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## Weights for relative motives; relation with mixed sheaves

by

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# Weights for relative motives; relation with mixed sheaves

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**Abstract**

The main goal of this paper is to define the *Chow weight structure* $w_{Chow}$ for the category $DM^c(S)$ of (constructible) Beilinson motives over any 'reasonable' base scheme $S$ (this is the version of Voevodsky's motives over $S$ defined by Cisinski and Deglise). We also study the functoriality properties of $w_{Chow}$ (they are very similar to those for weights of mixed complexes of sheaves, as established in §5 of [BBD82]).

As shown in a preceding paper, (the existence of) $w_{Chow}$ automatically yields the weight complex functor (it is a conservative exact functor $DM^c(S) \to K^b(Chow(S))$). Here $Chow(S)$ is the *heart* of $w_{Chow}$; it is 'generated' by motives of regular schemes that are projective over $S$. We also obtain that $K_0(DM^c(S)) \cong K_0(Chow(S))$ (and define a certain 'motivic Euler characteristic' for $S$-schemes).

Besides, we obtain (Chow)-weight spectral sequences and filtrations for any cohomology of motives; we discuss their relation with Beilinson's 'integral part' of motivic cohomology and with weights of mixed complexes of sheaves. For the study of the latter we also introduce a new formalism of *relative weight structures*.

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# Introduction

The goal of this paper is to prove that the *Chow weight structure* $w_{Chow}$ (as introduced in [Bon07] for Voevodsky's motives over a perfect field $k$) could also be defined for the category $DM^c(S)$ of motives with rational coefficients over any 'reasonable' base scheme $S$ (in [CiD09] where this category was constructed and studied, $DM^c(S)$ was called the category of Beilinson motives; one could also consider the 'big' category of $S$-motives $DM(S) \supset DM^c(S)$ here). The *heart* $\underline{Hw}_{Chow}$ of $w_{Chow}$ is 'generated' by the motives of regular schemes that are projective over $S$ (tensored by $\mathbb{Q}(n)[2n]$ for all $n \in \mathbb{Z}$). We also study the functoriality properties of $w_{Chow}$ (they are very similar to the functoriality of weights for mixed complexes of sheaves, as established in §5 of [BBD82]).

As was shown in [Bon07], the existence of $w_{Chow}$ yields several nice consequences. In particular, there exists a *weight complex* functor $t : DM^c(S) \to K^b(Chow(S))$, as well as *Chow-weight* spectral sequences and filtrations, and *virtual $t$-truncations* for any cohomological functor $H : DM^c(S) \to \underline{A}$.

We also relate the weights for $S$-motives with the 'integral part' of motivic cohomology (as constructed in [Sch00]; cf. §2.4.2 of [Bei85]), and with the weights of mixed complexes of sheaves (as defined in [BBD82] and in [Hub97]). In order to study the latter we introduce a new formalism of *relative weight structures.*

We also obtain $K_0(DM^c(S)) \cong K_0(Chow(S))$, and define a certain 'motivic Euler characteristic' for $S$-schemes.

Now we (try to) explain why the concept of a weight structure is important for motives. Recall that weight structures are natural counterparts of $t$-structures for triangulated categories; they allow to 'decompose' objects of a triangulated $\underline{C}$ into Postnikov towers whose 'factors' belong to the *heart* $\underline{Hw}$ of $w$. Weight structures were introduced in [Bon07] (and independently in [Pau08]). They were thoroughly studied and applied to motives (over perfect fields) in [Bon07]; in [Bon10a] a *Gersten* weight structure for a certain category $\mathfrak{D}_s \supset DM^{eff}_{gm}(k)$ was constructed; see also the survey preprint [Bon09s].

The Chow weight structure yields certain weights for (any cohomology of) motives. Note here: 'classical' methods of working with motives often fail (at our present level of knowledge) since they usually depend on (various) 'standard' motivic conjectures. In particular, the 'classical' method to define weights for a motif $M$ is to construct a motif $M_s$ such that $H^i(M_s) \cong W_sH^i(X)$ (for all $i \in \mathbb{Z}$ and a fixed $s$). It is scarcely possible to do this without constructing the so-called motivic $t$-structure for $DM(-)$. For instance, in order to find such $M_s$ for motives of smooth projective varieties one requires the so-called Chow-Kunneth decomposition; hence this completely out of reach at the moment.

The usage of weight structures (for motives) allows to avoid these difficulties completely. To this end one instead of $H^i(M_s)$ one considers $\operatorname{Im}(H^i(w_{Chow\le s+i}M) \to H^i(w_{Chow\le s+i+1}M))$ (this is the corresponding virtual $t$-truncation of $H$ applied to $M$; see §4.3 below and §8.6 of [Bon07]). Here $w_{Chow\le r}M$ for $r \in \mathbb{Z}$ are certain motives which could (at least, when the base is a field) be described in terms of $M$; note in contrast that there are no general conjectures that allow to construct motivic $t$-truncations and Chow-Kunneth decompositions explicitly. Whereas this approach is somewhat 'cheating' for pure motives (since it usually gives no new information on cohomology of motives); yet it yields interesting results on mixed motives and their cohomology. The first paper somewhat related to this approach is [GiS96] (this result was generalized in [GiS09]); there a weight complex functor that is essentially a (very) partial case of 'our' one was introduced (and related to cohomology with compact support of varieties).

Another example when constructions naturally coming from weight structures yield interesting results is described in Remark 3.3.2(4) below.

Now we mention (other) papers on relative motives that are related with the current one.

This text was written independently from the recent article [Heb10] (that appeared somewhat earlier). The main results of loc.cit. are a little stronger than our (central) Theorems 2.1.1 and 2.2.1(II). In particular, in Proposition 3.8 of ibid. the functoriality properties of $w_{Chow}$ (as constructed in Corollary 3.2 of ibid.) with respect to motivic functors induced by not necessarily quasi-projective morphisms of schemes (and also with respect to tensor products and inner homomorphisms) were established. Quasi-projectivity was not required there since Theorem 3.1 of ibid. yields the necessary orthogonality property for a not necessarily quasi-projective morphism of schemes. Yet (to the opinion of the author) the proof the latter theorem is more complicated than the proof of (the parallel) Lemma 1.1.4 (below); besides, in loc.cit. we also calculate morphism groups between shifts of ('basic') objects of $\underline{Hw}_{Chow}$.

The author should also note that he would have probably not noticed that the category $Chow(S) = \underline{Hw}_{Chow}$ has a reasonable description if not for the papers [CoH00] and [GiS09]. In [CoH00] the definition of Chow motives over $S$ was given as a part of a large program of study of relative motives and intersection cohomology of varieties (that relies on several hard 'motivic' conjectures). In [GiS09] certain analogues of (our) Chow motives were used in order to define (a sort of) weight complexes for $S$-schemes (only for one-dimensional $S$; cf. §3.1 below). Yet (to the opinion of the author) the results of these two papers are somewhat difficult to apply since these articles do not treat (any) triangulated categories of 'mixed' motives over $S$; this prevents applying them to cohomology of 'general' (finite type) $S$-schemes.

This paper (also) benefited from [Sch10]. In ibid. a 'mixed motivic' description of Beilinson's 'integral part' of motivic cohomology (as constructed in [Sch00]; see also §2.4.2 of [Bei85]) was proposed. The formulation of the main result of [Sch10] uses the so-called intermediate extensions of mixed motives; so it heavily relies on the (conjectural!) existence of a 'reasonable' motivic $t$-structure for $DM^c(S)$; note that we describe an alternative construction of this 'part' that does depend on any conjectures (in Remark 3.3.2(4) below).

Now we list the contents of the paper. More details could be found at the beginnings of sections.

In section 1 we recall the basic properties of Beilinson motives and weight structures. Most of the results of the section are taken from [CiD09] and [Bon07]; yet we also deduce some new statements.

In section 2 we define the category $Chow(S)$ of Chow motives over $S$ (similar definitions could be found in [CoH00], [Heb10], and [GiS09]). By definition, $Chow(S) \subset DM^c(S)$; since $Chow(S)$ is also negative in it and generates it (if $S$ is 'reasonable') we immediately obtain (using Theorem 4.3.2 of

[Bon07]) that there exists a weight structure $w_{Chow}$ on $DM^c(S)$ whose heart is $Chow(S)$. Next we study the 'functoriality' of $w_{Chow}$ (with respect to the functors of the type $f^*, f_*, f^!, f_!$, for $f$ being a quasi-projective morphism of schemes). Our functoriality statements are parallel to the 'stabilities' 5.1.14 of [BBD82] (we 'explain' this similarity in the following section). We also prove that Chow motives could be 'lifted from open subschemes up to retracts'; this statement could be called (a certain) 'motivic resolution of singularities'. Next we prove that $w_{Chow}$ could be described 'pointwisely' (cf. §5.1.8 of [BBD82]). Besides, we describe an alternative method for the construction of $w_{Chow}$ (over arbitrary excellent finite-dimensional $\mathbb{Q}$-schemes; these don't have to be 'reasonable'). This method uses stratifications and 'gluing' of weight structures; this makes this part of the paper very much parallel to the study of weights of mixed complexes of sheaves in §5 of [BBD82].

Section 3 is dedicated to the applications of our main results. The existence of $w_{Chow}$ automatically yields the existence of a conservative exact *weight complex* functor $DM^c(S) \to K^b(Chow(S))$, and the fact that $K_0(DM^c(S)) \cong K_0(Chow(S))$. We also define a certain 'motivic Euler characteristic' for $S$-schemes.

Next we recall that $w_{Chow}$ yields functorial *Chow-weight* spectral sequences and filtrations. A very partial case of Chow-weight filtrations yields Beilinson's 'integral part' of motivic cohomology. Chow-weight spectral sequences yield the existence of weight filtrations for perverse cohomology of motives (that is not automatic in the case when $S$ is a Spec $\mathbb{Z}$-scheme). We study in more detail the perverse cohomology of motives when $S = X_0$ is a variety over a finite field $\mathbb{F}_q$. It is well-known that mixed complexes of sheaves start to behave better if we extend scalars from $\mathbb{F}_q$ to $\mathbb{F}$ i.e. pass to sheaves over $X = X_0 \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} \mathbb{F}$. We (try to) axiomatize this situation and introduce the concept of a relative weight structure. Relative weight structures have several properties that are parallel to properties of 'ordinary' weight structures. The category $D^b_m(X_0)$ (of mixed complexes of sheaves) possesses a relative weight structure whose heart is the class of (pure) complexes of sheaves of weight 0. Besides, the étale realization functor $DM^c(S) \to D^b_m(X_0)$ is *weight-exact*.

In section 4 we recall the definition of a $t$-structure *adjacent* to a weight structure. Then we prove the existence of a (Chow) $t$-structure $t_{Chow}$ for $DM(S)$ that is adjacent to the Chow weight structure for it. We also study the functoriality of $t_{Chow}$ and relate it with *virtual $t$-truncations* (for cohomological functors from $DM^c(S)$).

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**Notation.** *Ab* is the category of abelian groups.

For a category $C$, $A, B \in ObjC$, we denote by $C(A, B)$ the set of $C$-morphisms from $A$ into $B$.

For categories $C, D$ we write $C \subset D$ if $C$ is a full subcategory of $D$.

For a category $C$, $X, Y \in ObjC$, we say that $X$ is a *retract* of $Y$ if $id_X$ could be factorized through $Y$ (if $C$ is triangulated or abelian, then $X$ is a retract of $Y$ whenever $X$ is its direct summand).

For an additive $D \subset C$ the subcategory $D$ is called *Karoubi-closed* in $C$ if it contains all retracts of its objects in $C$. The full subcategory of $C$ whose objects are all retracts of objects of $D$ (in $C$) will be called the *Karoubi-closure* of $D$ in $C$.

$M \in ObjC$ will be called compact if the functor $C(M, -)$ commutes with all those small coproducts that exist in $C$. In this paper (in contrast with the previous ones) we will only consider compact objects in those categories that are closed with respect to arbitrary small coproducts.

$\underline{C}$ below will always denote some triangulated category; usually it will be endowed with a weight structure $w$ (see Definition 1.2.1 below).

We will use the term 'exact functor' for a functor of triangulated categories (i.e. for a functor that preserves the structures of triangulated categories). We will call a contravariant additive functor $\underline{C} \to A$ for an abelian $\underline{A}$ *cohomological* if it converts distinguished triangles into long exact sequences.

For $f \in \underline{C}(X, Y)$, $X, Y \in Obj\underline{C}$, we will call the third vertex of (any) distinguished triangle $X \xrightarrow{f} Y \to Z$ a cone of $f$; recall that distinct choices of cones are connected by (non-unique) isomorphisms.

We will often specify a distinguished triangle by two of its morphisms.

For a set of objects $C_i \in Obj\underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all $C_i$; for $D \subset \underline{C}$ we will write $\langle D \rangle$ instead of $\langle ObjD \rangle$. If $\underline{C}$ is the Karoubi-closure of $\langle D \rangle$, we will say that it is generated by $D$ (or by $\{C_i\}$).

For $X, Y \in Obj\underline{C}$ we will write $X \perp Y$ if $\underline{C}(X, Y) = \{0\}$. For $D, E \subset Obj\underline{C}$ we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. For $D \subset \underline{C}$ we will denote by $D^{\perp}$ the class

$$\{Y \in Obj\underline{C} : \ X \perp Y \ \forall X \in D\}.$$

Sometimes we will denote by $D^{\perp}$ the corresponding full subcategory of $\underline{C}$. Dually, ${}^{\perp}D$ is the class $\{Y \in Obj\underline{C} : \ Y \perp X \ \forall X \in D\}$.

We will say that some $C_i$ *weakly generate* $\underline{C}$ if for $X \in Obj\underline{C}$ we have $\underline{C}(C_i[j], X) = \{0\} \ \forall i \in I, \ j \in \mathbb{Z} \implies X = 0$ (i.e. if $\{C_i[j]\}^{\perp}$ contains only zero objects).

$D \subset Obj\underline{C}$ will be called *extension-stable* if for any distinguished triangle $A \to B \to C$ in $\underline{C}$ we have: $A, C \in D \implies B \in D$.

We will call the smallest Karoubi-closed extension-stable subclass of $Obj\underline{C}$ containing $D$ the *envelope* of $D$.

Below all schemes will be excellent of finite Krull dimension; morphisms of schemes will always be separated and by default will be of finite type.

We will sometimes need certain stratifications of a scheme $S$. Recall that a stratification $\alpha$ is a presentation of $S$ as $\cup S_l^\alpha$ where $S_l^\alpha$, $1 \le l \le n$, are pairwise disjunct locally closed subschemes of $S$; the closure of each $S_l^\alpha$ should be the union of some subset of $S_l^\alpha$. Omitting $\alpha$, we will denote by $j_l : S_l^\alpha \to S$ the corresponding immersions.

# 1 Preliminaries: relative motives and weight structures

In §1.1 we recall some of basic properties of Beilinson motives over $S$ (as defined in [CiD09]; we also deduce certain results that were not stated in ibid. explicitly).

In §1.2 we recall some basics of the theory of weight structures (as developed in [Bon07]); we also prove some new lemmas on the subject.

## 1.1 Beilinson motives (after Cisinski and Deglise)

We list some of the properties of the triangulated categories of Beilinson motives (this is the version of relative Voevodsky's motives with rational coefficients defined by Cisinski and Deglise).

**Definition 1.1.1.** We will call a scheme $S$ *reasonable* if there exists an excellent (noetherian) scheme $S_0$ of dimension lesser than or equal to 2 such that $S$ is (separated and) of finite type over $S_0$.

**Proposition 1.1.2.** *Let $X$ be an (excellent finite dimensional) scheme; $f : X \to Y$ is a (separated) finite type morphism.*

1. *For any $X$ a tensor triangulated $\mathbb{Q}$-linear category $DM(X)$ with the unit object $\mathbb{Q}_X$ is defined; it is closed with respect to arbitrary small coproducts.*

   *$DM(X)$ is the category of Beilinson motives over $X$, as defined (and thoroughly studied) in §14 of [CiD09].*

2. *The (full) subcategory $DM^c(X) \subset DM(X)$ of compact objects is tensor triangulated, and $\mathbb{Q}_X \in Obj DM^c(S)$. $DM^c(X)$ weakly generates $DM(X)$.*

3. *All $DM(X)$ and $DM^c(X)$ are idempotent complete.*

4. *For any $f$ the following functors are defined: $f^* : DM(Y) \leftrightarrows DM(X) : f_*$ and $f_! : DM(X) \leftrightarrows DM(Y) : f^!$; $f^*$ is left adjoint to $f_*$ and $f_!$ is left adjoint to $f^!$.*

   *We call these the **motivic image functors**. Any of them (when $f$ varies) yields a 2-functor from the category of (noetherian separated finite-dimensional excellent) schemes with separated morphisms of finite type to the category of triangulated categories. Besides, all motivic image functors preserve compact objects (i.e. they could be restricted to the subcategories $DM^c(-)$); they also commute with arbitrary (small) coproducts.*

5. *For a Cartesian square of finite type separated morphisms*

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{g'} & & \downarrow{g} \\ Y & \xrightarrow{f} & X \end{array}$$

   *we have $g^* f_! \cong f'_! g'^*$ and $g'_* f'^! \cong f^! g_*$.*

6. *For any $X$ there exists a Tate object $\mathbb{Q}(1) \in Obj DM^c(X)$; tensoring by it yields an exact Tate twist functor $-(1)$ on $DM(X)$. This functor is an auto-equivalence of $DM(X)$; we will denote the inverse functor by $-(-1)$.*

   *Tate twists commute with all motivic image functors mentioned (up to an isomorphism of functors).*

   *Besides, for $X = \mathbb{P}^1(Y)$ there is a functorial isomorphism $f_!(\mathbb{Q}_{\mathbb{P}^1(Y)}) \cong \mathbb{Q}_Y \bigoplus \mathbb{Q}_Y(-1)[-2]$.*

7. *$f^*$ is symmetric monoidal; $f^*(\mathbb{Q}_Y) = \mathbb{Q}_X$.*

8. *$f_* \cong f_!$ if $f$ is proper; $f^!(M) \cong f^*(M)(s)[2s]$ if $f$ is smooth and quasi-projective (everywhere) of relative dimension $s$, $M \in Obj DM(Y)$.*

   *If $f$ is an open immersion, we just have $f^! = f^*$.*

9. *If $i : S' \to S$ is an immersion of regular schemes everywhere of codimension $d$, then $\mathbb{Q}_{S'}(-d)[-2d] \cong i^!(\mathbb{Q}_S)$.*

10. *If $i : Z \to X$ is a closed immersion, $U = X \setminus Z$, $j : U \to X$ is the complementary open immersion, then the motivic image functors yield gluing data for $DM(-)$ (in the sense of §1.4.3 of [BBD82]; see also Definition 8.2.1 of [Bon07]). That means that (in addition to the adjunctions given by assertion 4) the following statements are valid.*

    *(i) $i_* \cong i_!$ is a full embeddings; $j^* = j^!$ is isomorphic to the localization (functor) of $DM(X)$ by $i_*(DM(Z))$.*

    *(ii) For any $M \in ObjDM(X)$ the pairs of morphisms $j_!j^!(M) \to M \to i_*i^*(M)$ and $i_!i^!(M) \to M \to j_*j^*(M)$ could be completed to distinguished triangles (here the connecting morphisms come from the adjunctions of assertion 4).*

    *(iii) $i^*j_! = 0$; $i^!j_* = 0$.*

    *(iv) All of the adjunction transformations $i^*i_* \to 1_{DM(Z)} \to i^!i_!$ and $j^*j_* \to 1_{DM(U)} \to j^!j_!$ are isomorphisms of functors.*

11. *For the subcategories $DM^c(-) \subset DM(-)$ the obvious analogue of the previous assertion is fulfilled.*

12. *Let $S_{red}$ be the reduced scheme associated to $S$. Then for the canonical immersion $v : S_{red} \to S$ the functor $v^*$ is an equivalence of categories.*

13. *If $S$ is reasonable (see Definition 1.1.1), $DM^c(S)$ (as a triangulated category) is generated by $\{g_*(\mathbb{Q}_X)(r)\}$, where $g : X \to S$ runs through all projective morphisms (of finite type) such that $X$ is regular, $r \in \mathbb{Z}$.*

14. *Let $S$ be a scheme which is the limit of an essentially affine projective system of schemes $S_\beta$. Then $DM^c(S)$ is isomorphic to the 2-limit of the categories $DM^c(S_\beta)$; in these isomorphism all the connecting functors are given by the corresponding motivic inverse image functors (cf. Remark 1.1.3(2) below).*

15. *If $X$ is regular (everywhere) of dimension $d$, $i : Z \to X$ is a closed embedding, $p, q \in \mathbb{Z}$, then $DM(X)(\mathbb{Q}_X, i_!i^!(\mathbb{Q}_Z)[p](q)) \cong DM(Z)(\mathbb{Q}_Z, i^!(\mathbb{Q}_X)(q)[p])$ is isomorphic to $Chow_{d-q}(Z, 2q-p) \otimes \mathbb{Q} (= Gr^\gamma_{d-q} K'_{2q-p}(Z) \otimes \mathbb{Q})$. In particular, this morphism group is zero if $p > 2q$.*

*Proof.* Almost all of these properties of Beilinson motives are stated in the introduction of ibid.; the proofs are mostly contained in §1, §2, and §14 of ibid.

So, we will only prove those assertions that are not stated in ibid. (explicitly).

For (3): Since $DM(X)$ is closed with respect to arbitrary small coproducts, it is idempotent complete by Proposition 1.6.8 of [Nee01]. Since a retract of a compact object is compact also, $DM^c(X)$ is also idempotent complete.

Since $i^! = i^*$ if $i$ is an open immersion, and $i^*(\mathbb{Q}_S) = \mathbb{Q}_{S'}$, it suffices to prove (9) for $i$ being a closed immersion. In this case it is exactly Theorem 3 of [CiD09].

We should also prove (11). Assertion 10 immediately yields everything expect the fact that the (categoric) kernel of $j^* : DM^c(X) \to DM^c(Y)$ is contained $i_*(DM^c(Z))$. So, we should prove that $i_*(ObjDM(Z)) \cap ObjDM^c(X) = i_*(ObjDM^c(Z))$. This is easy, since $i^*i_* \cong id_{DM(Z)}$ and $i_*i^*$ preserves compact objects.

Assertion 13 is immediate from Corollary 14.3.9 of ibid. (cf. Corollary 14.3.6 of ibid.).

It remains to prove (15). Combining (12.4.1.3) and Corollary 13.2.14 of ibid., we obtain that the groups in question are isomorphic to the $q$-th factor of the $\gamma$-filtration of $K^Z_{2q-p} \otimes \mathbb{Q}$ (of the $K$-theory of $X$ with support in $Z$). By Theorem 7 of [Sou85], this is the exactly the $d-q$-th factor of the $\gamma$-factor of $K'_{2q-p}(Z) \otimes \mathbb{Q}$.

□

*Remark* 1.1.3. 1. In [CiD09] for a smooth $f : X \to Y$ the object $f_! f^!(\mathbb{Q}_Y)$ was denoted by $\mathcal{M}_Y(X)$ (cf. also Definition 1.3 of [Sch10]; yet note that in loc.cit. cohomological motives are considered, this interchanges $*$ with $!$ in the notation for motivic functors). We will not usually need this notation below (yet cf. Remarks 2.1.2(1) and 3.3.2(4)).

2. In [CiD09] the functor $f^*$ was constructed for any (separated) morphism $f$ not necessarily of finite type; it preserves compact objects (see Proposition 14.1.5 of ibid.). Besides, for such an $f$ and any separated finite type $g : X' \to X$ we have an isomorphism $f^* g_! \cong g'_! f'^*$ (for the corresponding $f'$ and $g'$; cf. part 5 of the proposition).

Below the only morphisms of infinite type that we will be interested in are limits of immersions (more precisely, we will need the natural morphism $j_K : K \to S$ from a Zariski point $K$ of some scheme $S$ to $S$).

Now note: if $f$ is a pro-open immersion, then one can define $f^! = f^*$. So, one can also define $j^!_K$ that preserves compact objects; the system of these functors satisfy the second assertion in part 5 of the proposition (for a finite type separated $g$).

3. A nice exposition of the properties of Beilinson motives (that also follows [CiD09]) could be found in §2 of [Heb10].

The following statements were not proved in [CiD09] explicitly; yet they follow from Proposition 1.1.2 easily. Below we will mostly need assertion I1 in the case when $g$ is projective; note that in this case $g_*(\mathbb{Q}_Y) \cong g_!(\mathbb{Q}_Y)$.

**Lemma 1.1.4.** *I1. Let $Y$ be a regular scheme everywhere of dimension $d$; let $f : X \to S$ and $g : Y \to S$ be finite type quasi-projective morphisms, $r, b, c \in \mathbb{Z}$.*

*Then $DM(S)(f_!(\mathbb{Q}_X)(b)[2b], g_*(\mathbb{Q}_Y)(c)[r+2c]) \cong CH_{d+b-c}(X \times_S Y, -r)$ (cf. Proposition 1.1.2(15) for the definition of the latter). In particular, $f_!(\mathbb{Q}_X)(b)[2b] \perp g_*(\mathbb{Q}_Y)(c)[r+2c]$ if $r > 0$.*

*2. Let $i : S' \to S$ be an immersion of regular schemes everywhere of codimension $d$; let $g$ be smooth. Denote $Y' = Y_{S'}$ and $g' = g_{S'}$. Then $i^! g_*(\mathbb{Q}_Y) \cong g'_*(\mathbb{Q}_{Y'})(-d)[-2d]$.*

*II Let $S = \cup S_l^\alpha$ be a stratification. Then for any $M, N \in Obj DM(S)$ there exists a filtration of $DM(S)(M, N)$ by certain subfactors of $DM(S_j^\alpha)(j_l^*(M), j_l^!(N))$.*

*Proof.* I1. By Proposition 1.1.2(6), we can assume that $b = c = 0$ (to this end we should possibly replace $X$ and $Y$ by $(\mathbb{P}^1)^n(X)$ and $(\mathbb{P}^1)^m(Y)$ for some $n, m \geq 0$).

Next, we have $DM(S)(f_!(\mathbb{Q}_X), g_*(\mathbb{Q}_Y)[r]) \cong DM(Y)(g^* f_!(\mathbb{Q}_X), \mathbb{Q}_Y[r])$ since $f^*$ is left adjoint to $f_*$. Applying part 5 of loc.cit., we obtain that the group in question is isomorphic to $DM(Y)(f'_! g'^*(\mathbb{Q}_X), \mathbb{Q}_Y[r]) = DM(Y)(f'_!(\mathbb{Q}_{X \times_S Y}), \mathbb{Q}_Y[r])$ (here $f' = f_Y$).

We denote $X \times_S Y$ by $Z$. Let $P$ be a smooth quasi-projective $Y$-scheme containing $Z$ as a closed subscheme; we denote by $i : Z \to Y$ and $p : P \to Y$ the corresponding morphisms. We can assume that $P$ is everywhere of some dimension $d'$ over $Y$.

Then we have $DM(Y)(f'_!(\mathbb{Q}_Z), \mathbb{Q}_Y[r]) = DM(S)(p_! i_!(\mathbb{Q}_Z), \mathbb{Q}_Y[r]) \cong DM(P)(i_!(\mathbb{Q}_Z), p^!(\mathbb{Q}_Y)[r])$ (here we apply the adjunction of $p_!$ with $p^!$). By part 8 of loc.cit., the group in question is isomorphic to $DM(P)(i_!(\mathbb{Q}_Z), p^*(\mathbb{Q}_Y)(d')[r+2d']) \cong DM(P)(i_!(\mathbb{Q}_Z), \mathbb{Q}_P(d')[r+2d'])$. It remains to apply part 15 of loc.cit.

2. $i^! g_*(\mathbb{Q}_Y) \cong g'_* i'^!(\mathbb{Q}_Y)$ by part 5 of loc.cit. (here $i' = i_Y$). Hence using part 9 of loc.cit. we obtain the result.

II We prove the statement by induction on the number of stratification components.

Suppose that $S_0^\alpha$ is open in $S$. Then the remaining $S_l^\alpha$ yield a stratification of $S \setminus S_0^\alpha$. We denote $S \setminus S_0^\alpha$ by $Z$, the (open) immersion $S_0^\alpha \to S$ by $j$ and the (closed) immersion $Z \to S$ by $i$.

Now we apply part 10 of loc.cit. We obtain a (long) exact sequence $\dots \to DM(S)(i_* i^*(M), N) \to DM(S)(M, N) \to DM(S)(j_! j^!(M), N) \to \dots$. The adjunctions of functors yield $DM(S)(i_*i^*(M),N) \cong DM(Z)(i^*(M),i^!(N))$ and $DM(S)(j_!j^!(M),N') \cong DM(S_0^\alpha)(j^*(M),j^!(N))$.

Now, by the inductive assumption the group $DM(Z)(i^*(M),i^!(N))$ has a filtration by some subquotients of $DM(S_l^\alpha)(j_l^*(M),j^!(N))$ (for $l\neq 0$). This concludes the proof.

$\square$

## 1.2 Weight structures: short reminder

**Definition 1.2.1.** I A pair of subclasses $\underline{C}^{w\le 0},\underline{C}^{w\ge 0}\subset Obj\underline{C}$ will be said to define a weight structure $w$ for $\underline{C}$ if they satisfy the following conditions:

(i) $\underline{C}^{w\ge 0},\underline{C}^{w\le 0}$ are additive and Karoubi-closed in $\underline{C}$ (i.e. contain all $\underline{C}$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**

$\underline{C}^{w\ge 0}\subset \underline{C}^{w\ge 0}[1]$, $\underline{C}^{w\le 0}[1]\subset \underline{C}^{w\le 0}$.

(iii) **Orthogonality.**

$\underline{C}^{w\ge 0}\perp \underline{C}^{w\le 0}[1]$.

(iv) **Weight decompositions**.

For any $M\in Obj\underline{C}$ there exists a distinguished triangle

$$B[-1]\to M\to A\xrightarrow{f} B \qquad (1)$$

such that $A\in \underline{C}^{w\le 0}, B\in \underline{C}^{w\ge 0}$.

II The category $\underline{Hw}\subset \underline{C}$ whose objects are $\underline{C}^{w=0}=\underline{C}^{w\ge 0}\cap\underline{C}^{w\le 0}$, $\underline{Hw}(Z,T)=\underline{C}(Z,T)$ for $Z,T\in\underline{C}^{w=0}$, will be called the *heart* of $w$.

III $\underline{C}^{w\ge i}$ (resp. $\underline{C}^{w\le i}$, resp. $\underline{C}^{w=i}$) will denote $\underline{C}^{w\ge 0}[-i]$ (resp. $\underline{C}^{w\le 0}[-i]$, resp. $\underline{C}^{w=0}[-i]$).

IV We denote $\underline{C}^{w\ge i}\cap\underline{C}^{w\le j}$ by $\underline{C}^{[i,j]}$ (so it equals $\{0\}$ for $i>j$).

V We will say that $(\underline{C},w)$ is *bounded* if $\cup_{i\in\mathbb{Z}}\underline{C}^{w\le i}=Obj\underline{C}=\cup_{i\in\mathbb{Z}}\underline{C}^{w\ge i}$.

VI Let $\underline{C}$ and $\underline{C}'$ will be triangulated categories endowed with weight structures $w$ and $w'$, respectively; let $F:\underline{C}\to\underline{C}'$ be an exact functor.

$F$ will be called left weight-exact (with respect to $w,w'$) if it maps $\underline{C}^{w\le 0}$ to $\underline{C}'^{w'\le 0}$; it will be called right weight-exact if it maps $\underline{C}^{w\ge 0}$ to $\underline{C}'^{w'\ge 0}$. $F$ is called weight-exact if it is both left and right weight-exact.

VII Let $H$ be a full subcategory of a triangulated $\underline{C}$.

We will say that $H$ is *negative* if $ObjH\perp(\cup_{i>0}Obj(H[i]))$.

VIII We call a category $\frac{A}{B}$ a *factor* of an additive category $A$ by its (full) additive subcategory $B$ if $Obj\left(\frac{A}{B}\right)=ObjA$ and $(\frac{A}{B})(M,N)=A(M,N)/(\sum_{O\in ObjB}A(O,N)\circ A(M,O))$.

IX For an additive $B$ we will consider the category of 'formal coproducts' of objects of $B$: its objects are (formal) $\coprod_{j\in J}B_j:\ B_j\in ObjB$, and

$Mor(\coprod_{l\in L} B_l, \coprod_{j\in J} C_j) = \prod_{l\in L}(\bigoplus_{j\in J} \underline{C}(B_l, C_j))$; here $L, J$ are index sets. We will call the idempotent completion of this category the *big hull* of $B$.

*Remark* 1.2.2. 1. If $B$ is a full subcategory of an additive $C$, and $C$ is idempotent complete and closed with respect to arbitrary small coproducts, then there exists a natural full embedding of the big hull of $B$ into $C$. Note here: if $C$ is triangulated and closed with respect to arbitrary small coproducts, then it is necessarily idempotent complete (see Proposition 1.6.8 of [Nee01]).

2. A simple (and yet useful) example of a weight structure is given by the stupid filtration on $K(B) \supset K^b(B)$ for an arbitrary additive category $B$. In this case $K(B)^{w\le 0}$ (resp. $K(B)^{w\ge 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees $\le 0$ (resp. $\ge 0$). The heart of this weight structure (either for $K(B)$ or for $K^b(B)$) is the the Karoubi-closure of $B$ in the corresponding category.

3. A weight decomposition (of any $M \in Obj\underline{C}$) is (almost) never unique; still we will sometimes denote any pair $(A, B)$ as in (1) by $(M^{w\le 0}, M^{w\ge 1})$.

$M^{w\le l}$ (resp. $M^{w\ge l}$) will denote $(M[l])^{w\le 0}$ (resp. $(M[l-1])^{w\ge 1}$); we will also sometimes need $w_{\le l}M = M^{w\le l}[-l]$ and $w_{\ge l}M = M^{w\ge l}[-l]$.

We will call (any choices of) $w_{\le l}M$, $w_{\ge l}M$, $M^{w\le l}$, and $M^{w\ge l}$ *weight truncations* of $M$.

Now we recall those properties of weight structures that will be needed below (and that could be easily formulated). We will not mention more complicated matters (weight spectral sequences and weight complexes) here; instead we will just formulate the corresponding 'motivic' results below.

**Proposition 1.2.3.** *Let $\underline{C}$ be a triangulated category.*

1. *$(C_1, C_2)$ ($C_1, C_2 \subset Obj\underline{C}$) define a weight structure for $\underline{C}$ whenever $(C_2^{op}, C_1^{op})$ define a weight structure for $\underline{C}^{op}$.*

2. *Let $w$ be a weight structure for $\underline{C}$. Then $\underline{C}^{w\le 0}$, $\underline{C}^{w\ge 0}$, and $\underline{C}^{w=0}$ are extension-stable.*

3. *Let $w$ be a weight structure for $\underline{C}$. Then $\underline{C}^{w\le 0} = (\underline{C}^{w\ge 1})^{\perp}$ and $\underline{C}^{w\ge 0} = {}^{\perp}\underline{C}^{w\le -1}$ (see Notation).*

4. *Suppose that $v, w$ are weight structures for $\underline{C}$; let $\underline{C}^{v\le 0} \subset \underline{C}^{w\le 0}$ and $\underline{C}^{v\ge 0} \subset \underline{C}^{w\ge 0}$. Then $v = w$ (i.e. the inclusions are equalities).*

5. *Let $w$ be a bounded weight structure for $\underline{C}$. Then $w$ extends to a bounded weight structure for the idempotent completion $\underline{C}'$ of $\underline{C}$ (i.e. there exists a weight structure $w'$ for $\underline{C}'$ such that the embedding $\underline{C} \to \underline{C}'$ is weight-exact); its heart is the idempotent completion of $\underline{Hw}$.*

6. *Let $H \subset Obj\underline{C}$ be negative; let $\underline{C}$ be idempotent complete. Then there exists a unique weight structure $w$ on the Karoubi-closure $T$ of $\langle H \rangle$ in $\underline{C}$ such that $H \subset T^{w=0}$. Its heart is the Karoubi-closure of the closure of $H$ in $\underline{C}$ with respect to (finite) direct sums.*

7. *For the weight structure mentioned in the previous assertion, $T^{w\le 0}$ is the envelope (see the Notation) of $\cup_{i\ge 0} H[i]$; $T^{w\ge 0}$ is the envelope of $\cup_{i\le 0} H[i]$.*

8. *A composition of left (resp. right) weight-exact functors is left (resp. right) weight-exact.*

9. *Let $\underline{C}$ and $\underline{D}$ be triangulated categories endowed with weight structures $w$ and $v$, respectively. Let $F : \underline{C} \leftrightarrows \underline{D} : G$ be adjoint functors. Then $F$ is right weight-exact whenever $G$ is left weight-exact.*

10. *Let $\underline{C}$ and $\underline{D}$ be triangulated categories endowed with weight structures $w$ and $v$, respectively; let $w$ be bounded. Then an exact functor $F : \underline{C} \to \underline{D}$ is left (resp. right) weight-exact whenever $F(\underline{C}^{w=0}) \subset \underline{D}^{v\le 0}$ (resp. $F(\underline{C}^{w=0}) \subset \underline{D}^{v\ge 0}$).*

11. *Let $w$ be a weight structure for $\underline{C}$; let $\underline{D} \subset \underline{C}$ be a triangulated subcategory of $\underline{C}$. Suppose that $w$ induces a weight structure $w_{\underline{D}}$ for $\underline{D}$ (i.e. $Obj\underline{D} \cap \underline{C}^{w\le 0}$ and $Obj\underline{D} \cap \underline{C}^{w\ge 0}$ give a weight structure for $\underline{D}$).*

    *Then $w$ induces a weight structure on $\underline{C}/\underline{D}$ (the localization i.e. the Verdier quotient of $\underline{C}$ by $\underline{D}$) i.e.: the Karoubi-closures of $\underline{C}^{w\le 0}$ and $\underline{C}^{w\ge 0}$ (considered as classes of objects of $\underline{C}/\underline{D}$) give a weight structure $w'$ for $\underline{C}/\underline{D}$ (note that $Obj\underline{C} = Obj\underline{C}/\underline{D}$). Besides, there exists a full embedding $\frac{\underline{Hw}}{\underline{Hw}_{\underline{D}}} \to \underline{Hw}'$; $\underline{Hw}'$ is the Karoubi-closure of $\frac{\underline{Hw}}{\underline{Hw}_{\underline{D}}}$ in $\underline{C}/\underline{D}$.*

12. *Suppose that $\underline{D} \subset \underline{C}$ is a full category of compact objects endowed with bounded a weight structure $w'$. Suppose that $\underline{D}$ weakly generates $\underline{C}$; let $\underline{C}$ admit arbitrary (small) coproducts. Then $w'$ could be extended to a weight structure $w$ for $\underline{C}$. Its heart is the big hull of $\underline{Hw}$ (as defined in Definition 1.2.1(IX)).*

13. *Let $\underline{D} \xrightarrow{i_*} \underline{C} \xrightarrow{j^*} \underline{E}$ be a part of gluing data. This means that $\underline{D}, \underline{C}, \underline{E}$ are triangulated categories, $i^*$ and $j^*$ are exact functors; $j^*$ is a localization functor, $i_*$ is an inclusion of the categorical kernel of $j^*$ into $\underline{C}$; $i_*$ possesses both a left adjoint $i^*$ and a right adjoint $i^!$ (see Chapter 9 of [Nee01]; note that this piece of data extends to data similar to those described in Proposition 1.1.2(10)).*

*Then for any pair of weight structures on $\underline{D}$ and $\underline{E}$ (we will denote them by $w_{\underline{D}}$ and $w_{\underline{E}}$, respectively) there exists a weight structure $w$ on $\underline{C}$ such that both $i_*$ and $j^*$ are weight-exact (with respect to the corresponding weight structures). Besides, $i^!$ and $j_*$ are left weight-exact (with respect to the corresponding weight structures); $i^*$ and $j_!$ are right weight-exact. Moreover, $\underline{C}^{w\le 0} = C_1 = \{M \in Obj\underline{C} :\ i^!(M) \in \underline{D}^{w_{\underline{D}}\le 0},\ j^*(M) \in \underline{E}^{w_{\underline{E}}\le 0}\}$ and $\underline{C}^{w\ge 0} = C_2 = \{M \in Obj\underline{C} :\ i^*(M) \in \underline{D}^{w_{\underline{D}}\ge 0},\ j^*(M) \in \underline{E}^{w_{\underline{E}}\ge 0}\}$. Lastly, $C_1$ (resp. $C_2$) is the envelope of $Obj j_*(\underline{E}^{w\le 0}) \cup Obj i_*(\underline{D}^{w\le 0})$ (resp. of $Obj j_!(\underline{E}^{w\ge 0}) \cup Obj i_*(\underline{D}^{w\ge 0})$; envelopes are defined in the Notation).*

14. *In the setting of the previous assertion, if $w_{\underline{D}}$ and $w_{\underline{E}}$ are bounded, then: $w$ bounded also; besides, $\underline{C}^{w\le 0}$ is the envelope of $\{i_*(\underline{D}^{w_{\underline{D}}=l}),\ j_*(\underline{E}^{w_{\underline{E}}=l}),\ l \le 0\}$; $\underline{C}^{w\ge 0}$ is the envelope of $\{i_!(\underline{D}^{w_{\underline{D}}=l}),\ j_!(\underline{E}^{w_{\underline{E}}=l}),\ l \ge 0\}$.*

15. *In the setting of assertion 13, the weight structure $w$ described is the only weight structure for $\underline{C}$ such that both $i_*$ and $j^*$ are weight-exact.*

*Proof.* Most of the assertions were proved in [Bon07]; more precise references to most of the proofs could be found in the proof of Proposition 1.3.3 of [Bon10b].

We only have to prove assertions 9, 10, 14, and 15.

(9) follows immediately from assertion 3 (using the definition of adjoint functors).

(10) is immediate from assertion 7 by assertion 2.

If $w_{\underline{C}}$ and $w_{\underline{D}}$ are bounded, then $w$ also is by definition. The remaining part of assertion 14 is immediate from Remark 8.2.4(1) of [Bon07] and assertion 7.

(15): Suppose that the assumptions of assertion 13 are fulfilled, and consider some weight structure $v$ for $\underline{C}$ such that $i_*$ and $j^*$ are weight-exact.

Since $i_*$ and $j^*$ are weight exact, by assertion 9 we obtain: $i^!$ and $j_*$ are left weight-exact; $i^*$ and $j_!$ are right weight-exact (with respect to the corresponding weight structure). Hence the class $\underline{C}^{v\le 0}$ (resp. $\underline{C}^{v\ge 0}$) is contained in $C_1$ (resp. in $C_2$) in the notation of assertion 13. Since $(C_1, C_2)$ does yield a weight structure $w$ for $\underline{C}$ (by loc.cit.), by assertion 4 we obtain that $v = w$. $\square$

*Remark* 1.2.4. Part 11 of the proposition could be re-formulated is follows. If $i_* : \underline{D} \to \underline{C}$ is an embedding of triangulated categories that is weight-exact (with respect to certain weight structures for $\underline{D}$ and $\underline{C}$), an exact functor $j^* : \underline{C} \to \underline{E}$ is equivalent to the localization of $\underline{C}$ by $i_*(\underline{D})$, then there exists a unique weight structure $w'$ for $\underline{E}$ such that $j^*$ is weight exact; $\underline{Hw}_{\underline{E}}$ is the Karoubi-closure of $\frac{\underline{Hw}}{i_*(\underline{Hw}_{\underline{D}})}$ (with respect to the natural functor $\frac{\underline{Hw}}{i_*(\underline{Hw}_{\underline{D}})} \to \underline{E}$).

# 2 The Chow weight structure: two constructions and basic properties

In §2.1 we define the category $Chow(S)$ of Chow motives over $S$ (similar definitions could be found in [CoH00], [Heb10], and [GiS09]). By our definition, $Chow(S) \subset DM^c(S)$; since $Chow(S)$ is also negative in it and generates it (if $S$ is reasonable; here we use the properties of $DM^c(S)$ proved in §1.1) we immediately obtain (by Proposition 1.2.3(7)) that there exists a weight structure on $DM^c(S)$ whose heart is $Chow(S)$.

In §2.2 we study the 'functoriality' of $w_{Chow}$ (with respect to the functors of the type $f^*, f_*, f^!$, and $f_!$, for $f$ being a quasi-projective morphism of schemes). Our functoriality statements are parallel to the 'stabilities' 5.1.14 of [BBD82]; we will explain this similarity in the next section. We also prove that $w_{Chow}$ could be described 'pointwisely' (similarly to §5.1.8 of [BBD82]).

In §2.3 we describe an alternative method for the construction of $w_{Chow}$ for $DM^c(S)$ (for any Spec $\mathbb{Q}$-scheme $S$ that is not necessarily reasonable). This method uses stratifications and 'gluing' of weight structures; this makes this part of the paper very much parallel to the study of weights of mixed complexes of sheaves in §5 of [BBD82]. Actually, this method is the first one developed by the author (it was first proposed in Remark 8.2.4(3) of [Bon07], that was in its turn inspired by [BBD82]). We prove that this 'alternative' method yields the same result as the method of §2.1 if $S$ is reasonable. This yields two 'new' descriptions of $w_{Chow}$ (in this case).

## 2.1 Relative Chow motives; the 'basic' construction of $w_{Chow}$

We define $Chow(S)$ as the Karoubi-closure of $\{f_!(\mathbb{Q}_X)(r)[2r]\} = \{f_*(\mathbb{Q}_X)(r)[2r]\}$ in $DM^c(S)$; here $f : X \to S$ runs through all finite type projective morphisms such that $X$ is regular, $r \in \mathbb{Z}$.

Till §2.3 we will assume that all schemes that we consider are reasonable (see Definition 1.1.1).

**Theorem 2.1.1.** *I There exists a (unique) weight structure $w_{Chow}$ for $DM^c(S)$ whose heart is $Chow(S)$.*

*II $w_{Chow}(S)$ could be extended to a weight structure $w_{Chow}^{big}$ for the whole $DM(S)$. $\underline{Hw}_{Chow}^{big}$ is the big hull of $Chow(S)$ (as defined in Definition 1.2.1(IX); see Remark 1.2.2).*

*Proof.* I By Proposition 1.2.3(6) it suffices to verify that $Chow(S)$ is negative and generates $DM^c(S)$. Negativity of $Chow(S)$ is immediate from Lemma 1.1.4(I). $Chow(S)$ generates $DM^c(S)$ by Proposition 1.1.2(13).

II Since $Chow(S)$ generates $DM^c(S)$, and $DM^c(S)$ weakly generates $DM(S)$ (by part 2 of loc.cit.), $Chow(S)$ weakly generates $DM(S)$.

Hence the assertion follows immediately from assertion I and Proposition 1.2.3(12).

□

*Remark* 2.1.2. 1. In particular, the theorem holds for $S$ being the spectrum of a (not necessarily perfect) field $k$. For a perfect $k$ this statement was already proved in §6 of [Bon07]. Note here that $DM^c(\operatorname{Spec} k) \cong DM_{gm}\mathbb{Q}(k)$ for a perfect $k$ (in the notation of Voevodsky and loc.cit.), whereas $p_!\mathbb{Q}_P(r)[2r]$ yields a Chow motif over $k$ (for any $r \in \mathbb{Z}$ and $p : P \to \operatorname{Spec} k$ being a smooth projective morphism; recall here that the 'ordinary' category of Chow motives over $k$ can be fully embedded into $DM_{gm}$).

Besides, in [Bon09a] a related differential graded 'description' of motives over a characteristic zero $k$ was given. It was generalized in [Lev09] to a description of a certain category of 'smooth motives' over $S$, when $S$ is a smooth variety over (a characteristic 0 field) $k$; the category of smooth motives is the triangulated category generated by motives of smooth projective $S$-schemes.

2. Our results would certainly look nicer if we had a description of the composition of morphisms in $Chow(S)$ (note here that the morphism groups between 'generating objects' of $Chow(S)$ can be immediately computed using Lemma 1.1.4(I)). The author conjectures that this composition is compatible with the ones described §2 of [CoH00] and in §5.2 of [GiS09]. In order to prove this Levine's method could be quite useful, as well as the description of $DM(S)$ in terms of $qfh$-sheaves (see Theorem 15.1.2 of [CiD09]). Moreover, the methods of [Lev09] could possibly allow to give a 'differential graded' description of the whole $DM^c(S)$ (extending the main result of [Lev09]).

The author plans to study these matters further.

3. In Theorem 3.1 of [Heb10] an orthogonality property (similar to those in Lemma 1.1.4(I1)) was established for not necessarily quasi-projective $f$ and $g$. This yielded that $\{f_!(\mathbb{Q}_X)(r)[2r]\} \in DM^c(S)^{w=0}$ for any proper (not necessarily projective!) $f$ such that $X$ is regular, $r \in \mathbb{Z}$, and allowed to generalize Theorem 2.2.1(II) (below) to not necessarily quasi-projective morphisms.

4. If $S$ is not reasonable, we still obtain that $Chow(S)$ is negative. Hence, there exists a weight structure on $\langle Chow(S)\rangle$ whose heart is $Chow(S)$ (since $Chow(S)$ is idempotent complete). The problem is that we do not know whether $Chow(S)$ is the whole $DM^c(S)$.

One can also prove the existence of a certain analogue of the Chow weight structure over any not necessarily reasonable Spec $\mathbb{Q}$-scheme $S$; see §2.3 below. The main disadvantage of this method is that it does not yield an 'explicit' description of $\underline{Hw}_{Chow}$ (though $\underline{Hw}_{Chow} \supset Chow(S)$).

## 2.2 Functoriality of $w_{Chow}$

Now we study (left and right) weight-exactness of motivic image functors. These statements are very similar to the properties of pure complexes of constructible sheaves. This is no surprise at all, see §3.6 below. Below $S, X, Y$ (and hence also $Z$, $U$, and all $S_l^\alpha$) will be reasonable.

**Theorem 2.2.1.** *I The functor $-(1)[2](=\otimes\mathbb{Q}(1)[2])$ and its inverse $-(-1)[-2]: DM^c(S) \to DM^c(S)$ are weight-exact with respect to $w_{Chow}$ for any $S$.*

*II Let $f: X \to Y$ be a (separated finite type) quasi-projective morphism of schemes.*

*1. $f^!$ and $f_*$ are left weight-exact; $f^*$ and $f_!$ are right weight-exact.*

*2. Suppose moreover that $f$ is smooth. Then $f^*$ and $f^!$ are also weight-exact.*

*III Let $i: Z \to X$ be a closed immersion; let $j: U \to X$ be the complimentary open immersion.*

*1. $Chow(U)$ is the idempotent completion of the factor (in the sense of Definition 1.2.1(VIII)) of $Chow(X)$ by $i_*(Chow(Z))$.*

*2. For $M \in Obj DM^c(X)$ we have: $M \in DM^c(X)^{w_{Chow}\le 0}$ (resp. $M \in DM^c(X)^{w_{Chow}\ge 0}$) whenever $j^!(M) \in DM^c(U)^{w_{Chow}\le 0}$ and $i^!(M) \in DM^c(Z)^{w_{Chow}\le 0}$ (resp. $j^*(M) \in DM^c(U)^{w_{Chow}\ge 0}$ and $i^*(M) \in DM^c(Z)^{w_{Chow}\ge 0}$).*

*IV Let $S = \cup S_l^\alpha$ be a stratification, $i_l: S_l^\alpha \to S$ are the corresponding immersions. Then for $M \in Obj DM^c(X)$ we have: $M \in DM^c(X)^{w_{Chow}\le 0}$ (resp. $M \in DM^c(X)^{w_{Chow}\ge 0}$) whenever $i_l^!(M) \in DM^c(S_l^\alpha)^{w_{Chow}\le 0}$ (resp. $i_l^*(M) \in DM^c(S_l^\alpha)^{w_{Chow}\ge 0}$) for all $l$.*

*V 1. For any $S$ we have $\mathbb{Q}_S \in DM^c(S)^{w_{Chow}\ge 0}$.*

*2. If $S_{red}$ is regular, then $\mathbb{Q}_S \in DM^c(S)^{w_{Chow}=0}$.*

*Proof.* I Since $w_{Chow}$ is bounded for any base scheme, in order to prove that a motivic image functor is weight-exact it suffices to prove that it preserves Chow motives; see Proposition 1.2.3(10). The assertion follows immediately.

II Let $f$ be smooth. Then we obtain: $f^*(DM^c(Y)^{w_{Chow}=0}) \subset DM^c(X)^{w_{Chow}=0}$ by Proposition 1.1.2(5). Hence $f^*$ is weight-exact (by the same argument as above). We also obtain that $f^!$ is weight-exact using assertion I and Proposition 1.1.2(8) i.e we proved assertion II2. Besides, the adjunctions yield (by Proposition 1.2.3(9)): $f_*$ is left weight-exact, $f_!$ is right weight-exact; i.e. assertion II1 for $f$ is fulfilled.

Now let $f$ be projective. Then $f_!(DM^c(X)^{w_{Chow}=0}) \subset DM^c(Y)^{w_{Chow}=0}$ (since $f_! \circ g_! = (f \circ g)_!$ for any $g$, and $f_!$ commutes with Tate twists). By Proposition 1.2.3(10) we obtain that $f_! = f_*$ is weight-exact. Using the adjunctions and Proposition 1.2.3(9) again, we obtain that $f^!$ is left weight-exact and $f^*$ is right weight-exact. So, assertion II1 is fulfilled also in the case when $f$ is projective.

Assertion II1 in the general case follows since any quasi-projective morphism is a composition of a closed (i.e. projective) immersion with a smooth quasi-projective morphism.

III Since $i_* \cong i_!$ in this case, $i_*$ is weight-exact by assertion II1. $j^*$ is weight-exact by assertion II2.

1. $DM^c(U)$ is the localization of $DM^c(X)$ by $i_*(DM^c(Z))$ by Proposition 1.1.2(11). Hence Proposition 1.2.3(11) yields the result (see Remark 1.2.4).

2. Proposition 1.1.2(11) yields: $w_{Chow}(X)$ is exactly the weight structure obtained by 'gluing $w_{Chow}(Z)$ with $w_{Chow}(U)$' via Proposition 1.2.3(13) (here we use part 15 of loc.cit.). Hence loc.cit. yields the result (note that $j^* = j^!$).

IV The assertion could be easily proved by induction on the number of stratification components using assertion III2.

V Let $S_{red}$ be regular; denote by $v$ the canonical immersion $S_{red} \to S$. Then $v_*(\mathbb{Q}_{S_{red}}) \in DM^c(S)^{w_{Chow}=0}$ by the definition of $w_{Chow}$. Now, $v^*$ is an equivalence of categories (by Proposition 1.1.2(12)) that sends $\mathbb{Q}_S$ to $\mathbb{Q}_{S_{red}}$ (see part 7 of loc.cit.). Hence (applying the adjunction) we obtain $v_*(\mathbb{Q}_{S_{red}}) \cong \mathbb{Q}_S$. So, we proved assertion V2.

In order to verify assertion V1 we choose a stratification $S = \cup S_\alpha$ such that all $S^\alpha_{l\,red}$ are regular. Since we have $i_l^*(\mathbb{Q}_S) = \mathbb{Q}_{S_l} \in DM^c(S_l)^{w_{Chow} \geq 0}$ (by assertion V2), assertion IV implies the result.

$\square$

*Remark* 2.2.2. Assertion III1 yields that any object of $Chow(U)$ is a retract of some object coming from $Chow(X)$. This fact could be easily deduced from Hironaka's resolution of singularities (if we believe that the composition of morphisms in $Chow(-)$ could be described in terms of algebraic cycles; cf. Remark 2.1.2(2)) in the case when $X$ is a variety over a characteristic 0 field. Indeed, then any projective regular $U$-scheme $Y_U$ possesses a projective regular $X$-model $Y$ (since one can resolve the singularities of any projective model $Y'/X$ of $Y_U$ by a morphism that is an isomorphism over $U$). The author does not know any analogues of this argument in the case of a general (reasonable) $X$ (even with alterations instead of modifications, since it does not seem to be known whether there exists an alteration of $Y'$ that is étale over $U$).

So, assertion III1 could be called (a certain) *motivic resolution of singularities* (over a reasonable $X$).

Now we prove that positivity and negativity of objects of $DM^c(S)$ (with respect to $w_{Chow}$) could be 'checked at points'; this is a motivic analogue of §5.1.8 of [BBD82].

**Proposition 2.2.3.** *Let $\mathcal{S}$ denote the set of (Zariski) points of $S$; for a $K \in \mathcal{S}$ we will denote the corresponding morphism $K \to S$ by $j_K$.*

*Then $M \in DM^c(S)^{w_{Chow} \le 0}$ (resp. $M \in DM^c(S)^{w_{Chow} \ge 0}$) if and only if for any $K \in \mathcal{S}$ we have $j_K^!(M) \in DM^c(K)^{w_{Chow} \le 0}$ (resp. $j_K^*(M) \in DM^c(K)^{w_{Chow} \ge 0}$); see Remark 1.1.3(2).*

*Proof.* By Theorem 2.1.1(II) if $M \in DM^c(S)^{w_{Chow} \le 0}$ (resp. $M \in DM^c(S)^{w_{Chow} \ge 0}$) then for any immersion $f : X \to S$ we have we have $f^!(M) \in DM^c(X)^{w_{Chow} \le 0}$ (resp. $f^*(M) \in DM^c(X)^{w_{Chow} \ge 0}$). It remains to pass to the limits with respect to immersions corresponding to points of $S$ (see Remark 1.1.3(2)).

We prove the converse implication by noetherian induction. So, suppose that our assumption is true for motives over any closed subscheme of $S$, and that for some $M \in Obj DM^c(S)$ we have $j_K^!(M) \in DM^c(K)^{w_{Chow} \le 0}$ (resp. $j_K^*(M) \in DM^c(K)^{w_{Chow} \ge 0}$) for any $K \in \mathcal{S}$.

We should prove that $M \in DM^c(S)^{w_{Chow} \le 0}$ (resp. $M \in DM^c(S)^{w_{Chow} \ge 0}$). By Proposition 1.2.3(3) it suffices to verify: for any $N \in DM^c(S)^{w_{Chow} \ge 1}$ (resp. for any $N \in DM^c(S)^{w_{Chow} \le -1}$), and any $h \in DM^c(S)(N, M)$ (resp. any $h \in DM^c(S)(M, N)$) we have $h = 0$. We fix some $N$ and $h$.

By the 'only if' part of our assertion (that we have already proved) we have $j_K^*(N) \in DM^c(K)^{w_{Chow} \ge 1}$ (resp. $j_K^*(N) \in DM^c(K)^{w_{Chow} \le -1}$); hence $j_K^*(h) = 0$. By Proposition 1.1.2(14) we obtain that $j^*(h) = 0$ for some open embedding $j : U \to S$, where $K$ is a generic point of $U$.

Now suppose that $h \ne 0$; let $i : Z \to S$ denote the closed embedding that is complimentary to $j$. Then Lemma 1.1.4(II) yields that $DM^c(S)(i^*(N), i^!(M)) \ne \{0\}$ (resp. $DM^c(S)(i^*(M), i^!(N)) \ne \{0\}$). Yet $i^*(N) \in DM^c(Z)^{w_{Chow} \ge 1}$ (resp. $i^!(N) \in DM^c(Z)^{w_{Chow} \le -1}$) by Theorem 2.2.1(II), whereas $i^!(M) \in DM^c(Z)^{w_{Chow} \le 0}$ (resp. $i^*(M) \in DM^c(Z)^{w_{Chow} \ge 0}$) by the inductive assumption. The contradiction obtained proves our assertion. □

## 2.3 The 'gluing' construction of $w_{Chow}$ (over any excellent $S$ of characteristic $0$)

In this paragraph all schemes will be (excellent separated) Spec $\mathbb{Q}$-schemes; we do not assume them to be reasonable. Then we can define the Chow weight structure 'locally'. We explain how do this (using stratifications and gluing of weight structures (we call this approach to constructing $w_{Chow}$ the 'gluing method'); the constructions and results of this section are quite similar to those of §5 of [BBD82]).

First we will describe certain candidates for $DM^c(S)^{w_{Chow}\le 0}$ and $DM^c(S)^{w_{Chow}\ge 0}$; next we will prove that they yield a weight structure for $DM^c(S)$.

For a scheme $X$ we will denote by $\mathcal{ON}(X)$ (resp. $\mathcal{OP}(X)$) the envelope (see the Notation) of $p_*(\mathbb{Q}_P)(s)[i+2s](\cong p_!(\mathbb{Q}_P)(s)[i+2s])$ in $DM^c(X)$; here $p: P \to X$ runs through all smooth projective morphisms to $X$, $s \in \mathbb{Z}$, whereas $i \ge 0$ (resp. $i \le 0$).

*Remark* 2.3.1. It is easily seen (using Proposition 1.1.2) that for any morphism $g$ of schemes we have $g^*(\mathcal{ON}(Y)) \subset \mathcal{ON}(X)$ and $g^*(\mathcal{OP}(Y)) \subset \mathcal{OP}(X)$. Indeed, we have $g^*p_!(\mathbb{Q}_P) \cong p'_!g'^*(\mathbb{Q}_P) = p'_!(\mathbb{Q}'_P)$ (we use the usual notation for the base change of $g, p$, and $P$).

Besides, if $g$ is an immersion of regular schemes, we also have $g^!(\mathcal{ON}(Y)) \subset \mathcal{ON}(X)$, $g^!(\mathcal{OP}(Y)) \subset \mathcal{OP}(X)$; here we use Lemma 1.1.4(I2).

For a stratification $\alpha : S = \cup S_l^\alpha$ we denote by $\mathcal{ON}(\alpha)$ the class $\{M \in Obj DM^c(S) :\ j_l^!(M) \in \mathcal{ON}(S_l^\alpha), 1 \le l \le n\}$; $\mathcal{OP}(\alpha) = \{M \in Obj DM^c(S) : j_l^*(M) \in \mathcal{OP}(S_l^\alpha), 1 \le l \le n\}$.

We will call a stratification $\alpha$ of $S$ (i.e. $S = \cup S_l^\alpha$; see the Notation) *regular* if all $S_l^\alpha$ are regular. We define: $DM^c(S)^{\le 0} = \cup_\alpha \mathcal{ON}(\alpha)$; $DM^c(S)^{\ge 0} = \cup_\alpha \mathcal{OP}(\alpha)$; here $\alpha$ runs through all regular stratifications of $S$.

We will need the following statement.

**Lemma 2.3.2.** *1. Let $\delta$ be a (not necessarily regular) stratification of $S$; we denote the corresponding immersions $S_l^\delta \to S$ by $j_l$. Let $M \in Obj DM^c(S)$.*

*Suppose that $j_l^!(M) \in DM^c(S_l^\delta)^{\le 0}$ (resp. $j_l^*(M) \in DM^c(S_l^\delta)^{\le 0}$) for all $l$. Then $M \in DM^c(S)^{\le 0}$ (resp. $M \in DM^c(S)^{\ge 0}$).*

*2. $j_*(DM^c(V)^{\le 0}) \subset DM^c(S)^{\le 0}$ and $j_!(DM^c(V)^{\ge 0}) \subset DM^c(S)^{\ge 0}$ for any immersion $j : V \to S$.*

*Proof.* 1. We use induction on the number of components of $\delta$. The 2-functoriality of motivic upper image functors yields: it suffices to prove the statement for $\delta$ consisting of two components.

So, let $S = U \cup Z$, $Z$ and $U$ are disjoint, $U$ is open (dense) in $S$; we denote the immersion $U \to S$ and $Z \to S$ by $j$ and $i$, respectively.

By the assumptions on $M$, there exist regular stratifications $\beta$ of $Z$ and $\gamma$ of $U$ such that $i^!(M) \in \mathcal{ON}(\beta)$ and $j^!(M) \in \mathcal{ON}(\gamma)$ (resp. $i^*(M) \in \mathcal{OP}(\beta)$ and $j^*(M) \in \mathcal{OP}(\gamma)$).

We 'unify' $\beta$ with $\gamma$ and denote the regular stratification of $S$ obtained by $\alpha$. Then 2-functoriality of $-^!$ (resp. of $-^*$) yields that $M \in \mathcal{ON}(\alpha)$ (resp. $M \in \mathcal{OP}(\alpha)$).

2. We choose a stratification $\delta$ containing $V$; so we assume that $V = S_0^\delta$. Then it can be easily seen that $j_l^! j_* = 0 = j_l^* j_!$ for $l \neq 0$ and $j^! j_* \cong 1_{DM(V)} \cong j^* j_!$ (see Proposition 1.1.2(10)). Hence the result follows from assertion 1.

□

**Proposition 2.3.3.** *I1. $(DM^c(S)^{\le 0}, DM^c(S)^{\ge 0})$ yield a bounded weight structure $w_{Chow}$ for $DM^c(S)$.*

*2. $DM^c(S)^{w_{Chow} \le 0}$ (resp. $DM^c(S)^{w_{Chow} \ge 0}$) is the envelope of $p_*(\mathbb{Q}_P)(s)[2s+i]$ (resp. of $p_!(\mathbb{Q}_P)(s)[2s-i]$) for $s \in \mathbb{Z}$, $i \ge 0$, and $p : P \to S$ being the composition of a smooth projective morphism with the immersion of a regular subscheme into $S$.*

*II $w(S)$ could be extended to a weight structure $w_{Chow}^{big}$ for the whole $DM(S)$.*

*Proof.* I We prove the statement by Noetherian induction. So, we suppose that the statement is valid for all proper closed subschemes of $S$. We prove it for $S$.

Obviously, $(DM^c(S)^{\le 0}, DM^c(S)^{\ge 0})$ are Karoubi-closed in $DM^c(S)$ and are semi-invariant with respect to translations.

Now, any two regular stratifications have a common regular subdivision. We apply Remark 2.3.1 and obtain: in order to verify orthogonality it suffices to prove for any regular $\alpha$ that $\mathcal{OP}(\alpha) \perp \mathcal{ON}(\alpha)[1]$. The latter statement follows from Lemma 1.1.4 (parts I1 and II).

In order to verify assertion I1 it remains to prove: any $M \in Obj DM^c(S)$ has some weight decomposition (with respect to $(DM^c(S)^{\le 0}, DM^c(S)^{\ge 0})$), and that $M \in DM^c(S)^{\le m} \cap DM^c(S)^{\ge n}$ for some $m, n \in \mathbb{Z}$.

We choose some generic point $K$ of $S$, $j_K \to S$ is the corresponding morphism. Since $K$ is a reasonable scheme, we have $j_K^*(M) \in \langle Chow(K) \rangle$ (see Proposition 1.1.2(13)) We fix some smooth projective varieties $P'_i/K$, $1 \le i \le n$ (we denote the corresponding morphisms $P'_i \to K$ by $p'_i$) and some $s \in \mathbb{Z}$ such that $j_K^*(M)$ belongs to the triangulated subcategory of $DM^c(K)$ generated by $\{p'_{i!}(\mathbb{Q}_{P'_i})(s)[2s]\}$.

Now we apply Proposition 1.1.2(14). We obtain: there exists an open embedding $j : U \to S$ such that: $U$ is regular, the generic point of $U$ is $K$, and $j^*(M)$ belongs to $D$. Here we denote by $D$ the triangulated subcategory of $DM^c(U)$ generated by $\{p_{i!}(\mathbb{Q}_{P_i})(s)[2s]\}$; $p_i : P_i \to U$ are some smooth projective 'models' of $p'_i$ (that exist if $U$ is small enough).

Since $id_U$ yields a regular stratification of $U$, $\{p_{i!}(\mathbb{Q}_{P_i})(s)[2s]\}$ is negative in $DM(S)(U)$ (since $\mathcal{OP}(\alpha) \perp \mathcal{ON}(\alpha)[1]$ for any regular $\alpha$, as we have just proved). Therefore (by Proposition 1.2.3(6–7)) there exists a weight structure $d$ for $D$ such that $D^{d\le 0}$ (resp. $D^{d\ge 0}$) is the envelope of $\cup_{n\ge 0}\{p_{i!}(\mathbb{Q}_{P_i})(s)[2s+n]\}$ (resp. of $\cup_{n\ge 0}\{p_{i!}(\mathbb{Q}_{P_i})(s)[2s-n]\}$). We also obtain that $D^{d\le 0} \subset DM^c(U)^{\le 0}$ and $D^{d\ge 0} \subset DM^c(U)^{\ge 0}$.

We denote $S \setminus U$ by $Z$ ($Z$ could be empty); $i : Z \to S$ is the corresponding closed immersion. By the inductive assumption, our method defines a bounded (Chow) weight structure for $DM^c(Z)$.

We have the gluing data $DM^c(Z) \xrightarrow{i_*} DM^c(S) \xrightarrow{j^*} DM^c(U)$. We can 'restrict it' to a gluing data

$$DM^c(Z) \xrightarrow{i_*} j^{*-1}(D) \xrightarrow{j_0^*} D$$

(see Proposition 1.2.3(13)), whereas $M \in Obj(j^{*-1}(D))$; here $j_0^*$ is the corresponding restriction of $j^*$. Hence by loc. cit. there exists a weight structure $w'$ for $j^{*-1}(D)$ such that $i_*$ and $j_0^*$ are weight-exact (with respect to the weight structures mentioned). Hence there exists a weight decomposition $M \to A \to B$ with respect to $w'$. Besides, there exist $m, n \in \mathbb{Z}$ such that $j_0^*(M) \in DM^c(U)^{\le m}$, $j_0^*(M) \in DM^c(U)^{\ge n}$, $i^!(M) \in DM^c(Z)^{\le m}$, and $i^*(M) \in DM^c(Z)^{\ge n}$. Hence $A, M[m] \in DM^c(S)^{\le 0}$; $B, M[n] \in DM^c(S)^{\ge 0}$; see Lemma 2.3.2(1).

Now we apply Proposition 1.2.3(14) and obtain: if $w_{Chow}(Z)$ could be described as in assertion I2, then $M$ possesses a weight decomposition whose components belong to the corresponding envelopes over $S$. Hence assertion I2 follows from the description of $w_{Chow}$ over (characteristic zero) fields by Noetherian induction.

II: immediate from assertion I1; cf. the proof of Theorem 2.1.1.

□

**Proposition 2.3.4.** *For the version of $w_{Chow}$ constructed in this section, the analogues of all parts of Theorem 2.2.1 as well as of Proposition 2.2.3 are fulfilled.*

*Proof.* The proof of part I (of loc.cit.) carries over to our situation without changes. The same is true for part II for the case of a smooth $f$. Lemma 2.3.2 yields assertion II1 for the case when $f$ is an immersion. The general case follows from these two immediately.

The (analogues of) the remaining parts of Theorem 2.2.1 and Proposition 2.2.3 follow from (the analogue of) part II by the same method as the one used in §2.2.

$\square$

**Corollary 2.3.5.** *1. We have $Chow(S) \subset \underline{Hw}_{Chow}$.*

*2. For a reasonable $S$ the 'alternative' version of $w_{Chow}$ (constructed above) coincides with the version given by Theorem 2.1.1(I).*

*Proof.* 1. It suffices to verify that $p_!(\mathbb{Q}_P) \in \underline{Hw}_{Chow}$ for any regular $P$ and a projective morphism $p : P \to S$. By the previous proposition, we obtain $\mathbb{Q}_p \in DM^c(P)^{w_{Chow}=0}$; since $p_! \cong p_*$, we obtain the result.

2. Indeed, denote the 'old' version of $w_{Chow}$ by $v$, and the 'alternative' one by $w$. The previous assertion along with Proposition 1.2.3(10) yields that $id_{DM^c(S)}$ is weight-exact with respect to $v$ and $w$. Hence part (4) of loc.cit. yields the result. $\square$

*Remark* 2.3.6. 1. Actually, in the first draft of this paper (only) the gluing method of constructing $w_{Chow}$ was used (this approach was first proposed in Remark 8.2.4(3) of [Bon07], that was in its turn inspired by [BBD82]). Next the author proved part 1 of the Corollary. Then (in order to deduce our main results) it remained to note that $Chow(S)$ generates $DM^c(S)$. Luckily, it was easy to prove the negativity of $Chow(S)$ (without relying on the gluing construction of $w_{Chow}$; see Lemma 1.1.4(I1)); so the proof was simplified (for a reasonable $S$; note still that the scheme of the proof of loc.cit. is similar to that for the chain of arguments that yield the first part of the Corollary). Yet in the case when $S$ is a scheme over Spec $\mathbb{Q}$, the gluing method gives us two ('new') descriptions of $w_{Chow}$.

A disadvantage of the gluing method is that it does not yield an explicit description of the whole $DM^c(S)^{w_{Chow}=0}$ (though we can describe it as the intersection of $DM^c(S)^{w_{Chow}\le 0}$ with $DM^c(S)^{w_{Chow}\ge 0}$).

2. Possibly, the results of this section could be extended (somehow) to motives over general excellent schemes. Yet $p_!(\mathbb{Q}_P)(s)[i+2s]$ for smooth projective $P/X$ are not sufficient to define the (corresponding) analogues of $\mathcal{OP}(X)$ and $\mathcal{ON}(X)$ in this case (even in the case when the base is an imperfect field). Probably, universal homeomorphisms of schemes should (somehow) be included in the construction. The main problem here is to verify that a 'weight-positive' (or 'weight-negative') motif over a generic point $K$ of $S$ could be 'expanded' to a weight-positive (resp. weight-negative) motif over some open $U \subset S$ (such that $K \in U$).

3. Motives with $\mathbb{Z}$-coefficients are more 'mysterious' than those with $\mathbb{Q}$-ones; yet possibly one can construct Chow weight structure(s) for them also. At least, the author hopes to achieve this for motives over (excellent separated) $\mathbb{Q}$-schemes (and also for motives with $\mathbb{Z}[\frac{1}{p}]$-coefficients over varieties over characteristic $p$ fields; cf. [Bon10b]).

4. Actually, most of the arguments of this section work for motives over schemes all of whose residue fields are separable (so, we don't have to assume residue fields to be of characteristic 0). In particular, the gluing method could be applied for $S = \text{Spec } \mathbb{Z}$. Yet we will obtain functoriality of $w_{Chow}$ only for motivic image functors coming from smooth (quasi-projective) morphisms. The problem here is the following one: when we decompose a morphism $X \to Y$ into a composition of an immersion $i : X \to P$ with a smooth projective morphism $P \to Y$, we cannot assume that all the residue field of $P$ are perfect even if this is true for $X$ and $Y$.

5. The author plans to (try to) reduce the conjecture on the existence of the motivic $t$-structure for $DM^c(S)$ to the case when $S$ is a field. To this end a certain gluing argument (as well as the methods applied in [BBD82] to the study of mixed complexes of sheaves) could be helpful.

# 3 Applications to cohomology and other matters

In §3.1 we study weight complexes for $S$-motives (and their compatibility with weight-exact motivic image functors).

In §3.2 we prove that $K_0(DM^c(S)) \cong K_0(Chow(S))$ (following [Bon07]), and define a certain 'motivic Euler characteristic' for (separated finite type) $S$-schemes.

In §3.3 we consider Chow-weight spectral sequences and filtrations for cohomology of $S$-motives (following §2.4 of [Bon07]). We observe that Chow-weight filtrations yield Beilinson's 'integral part' of motivic cohomology (see §2.4.2 of [Bei85] and [Sch00]).

In §3.4 we verify that Chow-weight spectral sequences (in particular) yield the existence of weight filtrations for the perverse cohomology of motives (that is not automatic in the case when $S$ is a Spec $\mathbb{Z}$-scheme).

In §3.5 we introduce the notion of a relative weight structures. The axiomatics of those was chosen to be an abstract analogue of Proposition 5.1.15 of [BBD82]. Several properties of relative weight structures are parallel to those for 'ordinary' weight structures.

In §3.6 we study the case when $S = X_0$ is a variety over a finite field. In this case the category $D^b_m(X_0, \mathbb{Q}_l)$ of mixed complexes of sheaves possesses a relative weight structure whose heart is the class of pure complexes of sheaves. Since the étale realization of motives preserves weights, we obtain that (Chow)-weight filtrations for some cohomology theories could be described in terms of the category $D^b_m(X_0, \mathbb{Q}_l)$.

In this section we will always assume that our base schemes are reasonable. Yet for $S$ as in §2.3 we also could have used the 'gluing' version of $w_{Chow}$ (the main difference is that we would have to put $\underline{Hw}_{Chow}$ instead of $Chow(S)$ everywhere).

## 3.1 The weight complex for $DM^c(S)$

We prove that the weight complex functor (whose 'first ancestor' was defined by Gillet and Soule) could be defined for $DM^c(S)$.

**Proposition 3.1.1.** *1. The embedding $Chow(S) \to K^b(Chow(S))$ factorizes through a certain exact conservative weight complex functor $t_S : DM^c(S) \to K^b(Chow(S))$.*

*2. For $M \in Obj DM^c(S)$, $i, j \in \mathbb{Z}$, we have $M \in DM^c(S)^{[i,j]}$ whenever $t(M) \in K(Chow(S))^{[i,j]}$ (see Remark 1.2.2).*

*3. For schemes $X, Y$ let $F : DM^c(X) \to DM^c(Y)$ be a weight-exact functor of triangulated categories (with respect to the Chow weight structures for these categories; so $F$ could be equal to $i_!$ for a finite type separated projective morphism $i : X \to Y$, or to $j^*$ for a finite type smooth morphism $j : Y \to X$) that possesses a differential graded enhancement. Denote by $F_{K^b(Chow)}$ the corresponding functor $K^b(Chow(X)) \to K^b(Chow(Y))$. Then there exists a choice of $t_X$ and $t_Y$ that makes the diagram*

$$\begin{array}{ccc} DM^c(X) & \xrightarrow{F} & DM^c(Y) \\ \downarrow{\scriptstyle t_X} & & \downarrow{\scriptstyle t_Y} \\ K^b(Chow(X)) & \xrightarrow{F_{K^b(Chow)}} & K^b(Chow(Y)) \end{array}$$

*commutative up to an equivalence of categories.*

*Proof.* 1. By Proposition 5.3.3 of [Bon07], this follows from the existence of a bounded Chow weight structure for $DM^c(S)$ along with the fact that it admits a differential graded enhancement (see Definition 6.1.2 of ibid.). The latter property of $DM(S)$ could be easily verified since it could be described in terms of the derived category of $qfh$-sheaves over $S$; see Theorem 15.1.2 of [CiD09] (and also cf. §6.1 of [BeV08]).

2. Immediate from Theorem 3.3.1(IV) of ibid.

3. We use the notation and definitions of §2 of [Bon09a] (that originate mostly from [BoK90]).

Since $DM^c(X) = \langle Chow(S) \rangle$, we can assume that $DM^c(X) = Tr^+(C_X)$, where $C_X$ is a negative triangulated category such that $H(C_X) = Chow(X)$

(see Remark 2.7.4(2) of ibid.). Replacing $DM^c(Y)$ by an equivalent category, we may also assume (similarly) that $DM^c(Y) = Tr^+(C_Y)$ where $C_Y$ is a negative triangulated category such that $H(C_Y) = Chow(Y)$, and $F = \text{Pre-Tr}(F')$ for some differential graded functor $C_X \to C_Y$. Arguing as in §6.1 of ibid, we obtain that it suffices to apply $Tr^+$ to the following diagram:

$$\begin{array}{ccc} C_X & \xrightarrow{F'} & C_Y \\ \downarrow & & \downarrow \\ H(C_X) & \xrightarrow{H(F')} & H(C_Y) \end{array}$$

□

*Remark* 3.1.2. 1. The 'first ancestor' of our weight complex functor was defined by Gillet and Soule in [GiS96]. Weight complex for a general triangulated category $\underline{C}$ endowed with a weight structure was defined in [Bon07]. Even in the case when $\underline{C}$ does not admit a differential graded enhancement, one can still define a certain 'weak' version of the weight complex; see §3 of ibid. (and this version does not depend on any choices). It follows that for $M \in Obj DM^c(S)$ the isomorphism class of $t_S(M)$ (in $K^b(Chow(S))$) does not depend on any choices (see ibid.).

2. In [GiS09] a functor $h$ from the category of Deligne-Mumford stacks over $S$ (with morphisms being proper morphisms over $S$) to the category of complexes over a certain category of $K_0$-*motives* was constructed; Gillet and Soule considered base schemes satisfying rather restrictive conditions (mostly, of dimension $\le 1$). We conjecture: for a finite type separated morphism $p : X \to S$ there is a functorial isomorphism $h(X) \to t(\mathcal{M}_c(X))$, where $\mathcal{M}_c(X) = p_* p^!(\mathbb{Q}_S)$. For $S$ being the spectrum of a characteristic 0 field this was (essentially) proved in §6.6 of [Bon09a]. Note here: though the category of $K_0$-*motives* is somewhat 'larger' than $Chow(S)$, it very probably suffices to consider its 'Chow' part (this would be the category $Chow(S)$ considered in [CoH00]).

Note that our definition of a weight complex (for $\mathcal{M}^c(X)$) gives it much more functoriality in $X$ than it was established [GiS09]; we also study its functoriality with respect to $S$, and relate it with cohomology (below).

Besides, we can restrict our definition of weight complexes to (motives with compact support of) quotient stacks (cf. Definition 1.2 of [GiS09]). For a finite $G$, $\#G = n$, acting on a finite type scheme $X/S$ one can take $\mathcal{M}^c(X/G) = a_{G*}\mathcal{M}^c(X) \in Obj DM^c(S)$ Here $a_G$ is the idempotent morphism (correspondence) $\frac{\sum_{g\in G} g}{n} : X \to X$. Certainly, for $G = \{e\}$ we will have $t_{\mathbb{Q}}(\mathcal{M}^c(X/G)) = t(\mathcal{M}^c(X))$.

3. Theorem 2.1.1 along with the results of [Bon07] also imply: $t_S$ could be extended to an exact functor $DM(S) \to K(BChow(S))$, where $BChow(S)$ is the big hull of $Chow(S)$ (see Definition 1.2.1(IX)).

4. One can also define exact (and conservative) *higher truncations functors* $t_{S,N}$ from $DM^c(S)$ to certain triangulated $DM^c(S)_N$ for all $N \geq 0$; cf. §6.1 of [Bon09a]. Here $t_{S,0} = t_S$; $DM^c(S)_N$ is obtained from a ('$Chow(S)$-negative') differential graded description of $DM^c(S)$ by killing all morphisms from $DM^c(S)^{w_{Chow}=0}$ to $DM^c(S)^{w_{Chow}=i}$ for $i < -N$. So, $DM^c(S)_N$ 'approximate' $DM^c(S)$ (when $N$ grows). $t_{S,N}$ would satisfy the analogue of Theorem 6.2.1 of ibid. Yet it seems that $t_S = t_{S,0}$ is the most interesting of the (higher) truncation functors.

## 3.2 $K_0(DM^c(S))$ and motivic Euler characteristic

Now we calculate $K_0(DM^c(S))$ and study a certain Euler characteristic for (finite type separated) $S$-schemes.

**Proposition 3.2.1.** *1. We define $K_0(Chow(S))$ as the groups whose generators are $[M]$, $M \in ObjChow(S)$ and the relations are $[M \bigoplus N] = [M]+[N]$ for $M, N \in ObjChow(S)$. For $K_0(DM^c(S))$ we take similar generators and set $[B] = [A] + [C]$ if $A \to B \to C \to A[1]$ is a distinguished triangle.*

*Then the embedding $Chow(S) \to DM^c(S)$ yields an isomorphism $K_0(Chow(S)) \cong K_0(DM^c(S))$.*

*2. For the correspondence $\chi : X \mapsto [p_*p^!\mathbb{Q}_S]$ (here $p : X \to S$ is a finite type separated morphism) from the class of finite type separated $S$-schemes to $K_0(DM^c(S)) \cong K_0(Chow(S))$ we have: $\chi(X \setminus Z) = \chi(X) - \chi(Z)$ if $Z$ is a closed subscheme of $X$.*

*Proof.* 1. Immediate from (part I of) Theorem 2.1.1 and Proposition 5.3.3(3) of [Bon07].

2. Denote the immersion $Z \to X$ by $i$, and the complimentary immersion by $j$. By Proposition 1.1.2(10) for any $M \in ObjDM^c(X)$ we have a distinguished triangle $i_*i^!M \to M \to j_*j^!M$ (note that $i_! \cong i_*$ and $j^! = j^*$). Now for $M = p^!\mathbb{Q}_S$ this triangle specializes to the triangle $i_*(p \circ i)^!\mathbb{Q}_S \to p^!\mathbb{Q}_S \to j_*(p \circ j)^!\mathbb{Q}_S$. It remains to apply $[p_*(-)]$ and the definition of $K_0(DM^c(S))$ to obtain the result.

□

*Remark* 3.2.2. 1. Assertion 2 is a vast extension of Corollary 5.13 of [GiS09]. It allows to define certain motivic Euler characteristics for (finite type separated) $S$-schemes.

2. We hope that our results could be useful for the theory of motivic integration.

Note in particular: we obtain that any (not necessarily weight-exact!) motivic image functor $DM^c(X) \to DM^c(Y)$ induces a homomorphism $K_0(Chow(X)) \to K_0(Chow(Y))$.

Besides, in contrast to the 'classical' case (when $S$ is a spectrum of a perfect field) there does not seem to exist a 'reasonable' (tensor) product for $Chow(S)$. Yet $DM^c(S)$ is a tensor triangulated category; hence one can use assertion 1 in order to define a ring structure on $K_0(Chow(S))$.

## 3.3 Chow-weight spectral sequences and filtrations

Now we discuss (Chow)-weight spectral sequences and the corresponding filtrations for cohomology of motives. One could also easily dualize this to obtain similar results for homological functors (see Theorem 2.3.2 of [Bon07]). We note that any weight structure yields certain weight spectral sequences for any cohomology theory; the main difference of the result below from the general case (as in Theorem 2.4.2 of ibid.) is that $T(H, M)$ converges always (since $w_{Chow}$ is bounded).

**Proposition 3.3.1.** *Let $\underline{A}$ be an abelian category, let $H : DM^c(S) \to \underline{A}$ be a cohomological functor.*

*For some $M \in Obj DM^c(S)$ we denote by $(M^i)$ the terms of $t(M)$ (so $M^i \in Obj Chow(S)$; here we can take any possible choice of $t(M)$).*

*Then the following statements are valid.*

*1. There exists a spectral sequence $T = T(H, M)$ with $E_1^{pq} = H^q(M^{-p}) \implies H^{p+q}(M)$; the differentials for $E_1(H, M)$ come from $t(M)$.*

*2. $T(H, M)$ is $DM^c(S)$-functorial in $M$ (and does not depend on any choices) starting from $E_2$.*

*3. Denote the step of filtration given by $(E_1^{l,m-l} : l \geq k)$ on $H^m(M)$ by $F^k H^m(M)$. Then $F^k H^m(M) = \operatorname{Im}(H^m(w_{Chow \leq -k} M) \to H^m(M))$; here for $w_{Chow \leq -k} M$ one can take arbitrary choices of the corresponding weight truncations of $M$.*

*Proof.* Immediate from Theorem 2.4.2 of [Bon07].

□

*Remark* 3.3.2. 1. We obtain certain *Chow-weight* spectral sequences and filtrations for any cohomology of motives. In particular, we have them for (rational) étale and motivic cohomology of motives.

2. $T(H, M)$ could be naturally described in terms of the virtual $t$-truncations of $H$ (starting from $E_2$); see §4.3 below.

3. We obtain that any cohomology of any $M \in Obj DM^c(S)$ possesses a filtration by subfactors of cohomology of regular projective $S$-schemes.

4. The fact that $\operatorname{Im}(H^m(w_{Chow \le -k}M) \to H^m(M))$ is $DM^c(S)$-functorial in $M$ follows from the axioms of weight structures very easily (see §2.1 of [Bon07]). Yet it has quite interesting consequences.

Let a scheme $X$ be reasonable; in the notation of Proposition 1.1.2(10) let $M \in DM^c(X)^{w_{Chow}=0}$, and denote $j^!(M)$ by $N$. Then by Theorem 2.2.1 we have $j_!(N), i_*i^*(M) \in DM^c(X)^{w_{Chow}\ge 0}$. Hence the distinguished triangle $j_!(N) \to M \to i_*i^*(M)$ (see Proposition 1.1.2(10)) yields a weight decomposition of $j_!(N)$. Therefore for any cohomological theory $H : DM^c(S) \to \underline{A}$ one has

$$F^0H^*(j_!(N)) = \operatorname{Im}(H^*(M) \to H^*(j_!(N))). \qquad (2)$$

In particular, the right hand side of (2) is $DM^c(U)$-functorial in $N$ (and does not depend on the choice of $M$ if we fix $N$). Moreover, $F^0H^*(j_!(N))$ yields a $DM^c(U)$-functorial extension of the right hand side of (2) (considered for $N$ of the form $j^!(M)$, $M \in DM^c(X)^{w_{Chow}=0}$) to the whole $DM^c(U)$. For $N \in DM^c(U)^{w_{Chow}\ge 0}$ we also obtain that $F^0H^*(j_!(N))$ could be described as the image $H^*(M') \to H^*(j_!(N))$ for certain $M' \in DM^c(X)^{w_{Chow}=0}$. Note here: $N \neq j^!(M')$ for any $M' \in DM^c(X)^{w_{Chow}=0}$ if $N \notin DM^c(U)^{w_{Chow}=0}$; yet cf. Remark 2.2.2.

One may use this observation in order to define the 'integral part' (i.e. the subobject of $H^*(j_!N)$ that 'comes from a nice $X$-model' of $N$) in the cohomology of motives over $U$ (cf. [Bei85], [Sch00], and [Sch10]). Note here that one could also consider $N \in Chow(K)$ for $K$ being a generic point of $U$, since any such $N$ could be lifted to a Chow motif over some $U$ ($K \in U$, $U$ is open in $S$), by Theorem 2.2.1(III1) combined with Proposition 1.1.2(14).

Suppose now that $M = p_*(\mathbb{Q}_P)$, where $P$ is regular, $p : P \to X$ is a projective morphism. Then $N(= j^!(M)) \cong p_{U*}(\mathbb{Q}_{P_U})$ (by Proposition 1.1.2(5)). Hence, if a scheme $P_U/U$ possesses a 'nice model' over $X$, then (2) (for $N = p_{U*}(\mathbb{Q}_{P_U})$) yields that the image of the $H$-cohomology of $P$ in the $H$-cohomology of $P_U$ is canonical and functorial. For a general $P_U$ one still obtains a certain subobject of $H^*(N)$ that is functorial in $P_U$ and equals the image in $H^*(N)$ of the $H$-cohomology of some regular $X$-projective scheme.

Arguing this way one obtains a description of the 'integral part' of motivic cohomology of $P_U$; this is an alternative to Theorem 1.1.6 of [Sch00]. One still has to do some work here in order to verify that $H^*(M)$ and $H^*(j_!(N))$ would become the motivic cohomology groups desired; yet this could be easily verified using Proposition 1.1.2(15) (in order to establish the functoriality of the isomorphism in loc.cit. also certain results of §13 of [CiD09] should be recalled). Note still that our description of the 'integral part' of cohomology is very short and does not rely on any conjectures (in contrast to the description given in [Sch10]). The author plans to write down this reasoning in more detail (later).

It could also be interesting to consider $F^lH^*(j_!(N))$ for $l \neq 0$.

## 3.4 Application to mixed sheaves: the 'arithmetic' case

Suppose that $S$ is a finite type Spec $\mathbb{Z}$-scheme. Denote by $\mathbb{H}$ the étale realization functor $DM^c(S) \to DSH$, where $DSH = DSH(S)$ is the category $D^b_m(S, \mathbb{Q}_l)$ of mixed complexes of $\mathbb{Q}_l$-étale sheaves as considered in [Hub97] and in [BBD82]. We will assume below that $\mathbb{H}$ converts the motivic image functors into the corresponding functors for $DSH(-)$ (it seems that the existence of such a realization is not fully established in the existing literature; yet a forthcoming paper of Cisinski and Deglise should close this gap).

We obtain that $\mathbb{H}$ sends Chow motives over $S$ to pure complexes of sheaves (of weight 0; see Definition 3.3 of [Hub97]). Indeed, it suffices to note that $H$ sends $\mathbb{Q}_X$ for a regular $X$ to a sheaf of weight 0, whereas $f_!$ for a projective $f$ preserves weights of sheaves (here it suffices to apply the corresponding results of §5 of [BBD82]; cf. Proposition 3.9 of [Hub97]).

Now we take $H_{per}$ being the perverse étale cohomology theory i.e. $H^i_{per}(M)$ (for $M \in ObjDM^c(S)$, $i \in \mathbb{Z}$) is the $i$-th cohomology of $\mathbb{H}(M)$ with respect to the perverse $t$-structure of $DSH$ (see Proposition 3.2 of [Hub97]). Then $T_{w_{Chow}}(H_{per}, M)$ for any $M \in ObjDM^c(S)$ yields: all $H^i_{per}(M)$ have weight filtrations (defined using Definition 3.3 of loc.cit., for all $i \in \mathbb{Z}$). Note that this is not at all automatic (for perverse sheaves over $S$); see Remark 6.8.4(i) of [Jan90]. Certainly, one can replace perverse sheaves over $S$ here by $\mathbb{Q}_l$-adic representations of the absolute Galois group of the function field of $S$; cf. §6.8 of loc.cit.

## 3.5 Relative weight structures

In order to define weights for mixed complexes of sheaves (over a finite field), we have to generalize the definition of a weight structure.

**Definition 3.5.1.** I Let $F : \underline{C} \to \underline{D}$ be an exact functor (of triangulated categories).

A pair of extension-stable Karoubi-closed subclasses $\underline{C}^{w\le 0}, \underline{C}^{w\ge 0} \subset Obj\underline{C}$ for a triangulated category $\underline{C}$ will be said to define a relative weight structure $w$ for $\underline{C}$ with respect to $F$ (or just and $F$-weight structure) if they satisfy the following conditions.

(i) **'Semi-invariance' with respect to translations**.

$\underline{C}^{w\ge 0} \subset \underline{C}^{w\ge 0}[1]$, $\underline{C}^{w\le 0}[1] \subset \underline{C}^{w\le 0}$.

(ii) **Weak orthogonality**.

$\underline{C}^{w\ge 0} \perp \underline{C}^{w\le 0}[2]$.

(iii) $F$-**orthogonality**.

$F$ kills all morphisms between $\underline{C}^{w\ge 0}$ and $\underline{C}^{w\le 0}[1]$.

(iv) **Weight decompositions**.

For any $M \in Obj\underline{C}$ there exists a distinguished triangle

$$B[-1] \to M \to A \xrightarrow{f} B \tag{3}$$

such that $A \in \underline{C}^{w\le 0}, B \in \underline{C}^{w\ge 0}$.

II We define $\underline{C}^{w\ge i}$, $\underline{C}^{w\le i}$, $\underline{C}^{w=i}$, $\underline{C}^{[i,j]}$, bounded relative weight structures, and $\underline{C}^b$ similarly to definition 1.2.1.

We will call the class $\underline{C}^{w=0}$ the heart of $w$ (we will not define the category $\underline{Hw}$).

We will use the same notation for weight truncations with respect to $w$ as the one introduced in Remark 1.2.2. We define weight-exact functors for relative weight structures as in Definition 1.2.1(VI) (i.e. we do not mention the corresponding $F$'s in the definition).

III Let $H$ be a full subcategory of a triangulated $\underline{C}$.

We will say that $H$ is $F$-*negative* if $ObjH \perp (\cup_{i>1} Obj(H[i]))$ and $F$ kills all morphisms between $H$ and $H[1]$.

*Remark* 3.5.2. 1. A weight structure is a relative weight structure with respect to $F = id_{\underline{C}}$.

2. An $F$-weight structure is also a $G \circ F$-weight structure for any exact functor $G : \underline{D} \to \underline{E}$ (for any triangulated $\underline{E}$). In particular, one can always take $F = 0$. Hence we do not lose in generality by adding the $F$-orthogonality axiom to the definition of relative weight structures.

Yet those properties of relative weight structures that do not depend on the choice of $F$ are certainly valid without this axiom. The main reason to put the $F$-orthogonality axiom together with the weak orthogonality one is that these conditions could be tracked down using similar methods.

3. The weak orthogonality axiom is a partial case of the higher Hom decomposition condition that was studied in Appendix B of [Pos10]. Respectively, Proposition 2 of loc.cit. generalizes our Proposition 3.5.3(8) considered in the case $F = 0$.

Now we will extend to relative weight structures several properties of weight structures. We will skip those parts of the proofs that do not differ much from the ones in [Bon07] (for 'usual' weight structures); we will concentrate on the distinctions.

**Proposition 3.5.3.** *Let $F : \underline{C} \to \underline{D}$ be an exact functor (of triangulated categories).*

*In all assertions expect 8 we will also assume that $w$ is a relative weight structure for $\underline{C}$ with respect to $F$.*

1. *$(C_1, C_2)$ ($C_1, C_2 \subset Obj\underline{C}$) define an $F$-weight structure for $\underline{C}$ whenever $(C_2^{op}, C_1^{op})$ define a relative weight structure for $\underline{C}^{op}$ with respect to $F^{op}$; here $F^{op} : \underline{C}^{op} \to \underline{D}^{op}$ is the functor obtained from $F$ by inverting all arrows.*

2. *All $\underline{C}^{[i,j]}$ are extension-stable.*

3. *Let $l \le m \in \mathbb{Z}$, $M, M' \in Obj\underline{C}$. Let weight decompositions of $M[m]$ and $M'[l]$ be fixed; we consider the corresponding triangles $w_{\ge m+1}M \xrightarrow{b} M \xrightarrow{a} w_{\le m}M$ and $w_{\ge l+1}M' \xrightarrow{b'} M' \xrightarrow{a'} w_{\le l}M'$.*

   *Then for any $g \in \underline{C}(M, M')$ there exists some morphism of distinguished triangles*

$$\begin{array}{ccccc} F(w_{\ge m+1}M) & \xrightarrow{F(b)} & F(M) & \xrightarrow{F(a)} & F(w_{\le m}M) \\ \downarrow & & \downarrow{\scriptstyle F(g)} & & \downarrow \\ F(w_{\ge l+1}M') & \xrightarrow{F(b')} & F(M') & \xrightarrow{F(a')} & F(w_{\le l}M') \end{array} \qquad (4)$$

4. *In addition to the assumptions of the previous assertion, suppose that $l < m$.*

   *Then there also exists a commutative diagram*

$$\begin{array}{ccccc} w_{\ge m+1}M & \xrightarrow{b} & M & \xrightarrow{a} & w_{\le m}M \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ w_{\ge l+1}M' & \xrightarrow{b'} & M' & \xrightarrow{a'} & w_{\le l}M' \end{array} \qquad (5)$$

   *Moreover, $(g, a, a')$ determine $F(d)$ uniquely; $(g, b, b')$ determine $F(c)$ uniquely.*

5. *For any $M \in Obj\underline{C}$ any choices of $w_{\le i}M$ (and of the arrows $a_i : M \to w_{\le i}M$ for all $i \in \mathbb{Z}$) could be completed to a weight Postnikov tower for $M$ (cf. Definition 1.5.8 of [Bon10a]) i.e. for all $j \in \mathbb{Z}$ we can choose some morphisms $c_j : w_{\le j+1}M \to w_{\le j}M$ that are compatible with $a_i$, and for any choice of $c_j$ we have: $M^j = \mathrm{Cone}(c_j(M))[-j] \in \underline{C}^{w=0}$.*

6. *We can choose a weight Postnikov tower for $M$ such that $w_{\le j}M = 0$ for $j < j_0$ and $= M$ for $j \ge j_1$ for $j_0, j_1 \in \mathbb{Z}$, whenever $M \in \underline{C}^{[j_0,j_1]}$.*

   *We will call such a weight Postnikov tower a bounded one.*

7. *Let $w$ be bounded, $G$ be an exact functor $\underline{C} \to \underline{C}'$; suppose that $\underline{C}'$ is endowed with a relative weight structure (with respect to some exact functor $F' : \underline{C}' \to \underline{D}'$).*

   *Then $G$ is left (resp. right) weight exact whenever $G(\underline{C}^{w=0}) \subset \underline{C}'^{w' \le 0}$ (resp. $G(\underline{C}^{w=0}) \subset \underline{C}'^{w' \ge 0}$).*

8. *Let $H \subset Obj\underline{C}$ be $F$-negative. Then there exists a bounded weight structure $w$ on $\langle H \rangle$ in $\underline{C}$ such that $H \subset T^{w=0}$.*

*Proof.* Assertions 1 and 2 are immediate from Definition 3.5.1.

The proof of assertions 3 and 4 is similar to those of Proposition 1.5.1 (parts 1 and 2) of [Bon07]. The axiom (iii) of relative weight structure yields that the composition morphism $F(w_{\ge m+1}M) \to F(w_{\le l}M')$ vanishes. Hence (the easy) Proposition 1.1.9 of [BBD82] yields the existence of (4).

Similarly, we obtain the existence of (5) if $m > l$. Moreover, any two distinct choices of $d$ (resp. $c$) are easily seen (see the proof of loc.cit.) to differ by $s \circ a$ (resp. by $(b' \circ s)[-1]$) for some $s \in \underline{C}(w_{\le l}M[1], w_{\ge m+1}M')$. Since $F(s) = 0$ (by axiom (iii) of relative weight structures), we conclude the proof of assertion 4.

The argument needed for the proof of assertion 5 very similar to the one used in the proof Theorem 2.2.1(11) of [Bon10a].

We put $M' = M$, $l = j$, $m = j+1$ in assertion 4; this yields the existence of some $c_j$. Since $\underline{C}^{w \le j}$ is extension-stable, it contains $\operatorname{Cone} c_j$. Completing the commutative triangle $M \xrightarrow{a_{j+1}} w_{\le j+1}M \xrightarrow{c_j} w_{\le j}M$ to an octahedral digram (as drawn in loc.cit.), we obtain that $\operatorname{Cone} c_j$ is also a cone of some morphism $w_{\ge j+2}M[1] \to w_{\ge j+1}M[1]$. Since $\underline{C}^{w \ge j}$ is extension-stable also, we obtain the result.

(6): If $w_{\le j}M = 0$ for some $j < j_0$ (resp. $= M$ for some $j \ge j_1$) then obviously $M \in \underline{C}^{w \ge j_0}$ (resp. $M \in \underline{C}^{w \le j_1}$). Conversely, if $M \in \underline{C}^{[j_0,j_1]}$, then nothing prevents us from choosing $w_{\le j}M = 0$ for all $j < j_0$ and $= M$ for all $j \ge j_1$.

(7): Certainly, if $G$ left (resp. right) weight exact then $G(\underline{C}^{w=0}) \subset \underline{C}'^{w' \le 0}$ (resp. $G(\underline{C}^{w=0}) \subset \underline{C}'^{w' \ge 0}$). Conversely, let $M \in \underline{C}^{w \le 0}$ (resp. $M \in \underline{C}^{w \ge 0}$). By the previous assertion, $M$ possesses a bounded weight Postnikov tower with $M^i = 0$ for $i > 0$ (resp. for $i < 0$). The structure of the tower yields that $M$ belongs to the envelope of $M^i[-i]$; this concludes the proof of the assertion.

The proof of assertion 8 is similar to that of Theorem 4.3.2(II1) of [Bon07] (also, one can assume that $F = 0$ here). We take the envelope of $H[i]$ for $i \geq 0$ (resp. for $i \leq 0$) for $\underline{C}^{w\leq 0}$ (resp. for $\underline{C}^{w\geq 0}$; see the Notation). Obviously, $\underline{C}^{w\leq 0}$ and $\underline{C}^{w\geq 0}$ are Karoubi-closed, extension-stable, and satisfy the condition (i) of Definition 3.5.1(I). $F$-orthogonality of $H$ easily yields conditions (ii) and (iii) of loc.cit. It remains to verify that any object of $\underline{C}$ possesses a weight decomposition with respect to $w$.

We define the notion complexity for objects of $\underline{C}$. For $M \in ObjH[i]$ we will say that $M$ has complexity $\leq 0$. If there exists a distinguished triangle $M \to N \to O$, and $M, O$ are of complexity $\leq i$ (they also could have smaller complexity) we will say that the complexity of $N$ is $\leq i+1$. Since any object of $\langle H\rangle$ has finite complexity, it suffices to verify: for a distinguished triangle $M \to N \to O$ if $M, O$ possess weight decompositions (with respect to our $(\underline{C}^{w\leq 0}, \underline{C}^{w\geq 0})$), then $N$ possesses a weight decomposition also.

By assertion 4, we can complete the morphism $O[-1] \to M$ to a commutative square

$$\begin{array}{ccc} O[-1] & \longrightarrow & (O^{w\leq 0})[-1] \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{w\leq 0} \end{array}$$

Hence by the $3 \times 3$-Lemma (i.e. Proposition 1.1.11 of [BBD82]) we can complete the distinguished triangle $M \to N \to O$ to a diagram

$$\begin{array}{ccccc} M & \longrightarrow & M^{w\leq 0} & \longrightarrow & M^{w\geq 1} \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & N' & \longrightarrow & N'' \\ \downarrow & & \downarrow & & \downarrow \\ O & \longrightarrow & O^{w\leq 0} & \longrightarrow & O^{w\geq 1} \end{array} \tag{6}$$

(cf. Lemma 1.5.4 of [Bon07]). We have $N' \in \underline{C}^{w\leq 0}$, $N'' \in \underline{C}^{w\geq 0}$ (by the definition of these classes). Hence $N$ possesses a weight decomposition indeed.

□

*Remark* 3.5.4. One also can glue relative weight structures similarly to Proposition 1.2.3(13), and define weight structures for 'pure' localizations as in part (11) of loc.cit.

**Proposition 3.5.5.** *I Let $H : \underline{C} \to \underline{A}$ be a cohomological functor, $M \in Obj\underline{C}$. Fix (any choice of) a bounded weight Postnikov tower for $H$ (see Proposition (5(3.5.3))*

*1. There exists a weight spectral sequence $T$ with $E_1^{pq}(T) = H^q(M^{-p}) \implies E_\infty^{p+q}(T) = H^{p+q}(M)$.*

*2. Denote the step of filtration given by $(E_1^{l,m-l} : l \geq k)$ on $H^m(M)$ by $F^kH^m(M)$. Then $F^kH^m(M) = \operatorname{Im}(H^m(w_{\leq -k}M) \to H^m(M))$.*

*3. Suppose that $H$ could be factorized through $F$. Then the weight filtration $F^kH^m(M)$ described above is $\underline{C}$-functorial in $M$ (and does not depend on the choice of the tower).*

*II Let $F : \underline{C} \to \underline{D}$, $F' : \underline{C} \to \underline{D}$, and $G : \underline{C} \to \underline{C}'$ be exact functors. Let $w$ be an $F$-weight structure for $\underline{C}$, $w'$ be an $F'$-weight structure for $\underline{C}'$; suppose that $G$ is weight-exact.*

*1. $G$ converts $w$-Postnikov towers into $w'$-Postnikov towers.*

*2. For a cohomological functor $H' : \underline{C} \to \underline{A}$ suppose that $H'$ could be factorized through $F'$ and that $H = H' \circ G$ could be factorized through $F$. Then in the notation of assertion I1, we have $F^kH'^m(G(-)) = F^kH^m(-)$.*

*Proof.* I 1,2: Immediate from the standard properties of the spectral sequence coming from a Postnikov tower; see the Exercises after §IV.2 of [GeM03].

3: Immediate from assertion 2 and Proposition 3.5.3(3).

II Obvious.

□

*Remark* 3.5.6. 1. Suppose that there exist $t$-structures $t_{\underline{C}}$ for $\underline{C}$ and $t_{\underline{D}}$ for $\underline{D}$ such that $F$ is $t$-exact. Suppose also that for $M \in \underline{C}^{t=0}$ there exists a choice of $w_{\leq 0}M$ and $w_{\geq 1}M$ belonging to $\underline{Ht}$. Then the morphism $F(M) \to F(w_{\leq 0}M)$ is epimorphic in $\underline{Ht}_{\underline{D}}$. It follows: for the functor $H = H_0^{t_{\underline{D}},op} \circ F$ the zeroth level of the weight filtration of $H(F) = F(M)$ is just $F(w_{\leq 0}M)$. Here $H_0^{t_{\underline{D}},op}$ is the zeroth cohomology with respect to $t$ with values in the category opposite to $\underline{Ht}_{\underline{D}}$ (we invert the arrow in order to make the functor cohomological). So, such weight truncations are '$F$-functorial when they exist'; cf. Remark 1.5.2(2) and §8.6 of [Bon07]. Hence the corresponding weight filtrations are functorial also.

Yet it seems that in order to obtain stronger results (similar to those of §5 of [BBD82]) on weight filtration for objects of $\underline{Ht}$ one would require a certain theory of $t$-structures compatible (in a certain sense) with relative weight structures.

2. Unfortunately, it seems that weight spectral sequences given by the Proposition don't have to be canonical (in general).

3. In part II2 of the Proposition we assumed that $H = G \circ F$ for some functor $G : \underline{D} \to \underline{A}$; yet we did not demand $G$ to be additive (or cohomological).

### 3.6 Mixed sheaves over a finite field

Now let $S = X_0$ be a variety over a finite field $\mathbb{F}_q$; let $X$ denote $X_0 \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\mathbb{F}$, where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$. Let $F$ denote the extension of scalars functor $DSH \to D^b_m(S, \mathbb{Q}_l)$. We consider the same $\mathbb{H}$ as in §3.4.

**Proposition 3.6.1.** *1. The category $DSH(= D^b_m(X_0, \mathbb{Q}_l))$ can be endowed with an $F$-weight structure $w_{DSH}$ such that $DSH^{w_{DSH}\le 0}$ (resp. $DSH^{w_{DSH}\ge 0}$) is the class of complexes of non-negative (resp. non-positive) weights in the sense of §5.1.8 of [BBD82] (note that we change the signs of weights here). The heart of $w_{DSH}$ is the class of pure complexes of sheaves of weight* 0.

*2. $\mathbb{H}$ is a weight-exact functor (with respect to $w_{Chow}$ and $w_{DSH}$).*

*Proof.* Proposition 5.1.14 of [BBD82] yields all axioms of $F$-weight structures in our situation expect the existence of weight decompositions. So, by Proposition 3.5.3(8), it suffices to verify that the category of pure complexes of sheaves of weight 0 (note that it is idempotent complete) generates $DSH$. This is immediate from Theorem 5.3.5 of [BBD82].

2. Immediate from Proposition 3.5.3(7) and the observations made in §3.4.

□

*Remark* 3.6.2. 1. In particular, we obtain that any object $M$ of $DSH$ possesses a weight Postnikov tower whose 'factors' are pure complexes of sheaves.

2. So, it is no surprise that Theorem 2.2.1 is a motivic analogue of the 'stability properties' 5.1.14 of [BBD82].

## 4 Supplements

In §4.1 we recall the notion of a $t$-structure adjacent to a weight structure (as introduced in §4.4 of [Bon07]).

In §4.2 we use Theorem 4.5.2 of ibid. to prove the existence of the *Chow $t$-structure* for $DM(S)$ that is adjacent to the Chow weight structure for it (cf. Theorem 2.1.1(II)); we also establish certain functoriality properties of this $t$-structure (with respect to the motivic image functors, when $S$ varies).

In §4.3 we recall the notion of *virtual $t$-truncations* (for cohomological functors from $DM^c(S)$), and relate virtual $t$-truncations with $t_{Chow}$.

### 4.1 Adjacent structures

We recall the notion of adjacent weight and $t$-structures (that was introduced in §4.4 of [Bon07]). For $t$-structures will will use notation and conventions similar to those of weight structures in §1.2 (see also §4.1 of [Bon07]). In particular, we will denote the heart of $t$ by $\underline{Ht}$ (recall that it is abelian); $Obj\,\underline{Ht} = \underline{C}^{t=0}$.

We will say that $t$ (for $\underline{C}$) is non-degenerate if $\cap_{n\in\mathbb{Z}}\underline{C}^{t\le n} = \cap_{n\in\mathbb{Z}}\underline{C}^{t\ge n} = \{0\}$.

**Definition 4.1.1.** We say that a weight structure $w$ is (left) *adjacent* to a $t$-structure $t$ if $\underline{C}^{w\le 0} = \underline{C}^{t\le 0}$.

We will also need the following properties of adjacent structures.

**Proposition 4.1.2.** *I Let $\underline{C}$ be endowed with a weight structure $w$ and also with an adjacent $t$-structure $t$.*

*1. The functor $\underline{C}(-,\underline{Ht}) : \underline{Ht} \to \mathrm{AddFun}(\underline{Hw}^{op}, Ab)$ that sends $N \in \underline{C}^{t=0}$ to $M \mapsto \underline{C}(M,N)$, ($M \in \underline{C}^{w=0}$), is an exact embedding of $\underline{Ht}$ into the abelian category $\mathrm{AddFun}(\underline{Hw}^{op}, Ab)$.*

*2. Let $t$ be non-degenerate. Then $\underline{C}^{t=0} = \{M \in Obj\underline{C} :\ \underline{C}^{w=i} \perp M\ \forall\, i \neq 0\}$.*

*II Moreover, let a triangulated category $\underline{C}'$ be endowed with a weight structure $w'$ and also with its adjacent $t$-structure $t'$. Let $F : \underline{C} \to \underline{C}'$ be an exact functor.*

*1. $F$ is left weight-exact whenever it is left $t$-exact.*

*2. Let $G : \underline{C}' \to \underline{C}$ be the right adjoint to $F$. Then $F$ is left (resp. right) weight-exact with respect to $w$ and $w'$ whenever $G$ is right (resp. left) $t$-exact with respect to $t'$ and $t$.*

*III Let $\underline{D} \subset \underline{C}$ be a full subcategory of compact objects endowed with a weight structure $w_{\underline{D}}$ (we denote its heart by $\underline{Hw}_{\underline{D}}$). Let $\underline{C}$ admit arbitrary (small) coproducts and suppose that $\underline{D}$ weakly generates $\underline{C}$. Then the following statements are valid.*

*1. For the weight structure $w$ for $\underline{C}$ given by Proposition 1.2.3(12) there exists an adjacent $t$-structure; it is non-degenerate. $\underline{Ht}$ is isomorphic to $\mathrm{AddFun}(\underline{Hw}_{\underline{D}}^{op}, Ab)$ (via the functor $N \mapsto (M \in \underline{D}^{\underline{Hw}_{\underline{D}}=0} \mapsto \underline{C}(M,N))$).*

*2. Suppose that $w_{\underline{D}'}$ and $\underline{D}' \subset \underline{C}'$ satisfy the conditions for $w_{\underline{D}}$ and $\underline{D} \subset \underline{C}$; denote the corresponding adjacent weight structure for $\underline{C}$ by $w'$ and $t'$.*

*Let $F : \underline{C} \to \underline{C}'$ be an exact functor that maps $\underline{D}$ into $\underline{D}'$; suppose that is possesses a right adjoint $G$ that maps $\underline{D}'$ in $\underline{D}$. Then the restriction of $F$ to $\underline{D}$ is left (resp. right) weight-exact with respect to $w_{\underline{D}}$ and $w'_{\underline{D}'}$ whenever $G$ is right (resp. left) $t$-exact with respect to $t'$ and $t$.*

*Proof.* I These are just parts 4 and 5 of Theorem 4.4.2 of ibid.

II1. Immediate from the definition of adjacent structures.

2. See Remark 4.4.6 of ibid.

III 1. Immediate from Theorem 4.5.2 of ibid.

2. Immediate from the previous assertions by adjunction (we use the description of $\underline{Ht}$).

$\square$

## 4.2 The Chow $t$-structure for $DM(S)$

Now we study the $t$-structure adjacent to $w_{Chow}^{big}$.

**Proposition 4.2.1.** *I Let $S$ be an (excellent finite dimensional) scheme that is either reasonable or a $\mathbb{Q}$-scheme.*

*1. There exists a non-degenerate $t$-structure $t_{Chow}(S)$ on $DM(S)$ that is adjacent to $w_{Chow}^{big}$ (the latter is given either by Theorem 2.1.1(II) or by Proposition 2.3.3(II)).*

*2. $\underline{Ht}_{Chow}(S) \cong \mathrm{AddFun}(Chow(S)^{op}, Ab)$ (via the functor $N \mapsto (M \in DM^c(S)^{w_{Chow}=0} \mapsto DM(S)(M,N))$).*

*II Let $f : X \to Y$ be a quasi-projective finite type morphism of schemes that are easier reasonable or are* Spec $\mathbb{Q}$*-schemes.*

*1. $f^!$ and $f_*$ are left $t_{Chow}$-exact (with respect to the corresponding Chow $t$-structures).*

*2. Suppose that $f$ is smooth. Then $f_*$ is (also) $t_{Chow}$-exact.*

*Proof.* I Immediate from the definition of $w_{Chow}$ and $w_{Chow}^{big}$, and Proposition 4.1.2(I).

II The assertion follows easily either from Theorem 2.2.1(II) or from Proposition 2.3.4 (depending on our assumptions on $X$ and $Y$) by applying the adjunctions; see Proposition 4.1.2(III).

$\square$

*Remark* 4.2.2. So, for any $N \in ObjDM(S)$ the Chow $t$-structure for $DM(S)$ allows to 'slice' the cohomology theory $H : M \mapsto DM(S)(M,N)$, into 'pieces' $H^i : M \to DM(S)(M, t_{Chow=i}N)$; note that $H^i(N[j]) = \{0\}$ for any $N \in ObjChow(S) \subset ObjDM^c(S)$, $j \neq i$ (see Proposition 4.1.2(I2)). One may call these $H^i$ *pure* cohomology theories.

We will describe another (more general) method for slicing a cohomology theory into pure pieces below; yet this method does not demonstrate that the pieces of a representable cohomology theory are representable.

## 4.3 Virtual $t$-truncations with respect to $w_{Chow}$; 'pure' cohomology theories

Now suppose that we are given an arbitrary cohomological functor $H : DM^c(S) \to \underline{A}$, $\underline{A}$ is an abelian category. Virtual $t$-truncations (defined in §2.5 of [Bon07] and developed further in §2 of [Bon10a]) allow to 'slice' $H$ into pure pieces $H^i$. To this end we only use $w_{Chow}$ (and have no need to put $H$ into some 'category of cohomological functors' $DM^c(S) \to \underline{A}$, and define a $t$-structure for this category). Virtual $t$-truncations also yield a functorial description of Chow-weight spectral sequences (starting from $E_2$).

Now we just list the main properties of virtual $t$-truncations (in the case when $(\underline{C}, w) = (DM^c(S), w_{Chow})$; the properties are the same as in the general case).

**Proposition 4.3.1.** *Let $H : DM^c(S) \to \underline{A}$ and $i \in \mathbb{Z}$ be fixed.*

*1. For any $M \in ObjDM^c(X)$ there exist unique morphisms $i_1(M) \in DM^c(S)(w_{Chow \le i+1}M, w_{Chow \le i}M)$ and $i_2(M) \in DM^c(S)(w_{Chow \ge i}M, w_{Chow \ge i-1}M)$ that fit into a commutative diagram*

$$\begin{CD}
w_{Chow \ge i}M @>>> M @>>> w_{Chow \le i+1}M \\
@VV{i_2(M)}V @VV{id_M}V @VV{i_1(M)}V \\
w_{Chow \ge i-1}M @>>> M @>>> w_{Chow \le i}M
\end{CD} \tag{7}$$

*here the horizontal arrows are compatible with (arbitrary fixed) weight decompositions of $M[i+j]$ (for $-2 \le j \le 1$).*

*2. The correspondences $M \to \operatorname{Im} H(i_1(M))$ and $M \to \operatorname{Im} H(i_2(M))$ yield well-defined cohomological functors $\tau_{\le i}H$, $\tau_{\ge i}H : DM^c(S) \to \underline{A}$ (we call them virtual $t$-truncations of $H$).*

*3. $\tau_{\le i}H$ vanishes on $DM^c(S)^{w_{Chow} \ge i+1}$; $\tau_{\ge i}H$ vanishes on $DM^c(S)^{w_{Chow} \le i-1}$.*

*4. $H$ yields naturally an (infinite) sequence of transformations of functors*

$$\dots \to (\tau_{\ge i+1}H) \circ [1] \to \tau_{\le i}H \to H \to \tau_{\ge i+1}H \to (\tau_{\le i}H) \circ [-1] \to \dots$$

*that yields a long exact sequence when applied to any $M \in ObjDM^c(S)$.*

*5. For any $j \in \mathbb{Z}$ we have a natural isomorphism $\tau_{\le i}(\tau_{\ge j}H) \cong \tau_{\ge j}(\tau_{\le i}H)$.*

*6. We have a natural isomorphism $E_2^{-ii}(T(H, M) \cong \tau_{=i}H (= \tau_{\ge i}(\tau_{\le i}H))$ (see Proposition 3.3.1 for the definition of $T(H, M)$).*

*7. For $N \in ObjDM(S)$, $H = DM^c(-, N)$ we have $\tau_{\le i}H \cong (-, t_{Chow \le i}N)$, $\tau_{\ge i}H \cong (-, t_{Chow \ge i}N)$, and $\tau_{=i}H \cong (-, t_{Chow = i}N)$*

*Proof.* Assertions 1–5 are immediate from Theorem 2.3.1. Assertion 6 is contained in Theorem 2.4.2 of ibid. Assertion 7 follows from Proposition 2.5.4 of ibid. □

*Remark* 4.3.2. 1. Note that $H^i = \tau_{=i}H$ vanishes on $DM^c(S)^{w_{Chow}=j}$ for all $j \neq i$, so $H^i$ are 'pure' (cf. Remark 4.2.2).

2. One can also describe the whole $T(H, M)$ starting from $E_2$ in terms of (various) virtual $t$-truncations of $H$; see Theorem 2.4.2 of [Bon10a].

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