resp., to the groupoid algebra of a finite groupoid) [27].

Given a gen. C^*-bialgebra, $e = \Delta(1)$, let us define two sets: $N_s = \{x \in M | \Delta(x) = e(1 \otimes x) = (1 \otimes x)e\}$ and $N_t = \{x \in M | \Delta(x) = e(x \otimes 1) = (x \otimes 1)e\}$.

Proposition 1.3.2 *[26] For any gen. C^*-bialgebra we have:*

a) N_s and N_t are C^*-subalgebras of M.

b) N_s and N_t commute: $nn' = n'n$ $(\forall n \in N_s, n' \in N_t)$.

Proof. a) It is clear that N_s and N_t are linear $*$-invariant subspaces of M containing 1. For any $x, y \in N_t$ we have:

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y) = \Delta(x)e(y \otimes 1) \\ &= \Delta(x)(y \otimes 1) = e(xy \otimes 1), \end{aligned}$$

so N_t is a C^*-subalgebra of M; similarly for N_s.

b) For all $n \in N_s, n' \in N_t$ we have:

$$\begin{aligned} \Delta(nn') &= \Delta(n)e(n' \otimes 1) = \Delta(n)(n' \otimes 1) \\ &= e(1 \otimes n)(n' \otimes 1) = e(n' \otimes 1)(1 \otimes n) \\ &= \Delta(n')(1 \otimes n) = \Delta(n'n), \end{aligned}$$

and the injectivity of Δ gives the result.

N_s and N_t are said to be *source and target Cartan subalgebras* respectively. In any generalized coinvolutive C^*-bialgebra $S(N_s) = N_t$.

Definition 1.3.3 A gen. coinvolutive C^*-bialgebra is said to be *counital* if the *target counital map* $\varepsilon_t := \mu(id \otimes S)\Delta : M \to M$ (where $\mu : M \otimes M \to M$ is the multiplication in M) satisfies the following conditions:

1. $\varepsilon_t(1) = 1$.
2. $Im\varepsilon_t \subset N_t$.
3. $(id \otimes \varepsilon_t)\Delta(x) = e(x \otimes 1)$ $(\forall x \in N_s)$.

A definition of a morphism of generalized counital C^*-bialgebras is obvious.

Remark 1.3.4 a) Clearly, *source counital map* $\varepsilon_s := \mu(S \otimes id)\Delta : M \to M$ has similar properties:

1'. $\varepsilon_s(1) = 1$.

2'. $Im\varepsilon_s \subset N_s$.

3'. $(\varepsilon_s \otimes id)\Delta(x) = (1 \otimes x)e \ (\forall x \in N_t)$.

4. $S \circ \varepsilon_t = \varepsilon_s \circ S$.

b) From 3 and 3' we have $(id \otimes \varepsilon_t)e = e = (\varepsilon_s \otimes id)e$, together with 2 and 2' this gives $e \in N_s \otimes N_t$.

c) $(\Delta \otimes id)(e) = (id \otimes \Delta)(e) = (e \otimes 1)(1 \otimes e) = (1 \otimes e)(e \otimes 1)$.

Indeed, let $e = \sum_{i=1}^n x_i \otimes y_i$ (where $x_i \in N_s, y_i \in N_t$ - such a representation is possible by b)). Then, using the definition of N_s, we have:

$$(\Delta \otimes id)(e) = \sum_{i=1}^n \Delta(x_i) \otimes y_i = \sum_{i=1}^n e(1 \otimes x_i) \otimes y_i = (e \otimes 1)(1 \otimes e).$$

The other equalities are now obvious.

d) One can easily verify that:

$$\varepsilon_t(x)^* = \varepsilon_t(S(x)^*), \ \varepsilon_t(xy) = \varepsilon_t(x\varepsilon_t(y)), \ \varepsilon_t(xn) = \varepsilon_t(xS(n))$$

for all $x, y \in M, n \in N_t$, and similar relations for ε_s.

Proposition 1.3.5 $\varepsilon_t(nS(n')x) = n \cdot \varepsilon_t(x) \cdot n' \ (\forall x \in M, n, n' \in N_t)$.

Proof. The definitions of N_t, N_s and the properties of Δ, S give:

$$\begin{aligned}
\varepsilon_t(nS(n')x) &= \mu(id \otimes S)\Delta(nS(n')x) \\
&= \mu(id \otimes S)[(n \otimes 1)e(1 \otimes S(n'))\Delta(x)] \\
&= n \cdot \mu[(id \otimes S)\Delta(x)(1 \otimes n')] = n \cdot \varepsilon_t(x) \cdot n'.
\end{aligned}$$

Remark 1.3.6 a) Let us introduce the following structure of an (N_t, N_t)-bimodule ${}_tM_s$ on M : $n \cdot_t x \cdot_s n' = nS(n')x \ (\forall x \in M, n, n' \in N_t)$. Then Proposition 1.3.5 means that $\varepsilon_t : {}_tM_s \to N_t$ is the (N_t, N_t)-bimodule map.

b) One can prove the similar property for ε_s:

$$\varepsilon_s(xnS(n')) = S(n) \cdot \varepsilon_s(x) \cdot S(n') \ (\forall x \in M, n, n' \in N_t).$$

c) Proposition 1.3.5 and axiom 1 imply that ε_t equals to identity on N_t and to S on N_s; similarly ε_s equals to identity on N_s and to S on N_t.

Proposition 1.3.7 *In any gen. counital C^*-bialgebra*

$$e(S(n) \otimes 1) = e(1 \otimes n), \ (S(n) \otimes 1)e = (1 \otimes n)e \ (\forall n \in N_s).$$

Proof. Clearly, the second relation can be deduced from the first. Then, using successively the axiom 3, the definition of N_t, Proposition 1.3.5 and axiom 1, we have for any $n \in N_t$:

$$e(S(n) \otimes 1) = (id \otimes \varepsilon_t)\Delta(S(n)) = (id \otimes \varepsilon_t)e(1 \otimes S(n)) = e(1 \otimes n).$$

Definition 1.3.8 Let (M, Δ) be a gen. counital C^*-bialgebra.

a) A projection $\Lambda \in M$ is said to be *a Haar projection* if

$$x\Lambda = \varepsilon_t(x)\Lambda \qquad (\forall x \in M), \quad S(\Lambda) = \Lambda, \quad \varepsilon_t(\Lambda) = 1.$$

b) A functional (resp., a faithful trace) ϕ on M is said to be *a Haar functional* (resp., *a Haar trace*) if

$$(id \otimes \phi)\Delta = (\varepsilon_t \otimes \phi)\Delta, \quad \phi \circ S = \phi,$$

ϕ is said to be *normalized* if $(id \otimes \phi)e = 1$.

c) A conditional expectation $E_t : M \to N_t$ (resp., $E_s : M \to N_s$) is said to be *target (resp., source) Haar conditional expectation* on (M, Δ) if $(id \otimes E_t) \circ \Delta = \Delta \circ E_t$ (resp., $(E_s \otimes id) \circ \Delta = \Delta \circ E_s$).

Obviously, the identities symmetric to a) and b) hold true for ε_s.

1.4 Counital Hopf bimodules

Definition 1.4.1 [24], [8]. A collection $B = (N, M, t, s, \Delta)$ is said to be a *Hopf bimodule*, if N, M are C^*-algebras, s (resp., t, Δ) is an injective unital anti-$*$-homomorphism (resp., $*$-homomorphism) of $N \to M$ (resp., $N \to M$, $M \to M \underset{N}{{}_s*_t} M$) such that

$$s(n)t(n') = t(n')s(n) \ \ \forall n, n' \in N,$$

$$\Delta(t(n)) = t(n) \underset{N}{{}_s\otimes_t}(1), \qquad \Delta(s(n)) = (1) \underset{N}{{}_s\otimes_t} s(n) \ (\forall n \in N);$$

$$(\Delta \underset{N}{{}_s*_t} id) \circ \Delta = ((id) \underset{N}{{}_s*_t} \Delta) \circ \Delta.$$

Let us clarify this definition. The structures of left, right N-module and (N,N)-bimodule on M (${}_tM, M_s$ and ${}_tM_s$ respectively), can be defined by:

$$n\cdot_t m = t(n)m,\ m\cdot_s n' = ms(n'),\ n\cdot_t m\cdot_s n' = t(n)ms(n')\ \forall m \in M, n, n' \in N.$$

Then the fibered product $M \underset{N}{{}_s*_t} M$ is the (N,N)-bimodule

$${}_tM \underset{N}{{}_s*_t} M_s:\ n\cdot_t (m \underset{N}{{}_s*_t} m')\cdot_s n' = (n\cdot_t m) \underset{N}{{}_s*_t} (m'\cdot_s n'),$$

for all $m, m' \in M$, $n, n' \in N$. So one can construct two fibered products: $(M \underset{N}{{}_s*_t} M) \underset{N}{{}_s*_t} M$ and $M \underset{N}{{}_s*_t} (M \underset{N}{{}_s*_t} M)$; both of them are isomorphic to $p(M \otimes M \otimes M)p$, where $p = (e_{s,t} \otimes 1)(1 \otimes e_{s,t}) = (1 \otimes e_{s,t})(e_{s,t} \otimes 1)$. Two first conditions mean that $\Delta :_t M_s \to_t M \underset{N}{{}_s*_t} M_s$ is an (N,N)-bimodule morphism; after that the last one is clear.

If $B = (N, M, t, s, \Delta)$ is a Hopf bimodule, then $B^o = (N^o, M, s, t, \Sigma_{s,t}\circ\Delta)$ is also a Hopf bimodule – *symmetrized* to the first one; obviously, $\Sigma_{s,t} \circ \Delta : M \to M \underset{N^o}{{}_t*_s} M$. If N is abelian and $t = s$, then the two above Hopf bimodules coincide; we call such a Hopf bimodule *symmetric*.

Definition 1.4.2 Let $B = (N, M, t, s, \Delta)$ be a Hopf bimodule. A *coinvolution* is such a unital anti-$*$-automorphism $S : M \to M$ that $S^2 = id$, $S \circ t = s$, and $(S \underset{N}{{}_s*_t} S) \circ \Delta = \Sigma_{s,t} \circ \Delta \circ S$, where $\Sigma_{s,t} : M \underset{N}{{}_s*_t} M = M \underset{N^o}{{}_t*_s} M$ is a flip introduced in **1.1**, **1.2**. Then the collection $B = (N, M, t, \Delta, S)$ is said to be a coinvolutive Hopf bimodule.

Let us clarify this definition. The right-hand side of the latest equality is the unital injective anti-$*$-homomorphism of C^*-algebras M and $M \underset{N^o}{{}_t*_s} M$. If one considers S as an isomorphism between M and M^o, then the map $(S \underset{N}{{}_s*_t} S)$ in the left-hand side of the above equality is, according to **1.2**, the isomorphism of C^*-algebras $M \underset{N}{{}_s*_t} M$ and $M^o \underset{N}{{}_t*_s} M^o$, the last algebra coincides with $[M \underset{N^o}{{}_t*_s} M]^o$. Thus, the left-hand side of the discussed equality is the unital injective anti-$*$-homomorphism of C^*-algebras M and $M \underset{N^o}{{}_t*_s} M$.

Definition 1.4.3 A coinvolutive Hopf bimodule $B = (N, M, t, \Delta, S)$ is said to be *counital* if the *target counital map* $\epsilon_t := \mu[(id) \underset{N}{{}_s*_t} S] \circ \Delta : M \to M$

(where $(id) \underset{N}{{}_s*_t} S : M \underset{N}{{}_s*_t} M \to M \underset{N}{{}_s*_s} M^o$, the last one is a subalgebra in $M \otimes M^o$ (see **1.2**), and $\mu : M \otimes M^o \to M$ is the linear map given by the multiplication in M) satisfies the following conditions:

1. $\epsilon_t(1) = 1$.
2. $Im\epsilon_t \subset t(N)$.
3. $\epsilon_t : {}_tM_s \to t(N)$ is the (N, N)-bimodule map.

A definition of a morphism of counital Hopf bimodules is clear.

Lemma 1.4.4 *The categories of counital Hopf bimodules and gen. counital C^*-bialgebras are equivalent.*

Proof. a) Given a counital Hopf bimodule $B = (N, M, t, \Delta, S)$, let us consider the projection $e_{s,t} \in s(N) \otimes t(N)$ such that $e_{s,t}(S(n) \otimes 1) = e_{s,t}(1 \otimes t(n))$ $(\forall n \in N)$ and $M \underset{N}{{}_s*_t} M = (M \otimes M)_{e_{s,t}}$, and the canonical injection of C^*-algebras $i_{s,t} : M \underset{N}{{}_s*_t} M \to M \otimes M$ (see **1.2**). Then $\tilde{\Delta} = i_{s,t} \circ \Delta : M \to M \otimes M$ is an injective $*$-homomorphism (not necessarily unital) such that:

$$(\tilde{\Delta} \otimes id) \circ \tilde{\Delta} = (id \otimes \tilde{\Delta}) \circ \tilde{\Delta},$$

$$\tilde{\Delta}(t(n)) = (t(n) \otimes 1) \cdot e_{s,t},\ \tilde{\Delta}(s(n)) = (1 \otimes s(n)) \cdot e_{s,t},\ (\forall n \in N),$$

from where $\tilde{\Delta}(1) = e_{s,t} \in M \otimes M$. Now it is clear that $(M, \tilde{\Delta}, S)$ is a gen. coinvolutive C^*-bialgebra such that $\epsilon_t = \varepsilon_t$ and $t(N) \subset N_t$, so the conditions 1 and 2 of Definition 1.3.3 are satisfied. But the definition of N_t, the structure of ε_t and the property $\varepsilon_t(1) = 1$ imply that ε_t is identical on N_t; from $Im\varepsilon_t \subset t(N)$ we have $t(N) = N_t$, so $s(N) = N_s$.

Finally, using the above properties of $e_{s,t}, \tilde{\Delta}$ and ϵ_t, we have:

$$\begin{aligned}(id \otimes \varepsilon_t)\tilde{\Delta}(s(n)) &= (id \otimes \epsilon_t)e_{s,t}(1 \otimes s(n)) = e_{s,t}(1 \otimes t(n)) \\ &= e_{s,t}(s(n) \otimes 1)\ (\forall n \in N).\end{aligned}$$

b) Given a gen. counital C^*-bialgebra $(M, \tilde{\Delta}, S)$ with Cartan subalgebras N_t and N_s, put $N = N_t, t = id_N, s = S$. Let $H = L_2(M)$ be the Hilbert space given by the GNS-construction for (M, ϕ), where ϕ is a fixed faithful trace on M. Then, using the properties of e (Remark 1.3.4 b), Proposition 1.3.7), one can establish the following isomorphisms (see 1.1,1.2): $e(H \otimes H) \cong H \underset{N}{{}_s\otimes_t} H$, $(e \otimes 1)(1 \otimes e)(H \otimes H \otimes H) \cong H \underset{N}{{}_s\otimes_t} H \underset{N}{{}_s\otimes_t} H$, $(M' \otimes$

$M')e \cong M' \underset{N}{{}_s\otimes_t} M'$, $(M' \otimes M' \otimes M')(e \otimes 1)(1 \otimes e) \cong M' \underset{N}{{}_s\otimes_t} M' \underset{N}{{}_s\otimes_t} M'$ (spatially). Then $M \underset{N}{{}_s*_t} M$ is isomorphic to the reduced C^*-algebra $(M \otimes M)_e$ and $M \underset{N}{{}_s*_t} M \underset{N}{{}_s*_t} M$ is isomorphic to the reduced C^*-algebra $(M \otimes M \otimes M)_{(e\otimes 1)(1\otimes e)}$.

Taking in mind the above isomorphisms, Definition 1.3.3 and Proposition 1.3.5, one can see that the collection (N, M, t, Δ, S) is a counital Hopf bimodule. The statements concerning morphisms are obvious.

Obviously, $e_{s,t} = 1 \otimes 1$ iff $N = \mathbb{C}$.

Definition 1.4.5 Let $B = (N, M, t, s, \Delta)$ be a Hopf bimodule. A conditional expectation $E_t : M \to t(N)$ (resp., $E_s : M \to s(N)$) is said to be a *target (resp., source) Haar conditional expectation* on B if:

$$[(id) \underset{N}{{}_s*_t} E_t] \circ \Delta = \Delta \circ E_t, \quad (resp.,[(E_s) \underset{N}{{}_s*_t}(id)] \circ \Delta = \Delta \circ E_s).$$

Here $(id) \underset{N}{{}_s*_t}(E_t)$ is the restriction of the map $id \otimes E_t : M \otimes M \to M \to t(N)$ to $M \underset{N}{{}_s*_t} M$; since E_t is a conditional expectation, this restriction is a well defined map from $M \underset{N}{{}_s*_t} M = (M \otimes M)_{e_{s,t}}$ to $M \underset{N}{{}_s*_t} t(N) = (M \otimes t(N))_{e_{s,t}}$. Similarly for $(E_s) \underset{N}{{}_s*_t}(id)$.

Clearly E is a source (resp., target) Haar conditional expectation on a counital Hopf bimodule B iff it is a source (resp., target) Haar conditional expectation on the corresponding gen. counital C^*-bialgebra.

Remark 1.4.6 There exists a duality theory for gen. Kac algebras [27]: for any $\mathcal{K} = (M, \Delta, S, \phi)$ one can construct a *dual* gen. Kac algebra $\hat{\mathcal{K}} = (\hat{M}, \hat{\Delta}, \hat{S}, \hat{\phi})$ in such a way that the algebra dual for $\hat{\mathcal{K}}$ is isomorphic to $\mathcal{K}$. It is shown in [26] that for any gen. Kac algebra: a) $N_t = M \cap \hat{M}$ is a target Cartan subalgebra and the conditions of Proposition 1.2 are satisfied; b) there exists the unique right Haar conditional expectation $E : M \to N_t$, which can be defined as the orthogonal projection on N_t in the Hilbert space given by the GNS-construction for (M, ϕ) such that $\phi \circ E = \phi$.

2 Weak and generalized Kac algebras, Kac bimodules

2.1 Weak Kac algebras

Definition 2.1.1 A *counit* for a gen. coinvolutive C^*-bialgebra $(M,\ \Delta, S)$ is such a linear map $\varepsilon : M \to \mathbb{C}$ that $(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$ and:

1). $\varepsilon(S(x)) = \varepsilon(x),\ \varepsilon(x^*) = \overline{\varepsilon(x)}.$

2). $(\varepsilon \otimes \varepsilon)((x \otimes 1)e(1 \otimes y)) = \varepsilon(xy).$

3). $(\varepsilon_s \otimes id)\Delta(x) = (1 \otimes x)e \quad (recall\ that\ e := \Delta(1)).$

Here $x, y \in M, \varepsilon_s = \mu(S \otimes id)\Delta,\ \mu : M \otimes M \to M$ is a multiplication in M.

A collection $WK = (M, \Delta, S, \varepsilon)$ satisfying 1)-3) is said to be a *weak Kac algebra.* A homomorphism $\pi : WK_1 \to WK_2$ of weak Kac algebras is such a homomorphism of their gen. coinvolutive C^*-bialgebras (Definition 1.3.1 c)), that for the counits $\varepsilon_2 \circ \pi = \varepsilon_1$. Clearly weak Kac algebras form a category.

The above notion is the special case ($S^2 = id$) of the notion of a weak C^*-Hopf algebra introduced in [3]. It becomes a usual Kac algebra iff either $e = 1 \otimes 1$, or $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$, or $\mu(S \otimes id)\Delta(x) = \varepsilon(x)1$ $(\forall x \in M)$.

Proposition 2.1.2 *The set of axioms 2) and 3) is equivalent to the following set of axioms:*

A2. $(\varepsilon \otimes id)((x \otimes 1)e(1 \otimes y)) = (\varepsilon \otimes id)((x \otimes 1)\Delta(y)),$

A3. $(id \otimes \varepsilon \otimes id)[(e \otimes 1)(1 \otimes \Delta(x))] = e(1 \otimes x),$

A4. $\varepsilon_s(x) = (id \otimes \varepsilon)((1 \otimes x)e)$ $(\forall x, y \in M).$

Proof. Indeed, applying $(id \otimes \varepsilon)$ to 3), we get A4. This, in its turn, gives

A3'. $(id \otimes \varepsilon \otimes id)[(1 \otimes \Delta(x))(e \otimes 1)] = (1 \otimes x)e,$

which is clearly equivalent to A3.

Finally, using the above identities and the standard notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ $(\forall x \in M)$, we have an equality A2':

$$\begin{aligned}
(\varepsilon \otimes id)((1 \otimes x)e(y \otimes 1)) &= (\varepsilon \otimes \varepsilon \otimes id)((1 \otimes \Delta(x))(e \otimes 1)(y \otimes 1 \otimes 1)) \\
&= (\varepsilon \otimes \varepsilon)((1 \otimes x_{(1)})e(y \otimes 1))x_{(2)} \\
&= (\varepsilon \otimes id)(\Delta(x)(y \otimes 1)).
\end{aligned}$$

which is equivalent to A2. Vice versa, applying ε to A2, we get 2), and we get 3) combining A3' and A4.

The following relations are straightforward corollaries of A2-A4:

A3". $(id \otimes \varepsilon_t)\Delta(x) = e(x \otimes 1)$.

A4'. $\varepsilon_t(x) = (\varepsilon \otimes id)(e(x \otimes 1))$ (*here* $\varepsilon_t = \mu(id \otimes S)\Delta$).

Remark 2.1.3 a) Let us equip the dual linear space $\hat{M}$ with a product and a coproduct obtained by transposing the product and the coproduct of M by means of the canonical pairing $<,>$: $\hat{M} \times M \to \mathbb{C}$. The unit of $\hat{M}$ is $\hat{1} = \varepsilon$, the coinvolution $\hat{S}$ and the involution $*$ of $\hat{M}$ are defined by

$$< \hat{S}(\alpha), x >=< \alpha, S(x) >, \quad < \alpha^*, x >= \overline{< \alpha, S(x)^* >} \; (\forall x \in M, \alpha \in \hat{M}).$$

One can see that $(\hat{M}, \hat{\Delta}, \hat{S}, \hat{\varepsilon})$ is also a weak Kac algebra. Indeed, using linear maps $\eta : \lambda \to \lambda e$ from $\mathbb{C}$ to $M \otimes M$ and $\varepsilon \circ \mu : M \otimes M \to \mathbb{C}$, let us rewrite the above axioms as follows:

$\tilde{A}2$. $(\varepsilon \circ \mu \otimes \mu)(id \otimes \eta \otimes id) = (\varepsilon \circ \mu \otimes id)(id \otimes \Delta)$.

$\tilde{A}3$. $(id \otimes \varepsilon \circ \mu \otimes id)(\eta \otimes \Delta) = (id \otimes \mu)(\eta \otimes id)$.

$\tilde{A}4$. $\mu(S \otimes id)\Delta = (id \otimes \varepsilon \circ \mu)(id \otimes \varsigma \otimes id)(id \otimes \eta)$.

Now it is clear that A2 and A3 are dual to one another, A4 is selfdual.
b) The particular case of A3, in which $x = 1$:

*A3**. $(id \otimes \varepsilon \otimes id)[(e \otimes 1)(1 \otimes e)] = e$

is sufficient for getting A3. Indeed, using axioms A2, A3*, one has $\forall x \in M$:

$$\begin{aligned}
(id \otimes \varepsilon \otimes id)[(e \otimes 1)(1 \otimes \Delta(x))] &= 1_{(1)} \otimes (\varepsilon \otimes id)[(1_{(2)} \otimes 1)\Delta(x)] \\
&= 1_{(1)} \otimes (\varepsilon \otimes id)[(1_{(2)} \otimes 1)e(1 \otimes x)] \\
&= 1_{(1)} \otimes (\varepsilon \otimes id)[(1_{(2)} \otimes 1)e] \cdot x \\
&= e(1 \otimes x).
\end{aligned}$$

Example 2.1.4 As groups and their duals are trivial examples of usual Kac algebras, groupoids and their duals give trivial examples of weak Kac algebras. Let G be a finite groupoid (see [20] for definitions and terminology).
(a) $\mathbb{C}G$, the groupoid C^*-algebra of G, has a structure of a cocommutative weak Kac algebra given by

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \varepsilon(g) = 1 \quad \text{for all } g \in G.$$

In this case $\varepsilon_s(g) = g^{-1}g$ and $\varepsilon_t(g) = gg^{-1}$ are familiar source and target maps. The source and target Cartan subalgebras coincide with $\mathbb{C}G^0$, where

G^0 is the unit space of G.
(b) $\mathbb{C}(G)$, the C^*-algebra of complex-valued functions on G, has a structure of a commutative weak Kac algebra given by

$$\Delta(\delta_g) = \sum_{xy=g} \delta_x \otimes \delta_y, \quad S(\delta_g) = \delta_{g^{-1}}, \quad \varepsilon(\delta_g) = \delta_{g,t(g)} = \delta_{g,s(g)} \quad \text{for all } g \in G.$$

In this case $\varepsilon_s(\delta_g) = \sum_{s(x)=g} \delta_x$ and $\varepsilon_t(\delta_g) = \sum_{t(x)=g} \delta_x$. The source and target Cartan subalgebras are $N_s = \{f \in \mathbb{C}(G) \mid f(g) = f(s(g)) \, \forall g \in G\}$ and $N_t = \{f \in \mathbb{C}(G) \mid f(g) = f(t(g)) \, \forall g \in G\}$.

The above weak Kac algebras are clearly dual to each other.

Remark 2.1.5 It follows from [27] and Theorem 2.3.5 below that every cocommutative (resp. commutative) weak Kac algebra is isomorphic to $\mathbb{C}G$ (resp. $\mathbb{C}(G)$) for some groupoid G.

Given two weak Kac algebras M_1 and M_2, one can construct their tensor product $M_1 \otimes M_2$ and direct sum $M_1 \oplus M_2$ in an obvious way.

Let us write $e = \sum_{i=1}^n x_i \otimes y_i$ with minimal possible n (i.e., with both $\{x_i\}$ and $\{y_i\}$ linearly independent). Obviously, one can choose x_i and y_i in such a way that $x_i = x_i^*$, $y_i = y_i^*$.

Proposition 2.1.6 $\varepsilon(y_i x_k) = \varepsilon(x_k y_i) = \delta_{ik}$.

Proof. From $A3^*$ one gets:

$$\sum_{i,j=1}^n x_i \otimes \varepsilon(y_i x_j) y_j = \sum_{i=1}^n x_i \otimes y_i, \quad \sum_{i,j=1}^n x_i \otimes \varepsilon(x_j y_i) y_j = \sum_{i=1}^n x_i \otimes y_i,$$

so $\sum_{j=1}^n \varepsilon(y_k x_j) y_j = y_k = \sum_{j=1}^n \varepsilon(x_j y_k) y_j$, from where the result follows.

Let us denote $N_s = \mathrm{span}\{x_1, \dots x_n\}$ and $N_t = \mathrm{span}\{y_1, \dots y_n\}$. Then it is clear that $e \in N_s \otimes N_t$, that both N_s, N_t are $*$-invariant and contain 1.

Proposition 2.1.7 *a)* ε_s *(resp.* ε_t*) is a linear map from* M *to* N_s *(resp.* N_t*); b)* $\varepsilon_s|_{N_s} = id$, $\varepsilon_t|_{N_t} = id$.

Proof. a) is clear from A4 and A4'. b): $\varepsilon_s(x_k) = \sum_{i=1}^n x_i \varepsilon(x_k y_i) = x_k$, similarly $\varepsilon_t(y_l) = y_l$ $(\forall k, l = 1, ..., n)$.

Proposition 2.1.8 $\Delta(\varepsilon_s(x)) = (1 \otimes \varepsilon_s(x))e, \quad \Delta(\varepsilon_t(x)) = (\varepsilon_t(x) \otimes 1)e$.

Proof. The equality $(\Delta \otimes id)(e) = (id \otimes \Delta)(e)$ gives:

$$\sum_{i=1}^{n} \Delta(x_i) \otimes y_i = \sum_{i=1}^{n} x_i \otimes \Delta(y_i),$$

from where, using Proposition 2.1.6

$$\Delta(x_k) = \sum_{i=1}^{n} \Delta(x_i)\varepsilon(x_k y_i) = \sum_{i=1}^{n} x_i \otimes (id \otimes \varepsilon)[(1 \otimes x_k)\Delta(y_i)] \in N_s \otimes M.$$

Similarly $\Delta(y_k) \in M \otimes N_t$. Now Axioms 3), A3" and Proposition 2.1.7 give the result.

Corollary 2.1.9 *Any weak Kac algebra is a gen. counital C^*-bialgebra.*

Proof. Relations 1 and 3 of Definition 1.3.3 follow from A4 and A4'. Since N_s, N_t are $*$-invariant, Proposition 2.1.8 shows that they belong to the corresponding Cartan subalgebras, then Proposition 2.1.7 a) gives the relation 2 of Definition 1.3.3.

Moreover, N_s and N_t coincide with the corresponding Cartan subalgebras, so they do not depend on the choice of a representation of e by means of $\{x_i, y_i\}_{i=1}^{n}$. Indeed, applying $\mu(S \otimes id)$ to both sides of the equality in the definition of N_s, one can see from A4 that ε_s is identical on this algebra. Similarly ε_t is identical on N_t.

Proposition 2.1.10 $\varepsilon(x) = Tr\, L_x$ $(\forall x \in N_t)$, *where the linear operator L_x acting in N_t is defined by $L_x(n) = xn$ $(\forall n \in N_t)$, $Tr\, L_x$ is its usual trace.*

Proof. A4' shows that $\varepsilon \circ \varepsilon_t = \varepsilon$, then from Corollary 2.1.9 and Remark 1.3.4 d) we have for all $x \in M, y \in N_t$: $\varepsilon(xy) = \varepsilon(yS(x))$.

Then in the representation $(S \otimes id)e = \sum_{i=1}^{n} y_i' \otimes y_i = \sum_{i=1}^{n} y_i \otimes y_i'$, where $y_i' = S(x_i)$, $y_i \in N_t$, we have from Proposition 2.1.6 : $\varepsilon(y_k' y_i) = \varepsilon(y_i y_k') = \delta_{i,k}$ $(\forall i, k = 1, ..., n)$. This means that ε is central on N_t and that $\sum_{i=1}^{n} y_i' \cdot y_i = \sum_{i=1}^{n} y_i \cdot y_i' = 1$. Now one can identify N_t with its dual linear space by means of the non-degenerated duality given by $< y, z >= \varepsilon(yz) = \varepsilon(zy)$ $(\forall y, z \in N_t)$. Then, by the definition of a trace, for every $x \in N_t$ we have (using the centrality of ε): $Tr(L_x) = \sum_{i=1}^{n} < xy_i, y_i' >= \varepsilon(x(\sum_{i=1}^{n} y_i \cdot y_i')) = \varepsilon(x)$. In particular, $\varepsilon(1) = \dim N_t$.

So the restriction of ε to N_t is a faithful trace. On the other hand, using Axioms 1),2) of Definition 2.1.1, one can easily show that $\varepsilon(\varepsilon_t(x)^* \varepsilon_t(y)) = \varepsilon(x^* y)$ $(\forall x, y \in M)$, so ε itself is a positive functional on M.

Proposition 2.1.11 *Let $N_s \cong \oplus_{\alpha=1}^{K} M_{n_\alpha}\mathbb{C}$ and $\{ f_{pq}^\alpha \}_{p,q=1\ldots n_\alpha}^{\alpha=1\ldots K}$ be a system of matrix units in N_s. Then*

$$e = \sum\nolimits_\alpha \frac{1}{n_\alpha} \sum\nolimits_{pq} f_{pq}^\alpha \otimes S(f_{qp}^\alpha).$$

Proof. Let us write $e = \sum_{\alpha pq} f_{pq}^\alpha \otimes g_{pq}^\alpha$, where $g_{pq}^\alpha \in N_t$. Using Proposition 1.3.7 we can write

$$e = \sum_{\alpha pq} f_{pq}^\alpha \otimes S(f_{pp}^\alpha) g_{pq}^\alpha S(f_{qq}^\alpha) = \sum_{\alpha pq} c_{pq}^\alpha f_{pq}^\alpha \otimes S(f_{qp}^\alpha),$$

for some scalars c_{pq}^α. Proposition 2.1.6 and Remark 1.3.4(d) imply that

$$1 = c_{pq}^\alpha \varepsilon(f_{pq}^\alpha S(f_{qp}^\alpha)) = c_{pq}^\alpha \varepsilon(f_{pq}^\alpha f_{qp}^\alpha) = c_{pq}^\alpha \varepsilon(f_{pp}^\alpha),$$

but $\varepsilon(f_{pp}^\alpha) = n_\alpha$ by Proposition 2.1.10, therefore $c_{pq}^\alpha = \frac{1}{n_\alpha}$, from where the result follows.

Remark 2.1.12 *Morphisms of weak Kac algebras preserve Cartan subalgebras.*

Indeed, let $\rho : M \to \tilde{M}$ be a morphism and $\tilde{N}_s$, $\tilde{N}_t$ be Cartan subalgebras of $\tilde{M}$. Let us write $\Delta(1) = \sum_{i=1}^n x_i \otimes y_i$ with $\{x_i\}_{i=1}^n$ linearly independent. Then $N_s = \mathrm{span}\{x_i\}$ and since $\Delta(1_{\tilde{M}}) = \sum_{i=1}^n \rho(x_i) \otimes \rho(y_i)$ we have $\tilde{N}_s = \mathrm{span}\{\rho(x_i)\}$, i.e., $\rho|_{N_s}$ is surjective. On the other hand, by Proposition 2.1.10 $n = \dim N_s = \varepsilon_M(1) = \varepsilon_{\tilde{M}}(1) = \dim \tilde{N}_s$, so $\rho|_{N_s}$ is injective and $N_s \cong \tilde{N}_s$. Similarly, $N_t \cong \tilde{N}_t$.

Corollary 2.1.13 *Weak Kac algebras with Cartan subalgebras isomorphic to the given C^*-algebra N form a subcategory of the category of weak Kac algebras. This subcategory is closed under duality.*

We will show in Section 3 that any C^*-algebra N can appear as a Cartan subalgebra of some weak Kac algebra.

2.2 The counital representation

Let $M = \oplus_{i \in \mathcal{I}} M_{d_i}(\mathbb{C})$ be the C^*-algebra of a weak Kac algebra, P_i ($i \in \mathcal{I}$, $\mathcal{I}$ is a finite set) be the minimal central projections (i.e., selfadjoint idempotents) of M; π_i be the class of the irreducible representation $x \mapsto$

$P_i x \in M_{d_i}(\mathbb{C}),\ x \in M$. Clearly, $\{\pi_i,\ i \in \mathcal{I}\}$ is the set of all classes of irreducible representations of M. Let χ_i be the character of π_i.

Observe that the classes of non-degenerate representations of M form a ring $K_0(M)$: if $\rho_i : M \to \mathcal{B}(\mathcal{H}_i),\ i = 1, 2$ are two representations, then

$$(\rho_1 \oplus \rho_2)(x) := \rho_1(x) \oplus \rho_2(x) \quad \text{and} \quad (\rho_1 \times \rho_2)(x) := (\rho_1 \otimes \rho_2)\Delta(x)$$

are representations of M in Hilbert spaces $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $(\rho_1 \otimes \rho_2)e(\mathcal{H}_1 \otimes \mathcal{H}_1)$ respectively. The set of all irreducible classes $\{\pi_i,\ i \in \mathcal{I}\}$ forms a linear basis of $K_0(M)$. For the properties of K_0-rings of usual Kac algebras and semisimple Hopf algebras see [16].

The class of a representation ρ is completely determined by its (normalized) character χ_ρ, the product and the direct sum of representations correspond to the product and the direct sum of characters respectively. Let ρ^* be the representation corresponding to the character $\chi_\rho \circ S$, then the map $\rho \mapsto \rho^*$ defines an antimultiplicative involution in $K_0(M)$.

Consider the *counital representation* π_ε of M associated by the GNS-construction with the positive functional ε. Proposition 2.1.10 shows that ε is faithful on N_t and we also have $\varepsilon((\varepsilon_t(x) - x)^*(\varepsilon_t(x) - x)) = 0$ $(\forall x \in M)$, therefore, π_ε acts in the Hilbert space N_t equipped with a scalar product $(x,\ y) := \varepsilon(y^* x)$ in the following way:

$$\pi_\varepsilon(x) y = \varepsilon_t(xy) \quad \text{for all} \quad x \in M,\ y \in N_t.$$

Proposition 2.2.1 *(cf. [3]) The class of π_ε is a unit for $K_0(M)$, i.e., $\pi_\varepsilon \times \rho$ and $\rho \times \pi_\varepsilon$ are equivalent to ρ for any representation ρ.*

Proof. Let us compute the character $\chi_\varepsilon = \operatorname{Tr} \pi_\varepsilon$. Let $e = \sum_k x_k \otimes y_k$, then the set $\{y_k\}$ forms a basis of N_t. Therefore, for all $x \in M$ we have

$$\pi_\varepsilon(x) y_k = \varepsilon_t(x y_k) = \sum_i \varepsilon(x_i x y_k) y_i,$$

and

$$\begin{aligned} \chi_\varepsilon(x) &= \textstyle\sum_k \varepsilon(x_k x y_k) = \varepsilon(\mu(e(x \otimes 1))) \\ &= \varepsilon \otimes \varepsilon((1 \otimes x_{(1)}) e (x_{(2)} \otimes 1)) = \varepsilon(\mu \circ \Delta(x)). \end{aligned}$$

Using this formula we compute

$$\begin{aligned} (\chi_\varepsilon \otimes \chi)\Delta(x) &= (\varepsilon \otimes \chi)((\mu \otimes id)(\Delta \otimes id)\Delta(x)) \\ &= (\varepsilon \otimes \chi)((x_{(1)} \otimes 1) e (1 \otimes x_{(2)})) = \chi(x), \end{aligned}$$

for all characters χ. Similarly, one can show that $(\chi \otimes \chi_\varepsilon)\Delta(x) = \chi(x)$. Hence, $\pi_\varepsilon \times \rho$, $\rho \times \pi_\varepsilon$, and ρ are equivalent for all representations ρ.

As a representation of a f-d C^*-algebra, π_ε is equivalent to the sum of irreducible representations with some multiplicities:

$$\pi_\varepsilon \simeq \oplus_{i\in\mathcal{S}}\, \nu_i \pi_i,$$

where $\mathcal{S} \subset \mathcal{I}$ and $\nu_i \geq 1$ for all $i \in \mathcal{S}$.

Proposition 2.2.2 *(cf. [3], 2.4.) The representation π_ε is multiplicity free, i.e. $\nu_i = 1$ for all $i \in \mathcal{S}$. We have $\pi_i^* = \pi_i$ for any $i \in \mathcal{S}$ and $\pi_\varepsilon^* = \pi_\varepsilon$.*

Proof. Since π_ε is the unit of $K_0(M)$, we have $\pi_\varepsilon^* = \pi_\varepsilon$ and $\pi_j = \pi_j \times \pi_\varepsilon = \sum_{i\in\mathcal{S}} \pi_j \times \nu_i \pi_i$ for all $j \in \mathcal{I}$. In particular, the right-hand side of the last equality must be irreducible. This is the case iff there exists a unique index $u(j) \in \mathcal{S}$ such that $\nu_{u(j)} = 1$ and $\pi_j \times \pi_i = \delta_{i\,u(j)}\,\pi_j$ for all $i \in \mathcal{S}$.

On the other hand, for any irreducible π_j one has $\pi_j \times \pi_j^* \neq 0$. Indeed, if $(P_j \otimes S(P_j))e = 0$, then applying $\mu(id \otimes S)$ we get $P_j = 0$, a contradiction. This implies $\pi_{u(j)} = \pi_j^*$ and, hence, $\pi_j \times \pi_j^* = \pi_j$ for all $j \in \mathcal{S}$. Applying the involution to the last equality we get $\pi_j = \pi_j^*$ and $\nu_{u(j)} = \nu_j = 1$ ($\forall j \in \mathcal{S}$).

Remark 2.2.3 Let p_ε be the support of ε, i.e., a non-zero projection p minimal with respect to the property $\varepsilon(xp) = \varepsilon(px) = \varepsilon(x)$ ($\forall x \in M$) (which is equivalent to $\varepsilon_s(px) = \varepsilon_s(x)$ and $\varepsilon_t(xp) = \varepsilon_t(x)$). Proposition 2.2.2 shows that $p_\varepsilon = \sum_{i\in\mathcal{S}} p_i$, where p_i are minimal in M projections whose central supports P_i are mutually orthogonal. Obviously $\varepsilon \circ S = \varepsilon$ implies $S(p_\varepsilon) = p_\varepsilon$.

Since π_ε is multiplicity free, we have

$$\dim N_t = \dim \mathcal{H}_\varepsilon = \sum\nolimits_{i\in\mathcal{S}} (\dim P_i M)^{\frac{1}{2}} = \sum\nolimits_{i\in\mathcal{S}} \dim M p_i = \dim M p_\varepsilon.$$

Therefore, the map $\varepsilon_t : Mp_\varepsilon \to N_t$ is a linear isomorphism of vector spaces. In other words, $\{x \in M | \varepsilon_t(x) = 0\} = \{x \in M | xp_\varepsilon = 0\}$, this implies $(x - \varepsilon_t(x))p_\varepsilon = 0$ and $xp_\varepsilon = \varepsilon_t(x)p_\varepsilon$ ($\forall x \in M$). Replacing x by $S(x)$ and applying S to the last equality, we also get $p_\varepsilon x = p_\varepsilon \varepsilon_s(x)$.

Proposition 2.2.4 *Let $I_s = \{\, y \in M \mid yx = y\varepsilon_s(x),\, \forall x \in M \,\}$ and $I_t = \{\, y \in M \mid xy = \varepsilon_t(x)y,\, \forall x \in M \,\}$. Then $I_s \cap I_t = \{\, y \in M \mid y = p_\varepsilon y p_\varepsilon \,\}$.*

Proof. Clearly, I_s is a left ideal in M, so there exists a projection $p \in M$ such that $I_s = Mp$. Since $p_\varepsilon \in I_s$, we have $p_\varepsilon = p_\varepsilon p = p_\varepsilon p p_\varepsilon$ and $p' = p - p_\varepsilon \in I_s$ is a projection orthogonal to p_ε. Hence,

$$p' = p'^2 = p'\varepsilon_s(p') = p'\varepsilon_s(p_\varepsilon p') = 0.$$

So $p = p_\varepsilon$ and $I_s = Mp_\varepsilon$, similarly $I_t = p_\varepsilon M$; these equalities give the result.

Theorem 2.2.5 *Given a weak Kac algebra, there exist (see Definition 1.3.8)*
a) a unique Haar projection $\Lambda = p_\varepsilon \in M$;
b) a unique normalized Haar functional $\phi_\varepsilon = \hat{p}_\varepsilon \in \hat{M}$.

Proof. It suffices to prove part a) since b) is just its dual version. Proposition 2.2.4 implies that any Λ satisfying 1.3.8 a) has the form $\Lambda = p_\varepsilon \Lambda p_\varepsilon = \sum_{i\in\mathcal{S}} \lambda_i p_i$ for some scalars λ_i; then, due to the above relations, $\varepsilon(p_i) = \varepsilon(\Lambda p_i) = \lambda_i \varepsilon(p_i)$ from where $\lambda_i = 1$ and $\Lambda = p_\varepsilon$. Since p_ε satisfies 1.3.8 a), the proof is completed.

Note that ϕ_ε is the Haar projection in $\hat{M}$.

Now let us introduce and study a *counital quotient* of a weak Kac algebra arising from its counital representation π_ε.

Proposition 2.2.6 *(a) Let $\{\pi_i\}_{i\in\mathcal{S}}$ be the set of irreducible representations of M contained in the decomposition of π_ε. Then for any $i \in \mathcal{S}$ $M_i = P_i M$ is a weak Kac algebra with a comultiplication $\Delta_i(x) = (P_i \otimes P_i)\Delta(x)$, an antipode $S_i(x) = S(x)$, and a counit $\varepsilon_i(x) = \varepsilon(x)$.*

(b) Let $P_\varepsilon = \sum_{i\in\mathcal{S}} P_i$ be the central support of ε. Then $x \mapsto P_\varepsilon x$ is a surjective morphism of weak Kac algebras M and $M_\varepsilon = \oplus_{i\in\mathcal{S}} M_i$, i.e., M_ε is a quotient of M.

Proof. Proposition 2.2.2 implies that $\pi_i = \pi_i^*$ and $\pi_i \times \pi_j = \delta_{ij}\pi_i$ for all $i, j \in \mathcal{S}$. This means that $(P_i \otimes P_i)\Delta = (P_i \otimes P_i)\Delta \circ P_i$ and $S \circ P_i = P_i \circ S$. In particular, $\Delta_i(x) \in P_i M \otimes P_i M$ and $S_i(x) \in P_i M$ $(\forall x \in P_i M)$. We also have that $(P_\varepsilon \otimes P_\varepsilon)\Delta = \Delta_i$. $x \in P_i M$.

Clearly, $(P_i M, \Delta_i, S_i)$ is a gen. coinvolutive C^*-bialgebra. Due to the definitions of $\Delta_i, S_i, \varepsilon_i$ and the centrality of P_i, ε_i is a counit for $P_i M$ satisfying the axioms of Definition 2.1.1, which proves (a). Obviously, $M_\varepsilon = \oplus_i M_i$ is a weak Kac algebra. To see that $x \mapsto P_\varepsilon x$ is a (clearly surjective) morphism of weak Kac algebras, it suffices to note that $(P_\varepsilon \otimes P_\varepsilon)\Delta(x) = \sum_i \Delta_i(P_\varepsilon x)$, $P_\varepsilon S(x) = \sum_i S_i(P_\varepsilon x)$, and $\varepsilon(x) = \sum_i \varepsilon_i(P_\varepsilon x)$ $\forall x \in M$.

Definition 2.2.7 *The weak Kac algebra M_ε defined in Proposition 2.2.6 is said to be a counital quotient of M.*

By Proposition 2.2.6(a), M_ε is a direct sum of weak Kac algebras which are simple as algebras. We will classify all such algebras in Section 3.

Remark 2.2.8 *M_ε is a minimal quotient weak Kac algebra of M, i.e. if $\rho : M \to \tilde{M}$ is a surjective morphism of weak Kac algebras, then $\rho(M_\varepsilon) \cong \tilde{M}_\varepsilon$.*

Indeed, $\varepsilon(x) = \tilde{\varepsilon}(\rho(x))$ $(\forall x \in M)$ implies $\rho(P_\varepsilon) = \tilde{P}_\varepsilon$, so $\rho|_{M_\varepsilon} : M_\varepsilon \to \tilde{M}_\varepsilon$ is surjective. If it is not injective then $\rho(P_i) = 0$ for some $i \in \mathcal{S}$, which contradicts to $\varepsilon(P_i) \neq 0$. Therefore, $\rho|_{M_\varepsilon}$ is an isomorphism.

2.3 Haar traces

In fact, $\phi_\varepsilon = p_{\hat{\varepsilon}}$ is a normalized Haar trace, in order to show that we need some preliminary statements. We use the notations $L_x : y \to xy$, $R^*_\alpha : y \to (id \otimes \alpha)\Delta(y)$ $(\forall x, y \in M, \alpha \in \hat{M})$ [14]. These two maps are clearly $*$-representations of the $*$-algebras M and $\hat{M}$ respectively in $End(M)$.

Proposition 2.3.1 *a) $R^*_\alpha L_x = \alpha_{(1)}(x_{(2)}) L_{x_{(1)}} R^*_{\alpha_{(2)}}$, b) $R^*_{\hat{\varepsilon}_t(\alpha)} = L_{(id\otimes\alpha)e}$, where $\hat{\varepsilon}_t(\alpha) = (\hat{\varepsilon} \otimes id_{\hat{M}})(\hat{\Delta}(\varepsilon)(\alpha \otimes \varepsilon))$.*

Proof. a) We have

$$\begin{aligned} R^*_\alpha L_x y &= (id \otimes \alpha)\Delta(xy) = x_{(1)} y_{(1)} \alpha(x_{(2)} y_{(2)}) \\ &= x_{(1)} y_{(1)} \hat{\Delta}(\alpha)(x_{(2)} \otimes y_{(2)}) = x_{(1)} y_{(1)} \alpha_{(1)}(x_{(2)}) \alpha_{(2)}(y_{(2)}) \\ &= \alpha_{(1)}(x_{(2)}) L_{x_{(1)}} R^*_{\alpha_{(2)}} y. \end{aligned}$$

b) Clearly the map $\hat{\varepsilon}_t$ is dual to the map ε_t, and we have, using A3":

$$\begin{aligned} R^*_{\hat{\varepsilon}_t(\alpha)}(x) &= (id \otimes \hat{\varepsilon}_t(\alpha))\Delta(x) = (id \otimes \alpha \circ \varepsilon_t)(\Delta(x)) \\ &= (id \otimes \alpha)(e(x \otimes 1)) = L_{(id\otimes\alpha)e}(x). \end{aligned}$$

Corollary 2.3.2 $R^*_{\alpha_{(1)}} L_x R^*_{\hat{S}(\alpha_{(2)})} = L_{x_{(1)}} \alpha(x_{(2)})$.

Proof. Indeed, using the notation $(\hat{\Delta} \otimes id_{\hat{M}})\hat{\Delta}(\alpha) = (id_{\hat{M}} \otimes \hat{\Delta})\hat{\Delta}(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}$ and Proposition 2.3.1, we have:

$$\begin{aligned} R^*_{\alpha_{(1)}} L_x R^*_{\hat{S}(\alpha_{(2)})} &= L_{x_{(1)}} \alpha_{(1)}(x_{(2)}) R^*_{\alpha_{(2)} \hat{S}(\alpha_{(3)})} = L_{x_{(1)}} \alpha_{(1)}(x_{(2)}) R^*_{\hat{\varepsilon}_t(\alpha_{(2)})} \\ &= L_{x_{(1)}} \alpha_{(1)}(x_{(2)}) L_{(id\otimes\alpha_{(2)})e} = L_{x_{(1)}} \alpha_{(1)}(x_{(2)}) L_{1_{(1)}} \alpha_{(2)}(1_{(2)}) \\ &= L_{x_{(1)} 1_{(1)}} \alpha(x_{(2)} 1_{(2)}) = L_{x_{(1)}} \alpha(x_{(2)}). \end{aligned}$$

Proposition 2.3.3 *Let us consider the following faithful trace on M:*

$$\theta(x) = Tr(L_x),$$

where Tr is the usual trace of a linear operator. Then we have:

$$(\theta \otimes id)(\Delta(x)) = (\theta \otimes S)(e(x \otimes 1)) = (\theta \otimes \varepsilon_s \circ S)(\Delta(x)) \ (\forall x \in M).$$

Proof. Using Proposition 2.3.1, Corollary 2.3.2 and the properties of Tr, one has for any $x \in M, \alpha \in \hat{M}$:

$$\begin{aligned}
(\theta \otimes \alpha)(\Delta(x)) &= Tr[L_{x_{(1)}}\alpha(x_{(2)})] = Tr[R^*_{\alpha_{(1)}} L_x R^*_{\hat{S}(\alpha_{(2)})}] \\
&= Tr[R^*_{\hat{S}(\alpha_{(2)})\alpha_{(1)}} L_x] = Tr[R^*_{\hat{\varepsilon}_t(\hat{S}(\alpha))} L_x] \\
&= Tr[L_{(id \otimes \hat{S}(\alpha))e} L_x] = Tr[L_{(id \otimes \alpha \circ S)(e(x \otimes 1))}] \\
&= (\theta \otimes \alpha \circ S)(e(x \otimes 1)).
\end{aligned}$$

Remark 2.3.4 Since $(\theta \otimes id)(\Delta(x)) \in N_s$, θ is a Haar trace ($\theta \circ S = \theta$ from $S^2 = id$ - see [11], Lemma 3.1), but generally it is not normalized.

Theorem 2.3.5 *Let $(M, \Delta, S, \varepsilon)$ be a weak Kac algebra and ϕ_ε a Haar trace on it. Then the collection $(M, \Delta, S, \phi_\varepsilon)$ is a gen. Kac algebra.*

Proof. It is enough to show that

$$(\phi_\varepsilon \otimes id)[(y \otimes 1)\Delta(x)] = S[(\phi_\varepsilon \otimes id)(\Delta(y)(x \otimes 1))] \ (\forall x, y \in M).$$

Indeed, using successively A3", A4' and Definition of ε_t, one has:

$$\begin{aligned}
(\phi_\varepsilon \otimes id)[(y \otimes 1)\Delta(x)] &= (\phi_\varepsilon \otimes id)[\Delta(x)e(y \otimes 1)] \\
&= (\phi_\varepsilon \otimes id)[\Delta(x)(id \otimes \varepsilon_t)\Delta(y)] \\
&= (\phi_\varepsilon \otimes id)[\Delta(x)(id \otimes \mu(id \otimes S)\Delta)\Delta(y)] \\
&= (\phi_\varepsilon \otimes id)[\Delta(xy_{(1)})]S(y_{(2)}) \\
&= (\phi_\varepsilon \otimes id)[e(xy_{(1)} \otimes 1)]S(y_{(2)}) \\
&= S[(\phi_\varepsilon \otimes id)(\Delta(y)(x \otimes 1))].
\end{aligned}$$

Now let us compute $\Delta(p_\varepsilon)$.

Lemma 2.3.6 *Let $A = M_d(\mathbb{C})$, $B = M_d(\mathbb{C})$, and $S : A \to B$ be an antiautomorphism of algebras. Let $Q \in A \otimes B$ be a projection of rank 1 in $A \otimes B$ such that $\mu(S \otimes id_B)Q = 1_B$ and $\mu(id_A \otimes S^{-1})Q = 1_A$. Then $Q = \frac{1}{d}\sum_{km} e_{km} \otimes S(e_{mk})$ for any system $\{e_{km}\}_{k,m=1}^d$ of matrix units in A.*

Proof. If $\{e_{km}\}_{k,m=1}^{d}$ is a system of matrix units in A, then $\{S(e_{nl})\}_{l,n=1}^{d}$ is a system of matrix units in B. Since Q has rank 1, we can write it as

$$Q = \sum_{klmn} \gamma_{kl}\beta_{mn}\, e_{km} \otimes S(e_{nl}),$$

for some $\beta_{mn},\ \gamma_{kl} \in \mathbb{C}\ (k,l,m,n = 1\ldots d)$. By the hypothesis,

$$1_B = \mu(S \otimes \mathrm{id}_B)Q = \sum_{klmn} \gamma_{kl}\beta_{mn}\, S(e_{nl}e_{km}) = (\sum_k \gamma_{kk}) \sum_{mn} \beta_{mn}\, S(e_{nm}),$$

so $\beta_{mn} = \delta_{mn}\,\beta$ for some $\beta \in \mathbb{C}$. Similarly, $\gamma_{kl} = \delta_{kl}\,\gamma$ for some $\gamma \in \mathbb{C}$. Thus, $Q = \gamma\beta \sum_{km} e_{km} \otimes S(e_{mk})$ and condition $Q^2 = Q$ gives $\gamma\beta = \frac{1}{d}$.

Lemma 2.3.7 *Let $P_\varepsilon = \sum_{i\in\mathcal{S}} P_i$ be the central support of p_ε. Then*

$$(P_\varepsilon \otimes P_\varepsilon)\Delta(p_\varepsilon) = \sum_{i\in\mathcal{S}} \frac{1}{d_i} \sum_{kl} e_{kl}^{(i)} \otimes S(e_{lk}^{(i)}),$$

where $\{e_{kl}^{(i)}\}_{k,l=1}^{d_i}$ is any system of matrix units in P_iM.

Proof. The proof of Proposition 2.2.2 shows that $P_i = S(P_i)$, $i \in \mathcal{S}$ and $(P_\varepsilon \otimes P_\varepsilon)\Delta(p_\varepsilon) = \sum_{i\in\mathcal{S}} Q_i$, where $Q_i = (P_i \otimes P_i)\Delta(p_\varepsilon)$ is a projection of rank 1 in $P_iM \otimes P_iM$ such that $\mu(S \otimes id)Q_i = P_i\varepsilon_s(p_\varepsilon) = P_i$. Similarly $\mu(id \otimes S)Q_i = P_i$. Applying Lemma 2.3.6 we get $Q_i = \frac{1}{d_i}\sum_{kl} e_{kl}^{(i)} \otimes S(e_{lk}^{(i)})$, from where the result follows.

Lemma 2.3.8 $(\theta \otimes id)\Delta(p_\varepsilon) = (id \otimes \theta)\Delta(p_\varepsilon) = 1.$

Proof. Let us choose a system of matrix units $\{e_{kl}^{(i)}\}_{k,l=1}^{d_i}$ in $P_\varepsilon M$ in such a way that $p_i = e_{11}^{(i)}$ and $S(e_{lk}^{(i)}) = e_{kl}^{(i)}$ for all i, k, l. We have, using Remark 2.3.4, relation $\Delta(p_\varepsilon) = (P_\varepsilon \otimes P_\varepsilon)\Delta(p_\varepsilon)$ and Lemma 2.3.7:

$$\begin{aligned}
(\theta \otimes id)\Delta(p_\varepsilon) &= (\theta \otimes \varepsilon_s)\Delta(p_\varepsilon) = (\theta \otimes \varepsilon_s)((1 \otimes p_\varepsilon)\Delta(p_\varepsilon)) \\
&= \sum_{i\in\mathcal{S}} \frac{1}{d_i} \sum_{kl} \theta(e_{lk}^{(i)})\, \varepsilon_s(e_{11}^{(i)} e_{kl}^{(i)}) = \sum_{i\in\mathcal{S}} \frac{1}{d_i} \sum_{kl} d_i\, \delta_{kl}\, \varepsilon_s(\delta_{1k} e_{1l}^{(i)}) \\
&= \sum_{i\in\mathcal{S}} \varepsilon_s(e_{11}^{(i)}) = \varepsilon_s(p_\varepsilon) = \varepsilon_s(1) = 1.
\end{aligned}$$

The second part is similar.

Let us generalize now the formula known for Kac algebras ([7], 6.3.7).

Proposition 2.3.9 *For any system* $\{ e_{kl}^{(i)};\ k,l = \ldots d_i,\, i \in \mathcal{I}\}$ *of matrix units in* M

$$\Delta(p_\varepsilon) = \sum_{i\in\mathcal{I}} \frac{1}{d_i} \sum_{kl} e_{kl}^{(i)} \otimes S(e_{lk}^{(i)}).$$

In particular, $\varsigma\Delta(p_\varepsilon) = \Delta(p_\varepsilon)$.

Proof. Let us write $\Delta(p_\varepsilon) = \sum_{ij\in\mathcal{I}} R_{ij}$, where R_{ij} is a projection in $P_iM \otimes P_jM$. ¿From Lemma 2.3.8 we have

$$\begin{aligned} 1 &= (id\otimes\theta)\Delta(p_\varepsilon) = \Sigma_i\ (\Sigma_j\, d_j(id\otimes\chi_j)R_{ij}))\, P_i \\ 1 &= (\theta\otimes id)\Delta(p_\varepsilon) = \Sigma_j\ (\Sigma_i\, d_i(\chi_i\otimes id)R_{ij}))\, P_j. \end{aligned}$$

Applying χ_i to the first equality and χ_j to the second one, we get :

$$d_i = \Sigma_j\, d_j\, \mathrm{rank}(R_{ij}), \qquad d_j = \Sigma_i\, d_i\, \mathrm{rank}(R_{ij}).$$

Note that $\varepsilon_s(p_\varepsilon) = 1$ implies that $R_{ii^*} \neq 0$ (we can assume that the involution acts on the set $\mathcal{I}$). Using this fact and the above relations, we conclude that $\mathrm{rank}(R_{ij}) = \delta_{ij^*}$. Thus, $\mu(S\otimes id)R_{ii^*} = \mu(id\otimes S)R_{ii^*} = P_i$ and application of Lemma 2.3.6 completes the proof.

Corollary 2.3.10 ϕ_ε *is the normalized Haar trace.*

Proof. Since $\phi_\varepsilon = \hat{p_\varepsilon}$ is a Haar projection of $\hat{M}$, the centrality follows from Proposition 2.3.9. In order to see that ϕ_ε is positive and faithful let us take a system $\{ f_{kl}^{(i)} \}$ of matrix units in $\hat{M}$. Then for any $x \in M$:

$$\begin{aligned} \phi_\varepsilon(xx^*) &= \ < x\otimes x^*, \hat{\Delta}(\hat{p}_\varepsilon) > = \sum_{i\in\mathcal{I}} \frac{1}{d_i} \sum_{kl} f_{kl}^{(i)}(x)\, \hat{S} f_{lk}^{(i)}(x^*) \\ &= \sum_{i\in\mathcal{I}} \frac{1}{d_i} \sum_{kl} \left| f_{kl}^{(i)}(x) \right|^2 \geq 0, \end{aligned}$$

which equals to 0 iff $x = 0$.

Remark 2.3.11 Let us describe all Haar traces on a weak Kac algebra. Proposition 2.3.9 shows that $\varsigma\Delta(p_i) = \Delta(p_i)$ $(\forall i \in \mathcal{S})$, therefore, any linear combination $\sum_{i\in\mathcal{S}} \lambda_i p_i$ with positive coefficients λ_i defines a trace on $\hat{M}$. But the proof of Theorem 2.2.5 shows that any Haar trace is of this form. Such a trace is faithful iff all $\lambda_i > 0$.

2.4 Haar conditional expectations

Proposition 2.4.1 *Given a weak Kac algebra $(M, \Delta, S, \varepsilon)$ and a normalized Haar trace ϕ_ε on it, there exist unique faithful target and source Haar conditional expectations $E_t : M \to N_t$ and $E_s : M \to N_s$ such that $\phi_\varepsilon = \phi_\varepsilon \circ E_t$ and $\phi_\varepsilon = \phi_\varepsilon \circ E_s$. They could be defined as follows:*

$$E_t(x) = (id \otimes \phi_\varepsilon)\Delta(x) = (S \otimes \phi_\varepsilon)((1 \otimes x)e), E_s(x) = (\phi_\varepsilon \otimes id)\Delta(x) \ (\forall x \in M).$$

We also have $E_t \circ S = S \circ E_s$ and

$$(E_t \otimes E_t)((\Delta(x)(y \otimes z)) = \varsigma(E_t \otimes E_t)((S(y) \otimes x)\Delta(z)) \ (\forall x, y, z \in M).$$

Proof. Clearly both E_t and E_s are linear unital $*$-maps from M to N_t and N_s respectively such that $E_t|N_t = id_{N_t}, E_s|N_s = id_{N_s}$, and that $E_t \circ S = S \circ E_s$. We also have $\forall x \in M, n \in N_t$:

$$\begin{aligned} E_t(x) \cdot n &= (S \otimes \phi)((1 \otimes x)e) \cdot n = (S \otimes \phi)((1 \otimes x)(S(n) \otimes 1)e) \\ &= (S \otimes \phi)((1 \otimes xn)e) = E_t(xn) \end{aligned}$$

and

$$\phi_\varepsilon \circ E_t(x) = (\phi_\varepsilon \circ S \otimes \phi_\varepsilon)((1 \otimes x)e) = \phi_\varepsilon(x).$$

Now it follows from ([9], 2.6.2) that E_t is indeed a faithful conditional expectation. The relations $(id \otimes E_t) \circ \Delta = \Delta \circ E_t$ and $(E_s \otimes id) \circ \Delta = \Delta \circ E_s$ are obvious from the coassociativity of Δ. In order to prove the last relation, it suffices to show that $(\alpha \otimes \beta)[(E_t \otimes E_t)(\Delta(x)(y \otimes z))] = (\alpha \otimes \beta)[\varsigma(E_t \otimes E_t)(S(y) \otimes x)\Delta(z))]$ for any linear functionals α, β on N_t. But these functionals can be represented as $\alpha(\cdot) = \phi_\varepsilon(n\cdot)$, $\beta(\cdot) = \phi_\varepsilon(n'\cdot)$ for some $n, n' \in N_t$. Then, using the properties of traces, conditional expectations, the invariancy of ϕ_ε and Propositions 2.1.8, 1.3.7, we have $\forall x, y, z \in M$:

$$\begin{aligned} & (\alpha \otimes \beta)[(E_t \otimes E_t)(\Delta(x)(y \otimes z))] = \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[(n \otimes n')(E_t \otimes E_t)(\Delta(x)(y \otimes z))] \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[(E_t \otimes E_t)((n \otimes n')\Delta(x)(y \otimes z))] \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[(n \otimes n')\Delta(x)(y \otimes z)] \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[(n \otimes 1)e\Delta(x)(y \otimes zn')] \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[\Delta(nx)(y \otimes zn')] \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[(S(y) \otimes nx)\Delta(zn')] \\ = \ & (\phi_\varepsilon \otimes \phi_\varepsilon)[(S(y) \otimes nx)\Delta(z)(n' \otimes 1)] \end{aligned}$$

$$
\begin{aligned}
&= (\phi_\varepsilon \otimes \phi_\varepsilon)[(n' \otimes n)(S(y) \otimes x)\Delta(z)] \\
&= (\phi_\varepsilon \otimes \phi_\varepsilon)[(E_t \otimes E_t)((n \otimes n')\varsigma((S(y) \otimes x)\Delta(z))] \\
&= (\phi_\varepsilon \otimes \phi_\varepsilon)[(n \otimes n')\varsigma((E_t \otimes E_t)((S(y) \otimes x)\Delta(z))] \\
&= (\alpha \otimes \beta)[\varsigma(E_t \otimes E_t)((S(y) \otimes x)\Delta(z))]
\end{aligned}
$$

Remark 2.4.2 For a general Haar trace ϕ the conditional expectations E_t and E_s satisfying all the above properties are the orthogonal projectors on N_t and N_s respectively in the Hilbert space given by the GNS-construction for (M, ϕ), but they cannot be given by the above formulae (see [26]).

Proposition 2.4.3 *a) In a weak Kac algebra any of the relations* $e(1 \otimes x) = 0$, $e(x \otimes 1) = 0$ *implies* $x = 0$.

b) If $y \in M$ *and one of the elements* $e(1 \otimes y)$, $e(y \otimes 1)$, $(1 \otimes y)e$, $(y \otimes 1)e$ *is positive, then* y *is positive.*

Proof. a) Since $\mu(S \otimes id)(e(1 \otimes x)) = x$ $(\forall x \in M)$, the first of the statements a) is clear. The second one can be proved similarly.

b) If $e(y \otimes 1)$ is positive and ϕ_ε is the normalized Haar trace, then:

$$(\phi_\varepsilon \otimes \phi_\varepsilon)((x^* \otimes 1)e(y \otimes 1)(x \otimes 1)) = \phi_\varepsilon(x^* y x) \geq 0 \ (\forall x \in M).$$

This means that y is positive. Now all the other statements are clear.

The following statement for gen. Kac algebras was obtained in [26].

Proposition 2.4.4 *a) the map* $y \to E_t^o(y) = \mu(S \otimes id)((1 \otimes y)e)$ *is a faithful conditional expectation from* M *to* $M \cap N_t'$ *satisfying the relations*

$$e(1 \otimes y)e = e(1 \otimes E_t^o(y)) = (1 \otimes E_t^o(y))e.$$

b) For any fixed Haar trace ϕ, $E_t^o(y)$ *is the unique faithful conditional expectation from* M *to* $M \cap N_t'$ *such that* $\phi = \phi \circ E_t^o$.

Proof. a) For any representation $e = \sum_{i=1}^n x_i \otimes y_i$ with minimal possible n we have, using Proposition 1.3.7:

$$
\begin{aligned}
(1 \otimes E_t^o(y))e &= (1 \otimes \sum_{i=1}^n S(x_i) y y_i)e = \sum_{i=1}^n (1 \otimes S(x_i)y)(1 \otimes y_i)e \\
&= \sum_{i=1}^n (S(y_i) \otimes S(x_i)y)e = \varsigma(S \otimes S)(e)(1 \otimes y)e \\
&= e(1 \otimes y)e \ (\forall y \in M).
\end{aligned}
$$

Similarly one can prove the other relation. These relations and Proposition 1.3.7 imply for any $y \in M,\ n \in N_t$:

$$\begin{aligned}(1 \otimes E_t^o(y)n)e &= (1 \otimes E_t^o(y))(1 \otimes n)e = (S(n) \otimes 1)(1 \otimes E_t^o(y))e \\ &= (S(n) \otimes 1)e(1 \otimes y)e = (1 \otimes n)e(1 \otimes y)e \\ &= (1 \otimes nE_t^o(y))e.\end{aligned}$$

Using Proposition 2.4.3 a), we see that $E_t^o(y)n = nE_t^o(y)$, so $E_t^o(y) \in M \cap N_t'$. If $x, x' \in M \cap N_t', y \in M$, then

$$E_t^o(xyx') = \sum_{i=1}^n S(x_i)(xyx')y_i = x(\sum_{i=1}^n S(x_i)yy_i)x' = xE_t^o(y)x'.$$

If now $y = zz^* \in M$ is positive, then

$$e(1 \otimes E_t^o(zz^*)) = e(1 \otimes zz^*)e = e(1 \otimes z)[e(1 \otimes z)]^* \geq 0.$$

So, according to Proposition 2.4.3 b), $E_t^o(zz^*))$ is positive, and if $E_t^o(zz^*)) = 0$, then Proposition 2.4.3 a) implies $z = 0$. Since $E_t^o(1) = \varepsilon_t(e) = 1$, then E_t^o is indeed the faithful conditional expectation from M to $M \cap N_t'$.

b) For any $x \in M$ one has: $\phi \circ E_t^o(x) = \phi(\sum_{i=1}^n y_i S(x_i)x) = \phi(x)$, so the uniqueness of E_t^o is a consequence of ([9], 2.6.2.)

2.5 Generalized Kac algebras

Let $\mathcal{K} = (M, \Delta, S, \phi)$ be a gen. Kac algebra and $\hat{\mathcal{K}} = (\hat{M}, \hat{\Delta}, \hat{S}, \hat{\phi})$ the corresponding dual gen. Kac algebra [27]. As a linear space, $\hat{M}$ can be identified with M because any linear functional α on M can be represented as $\alpha(x) = \phi(xy) = \phi(yx)$ for some $y \in M$. Let us consider the following linear functional $\varepsilon : M \to C :\ \ \varepsilon(x) = \phi(\hat{1}x)$ $(\forall x \in M)$, where $\hat{1} \in M$ corresponds to the unit in $\hat{M}$. From [27], §5 we have $S(\hat{1}) = (\hat{1})^* = \hat{1}$,

$$(\phi \otimes \phi)[\Delta(\hat{1})(x \otimes y)] = \phi(xS(y))\ \ (\forall x, y \in M),$$

which implies respectively $\varepsilon \circ S(x) = \varepsilon(x),\ \varepsilon(x^*) = \overline{\varepsilon(x)}$ and $(id \otimes \varepsilon)[\Delta(x)] = x\ (\forall x \in M)$. The last one gives: $(\varepsilon \otimes id)[\Delta(x)] = x\ (\forall x \in M)$.

Now let us show that $(M, \Delta, S, \varepsilon)$ is a weak Kac algebra. According to Proposition 2.1.2 , it suffices to verify axioms A2-A4.

One can calculate, using the invariancy of ϕ and the latest relations:

$$\begin{aligned}(\phi \otimes id)[\Delta(\hat{1})(x \otimes y)] &= (\phi \otimes id)[\Delta(\hat{1})(x \otimes 1)]y \\ &= (\phi \otimes S)[(\hat{1} \otimes 1)\Delta(x)]y \\ &= [S \circ (\varepsilon \otimes id)\Delta(x)]y = S(x)y\ \ (\forall x, y \in M),\end{aligned}$$

which gives $(\phi\otimes id)[\Delta(\hat{1})A]=\mu(S\otimes id)(A)$ $(\forall A\in M\otimes M)$. Similarly we have: $(id\otimes\phi)[A\Delta(\hat{1})]=\mu(id\otimes S)(A)$ $(\forall A\in M\otimes M)$.

Let us first verify A4, using the invariancy of ϕ and the latest relations:

$$\begin{aligned}(id\otimes\varepsilon)[(1\otimes x)\Delta(1)] &= (id\otimes\phi)[(1\otimes\hat{1}x)\Delta(1)]=(S\otimes\phi)[\Delta(\hat{1}x)]\\ &= \mu(S\otimes id)\Delta(x)\quad(\forall x\in M).\end{aligned}$$

Then let us verify A2, using the same reasoning and also A4:

$$\begin{aligned}(\varepsilon\otimes id)[(x\otimes 1)\Delta(y)] &= (\phi\otimes id)[(\hat{1}x\otimes 1)\Delta(y)]\\ &= (\phi\otimes S)[\Delta(\hat{1}x)(y\otimes 1)]\\ &= (\phi\otimes S)[\Delta(\hat{1})\Delta(x)(y\otimes 1)]\\ &= S[\mu(S\otimes id)(\Delta(x)(y\otimes 1))]\\ &= S[\mu(S\otimes id)\Delta(x)]y\\ &= (\phi\otimes S)[\Delta(\hat{1}x)]\cdot y\\ &= (\phi\otimes id)[(\hat{1}x\otimes 1)\Delta(1)]\cdot y\\ &= (\varepsilon\otimes id)[(x\otimes 1)e(1\otimes y)].\end{aligned}$$

Finally, it suffices to verify A3* (see Remark 2.1.3 b)). But this is obvious if one uses the following results obtained in [26]: a) $N_t=M\cap\hat{M}$ is a target Cartan subalgebra; b) $e\in N_s\otimes N_t$. Indeed:

$$\begin{aligned}(id\otimes\varepsilon\otimes id)[(e\otimes 1)(1\otimes e)] &= (id\otimes\varepsilon)(e(1\otimes 1_{(1)}))\otimes 1_{(2)}\\ &= (id\otimes\varepsilon)(\Delta(1_{(1)}))\otimes 1_{(2)}=e.\end{aligned}$$

Thus, we get the following

Theorem 2.5.1 *Let (M,Δ,S,ϕ) be a gen. Kac algebra, $\varepsilon(x)=\phi(\hat{1}x)$ $(\forall x\in M)$, where $\hat{1}\in M$ is the unit in $\hat{M}$. Then (M,Δ,S,ε) is a weak Kac algebra.*

Corollary 2.5.2 *On any gen. weak Kac algebra there is a unique Haar normalized trace.*

Indeed, to every gen. Kac algebra, there corresponds (Theorem 2.5.1) a weak Kac algebra with (clearly unique) counit ε and then the unique normalized Haar trace on it (Theorem 2.2.5, Corollary 2.3.10).

Definition 2.5.3 A homomorphism π of gen. Kac algebras with normalized Haar traces ψ_1 and ψ_2 is such a homomorphism of their gen. coinvolutive C^*-bialgebras (Definition 1.3.1 c)), that $\psi_2\circ\pi=\psi_1$.

Clearly gen. Kac algebras form a category; Corollary 2.5.2 and Theorem 2.5.1 describe its equivalence to the category of weak Kac algebras.

2.6 Kac bimodules

The structure of a counital C^*-bialgebra is not selfdual: there is a unit for its algebra but not necessarily a counit for its coalgebra. Now let us show that the existence of a counit of certain structure (motivated by Lemma 2.1.10), gives a weak Kac algebra structure.

Let θ_t and θ_s be the traces of left regular representations of Cartan subalgebras N_t and N_s respectively. Since $S: N_s \to N_t$ is an anti-$*$-isomorphism, we have $\theta_t \circ S = \theta_s$. Also $\theta_s \circ \varepsilon_s(x) = \theta_s(Sx_{(1)}x_{(2)}) = \theta_t \circ S(Sx_{(1)}x_{(2)}) = \theta_t(Sx_{(2)}x_{(1)}) = \theta_t(x_{(1)}Sx_{(2)}) = \theta_t \circ \varepsilon_t(x)$ $(\forall x \in M)$.

Denoting $\varepsilon := \theta_s \circ \varepsilon_s = \theta_t \circ \varepsilon_t : M \to \mathbb{C}$, we have immediately $\varepsilon \circ S = \varepsilon$ and, using Remark 1.3.4 d) and self-adjointness of θ_t:

$$\varepsilon(x) = \theta_t \circ (\varepsilon_t(x^*)^*) = \overline{\theta_t \circ \varepsilon_t(x^*)} = \overline{\varepsilon(x^*)}.$$

Theorem 2.6.1 *A counital C^*-bialgebra (M, Δ, S) is a weak Kac algebra if and only if $(id \otimes \varepsilon) \circ \Delta = id$.*

Proof. Any weak Kac algebra is a counital C^*-bialgebra satisfying the above condition (Corollary 2.1.9, Proposition 2.1.10). Conversely:

1) Using Remark 1.3.4 c) and the condition of the theorem, we have the axiom A3* of a weak Kac algebra:

$$(id \otimes \varepsilon \otimes id)[(e \otimes 1)(1 \otimes e)] = (id \otimes \varepsilon \otimes id)(\Delta \otimes id)(e) = e.$$

2) From Remark 1.3.4 d) and Proposition 1.3.7 we have:

$$(\varepsilon \otimes id)[\Delta(y)(x \otimes 1)] = (\theta_t \circ \varepsilon_t \otimes id)[\Delta(y)(S \circ \varepsilon_t(x) \otimes 1)] = y\varepsilon_t(x) \ (x, y \in M);$$

for $y = 1$ this gives the axiom A4' of a weak Kac algebra equivalent to A4.

3) Using the above relation 2) and the axiom A4', we have the axiom A2' of a weak Kac algebra (which is equivalent to A2):

$$(\varepsilon \otimes id)[\Delta(y)(x \otimes 1)] = y\varepsilon_t(x) = (\varepsilon \otimes id)((1 \otimes y)e(x \otimes 1)),$$

The final result follows from Proposition 2.1.2 and Remark 2.1.3 b).

Let us remark that the condition of Theorem 2.6.1 makes sense also for counital Hopf bimodules (see step a) of the proof of Lemma 1.4.4).

Definition 2.6.2 A counital Hopf bimodule $B = (N, M, t, \Delta, S)$ is said to be a *Kac bimodule* if it satisfies the condition of Theorem 2.6.1.

Now, taking into account Lemma 1.4.4 and Theorem 2.6.1, we get

Theorem 2.6.3 *The categories of Kac bimodules and weak Kac algebras are equivalent.*

3 Examples

3.1 Actions of finite groups on weak Kac algebras

Definition 3.1.1 A right action of a finite group G on a weak Kac algebra (Definition 2.1.1) $WK = (M, \Delta, S, \varepsilon)$ is a right action $M \triangleleft G$ of G on a C^*-algebra M [14] such that, for all $m \in M,\ g \in G$:

$$\Delta(m \triangleleft g) = \Delta(m) \triangleleft (g \otimes g),\ S(m \triangleleft g) = S(m) \triangleleft g,\ \varepsilon(m \triangleleft g) = \varepsilon(m).$$

In other words, a right action of G is exactly a right action of a cocommutative Kac algebra $\mathcal{K}_s(G) = (\mathbb{C}G, \Delta_s, S_s, \varepsilon_s)$ with $\Delta_s(g) = g \otimes g, S_s(g) = g^{-1}, \varepsilon(g) = 1$ [7] on a C^*-algebra M such that $\Delta(m \triangleleft \tilde{g}) = \Delta(m) \triangleleft \Delta_s(\tilde{g}),\ S(m \triangleleft \tilde{g}) = S(m) \triangleleft \tilde{g},\ \varepsilon(m \triangleleft \tilde{g}) = \varepsilon(m)$ $(\forall m \in M, \tilde{g} \in \mathbb{C}G)$.

Given a finite group G acting from the right on a weak Kac algebra WK, let us consider a collection $WK_G = (M_G, \Delta_G, S_G, \varepsilon_G)$. Here $M_G = M\#G$ is a *crossproduct C^*-algebra* [14] generated by elements of the form $m \otimes g$ $(\forall m \in M, g \in G)$ with operations

$$(m \otimes g)(n \otimes h) = (m \triangleleft h)n \otimes gh,\ 1_{M_G} = 1 \otimes 1_G, (m \otimes g)^* = (m \triangleleft g^{-1})^* \otimes g^{-1},$$

where $m \in M,\ g \in G,\ 1_G$ is a unit of G. A *coproduct* Δ_G, an *antipode* S_G and a *counit* ε_G are defined respectively by: $\Delta_G(m \otimes g) = (id \otimes \varsigma \otimes id_G)(\Delta(m) \otimes g \otimes g)$, $S_G(m \otimes g) = S(m \triangleleft g^{-1}) \otimes g^{-1}$, where $\varsigma : M \otimes \mathbb{C}G \to \mathbb{C}G \otimes M$ is the usual flip, and $\varepsilon_G(m \otimes g) = \varepsilon(m)$.

Lemma 3.1.2 *$WK_G = (M_G, \Delta_G, S_G, \varepsilon_G)$ is a weak Kac algebra.*

Proof. a) Δ_G is coassociative as tensor product of two coassociative coproducts Δ and Δ_s [14] and obviously multiplicative. Let $m, n \in M, g, h \in G$.

Relations $S_G(1_{M_G}) = 1_{M_G}$ and $S_G^2 = id_{M_G}$ are obvious; we also have: $S_G[(m \otimes g)(n \otimes h)] = S_G((m \triangleleft h)n \otimes gh) = S(((m \triangleleft h)n) \triangleleft (gh)^{-1} \otimes (gh)^{-1} = S((m \triangleleft g^{-1})(n \triangleleft (gh)^{-1})) \otimes (gh)^{-1}$.

On the other hand : $S_G(n \otimes h)S_G(m \otimes g) = (S(n \triangleleft h^{-1}) \otimes h^{-1})(S(m \triangleleft g^{-1}) \otimes g^{-1}) = ([(S(n \triangleleft h^{-1})] \triangleleft g^{-1})(S(m \triangleleft g^{-1}) \otimes h^{-1}g^{-1}) = S((m \triangleleft g^{-1})(n \triangleleft (gh)^{-1})) \otimes (gh)^{-1}$. Thus, S_G is antimultiplicative.

$(\Delta_G \circ S_G)(m \otimes g) = \Delta_G(S(m \triangleleft g^{-1}) \otimes g^{-1}) = (id \otimes \varsigma \otimes id_G)(\varsigma_M(S \otimes S)\Delta(m \triangleleft g^{-1}) \otimes g^{-1} \otimes g^{-1}) = \varsigma_{M_G}(S_G \otimes S_G)\Delta_G(m \otimes g)$, where the usual flips act: ς_M in $M \otimes M$ and ς_{M_G} in $M_G \otimes M_G$.

Thus, (M_G, Δ_G, S_G) is a gen. coinvolutive C^*-bialgebra.

b) Clearly, ε_G is a counit with respect to Δ_G. Let us verify the axiom 2):

$$\begin{aligned}\varepsilon_G((m\otimes g)(n\otimes h)) &= \varepsilon((m\triangleleft h)n) = (\varepsilon\otimes\varepsilon)[((m\triangleleft h)\otimes 1)e(1\otimes n)]\\ &= (\varepsilon\otimes\varepsilon)([((m\otimes 1)e)\triangleleft(h\otimes h)](1\otimes n))\\ &= (\varepsilon_G\otimes\varepsilon_G)[(m\otimes g\otimes 1_{M_G})\Delta_G(1_{M_G})(1_{M_G}\otimes n\otimes h)].\end{aligned}$$

c) Finally, let us verify the axiom 3) (μ_G is the multiplication in M_G):

$$\begin{aligned} &(\mu_G\otimes id_{M_G}))(S_G\otimes id_{M_G}\otimes id_{M_G})(\Delta_G\otimes id_{M_G})\Delta_G(m\otimes g) =\\ =\ &(\mu_G\otimes id_{M_G})(S_G\otimes id_{M_G}\otimes id_{M_G})(m_{(1)}\otimes g\otimes m_{(2)}\otimes g\otimes m_{(3)}\otimes g)\\ =\ &(id\otimes\varsigma\otimes id_G)[(\mu\otimes id)(S\otimes id\otimes id)(\Delta\otimes id)\Delta(m)\otimes 1_G\otimes g]\\ =\ &(id\otimes\varsigma\otimes id_G)[(1\otimes m)e\otimes 1_G\otimes g] = (1\otimes 1_G\otimes m\otimes g)\Delta_G(1\otimes 1_G).\end{aligned}$$

Remark 3.1.3 One can show that the Haar trace on WK_G is just a *dual trace* for the Haar trace on WK with respect to the action of G:

$$\phi_G(m\otimes g) = \delta_{1_G,g}\phi(m)\ (\forall m\in M, g\in G).$$

Now we describe a series of concrete examples of nontrivial (non-commutative and non-symmetric) weak Kac algebras of dimensions n^3 ($n\geq 3$).

Let us start with a weak Kac algebra $\mathbb{C}(K_n)$ of functions on a transitive principal groupoid K_n on a set of n elements (for various examples of groupoids see [20]). Then $M = \mathrm{span}\{e_{ij}\}_{i,j=1}^n$ is a commutative C^*-algebra of functions on the set $\{e_{ij}\}$ equipped with a coproduct $\Delta: e_{ij}\to\sum_{k=1}^n e_{ik}\otimes e_{kj}$, an antipode $S: e_{ij}\to e_{ji}$ and a counit $\varepsilon: e_{ij}\to\delta_{ij}$ ($i,j,k\in\{1,...,n\}$, δ_{ij} is the Kronecker symbol).

Let us consider the action of the cyclic group $G = \mathrm{Z}/n\mathrm{Z}$ on WK given by $e_{ij}\triangleleft\alpha := e_{i+1,j+1}$ (α is a generator of G, $i,j\in\{1,...,n\}$ and the summation is modulo n). Applying Lemma 3.1.2 (all its conditions are satisfied), we get a description of the resulting weak Kac algebra in terms of matrix units $f_{ij}^k := e_{j,j+k}\otimes\alpha^{j-i}$ of C^*-algebra $M_G = M\#G\cong\oplus_{k=1}^n M_n(\mathbb{C})$:

$$\Delta_G(f_{ij}^k) = \sum_{r=1}^n f_{ij}^r\otimes f_{i+r,j+r}^{k-r},\ S_G(f_{ij}^k) = f_{i+k,j+k}^{n-k},\ \varepsilon_G(f_{ij}^k) = \delta_{n,k},$$

($i,j,k\in\{1,...,n\}$ and the summation is modulo n. Then one can show that:

a) the above weak Kac algebra is non-trivial;

b) its Haar trace is the canonical trace on M_G;

c) $\varepsilon_t(f_{ij}^k) = \delta_{n,k}\sum_{r=1}^n f_{j+r,j+r}^{n-r}$, $\varepsilon_s(f_{ij}^k) = \delta_{n,k}\sum_{r=1}^n f_{i,i}^r$;
d) $N_t = \mathrm{span}\{\sum_{r=1}^n f_{j+r,j+r}^{n-r}\}$, $N_s = \mathrm{span}\{\sum_{r=1}^n f_{i,i}^r\}$.

For $n = 2$ the above non-trivial weak Kac algebra is of dimension 8. A non-trivial Kac algebra of dimension 8 is constructed in [11]. Let us show that this dimension is minimal possible.

Proposition 3.1.4 *All weak Kac algebras of dimension < 8 are either commutative or cocommutative (therefore, come from groupoids or groups).*

Proof. This statement is well-known for usual Kac algebras, so we may assume that dimension of Cartan subalgebras ≥ 2. Let M be a non-commutative weak Kac algebra of dimension < 8, then $M \cong \mathbb{C}^k \oplus M_2(\mathbb{C})$, $0 \leq k \leq 3$ as a C^*-algebra. Let us show that M is cocommutative.

For $k = 0$ this follows from the classification of elementary weak Kac algebras in section 3.2. For $k > 0$ it suffices to prove that M is a direct sum of two weak Kac algebras, since any weak Kac algebra of dimension ≤ 3 is cocommutative. Clearly, the last property is equivalent to existence of a non-trivial projection q in $N_s \cap N_t \cap \mathrm{Center}(M)$, the hyper-center of M [18].

Due to Proposition 2.3.9 and the multiplicativity of Δ, for the counital quotient $M_\varepsilon \subset M$ (Proposition 2.2.6) we have $M_\varepsilon \neq M_2(\mathbb{C})$, and $M_\varepsilon \neq \mathbb{C}$ since it must contain a copy of a Cartan subalgebra by Remark 2.1.12. Using Proposition 2.3.9 again, we see that all other possibilities lead to the existence of a proper minimal subprojection p_i (Remark 2.2.3) of the Haar projection $p_\varepsilon \in M$, such that $\varsigma\Delta(p_i) = \Delta(p_i)$ (i.e. p_i is cocommutative). Since $S(p_i) = p_i$, Proposition 2.2.2 implies $(S \otimes \mathrm{id}_M)\Delta(p_i) = (\mathrm{id}_M \otimes S)\Delta(p_i)$ and $q = \varepsilon_s(p_i) = \varepsilon_t(p_i)$ is a non-trivial projection in the hyper-center of M.

3.2 Elementary weak Kac algebras

In Subsection 2.2 we have seen that every counital quotient of a weak Kac algebra is a direct sum of weak Kac algebras which are simple as algebras.

Definition 3.2.1 *A weak Kac algebra is said to be* elementary *if it is simple as an algebra, i.e., is isomorphic to the full matrix algebra $M_n(\mathbb{C})$.*

The next theorem classifies all elementary weak Kac algebras and shows that any f-d C^*-algebra can appear as a Cartan subalgebra of an elementary weak Kac algebra in a unique way.

Theorem 3.2.2 *Given a f-d C^*-algebra A, there exists a unique elementary weak Kac algebra $M(A) = (M_n(\mathbb{C}), \Delta, S, \varepsilon)$ $(n = \dim A)$ with Cartan subalgebras isomorphic to A.*

Proof. 1) For $A = \oplus_\alpha M_{n_\alpha}$, $n = \sum_\alpha n_\alpha^2$ let us write down $M(A)$ explicitly, if matrix units $\{E_{ij\alpha}^{kl\beta}\}_{i,j=1\dots n_\alpha,\ k,l=1\dots n_\beta}^{\alpha,\beta=1\dots N}$ in $M_n(\mathbb{C})$ are defined by

$$E_{ij\alpha}^{kl\beta}\, E_{\tilde{k}\tilde{l}\tilde{\beta}}^{\tilde{i}\tilde{j}\tilde{\alpha}} = \delta_{\beta\tilde{\beta}}\, \delta_{k\tilde{k}}\, \delta_{l\tilde{l}}\, E_{ij\alpha}^{\tilde{i}\tilde{j}\tilde{\alpha}}, \qquad (E_{ij\alpha}^{kl\beta})^* = E_{kl\beta}^{ij\alpha}.$$

Equip $M_n(\mathbb{C})$ with the following structure of a weak Kac algebra :

$$\begin{aligned}\Delta(E_{ij\alpha}^{kl\beta}) &= \frac{1}{\sqrt{n_\alpha n_\beta}} \sum_{uv} E_{iu\alpha}^{kv\beta} \otimes E_{uj\alpha}^{vl\beta},\\ S(E_{ij\alpha}^{kl\beta}) &= E_{lk\beta}^{ji\alpha},\\ \varepsilon(E_{ij\alpha}^{kl\beta}) &= \sqrt{n_\alpha n_\beta}\, \delta_{ij}\, \delta_{kl}.\end{aligned}$$

The verification of axioms is a straightforward computation. We have

$$e = \Delta(1) = \sum_\alpha \frac{1}{n_\alpha} \sum_{pq} (\sum_i E_{ip\alpha}^{iq\alpha}) \otimes (\sum_j E_{pj\alpha}^{qj\alpha}),$$

therefore, $F_{pq\alpha} = \sum_i E_{ip\alpha}^{iq\alpha}$ and $G_{pq\alpha} = \sum_j E_{qj\alpha}^{pj\alpha}$ are matrix units for the source and target Cartan subalgebras respectively, isomorphic to A.

2) The uniqueness. If $\tilde{M}(A) = (M_n(\mathbb{C}),\ \tilde{\Delta},\ \tilde{S},\ \tilde{\varepsilon})$ is another weak Kac algebra with $\tilde{N}_s \cong \tilde{N}_t \cong \oplus_\alpha M_{n_\alpha}(\mathbb{C})$, by the observation from Remark 2.2.3, $\sum_{i\in\mathcal{S}} n_\alpha^2 = \dim A = \deg \pi_\varepsilon = n$.

The inclusion $N_s \subset M$ is determined up to a conjugation by the vector $(d_1, \dots d_n)$ where d_α is the multiplicity of $M_{n_\alpha}(\mathbb{C})$ in M. By Proposition 1.3.2, $N_t \subset N_s' \cap M = \oplus_\alpha M_{d_\alpha}(\mathbb{C})$, therefore we may assume that $n_\alpha \le d_\alpha$. Since the inclusion $N_s \subset M$ is unital, $\sum d_\alpha n_\alpha = n$, so $0 = \sum d_\alpha n_\alpha - \sum n_\alpha^2 = \sum (d_\alpha - n_\alpha) n_\alpha$, and $d_\alpha = n_\alpha$ $(\forall \alpha)$. Thus, $N_t = N_s' \cap M$, and $N_s = N_t' \cap M$.

Hence, we can write $N_s = \oplus_\alpha N_{s\alpha}$ and $N_t = \oplus_\alpha N_{t\alpha}$, where $N_{s\alpha} \cong N_{t\alpha} \cong M_{n_\alpha}(\mathbb{C})$, $N_{s\alpha} N_{t\beta} = 0$ if $\alpha \neq \beta$, and $N_{s\alpha} N_{t\alpha} \cong M_{n_\alpha^2}(\mathbb{C})$. Let $\{F_{pq\alpha}\}_{p,q=1\dots n_\alpha}$ be a system of matrix units in $N_{s\alpha}$, $\alpha = 1 \dots K$. Proposition 2.1.11 gives

$$\tilde{e} = \sum\nolimits_\alpha \frac{1}{n_\alpha} \sum\nolimits_{pq} F_{pq\alpha} \otimes S(F_{qp\alpha}).$$

Note that $E_{ij\alpha}^{kl\alpha} = \tilde{S}(F_{ki\alpha}) F_{jl\alpha}$ are matrix units in $N_{s\alpha} N_{t\alpha}$ such that

$$F_{pq\alpha} = \sum\nolimits_i E_{ip\alpha}^{iq\alpha}, \qquad \tilde{S}(F_{qp\alpha}) = \sum\nolimits_j E_{pj\alpha}^{qj\alpha}$$

Let $\{E^{kl\beta}_{ij\alpha}\}$ be the system of matrix units in $M_n(\mathbb{C})$ defined as in 1) which extends the union of the above systems $\{E^{kl\alpha}_{ij\alpha}\}$, $\alpha = 1 \dots K$. Using the definition of a Cartan subalgebra we compute

$$\begin{aligned}
\tilde{\Delta}(F_{pq\alpha}) &= \frac{1}{n_\alpha} \sum_{\tilde{p}\tilde{q}} F_{\tilde{p}\tilde{q}\alpha} \otimes E^{\tilde{q}q\alpha}_{\tilde{p}p\alpha}, \\
\tilde{\Delta}(E^{mq\alpha}_{lp\alpha}) &= \tilde{\Delta}(\tilde{S}(F_{ml\alpha}))\tilde{\Delta}(F_{pq\alpha}) = \frac{1}{n_\alpha} \sum_{uv} E^{mv\alpha}_{lu\alpha} \otimes E^{vq\alpha}_{up\alpha} = \Delta(E^{mq\alpha}_{lp\alpha}).
\end{aligned}$$

Since $\tilde{\Delta}$ is a coassociative $*$-homomorphism, we get

$$\tilde{\Delta}(E^{kl\beta}_{ij\alpha}) = \frac{\lambda^\beta_\alpha}{\sqrt{n_\alpha n_\beta}} \sum_{uv} E^{kv\beta}_{iu\alpha} \otimes E^{vl\beta}_{uj\alpha}, = \lambda^\beta_\alpha \Delta(E^{kl\beta}_{ij\alpha})$$

for some $\{\lambda^\beta_\alpha\}_{\alpha,\beta=1\dots N}$ with $|\lambda^\beta_\alpha| = 1$, $\lambda^\alpha_\beta = \overline{\lambda^\beta_\alpha}$, and $\lambda^\beta_\alpha \lambda^\gamma_\beta = \lambda^\gamma_\alpha$ $(\forall \alpha, \beta, \gamma)$. The properties of antipode and counit imply: $\tilde{S}(E^{kl\beta}_{ij\alpha}) = (\lambda^\beta_\alpha)^{-2} S(E^{kl\beta}_{ij\alpha})$ and $\tilde{\varepsilon}(E^{kl\beta}_{ij\alpha}) = (\lambda^\beta_\alpha)^{-1} \varepsilon(E^{kl\beta}_{ij\alpha})$, so the map $E^{kl\beta}_{ij\alpha} \mapsto \lambda^\beta_\alpha E^{kl\beta}_{ij\alpha}$ defines an isomorphism of weak Kac algebras $M(A) = (M_n(\mathbb{C}),\, \Delta,\, S,\, \varepsilon)$ and $\tilde{M}(A) = (M_n(\mathbb{C}),\, \tilde{\Delta},\, \tilde{S},\, \tilde{\varepsilon})$.

Corollary 3.2.3 *The dual weak Kac algebra $\hat{M}(A)$ has an algebra structure $\oplus_{\alpha\beta} M_{n_\alpha n_\beta}(\mathbb{C})$; there are comatrix units $\{C^{kl\beta}_{ij\alpha}\}^{\alpha,\beta=1\dots N}_{i,j=1\dots n_\alpha,\ k,l=1\dots n_\beta}$ of $\hat{M}(A)$,*

$$\Delta(C^{kl\beta}_{ij\alpha}) = \sum_{pq\gamma} C^{pq\gamma}_{ij\alpha} \otimes C^{kl\beta}_{pq\gamma}, \qquad \varepsilon(C^{kl\beta}_{ij\alpha}) = \delta_{\alpha\beta}\delta_{ik}\delta_{jl},$$

such that

$$\begin{aligned}
C^{kv\beta}_{iu\alpha} \cdot C^{\tilde{v}l\tilde{\beta}}_{\tilde{u}j\tilde{\alpha}} &= \frac{1}{\sqrt{n_\alpha n_\beta}} \delta_{\alpha\tilde{\alpha}}\delta_{\beta\tilde{\beta}}\delta_{u\tilde{u}}\delta_{v\tilde{v}}\, C^{kl\beta}_{ij\alpha}, \\
1 &= \sum_{\alpha\beta\, ij} \sqrt{n_\alpha n_\beta}\, C^{kk\beta}_{ii\alpha},
\end{aligned}$$

$$(C^{kl\beta}_{ij\alpha})^* = C^{lk\beta}_{ji\alpha},\ \ S(C^{kl\beta}_{ij\alpha}) = C^{ji\alpha}_{lk\beta}.$$

Proof. A straightforward computation using the definition of dual weak Kac algebra.

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