Explicit formula for the action of the automorphism group of a formal group on the moduli space of its deformations.

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0. Introduction.

There are two different approaches to the study of deformations of a formal group over a finite field or, to be more precise, to the study of $\star$-isomorphism classes of such deformations, where a $\star$-isomorphism of deformations means an isomorphism with identity reduction. First of them is based on the Dieudonné theory (see [2] or [3]) that assigns to any formal group over a finite field a profinite module over the Dieudonné ring of this field providing in this manner an equivalence between the category of formal groups and the category of Dieudonné modules. For any deformation of a formal group over a finite field which is defined over the ring of Witt vectors of this field, Fontaine has constructed a submodule in the Dieudonné module of this formal group in such a way that $\star$-isomorphic deformations correspond to the same submodule (see [3]). Another approach is due to Lubin and Tate who introduced a universal deformation of a formal group over a finite field, i.e. a deformation Γ with a property that any deformation of the given formal group which is defined over any complete local algebra is $\star$-isomorphic to the formal group which results from Γ by applying some homomorphism to its coefficients (see [8]). Our aim is to compare these two approaches and to establish a kind of explicit connection between them.

Let Φ be a formal group over a finite field of finite height. We consider an action of the automorphism group of Φ on the set of $\star$-isomorphism classes of deformations of Φ which is defined by the formula $\varphi[F] = [g^{-1}(F(g,g))]$, where F is a deformation of Φ, φ is an automorphism of Φ and g is a formal power series with reduction equal to φ. One can easily check that the stabilizer and the orbit of a $\star$-isomorphism class $[F]$ are the set of the reductions of automorphisms of F and the set of $\star$-isomorphism classes of deformations isomorphic to F respectively.

According to the results of Honda [6] whose theory is parallel to the theory of Dieudonne and Fontaine, every endomorphism of Φ is equal to the reduction of the formal power series $f_0^{-1}(\sum_{i=0}^{h-1} \beta_i \Delta^i f_0)$ for some β_i belonging to the ring of Witt vectors, where h is the height of Φ, f_0 is the logarithm of a fixed deformation F_0 of Φ over the ring of Witt vectors, Δ operates on the ring of formal power series in the following way: $\Delta\left(\sum_{i=1}^{\infty} c_i x^i\right) = \sum_{i=1}^{\infty} \mathrm{Frob}(c_i) x^{pi}$,

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and different tuples $(\beta_0, ..., \beta_{h-1})$ give rise to different endomorphisms. Moreover, the endomorphism corresponding to such a tuple is the reduction of an endomorphism of F_0 if and only if $\beta_i = 0$ for all $i \neq 0$. Then the left cosets of the stabilizer of the $\star$-isomorphism class $[F_0]$ in the automorphism group of Φ can be represented by the reductions of the formal power series $f_0^{-1}(f_0 + \sum_{i=1}^{h-1} \beta_i \Delta^i f_0)$ with β_i from the ring of Witt vectors. That gives a parametrization of the orbit of the $\star$-isomorphism class $[F_0]$.

According to the results of Lubin and Tate [8], a universal deformation Γ of Φ is defined over the ring of formal power series in $h-1$ variables with coefficients in the ring of Witt vectors. A homomorphism from this ring to the ring of Witt vectors is uniquely determined by the images $\tau_1, ..., \tau_{h-1}$ of the variables which have to belong to the maximal ideal. Applying different homomorphisms to the coefficients of Γ, we get deformations of Φ belonging to different $\star$-isomorphism classes. Thus we have another parametrization of the orbit of the $\star$-somorphism class $[F_0]$. In this article, we obtain an explicit expression of one of these two parameters through another.

We use Hazewinkel's universal p-typical formal group to construct a special universal deformation which suits to further calculations. Then we consider two $\star$-isomorphic deformations of Φ, the first one is the formal group with the logarithm $f_0 + \sum_{i=1}^{h-1} \beta_i \Delta^i f_0$ and the second one is resulted from the special universal deformation by applying the homomorphism corresponding to a tuple $(\tau_1, ..., \tau_{h-1})$ to its coefficients, and use Honda's technique to prove some equalities modulo p involving the coefficients of the logarithms of these deformations. Further we define rigid analytic power series $\rho_0, ..., \rho_{h-1}$ that allow us to find exact equalities containing parameters β_i and τ_i. From them we deduce the formula $\beta_i = \frac{\rho_i}{\rho_0}(\tau_1, ..., \tau_{h-1})$ which is our main result (see Theorem 8.3).

Finally, we consider a special case when Φ is the reduction of the Artin-Hasse formal group. It has a benefit that all of the deformations of Φ over the ring of Witt vectors are isomorphic, and thus the orbit of the $\star$-isomorphism class $[F_0]$ coincides with the set of all $\star$-isomorphism classes of deformations of Φ. Moreover in this case, the general formula for β_i through τ_i can be reduced to a simpler one.

1. Lubin-Tate's moduli space of formal group deformations.

Let k be an algebraic extension of $\mathbf{F}_p$, R be the ring of Witt vectors of k and K be the fraction field of R. We consider a one-parameter formal group Φ over k of finite height h. Let A be a complete noetherian local R-algebra with maximal ideal $\mathcal{M} \supseteq pA$ and the residue field $A/\mathcal{M} \supseteq k$. A formal group F over A such that its reduction modulo $\mathcal{M}$ is equal to Φ is called a deformation of Φ over A. If an isomorphism between the deformations F and G over A has identity reduction modulo $\mathcal{M}$, we say that it is a $\star$-isomorphism. In this case the deformations F and G are called $\star$-isomorphic. The $\star$-isomorphism class of the deformation F will be denoted by $[F]$.

Lubin and Tate constructed a moduli space for $\star$-isomorphism classes of deformations of Φ. We review their main results.

The formal group Φ over k of height h is said to be in normal form if

$$\Phi(x,y) \equiv x + y + a\, C_{p^h}(x,y) \mod \deg (p^h + 1)$$

holds for some $a \in k$, where $C_{p^i}(x,y) = ((x+y)^{p^i} - x^{p^i} - y^{p^i})/p$.

Lemma 1.1 *Every formal group of finite height over k is isomorphic to a formal group in normal form.*

Proof. See [7], Lemma 6.

Suppose Φ to be in normal form.

Theorem 1.2 *There exists a formal group Γ over $R[[t_1, ..., t_{h-1}]]$ such that the reduction of $\Gamma(0, ..., 0)$ modulo p is equal to Φ and*

$$\Gamma(0, ..., 0, t_i, ..., t_{h-1})(x,y) \equiv x + y - t_i\, C_{p^i}(x,y) \mod \deg (p^i + 1)$$

for $1 \leq i \leq h-1$.

Proof. See [8], Proposition 1.1.

A formal group satisfying the conditions of Theorem 1.2 is called a generic formal group for Φ. It is not unique, because constructing it, one can make several arbitrary choices.

Theorem 1.3 *Let Γ be a generic formal group for Φ and F be a deformation of Φ over A. Then there is a unique $(h-1)$-tuple $(\tau_1, ..., \tau_{h-1})$, $\tau_i \in \mathcal{M}$, such that F is $\star$-isomorphic to $\Gamma(\tau_1, ..., \tau_{h-1})$ and the $\star$-isomorphism is uniquely defined.*

Proof. See [8], Theorem 3.1.

Theorem 1.3 means that Γ is a universal deformation of the formal group Φ, and the set of $\star$-isomorphism classes of the deformations of Φ over A is in one-to-one correspondence with $\mathcal{M} \times ... \times \mathcal{M}$ ($h-1$ times).

2. Hazewinkel's p-typical formal groups.

Hazewinkel used a universal p-typical formal group to get a parametrization of a large number of generic formal groups, which in particular allows to choose one of them canonically. We review here his construction.

Denote $E = \mathbf{Q}_p[t_1, t_2, ...]$, $D = \mathbf{Z}_p[t_1, t_2, ...]$. Define a $\mathbf{Q}_p$-endomorphism σ of E by $\sigma(t_i) = t_i^p$. Let σ operate on the ring $E[[x]]_0$ of formal power series over E without constant term by the formula

$$\sigma\left(\sum_{n=1}^{\infty} c_n x^n\right) = \sum_{n=1}^{\infty} \sigma(c_n) x^{pn}.$$

There exists a unique formal power series $f_T = \sum_{n=0}^{\infty} d_n x^{p^n} \in E[[x]]_0$ which satisfies the functional equation

$$p f_T - \sum_{i=1}^{\infty} t_i \sigma^i(f_T) = px.$$

It is clear that $d_0=1$, $d_n\in\mathbf{Q}_p[t_1,...,t_n]\subset E$ and $d_n\equiv t_n/p \bmod t_1,...,t_{n-1}$.

Proposition 2.1 *The coefficients of f_T satisfy the recurrent relation*

$$pd_n=\sum_{i=1}^{n}t_i^{p^{n-i}}d_{n-i}.$$

Proof. See [5], formula 3.3.9.

Propositon 2.2

i) *f_T is the logarithm of a formal group F_T defined over D.*

ii) *$F_T(x,y)\equiv x+y-t_i\,C_{p^i}(x,y) \bmod t_1,...,t_{i-1},\deg(p^i+1)$.*

Proof. The part i) is an immediate consequence of Hazewinkel's functional equation lemma, see [5], section 2.2.i) and formula 2.3.7. The part ii) follows from the fact that $f_T\equiv x+(t_i/p)x^{p^i} \bmod t_1,...,t_{i-1},\deg(p^i+1)$.

A formal group F over a $\mathbf{Z}_p$-algebra B is called p-typical if there is a homomorphism ξ from D to B such that $F=\xi_*F_T$. Evidently, F_T is a universal p-typical formal group.

Proposition 2.3 *Every formal group over a $\mathbf{Z}_p$-algebra B is isomorphic to a p-typical one.*

Proof. See [5], Theorem 15.2.9 or Theorem 21.5.6.i).

Lemma 2.4 *Let ξ be a homomorphism from D to k. Then the height of the formal group ξ_*F_T is equal to the minimal i satisfying $\xi(t_i)\neq 0$.*

Proof. It follows from $f_T\equiv x+(t_i/p)x^{p^i} \bmod t_1,...,t_{i-1},\deg(p^i+1)$.

Corollary 2.5 *Every p-typical formal group over k is in normal form.*

Proof. It follows from Proposition 2.2.ii) and Lemma 2.4.

Now suppose Φ to be a p-typical formal group. By Corollary 2.5, it is automatically in normal form. Take a homomorphism ξ from D to k such that $\Phi=\xi_*F_T$ and define a $\mathbf{Q}_p$-homomorphism Ξ from E to $K[[t_1,...,t_{h-1}]]$ as follows: $\Xi(t_i)=t_i$ for $i<h$; $\Xi(t_i)=r_i\in R$ for $i\geq h$, where the reduction of r_i modulo p is equal to $\xi(t_i)$. Then the formal group $\Gamma=\Xi_*F_T$ is defined over $R[[t_1,...,t_{h-1}]]$ and its reduction modulo $p,t_1,...,t_{h-1}$ is equal to Φ. Moreover by Proposition 2.2.ii), Γ is a generic formal group for Φ, and hence by Theorem 1.3, it is a universal deformation of Φ. The choice of r_i in the definition of Ξ is not unique. From now on, we fix r_i to be the multiplicative representatives of $\xi(t_i)$ in R. Then by Lemma 2.4, we have $r_i=0$ for $i<h$, r_h is invertible. This fixation determines Ξ and Γ uniquely.

It is clear that the formal power series $\gamma=\Xi_*f_T$ is the logarithm of Γ. Put $a_n=\Xi(d_n)\in K[t_1,...,t_{h-1}]$. Then we have $\gamma(x)=\sum_{n=1}^{\infty}a_nx^{p^n}$. Moreover, Proposition 2.1 implies

$$pa_n=\sum_{j=1}^{\min(n,h-1)}t_j^{p^{n-j}}a_{n-j}+\sum_{i=h}^{n}r_i^{p^{n-i}}a_{n-i}.$$

3. The action of the automorphism group on the moduli space of deformations.

Let l be an algebraic extension of k such that $A/\mathcal{M} \supseteq l$. The group of automorphisms of Φ over l acts on the set of $\star$-isomorphism classes of deformations of Φ over A in the following way. If $\varphi \in \mathrm{Aut}_l\Phi$ and F is a deformation of Φ over A, then $\varphi[F] = [g^{-1}(F(g,g))]$, where $g \in A[[x]]_0$ is a power series without constant term, the reduction of g modulo $\mathcal{M}$ is equal to φ. The definition is correct, i.e. it does not depend on the choice of F and g.

Proposition 3.1 *Let F be a deformation of Φ over A and $A/\mathcal{M} = l$. Then*
i) $\mathrm{Stab}[F]$ *is the set of the reductions of automorphisms of F over A;*
ii) $\mathrm{Orb}[F]$ *is the set of the $\star$-isomorphism classes of deformations of Φ which are isomorphic to F over A.*

Proof. i) If φ is the reduction of an automorphism g of F, then we have $g^{-1}(F(g,g)) = F$ and $\varphi[F] = [F]$. If $\varphi \in \mathrm{Stab}[F]$, then there exists a $\star$-isomorphism f from F to $g^{-1}(F(g,g))$, where $g \in A[[x]]_0$, the reduction of g is equal to φ. Then $g(f)$ is an automorphism of F and its reduction is equal to φ.
ii) If $\varphi \in \mathrm{Aut}_l\Phi$ and $g \in A[[x]]_0$, the reduction of g is equal to φ, then g provides an isomorphism between $g^{-1}(F(g,g)) \in \varphi[F]$ and F. If g is an isomorphism between a deformation G of Φ over A and F, then the reduction of g is an automorphism of Φ and it maps the class $[F]$ to the class $[G]$.

According to Theorem 1.3, we have an action of $\mathrm{Aut}_l\Phi$ on $\mathcal{M} \times ... \times \mathcal{M}$ ($h-1$ times). We are going to consider it for the case where A is the ring of Witt vectors of l, and to focus our attention on the orbit of the point $(0,...,0)$.

4. Honda's explicit description of the automorphism group.

Honda described explicitly the automorphism group of a formal group over a finite field. In this section, we review his theory.

From now on, we suppose A to be the ring of Witt vectors of l. Let L be the fraction field of A and Δ denote the Frobenius automorphism of L. The reduction from A to l modulo p will be denoted by overline.

We consider the non-commutative ring $A[[\Delta]]$ of formal power series over A with the multiplication rule $\Delta a = a^\Delta \Delta$, $a \in A$. This ring has several common properties with the standard power series ring $A[[x]]$, namely
1) a power series $s \in A[[\Delta]]$ is invertible if and only if the constant term of s is invertible in A;
2) the non-commutative version of Weierstrass' preparation lemma holds, i.e. for any $u \in A[[\Delta]]$ which is not divisible by p, there exists a unique invertible $s \in A[[\Delta]]$ such that su is a polynomial with the greatest coefficient equal to 1 and $\deg su$ is equal to the least power of Δ in the power series u having an invertible coefficient;

3) $A[[\Delta]]$ admits uniquely defined left division transformation, that means for any $w \in A[[\Delta]]$ and $u \in A[\Delta]$ with the greatest coefficient equal to 1, there exist unique $q \in A[[\Delta]]$ and $r \in A[\Delta]$ such that $w = qu + r$ and $\deg r < \deg u$.

Let Δ operate on the ring $L[[x]]_0$ of formal power series over L without constant term by the formula

$$\Delta \sum_{i=1}^{\infty} c_i x^i = \sum_{i=1}^{\infty} c_i^{\Delta} x^{pi}.$$

That determines an action of $A[[\Delta]]$ on $L[[x]]_0$.

Let $u \in A[[\Delta]]$ be such that $u \equiv p \bmod \Delta$. A power series $f \in L[[x]]_0$ is said to be of type u if $f(x) \equiv x \bmod x^2$ and $uf \equiv 0 \bmod p$. Evidently, for any $u \in A[[\Delta]]$, $u \equiv p \bmod \Delta$ and $g \in L[[x]]_0$, $g(x) \equiv x \bmod x^2$, there exists a unique $f \in L[[x]]_0$ of type u such that $uf = pg$.

Theorem 4.1 *Let $u \in A[[\Delta]]$, $u \equiv p \bmod \Delta$. Then*
i) *if $f, g \in L[[x]]_0$ are of type u, then $g^{-1}(f)$ has coefficients in A;*
ii) *if $f \in L[[x]]_0$ is of type u, then f is the logarithm of a formal group over A.*

Proof. See [6], Lemma 2.4 and Theorem 2.

Lemma 4.2 *Let $u_1, u_2 \in A[[\Delta]]$, $u_1 \equiv p \bmod \Delta$ and $f \in L[[x]]_0$ be of type u_1. If $u_2 f \equiv 0 \bmod p$, then there exists $s \in A[[\Delta]]$ such that $u_2 = s u_1$.*

Proof. See [6], Proposition 2.6.

Theorem 4.3 *Let $a \in A$, $u_i \in A[[\Delta]]$, $u_i \equiv p \bmod \Delta$ and $f_i \in L[[x]]_0$ be of type u_i for $i = 1, 2$. Then $f_2^{-1}(af_1)$ has coefficients in A if and only if there exists $z \in A[[\Delta]]$ such that $u_2 a = z u_1$.*

Proof. See [6], Theorem 3.

Theorem 4.4 *If F is a formal group over A with the logarithm f, then there exists $u \in A[[\Delta]]$ such that $u \equiv p \bmod \Delta$ and f is of type u.*

Proof. See [6], Proposition 3.3.

Lemma 4.5 *Let $u \in A[[\Delta]]$, $u \equiv p \bmod \Delta$, $f \in L[[x]]_0$ be of type u, $\psi_1 \in L[[x]]_0$ and $\psi_2 \in A[[x]]_0$. Then $f(\psi_1) \equiv f(\psi_2) \bmod p$ if and only if $\psi_1 \equiv \psi_2 \bmod p$.*

Proof. See [6], Lemma 4.2.

Theorem 4.6 *Let $w, w' \in A[[\Delta]]$, $u_i \in A[[\Delta]]$, $u_i \equiv p \bmod \Delta$, $f_i \in L[[x]]_0$ be of type u_i and F_i be the formal group with the logarithm f_i for $i = 1, 2, 3$. Then*
i) *$f_2^{-1}(wf_1)$ has coefficients in A if and only if there exists $z \in A[[\Delta]]$ such that $u_2 w = z u_1$;*
ii) *if $f_2^{-1}(wf_1)$ has coefficients in A, then $\overline{f_2^{-1}(wf_1)} \in \mathrm{Hom}_l(\overline{F_1}, \overline{F_2})$;*
iii) *if $f_2^{-1}(wf_1)$ and $f_3^{-1}(w'f_2)$ have coefficients in A, then $f_3^{-1}(w'wf_1)$ has coefficients in A and $\overline{f_3^{-1}(w'wf_1)} = \overline{f_3^{-1}(w'f_2)} \circ \overline{f_2^{-1}(wf_1)}$;*
iv) *if $\varphi \in \mathrm{Hom}_l(\overline{F_1}, \overline{F_2})$, then there exists $w \in A[[\Delta]]$ such that $f_2^{-1}(wf_1)$ has coefficients in A and $\overline{f_2^{-1}(wf_1)} = \varphi$.*

Proof. See [6], Theorem 5 and Theorem 6.

Let $u \in A[[\Delta]]$, $u \equiv p \bmod \Delta$, $f \in L[[x]]_0$ be of type u and F be the formal group with the logarithm f. Denote by $A[[\Delta]]^u$ the set of power series $w \in A[[\Delta]]$ satisfying $uw = zu$ for some $z \in A[[\Delta]]$. Evidently $A[[\Delta]]^u$ is a subring of $A[[\Delta]]$, and $A[[\Delta]]u$ is a two-sided ideal in $A[[\Delta]]^u$. Define a map μ from the ring $A[[\Delta]]^u/A[[\Delta]]u$ to $\mathrm{End}_l\overline{F}$ by the formula $\mu(w + A[[\Delta]]u) = \overline{f^{-1}(wf)}$. Due to Lemma 4.5, the definition is correct.

Proposition 4.7 *μ is a ring isomorphism.*

Proof. By Theorem 4.6.iii), μ is a homomorphism. Theorem 4.6.iv) implies that μ is surjective. If $\overline{f^{-1}(wf)} = 0$ holds for some $w \in A[[\Delta]]^u$, then due to Lemma 4.5, we have $wf \equiv 0 \bmod p$. Now Lemma 4.2 implies that there exists $s \in A[[\Delta]]$ such that $w = su$, i.e. w belongs to the zero coset in $A[[\Delta]]^u/A[[\Delta]]u$. Thus μ is an isomorphism.

Now suppose that u is not divisible by p and denote by η the least power of Δ in the power series u having an invertible coefficient. According to Weierstrass' preparation lemma, there exists a unique $s \in A[[\Delta]]$ such that su is a polynomial of degree η with the greatest coefficient equal to 1. Since $A[[\Delta]]$ admits uniquely defined left division transformation, for any $w \in A[[\Delta]]$, there exist unique $q \in A[[\Delta]]$ and $r \in A[\Delta]$ such that $w = qsu + r$ and $\deg r < \eta$. Moreover if $w \in A[[\Delta]]^u$, then $r \in A[[\Delta]]^u$; and if w is invertible, then r is also invertible. It means that the polynomials over A of degree less than η which belong to $A[[\Delta]]^u$ can be choosen as representatives of the cosets in $A[[\Delta]]^u/A[[\Delta]]u$, and the invertible cosets are represented by the polynomials with invertible constant term.

5. Correspondence between Honda's and Lubin-Tate's parameters.

Let $F_0 = \Gamma(0, ..., 0) \in R[[x, y]]$ and $f_0 = \gamma(0, ..., 0) \in K[[x]]_0$ be its logarithm. Evidently $f_0(x) = \sum_{n=0}^{\infty} a_n(0, ..., 0)x^{p^n}$. Define a $\mathbf{Q}_p$-homomorphism θ from E to $K \subseteq L$ by the formula $\theta(t_i) = r_i$. Then $F_0 = \theta_* F_T$, $f_0 = \theta_* f_T$ and $a_n(0, ..., 0) = \theta(d_n) = d_n(r_1, ..., r_n)$. Moreover we have

$$pf_0 - \sum_{i=1}^{\infty} r_i \Delta^i(f_0) = pf_0 - \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} r_i a_n(0, ..., 0)^{\Delta^i} x^{p^{i+n}} =$$

$$= pf_0 - \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} r_i d_n(r_1^{p^i}, ..., r_n^{p^i})x^{p^{i+n}} = \theta_*(pf_T - \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} t_i d_n(t_1^{p^i}, ..., t_n^{p^i})x^{p^{i+n}}) =$$

$$= \theta_*(pf_T - \sum_{i=1}^{\infty} t_i \sigma^i(f_T)) = px,$$

i.e. f_0 is of type $u_0 = p - \sum_{i=1}^{\infty} r_i \Delta^i$.

Since $\overline{F_0} = \Phi$, $r_i = 0$ for $i < h$ and r_h is invertible, Proposition 4.7 and the subsequent remark imply that $\mathrm{Aut}_l\Phi$ is in one-to-one correspondence with the set of polynomials from $A[[\Delta]]^{u_0}$ of degree less than h with invertible constant term. Let w be such a polynomial. Then w corresponds to the reduction of an automorphism of F_0 if and only if it is a constant. Indeed, if w is a constant, then by Theorem 4.3, $f_0^{-1}(wf_0)$ is an automorphism of F_0 and $\mu(w + A[[\Delta]]u_0) = \overline{f_0^{-1}(wf_0)}$. Conversely, if $\mu(w + A[[\Delta]]u_0) = \overline{g}$ for some automorphism g of F_0, we denote by a the first coefficient of g and observe that $\mu(a + A[[\Delta]]u_0) = \overline{f_0^{-1}(af_0)} = \overline{g}$. Then by Proposition 4.7, $w \in a + A[[\Delta]]u_0$, and hence $w = a$. Now applying Proposition 3.1.i), we get that the set of polynomials from $A[[\Delta]]^{u_0}$ of degree less than h with constant term equal to one is in one-to-one correspondence with the set of left cosets of $\mathrm{Stab}[F_0]$ in $\mathrm{Aut}_l\Phi$ which is in one-to-one correspondence with $\mathrm{Orb}[F_0]$. Taking into account Theorem 1.3 and Proposition 3.1.ii), we get

Theorem 5.1 *There is a canonical correspondence between the set of the polynomials of the form $1 + \sum_{\kappa=1}^{h-1} \beta_\kappa \Delta^\kappa$, $\beta_\kappa \in A$, which belong to $A[[\Delta]]^{u_0}$ and the set of the $(h-1)$-tuples $(\tau_1, ..., \tau_{h-1})$, $\tau_i \in pA$, such that $\Gamma(\tau_1, ..., \tau_{h-1})$ is isomorphic to F_0 over A.*

We introduce an additional assumption that we will call assumption $(*)$:

There exists an invertible $v \in A[[\Delta]]$ such that vu_0 has non-zero coefficients only at the powers of Δ which are multiples of the degree of l over $\mathbf{F}_p$.

The assumption $(*)$ is satisfied if, for example, $l = \mathbf{F}_p$ or if $u_0 = p - \Delta^h$ and $l = \mathbf{F}_{p^h}$.

If the assumption $(*)$ is satisfied, then vu_0 belongs to the center of $A[[\Delta]]$. It implies that $A[[\Delta]]^{u_0} = A[[\Delta]]$, because for any $w \in A[[\Delta]]$, we have $u_0 w = v^{-1}vu_0 w = v^{-1}wvu_0$. Moreover in this case, any deformation F of Φ over A is isomorphic to F_0. Indeed, by Theorem 4.4, the logarithm f of F is of type u for some $u \in A[[\Delta]]$, $u \equiv p \bmod \Delta$. Then by Theorem 4.6.iv), the identity morphism from $\overline{F} = \Phi$ to $\overline{F_0} = \Phi$ is equal to $\overline{f_0^{-1}(wf)}$ for some invertible $w, z \in A[[\Delta]]$ satisfying $u_0 w = zu$. We have $zu = v^{-1}vu_0 w = v^{-1}wvu_0$. Hence $u_0 f = v^{-1}w^{-1}vzuf \equiv 0 \bmod p$, i.e. f is of type u_0, and by Theorem 4.1.i), F is isomorphic to F_0. Thus we get

Theorem 5.2 *If the assumption $(*)$ is satisfied, there is a canonical correspondence between the set of the polynomials of the form $1 + \sum_{\kappa=1}^{h-1} \beta_\kappa \Delta^\kappa$, $\beta_\kappa \in A$, and $pA \times ... \times pA$ $(h-1$ times).*

Our aim is to calculate the correspondence mentioned in Theorem 5.1 and Theorem 5.2 explicitly.

6. The ring of rigid analytic functions.

We define the ring of rigid analytic function $L\{\{t_1, ..., t_{h-1}\}\}$ as the subring of $L[[t_1, ..., t_{h-1}]]$ consisting of the formal power series

$$f = \sum_{I=(i_1,...,i_{h-1}) \in \mathbf{N}^{h-1}} f_I t^I$$

such that $\lim\limits_{|I|\to\infty} \nu(f_I) + |I| = \infty$, where $f_I = f_{i_1,\dots,i_{h-1}}$, $t^I = t_1^{i_1} t_2^{i_2} \dots t_{h-1}^{i_{h-1}}$, $|I| = i_1 + \dots + i_{h-1}$ and ν is the normalized valuation of L. Such power series converge if the variables t_i, $1 \le i \le h-1$, take value in pA, i.e. rigid analitic functions can be evaluated on the set $pA \times \dots \times pA$ ($h-1$ times).

Define a norm on $L\{\{t_1, \dots, t_{h-1}\}\}$ by

$$\|f\| = p^{-\min_{I\in \mathbf{N}^{h-1}} \{\nu(f_I)+|I|\}}.$$

This norm provides to $L\{\{t_1, \dots, t_{h-1}\}\}$ the structure of a Banach L-algebra (see [1], section 6.1.5, Proposition 1). If a sequence of rigid analitic functions converges to $f \in L\{\{t_1, \dots, t_{h-1}\}\}$, then the sequence of their values at a point in $pA \times \dots \times pA$ ($h-1$ times) converges to the value of f at this point.

Proposition 6.1 *Let $f \in L\{\{t_1, \dots, t_{h-1}\}\}$, $f \equiv 1 \bmod t_1, \dots, t_{h-1}$ and $\|f - 1\| < 1$, i.e. $\nu(f_J) + |J| > 0$ for any $J \in \mathbf{N}^{h-1}$, $J \ne (0, \dots, 0)$. Then f is invertible in $L\{\{t_1, \dots, t_{h-1}\}\}$.*

Proof. We have $f = 1 + \sum\limits_{J\in\mathbf{N}^{h-1}, J\ne(0,\dots,0)} f_J t^J$. We are going to prove that the formal power series $g = 1 - \sum f_J t^J + (\sum f_J t^J)^2 - \dots \in L[[t_1, \dots, t_{h-1}]]$ belongs to $L\{\{t_1, \dots, t_{h-1}\}\}$. If we write g in the form $g = 1 + \sum\limits_{I\in\mathbf{N}^{h-1}, I\ne(0,\dots,0)} g_I t^I$, then we have

$$g_I = \sum_{J_1+\dots+J_n=I, J_i\ne(0,\dots,0)} (-1)^n f_{J_1} f_{J_2} \dots f_{J_n},$$

and hence

$$\nu(g_I) + |I| \le \min_{J_1+\dots+J_n=I, J_i\ne(0,\dots,0)} \{\sum_{i=1}^{n} (\nu(f_{J_i}) + |J_i|)\}.$$

Let $M > 0$ and choose N such that $\nu(f_J) + |J| > M$ for any $J \in \mathbf{N}^{h-1}$ with $|J| > N$. Consider $I \in \mathbf{N}^{h-1}$ satisfying $|I| > MN$ and let $J_1, \dots, J_n \in \mathbf{N}^{h-1}$ be such that $J_1 + \dots + J_n = I$, $J_i \ne (0, \dots, 0)$. If $n \le M$, then there exists j such that $|J_j| > N$ and hence $\sum_{i=1}^n (\nu(f_{J_i}) + |J_i|) \ge \nu(f_{J_j}) + |J_j| > M$. If $n > M$, then $\sum_{i=1}^n (\nu(f_{J_i}) + |J_i|) \ge n > M$. Thus we get that $\nu(g_I) + |I| > M$ holds for any $I \in \mathbf{N}^{h-1}$ with $|I| > MN$, i.e. g belongs to $L\{\{t_1, \dots, t_{h-1}\}\}$.

7. The rigid analytic function ρ.

Let $L\{y\}_p$ denote the subring of $L[[y]]$ consisting of the formal power series $\sum_{n=0}^\infty c_n y^{p^n}$ such that $\lim c_n = 0$. Let Δ operate on $L\{y\}_p$ by the formula

$$\Delta \sum_{n=0}^\infty c_n y^{p^n} = \sum_{n=0}^\infty c_{n+1}^\Delta y^{p^n}.$$

That determines an action of $A[[\Delta]]$ on $L\{y\}_p$. Because of Weierstrass' preparation lemma, there exists an invertible element $s_0 \in A[[\Delta]]$ such that $s_0 u_0 =$

$\Delta^h - \pi_{h-1}\Delta^{h-1} - \ldots - \pi_1\Delta - \pi_0$, where $\pi_i \in pA$. Take arbitrary elements $\zeta_0, \ldots, \zeta_{h-1} \in \mathbf{Z}_p$ and define $\zeta_n \in A$ for $n \geq h$ by the recurrent formula

$$\zeta_n = \sum_{i=1}^{h} \pi_{h-i}^{\Delta^{-h}} \zeta_{n-i}^{\Delta^{-i}}.$$

It is clear that $\nu(\zeta_n) \geq [n/h]$. In paticular, $\lim \zeta_n = 0$.

Proposition 7.1 $p\zeta_n = \sum_{i=1}^{\infty} r_i \zeta_{n+i}^{\Delta^i}$.

Proof. Denote $\psi = \sum_{n=0}^{\infty} \zeta_n y^{p^n} \in L\{y\}_p$. The definition of ζ_n implies

$$\zeta_{n+h}^{\Delta^h} - \pi_{h-1}\zeta_{n+h-1}^{\Delta^{h-1}} - \ldots - \pi_1\zeta_{n+1}^{\Delta} - \pi_0\zeta_n = 0$$

for any $n \geq 0$. It follows that $(\Delta^h - \pi_{h-1}\Delta^{h-1} - \ldots - \pi_1\Delta - \pi_0)\psi = 0$. Therefore $u_0\psi = s_0^{-1}s_0u_0\psi = 0$, i.e. $p\zeta_n - \sum_{i=0}^{\infty} r_i\zeta_{n+i}^{\Delta^i} = 0$ holds for any $n \geq 0$.

Define $\alpha_m^{(n)} \in A$ for $m, n \in \mathbf{N}$ as follows: $\alpha_0^{(n)} = p\zeta_n$, $\alpha_m^{(n)} = \alpha_{m-1}^{(n)} - r_m\zeta_{n+m}^{\Delta^m}$ for $m \geq 1$. Proposition 7.1 implies that $\alpha_m^{(n)} = \sum_{i=m+1}^{\infty} r_i\zeta_{n+i}^{\Delta^i}$, and hence $\nu(\alpha_m^{(n)}) \geq [(m+n)/h]$. Define

$$b_m^{(n)} = \sum_{i=0}^{m} \alpha_{m-i}^{(n+i)\Delta^i} a_i \in L[t_1, \ldots, t_{h-1}].$$

Proposition 7.2 $b_m^{(n)} = p\zeta_n + \sum_{i=1}^{m} \sum_{j=1}^{\min(i,h-1)} \zeta_{n+i}^{\Delta^i} t_j^{p^{i-j}} a_{i-j}$.

Proof. Evidently $b_0^{(n)} = \alpha_0^{(n)} a_0 = p\zeta_n$. Moreover calculating the difference $b_m^{(n)} - b_{m-1}^{(n)}$, we get

$$b_m^{(n)} - b_{m-1}^{(n)} = \alpha_0^{(n+m)\Delta^m} a_m + \sum_{i=1}^{m} \left(\alpha_i^{(n+m-i)} - \alpha_{i-1}^{(n+m-i)}\right)^{\Delta^{m-i}} a_{m-i} =$$

$$= p\zeta_{n+m}^{\Delta^m} a_m - \sum_{i=1}^{m} \left(r_i\zeta_{n+m}^{\Delta^i}\right)^{\Delta^{m-i}} a_{m-i} = \zeta_{n+m}^{\Delta^m}\left(pa_m - \sum_{i=1}^{m} r_i^{\Delta^{m-i}} a_{m-i}\right) =$$

$$= \zeta_{n+m}^{\Delta^m}\left(pa_m - \sum_{i=h}^{m} r^{p^{m-i}} a_{m-i}\right) = \zeta_{n+m}^{\Delta^m} \sum_{j=1}^{\min(m,h-1)} t_j^{p^{m-j}} a_{m-j}.$$

Summing up, we obtain the required equality.

Proposition 7.3 *The sequence of polynomials $b_m^{(0)}$ converges in $L\{\{t_1, \ldots, t_{h-1}\}\}$ to the formal power series*

$$\rho = p\zeta_0 + \sum_{i=0}^{\infty} \sum_{j=1}^{\min(i,h-1)} \zeta_i^{\Delta^i} t_j^{p^{i-j}} a_{i-j} = p\zeta_0 + \sum_{i=0}^{\infty} \sum_{j=1}^{h-1} \zeta_{i+j}^{\Delta^{i+j}} t_j^{p^i} a_i.$$

Proof. The recurrent formula for a_m implies that $p^m a_m \in A[t_1, ...t_{h-1}]$. Therefore $\| \sum_{j=1}^{\min(m,h-1)} \zeta_m^{\Delta^m} t_j^{p^{m-j}} a_{m-j} \| \le p^{m-p^{m-h+1}}$, i.e $b_m^{(0)}$ is a Cauchy sequence.

8. Explicit calculation of the correspondence.

Let $w = 1 + \sum_{\kappa=1}^{h-1} \beta_\kappa \Delta^\kappa \in A[[\Delta]]^{u_0}$. Then the power series wf_0 is of type u_0, and, by Theorem 4.1.ii), it is the logarithm of the formal group F_w defined over A. Due to Theorem 4.6.i),ii), the power series $f_0^{-1}(wf_0)$ has coefficients in A, and its reduction is an automorphism of Φ. By the definition of the action of the automorphism group of Φ on the set of $\star$-isomorphism classes of deformations of Φ, this automorphism maps the class $[F_0]$ to the class $[F_w]$. According to Theorem 1.3, there exists a unique $(h-1)$-tuple $(\tau_1, ..., \tau_{h-1})$, $\tau_i \in pA$, such that the formal group $F = \Gamma(\tau_1, ..., \tau_{h-1})$ belongs to the same $\star$-isomorphism class that F_w does. Denote by f the logarithm of F and by ε the first coefficient of the $\star$-isomorphism from F_w to F. Then we obtain $f^{-1}(\varepsilon w f_0) \equiv x \bmod p$. Now Lemma 4.5 implies that $f \equiv \varepsilon w f_0 \bmod p$. We have $f(x) = \sum_{i=0}^{\infty} a_i(\tau_1, ..., \tau_{h-1}) x^{p^i}$. On the other hand $wf_0(x) = \sum_{i=0}^{\infty} (a_i(0, ..., 0) + \sum_{\kappa=1}^{\min(i,h-1)} \beta_\kappa a_{i-\kappa}(0, ..., 0)^{\Delta^\kappa}) x^{p^i}$. Thus we get that $a_i(\tau_1, ..., \tau_{h-1}) \equiv \varepsilon(a_i(0, ..., 0) + \sum_{\kappa=1}^{\min(i,h-1)} \beta_\kappa a_{i-\kappa}(0, ..., 0)^{\Delta^\kappa}) \bmod p$ holds for any i.

Proposition 8.1 $\rho(\tau_1, ..., \tau_{h-1}) = p\varepsilon(\zeta_0 + \sum_{\kappa=1}^{h-1} \beta_\kappa \zeta_\kappa)$.

Proof. Since $\nu(\alpha_m^{(n)}) \ge [(m+n)/h]$, we can deduce that

$$\alpha_{m-i}^{(i)\Delta^i} a_i(\tau_1, ..., \tau_{h-1}) \equiv \varepsilon \alpha_{m-i}^{(i)\Delta^i} (a_i(0, ..., 0) + \sum_{\kappa=1}^{\min(i,h-1)} \beta_\kappa a_{i-\kappa}(0, ..., 0)^{\Delta^\kappa}) \bmod p^{[m/h]+1}$$

holds for any $i \le m$. Then we get

$$b_m^{(0)}(\tau_1, ..., \tau_{h-1}) = \sum_{i=0}^{m} \alpha_{m-i}^{(i)\Delta^i} a_i(\tau_1, ..., \tau_{h-1}) \equiv$$

$$\equiv \varepsilon \sum_{i=0}^{m} \alpha_{m-i}^{(i)\Delta^i} \left(a_i(0, ..., 0) + \sum_{\kappa=1}^{\min(i,h-1)} \beta_\kappa a_{i-\kappa}(0, ..., 0)^{\Delta^\kappa} \right) =$$

$$= \varepsilon \left(\sum_{i=0}^{m} \alpha_{m-i}^{(i)\Delta^i} a_i(0, ..., 0) + \sum_{\kappa=1}^{h-1} \sum_{i=0}^{m-\kappa} \beta_\kappa \alpha_{m-i-\kappa}^{(i+\kappa)\Delta^{i+\kappa}} a_i(0, ..., 0)^{\Delta^\kappa} \right) =$$

$$= \varepsilon \left(b_m^{(0)}(0, ..., 0) + \sum_{\kappa=1}^{h-1} \beta_\kappa b_{m-\kappa}^{(\kappa)}(0, ..., 0)^{\Delta^\kappa} \right) \bmod p^{[m/h]+1}.$$

Proposition 7.2 implies $b^{(\kappa)}_{m-\kappa}(0,...,0) = p\zeta_\kappa$. Therefore

$$b^{(0)}_m(\tau_1,...,\tau_{h-1}) \equiv p\varepsilon(\zeta_0 + \sum_{\kappa=1}^{h-1}\beta_\kappa\zeta_\kappa) \bmod p^{[m/h]+1}.$$

It means that the sequence $b^{(0)}_m(\tau_1,...,\tau_{h-1}) \in L$ converges to $p\varepsilon(\zeta_0+\sum_{\kappa=1}^{h-1}\beta_\kappa\zeta_\kappa)$. On the other hand, it converges to $\rho(\tau_1,..,\tau_{h-1})$. Thus we obtain the required equality.

Now we are going to specify the sequence ζ_n. For $0 \le \kappa \le h-1$, define the sequences $\zeta_{n,\kappa}$ by fixing their first h elements as follows: $\zeta_{n,\kappa} = \delta_{n,\kappa}$, $0 \le n \le h-1$, where $\delta_{n,\kappa}$ is Kronecker's delta. Then we can define $\alpha^{(n)}_{m,\kappa} \in A$, $b^{(n)}_{m,\kappa} \in L[t_1,...,t_{h-1}]$ and $\rho_\kappa \in L\{\{t_1,...,t_{h-1}\}\}$ for these specified sequences $\zeta_{n,\kappa}$.

Lemma 8.2 *ρ_0 is invertible in $L\{\{t_1,...,t_{h-1}\}\}$.*

Proof. $\rho_0 \equiv p\zeta_{0,0} = p \bmod t_1,...,t_{h-1}$. We will prove that ρ_0/p is invertible in $L\{\{t_1,...,t_{h-1}\}\}$. By Proposition 6.1, it is enough to show that $\|\rho_0/p-1\| < 1$. Since $\zeta_{n,0} = 0$ for $1 \le n \le h-1$ and $\nu(\zeta_{n,o}) \ge [n/h]$, we obtain $\nu(\zeta_{n,o}) \ge 1$ for any $n \ge 1$. From the recurrent formula for a_i, it follows that $\|a_i\| \le p^i$. Therefore $\|\sum_{j=1}^{h-1}\zeta^{\Delta^{i+j}}_{i+j,0}t_j^{p^i}a_i\| \le p^{-1-p^i+i} \le p^{-2}$ for any $i \ge 0$, and thus $\|\rho_0 - p\| \le p^{-2}$, i.e. $\|\rho_0/p-1\| \le p^{-1} < 1$.

Theorem 8.3 $\beta_\kappa = \frac{\rho_\kappa}{\rho_0}(\tau_1,...,\tau_{h-1})$. *The element that corresponds to the tuple $(\tau_1,...,\tau_{h-1}) \in pA \times ... \times pA$ in the sense of Theorem 5.1 is equal to $1+\sum_{\kappa=1}^{h-1}\frac{\rho_\kappa}{\rho_0}(\tau_1,..,\tau_{h-1})\Delta^\kappa$.*

Proof. Applying Proposition 8.1 to the sequence $\zeta_{n,0}$, we get the equality $p\varepsilon = \rho_0(\tau_1,...,\tau_{h-1})$, applying it to the sequence $\zeta_{n,\kappa}$ for $1 \le \kappa \le h-1$, we get $p\varepsilon\beta_\kappa = \rho_\kappa(\tau_1,...,\tau_{h-1})$. That implies $\beta_\kappa = \rho_\kappa(\tau_1,...\tau_{h-1})/\rho_0(\tau_1,...,\tau_{h-1})$.

9. The reduction of the Artin-Hasse formal group.

In this section, we consider a special case when Φ is the reduction of the Artin-Hasse formal group over $\mathbf{Z}_p$ of height h. We put $\Phi = \xi_* F_T$, where the $\mathbf{Q}_p$-homomorphism ξ from D to $\mathbf{F}_p$ is defined by $\xi(t_i) = \delta_{i,h}$. Then we obtain $r_i = \delta_{i,h}$, and the $\mathbf{Q}_p$-homomorphism Ξ from D to $\mathbf{Q}_p[[t_1,...,t_{h-1}]]$ is defined as follows: $\Xi(t_i) = t_i$ for $i < h$, $\Xi(t_h) = 1$ and $\Xi(t_i) = 0$ for $i > h$. The recurrent formula on the coefficients a_n of the logarithm γ of the formal group $\Gamma = \Xi_* F_T$ turns to be $pa_n = \sum_{j=1}^n t_j^{p^{n-j}}a_{n-j}$ for $n < h$ and $pa_n = \sum_{j=1}^{h-1} t_j^{p^{n-j}}a_{n-j} + a_{n-h}$ for $n \ge h$. It implies that $a_{mh}(0,...,0) = p^{-m}$, $a_n(0,...,0) = 0$ for $n \ne 0 \bmod h$. Then the logarithm f_0 of the formal group $F_0 = \Gamma(0,...,0)$ is equal to $\sum_{n=0}^\infty a_n(0,...,0)x^{p^n} = \sum_{m=0}^\infty p^{-m}x^{p^{mh}}$, i.e. F_0 is the Artin-Hasse formal group of height h.

In this case, it makes sense to consider $l = \mathbf{F}_{p^h}$, because all of the automorphisms of Φ over $\mathbf{F}_p^{\mathrm{alg}}$ are defined over $\mathbf{F}_{p^h}$ (that can be easily proven with the aid of Theorem 4.6.i),iv). Then the assumption $(*)$ is satisfied. Furthermore we note $u_0 = p - \Delta^h$, $s_0 = -1$, $\pi_i = p\delta_{i,0}$ and $\zeta_n = p\zeta_{n-h}^{\Delta^{-h}}$ if $n \geq h$. In particular for $0 \leq \kappa \leq h-1$, it implies $\zeta_{mh+\kappa,\kappa} = p^m$, $\zeta_{n,\kappa} = 0$ if $n \neq \kappa \bmod h$. Now calculating $b^{(0)}_{mh+\kappa,\kappa}$, we get

$$b^{(0)}_{mh+\kappa,\kappa} = p\zeta_{0,\kappa} + \sum_{i=1}^{mh+\kappa} \sum_{j=1}^{\min(i,h-1)} \zeta_{i,\kappa}^{\Delta^i} t_j^{p^{i-j}} a_{i-j} =$$

$$= p\delta_{0,\kappa} + \sum_{j=1}^{\kappa} t_j^{p^{\kappa-j}} a_{\kappa-j} + \sum_{i=1}^{m} \sum_{j=1}^{h-1} p^i t_j^{p^{ih+\kappa-j}} a_{ih+\kappa-j} =$$

$$= p\delta_{0,\kappa} + (1-\delta_{0,\kappa})pa_\kappa + \sum_{i=1}^{m} p^i (pa_{ih+\kappa} - a_{(i-1)h+\kappa}) =$$

$$= pa_\kappa + p^{m+1}a_{mh+\kappa} - pa_\kappa = p^{m+1}a_{mh+\kappa}.$$

Then $\rho_\kappa = \lim\limits_m p^{m+1}a_{mh+\kappa}$ and $\beta_\kappa = \lim\limits_m \frac{a_{mh+\kappa}}{a_{mh}}(\tau_1, ..., \tau_{h-1})$. Thus we obtain

Theorem 9.1 *If Φ is the reduction of the Artin-Hasse formal group of height h and $l = \mathbf{F}_{p^h}$, then $\beta_\kappa = \lim\limits_m \frac{a_{mh+\kappa}}{a_{mh}}(\tau_1, ..., \tau_{h-1})$ and the element that corresponds to the tuple $(\tau_1, ..., \tau_{h-1}) \in pA \times ... \times pA$ in the sence of Theorem 5.2 is equal to $1 + \sum_{\kappa=1}^{h-1} \lim\limits_m \frac{a_{mh+\kappa}}{a_{mh}}(\tau_1, ..., \tau_{h-1})\Delta^\kappa$.*

This result can be derived from the considerations of Gross and Hopkins [4].

References.

[1] S. Bosch, U. Güntzer, R. Remmert, Non-Archimedian Analysis, Springer-Verlag, 1984.

[2] J. Dieudonné, Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique $p > 0$, *Math. Ann. 134*, 1957, pp. 114-133.

[3] J.-M. Fontaine, Groupes p–divisibles sur les corps locaux, *Astérisque 47-48*, 1977.

[4] B.H. Gross, M.J. Hopkins, Equivariant vector bundles on the Lubin-Tate moduli space, *Contemp. Math. 158*, 1994, pp. 23-88.

[5] M. Hazewinkel, Formal groups and aplications, Academic Press, 1978.

[6] T. Honda, On the theory of commutative formal groups, *Jour. Math. Soc. Japan 22, No. 2*, 1970, pp.213-246.

[7] M. Lazard, Sur les groupes de Lie formels à un paramètre, *Bull. Soc. math. France 83*, 1955, pp. 251-274.

[8] J. Lubin, J. Tate, Formal moduli for one-parameter formal Lie group, *Bull. Soc. math. France 94*, 1966, pp. 49-60.