LEFT CELLS IN THE AFFINE WEYL GROUP OF TYPE $\widetilde{F}_4$

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ABSTRACT. By applying an algorithm designed before, we complete the description for all the left cells of the affine Weyl group W_a of type $\widetilde{F}_4$ by finding a representative set of its left cells together with all its left cell graphs (or with all the associated essential graphs) in each of its two-sided cells. The generalized τ-invariants of left cells of W_a are exhibited graphically. A group-theoretical interpretation is given on the numbers of left cells of W_a in some two-sided cells. Thus so far the left cells of all the affine Weyl groups of ranks less than or equal to 4 have been known explicitly. Some techniques are developed in applying the algorithm. As a consequence, we complete the verification of a conjecture concerning the characterization of left cells of Weyl groups and affine Weyl groups.

It is designed in [21] and then improved in [24] for an algorithm of finding a representative set of left cells (an l.c.r. set for short) of W in a two-sided cell, where W is a Coxeter group belonging to a certain family of crystallographic groups, the latter includes all the Weyl groups and all the affine Weyl groups. By applying this algorithm, I described all the left cells of the affine Weyl groups of types $\widetilde{C}_4$ and $\widetilde{D}_4$, and also all the left cells Γ with $a(\Gamma) = 3, 4, 5$ in the affine Weyl group of type $\widetilde{F}_4$ (see [24], [22], [21]). Subsequently, three of my students, Zhang Xin-fa, Rui He-bin and Tong Chang-qing, achieved some progress on this respect also by applying this algorithm, where Zhang

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gave an explicit description for all the left cells in the affine Weyl group of type $\widetilde{B}_4$ [27], Rui for all the left cells Γ with $a(\Gamma) = 3$ in any irreducible affine Weyl group [14], and Tong for all the left cells of the Weyl group of type E_6 [26]. In the present paper, we shall apply this algorithm to complete the description of the left cells of the affine Weyl group of type $\widetilde{F}_4$. This, together with the earlier results of the others [15], [10], [1], [6], completes the description of the left cells for all the affine Weyl groups of ranks ≤ 4. Some techniques are developed in applying the algorithm (see sections 3 and 4). We find a representative set of left cells together with its left cell graphs (or with all the associated essential graphs) in each two-sided cell of W_a. The generalized τ-invariants of left cells of W_a are exhibited graphically. A group-theoretical interpretation is given on the numbers of left cells of W_a in some two-sided cells, which involves both the Lusztig map and Bala-Carter correspondence among two-sided cells of W_a, unipotent conjugacy classes of the complex algebraic group G of type F_4, and the G-classes of pairs $(L, P_{L'})$, where L is a Levi subgroup of G, $P_{L'}$ is a distinguished parabolic subgroup of semisimple part L' of L. As a consequence, we shall complete the verification of a conjecture concerning the characterization of left cells of Weyl groups and affine Weyl groups which was proposed in [21] and partly verified in [23].

The content of the paper is organized as follows. Some known results on cells of a Coxeter group, in particular of an affine Weyl group W_a are stated in section 1. Then in section 2, we recall the algorithm of finding an l.c.r. set of W_a in a two-sided cell and also state some results and terminologies which are needed in applying the algorithm. In section 3, some techniques of applying the algorithm are developed, which will be frequently used for finding an l.c.r. set of the affine Weyl group $W_a(\widetilde{F}_4)$ in a given two-sided cell. We illustrate them by several examples. We find an l.c.r. set together with all the left cell graphs (or with the corresponding essential graphs) for $W_a(\widetilde{F}_4)$ in section 4. Finally, in section 5, we complete the verification of the above-mentioned conjecture.

§1. Some results on cells.

1.1 Let $W = (W, S)$ be a Coxeter group with S its Coxeter generator set. Let $\leq$ be the Bruhat order on W. For $w \in W$, we denote by $\ell(w)$ the length of w. Let $A = \mathbb{Z}[u]$ be the ring of polynomials in an indeterminate u with integer coefficients. For each ordered pair $y, w \in W$, there exists a unique polynomial $P_{y,w} \in A$, called a Kazhdan-Lusztig polynomial, which satisfies the conditions: $P_{y,w} = 0$ if $y \not\leq w$, $P_{w,w} = 1$, and $\deg P_{y,w} \leq (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$. Let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$ for $y < w$. We denote y—w if $\mu(y, w) \neq 0$.

Checking the relation y—w for $y, w \in W$ usually involves very complicated computation of Kazhdan-Lusztig polynomials. But it becomes easy in some special case: if $x, y \in W$ satisfy $y < x$ and $\ell(y) = \ell(x) - 1$, then we have y—x. Another result concerning this relation will be stated in Proposition 2.7.

1.2 The preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ on W and the associated equivalence relations $\underset{\mathrm{L}}{\sim}$, $\underset{\mathrm{R}}{\sim}$, $\underset{\mathrm{LR}}{\sim}$ on W are defined as in [7]. The equivalence classes for $\underset{\mathrm{L}}{\sim}$ (resp. $\underset{\mathrm{R}}{\sim}$, $\underset{\mathrm{LR}}{\sim}$) on W are called left cells (resp. right cells, two-sided cells).

1.3 An affine Weyl group W_a is a Coxeter group which can be realized geometrically as follows. Let G be a connected, adjoint reductive algebraic group over $\mathbb{C}$. We fix a maximal torus $T \subset G$. Let X be the character group of T and let $\Phi \subset X$ be the root set with $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ a choice of simple root system. Then $E = X \otimes_{\mathbb{Z}} \mathbb{R}$ is a euclidean space with an inner product $\langle\ ,\ \rangle$ such that the Weyl group (W_0, S_0) of G with respect to T acts naturally on E and preserves its inner product, where S_0 is the set of simple reflections s_i corresponding to the simple roots α_i, $1 \leq i \leq \ell$. We denote by N the group of all translations T_λ ($\lambda \in X$) on E: T_λ sends x to $x + \lambda$,. Then the semidirect product $W_a = W_0 \ltimes N$ is called an affine Weyl group. Let K be the dual of the type of G. Then we define the type of W_a by $\widetilde{K}$. Sometimes we denote W_a by $W_a(\widetilde{K})$ to indicate its type $\widetilde{K}$. There is a canonical homomorphism from W_a to W_0: $w \mapsto \bar{w}$.

Let $-\alpha_0$ be the highest short root in Φ. We define $s_0 = s_{\alpha_0} T_{-\alpha_0}$, where s_{α_0} is the reflection corresponding to α_0. Then the generator set of W_a can be taken as $S = S_0 \cup \{s_0\}$.

1.4 The alcove form of an element $w \in W_a$ is, by definition, a Φ-tuple $(k(w, \alpha))_{\alpha\in\Phi}$

over $\mathbb{Z}$ subject to the following conditions.

(a) $k(w, -\alpha) = -k(w, \alpha)$ for any $\alpha \in \Phi$;

(b) $k(e, \alpha) = 0$ for any $\alpha \in \Phi$, where e is the identity element of W_a;

(c) If $w' = ws_i$ ($0 \leq i \leq \ell$), then

$$k(w', \alpha) = k(w, (\alpha)\bar{s}_i) + \epsilon(\alpha, i)$$

with

$$\epsilon(\alpha, i) = \begin{cases} 0 & \text{if } \alpha \neq \pm\alpha_i; \\ -1 & \text{if } \alpha = \alpha_i; \\ 1 & \text{if } \alpha = -\alpha_i, \end{cases}$$

where $\bar{s}_i = s_i$ if $1 \leq i \leq \ell$, and $\bar{s}_0 = s_{\alpha_0}$ (see [16, Proposition 4.2]).

By condition (a), we can also denote the alcove form of $w \in W_a$ by a Φ^+-tuple $(k(w, \alpha))_{\alpha \in \Phi^+}$.

1.5 Condition 1.4, (c) actually defines a set of operators $\{s_i \mid 0 \leq i \leq \ell\}$ on the alcove forms of elements of W_a:

$$s_i : \quad (k_\alpha)_{\alpha \in \Phi} \quad \longmapsto \quad (k_{(\alpha)\bar{s}_i} + \epsilon(\alpha, i))_{\alpha \in \Phi}.$$

These operators could be described graphically. Assume that W_a has type $\widetilde{F}_4$ and that the indices of simple roots are compatible with the following Dynkin diagram:

We denote a root $\alpha = \sum_{i=1}^{4} a_i \alpha_i$ by its coordinate form (a_1, a_2, a_3, a_4) and arrange the entries of a Φ^+-tuple $(k_\alpha)_{\alpha \in \Phi^+}$ in the following way.

$$(1.5.1) \qquad \begin{array}{cccc} & k_{(2,4,3,2)} & & k_{(2,3,2,1)} \\ & k_{(2,4,3,1)} & & k_{(1,3,2,1)} \\ & k_{(2,4,2,1)} & & k_{(1,2,2,1)} \\ & k_{(2,2,2,1)} & & k_{(1,2,1,1)} \\ k_{(2,2,1,1)} & k_{(0,2,2,1)} & k_{(1,2,1,0)} & k_{(1,1,1,1)} \\ k_{(0,2,1,1)} & k_{(2,2,1,0)} & k_{(0,1,1,1)} & k_{(1,1,1,0)} \\ k_{(0,0,1,1)} & k_{(0,2,1,0)} & k_{(0,1,1,0)} & k_{(1,1,0,0)} \\ k_{(0,0,0,1)} & k_{(0,0,1,0)} & k_{(0,1,0,0)} & k_{(1,0,0,0)} \end{array}$$

Then the actions of s_i, $0 \leq i \leq 4$, on a Φ^+-tuple

$$w = \begin{array}{cccc} \multicolumn{2}{c}{a} & \multicolumn{2}{c}{m} \\ \multicolumn{2}{c}{b} & \multicolumn{2}{c}{n} \\ \multicolumn{2}{c}{c} & \multicolumn{2}{c}{p} \\ \multicolumn{2}{c}{d} & \multicolumn{2}{c}{q} \\ e & f & r & s \\ g & h & t & u \\ i & j & v & w \\ k & l & x & y \end{array}$$

are listed as in the following table.

s	s_0	s_1	s_2	s_3	s_4
ws	$\begin{array}{cccc} \multicolumn{2}{c}{-h} & \multicolumn{2}{c}{-m+1} \\ \multicolumn{2}{c}{-e} & \multicolumn{2}{c}{-y} \\ \multicolumn{2}{c}{-d} & \multicolumn{2}{c}{-w} \\ \multicolumn{2}{c}{-c} & \multicolumn{2}{c}{-u} \\ -b & * & -s & -r \\ * & -a & * & -q \\ * & * & * & -p \\ * & * & * & -n \end{array}$	$\begin{array}{cccc} \multicolumn{2}{c}{*} & \multicolumn{2}{c}{n} \\ \multicolumn{2}{c}{*} & \multicolumn{2}{c}{m} \\ \multicolumn{2}{c}{*} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{f} & \multicolumn{2}{c}{*} \\ g & d & * & t \\ e & j & s & v \\ * & h & u & x \\ * & * & w & -y-1 \end{array}$	$\begin{array}{cccc} \multicolumn{2}{c}{*} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{*} & \multicolumn{2}{c}{p} \\ \multicolumn{2}{c}{d} & \multicolumn{2}{c}{n} \\ \multicolumn{2}{c}{c} & \multicolumn{2}{c}{s} \\ * & * & u & q \\ i & * & * & r \\ g & l & * & y \\ * & j & -x-1 & w \end{array}$	$\begin{array}{cccc} \multicolumn{2}{c}{*} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{c} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{b} & \multicolumn{2}{c}{q} \\ \multicolumn{2}{c}{e} & \multicolumn{2}{c}{p} \\ d & g & * & * \\ f & * & * & w \\ k & * & x & u \\ i & -l-1 & v & * \end{array}$	$\begin{array}{cccc} \multicolumn{2}{c}{b} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{a} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{*} & \multicolumn{2}{c}{*} \\ \multicolumn{2}{c}{*} & \multicolumn{2}{c}{r} \\ h & * & q & u \\ j & e & v & s \\ l & g & t & * \\ -k-1 & i & * & * \end{array}$

where the entries in the $*$ positions remain unchanged.

1.6 For $w, w' \in W_a$, we say that w' is a left extension of w if $\ell(w') = \ell(w) + \ell(w'w^{-1})$. Then we have the following results on the alcove form $(k(w,\alpha))_{\alpha\in\Phi}$ of $w \in W_a$.

Proposition [16, Propositions 4.1, 4.3]. *(1)* $\ell(w) = \sum_{\alpha\in\Phi^+} |k(w,\alpha)|$, *where the notation* $|x|$ *stands for the absolute value of* x*;*

(2) $\mathcal{R}(w) = \{s_i \mid k(w,\alpha_i) < 0\}$.

(3) w' *is a left extension of* w *if and only if the inequalities* $k(w',\alpha)k(w,\alpha) \geq 0$ *and* $|k(w',\alpha)| \geq |k(w,\alpha)|$ *hold for any* $\alpha \in \Phi$.

1.7 Now let W be either an affine Weyl group or a finite Weyl group. Lusztig defined a function $a: W \longrightarrow \mathbb{N}$ which satisfies the following properties:

(1) $a(z) \leq \nu = |\Phi|/2$, for any $z \in W$, where Φ is the root system associated to W as in 1.3;

(2) $x \underset{LR}{\leq} y \Longrightarrow a(x) \geq a(y)$. In particular, $x \underset{LR}{\sim} y \Longrightarrow a(x) = a(y)$. So we may define the a-value $a(\Gamma)$ on a (left, right or two-sided) cell Γ of W by $a(x)$ for any $x \in \Gamma$.
(3) $a(x) = a(y)$ and $x \underset{L}{\leq} y$ (resp. $x \underset{R}{\leq} y$) $\Longrightarrow x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$).
(4) Let w_I be the longest element in the subgroup W_I of W generated by I for any $I \subseteq S$ with W_I finite. Then $a(w_I) = \ell(w_I)$.

The above properties of function a were shown by Lusztig in his papers [9], [11]. Now we state some more properties of this function, the first two of which are simple consequences of properties (2), (3) and (4).

Let $W_{(i)} = \{w \in W \mid a(w) = i\}$ for any non-negative integer i. Then by (2), $W_{(i)}$ is a union of some two-sided cells of W.

To each element $x \in W$, we associate two subsets of S as below.

$$\mathcal{L}(x) = \{s \in S \mid sx < x\} \quad \text{and} \quad \mathcal{R}(x) = \{s \in S \mid xs < x\}.$$

(5) If $W_{(i)}$ contains an element of the form w_I for some $I \subset S$, then $\{w \in W_{(i)} \mid \mathcal{R}(w) = I\}$ forms a single left cell of W.
(6) By the notation $x = y \cdot z$ $(x, y, z \in W)$, we mean $x = yz$ and $\ell(x) = \ell(y) + \ell(z)$. In this case, we have $x \underset{L}{\leq} z$, $x \underset{R}{\leq} y$ and hence $a(x) \geq a(y), a(z)$. In particular, if $I = \mathcal{R}(x)$ (resp. $I = \mathcal{L}(x)$), then $a(x) \geq \ell(w_I)$.
(7) $W_{(i)}$ is a single two-sided cell of W if $i \in \{0, 1, \nu\}$ (see (1)). As sets, $W_{(i)}$ $(i = 0, 1, \nu)$ can be described as below. $W_{(0)} = \{e\}$, e, the identity element of W. $W_{(1)}$ consists of all the non-identity elements of W each of which has a unique reduced expression (see [8]). $W_{(\nu)}$ consists of all the elements of W which have no zero entry in their alcove forms (see 1.4). $W_{(\nu)}$ can also be described to be the lowest two-sided cell of W with respect to the partial order $\underset{\mathrm{LR}}{\leq}$ (see [18], [19]).
(8) Now let $W = W_a$ be an affine Weyl group. Call an element $s \in S$ special, if the subgroup of W_a generated by $S \setminus \{s\}$ is isomorphic to W_0 (see 1.3). Thus the element s_0 is always special. When W_a is of type $\widetilde{F}_4$, there is no other special element in S. It is known that for any two-sided cell $\Omega \neq \{e\}$ of W_a and any special $s \in S$, the set $Y_s = \{w \in \Omega \mid \mathcal{R}(w) = \{s\}\}$ is non-empty and is a single left cell of W_a (see [13]).

1.8 Let G and W_a be as in 1.3. Then the following result of Lusztig is important to our purpose.

Theorem [12, Theorem 4.8]. *There exists a bijection* $\mathbf{u} \mapsto \mathbf{c}(\mathbf{u})$ *from the set of unipotent conjugacy classes in* G *to the set of two-sided cells in* W_a. *This bijection satisfies the equation* $a(\mathbf{c}(\mathbf{u})) = \dim \mathfrak{B}_u$, *where* u *is any element in* $\mathbf{u}$, *and* $\dim \mathfrak{B}_u$ *is the dimension of the variety of Borel subgroups of* G *containing* u.

1.9 Let G, W_0 and W_a be as in 1.3. Then W_0 is a stardard parabolic subgroup of W_a. It is known that for any $w \in W_0$, the value $a(w)$ computed with respect to W_0 is equal to that computed with respect to W_a [9, Corollary 6.3]. From the results of [2], [3], [9] and [12], we know that the bijection in Theorem 1.8 induces a bijection between the set of special unipotent classes of G and the set of two-sided cells of W_0. Let w_0 be the longest element of W_0. Then the permutation $x \mapsto w_0x$ of W_0 induces an order-reversing bijective map on the set of two-sided cells of W_0 (see [7, 3.3]). Under this map, the two-sided cells $W_{(0)} = \{e\}$ and $W_{(\nu)} = \{w_0\}$ are transposed, and the two-sided cell $W_{(1)}$ of W_0 is sent to the second lowest one. Here $\nu = \ell(w_0)$.

1.10 Let $W_a = W_a(\widetilde{F}_4)$ be the affine Weyl group of type $\widetilde{F}_4$. Then according to the knowledge of the unipotent classes of the complex simple algebraic group of type F_4, we see from Theorem 1.8 that in W_a, the set $W_{(i)}$ is non-empty if and only if $i \in \Lambda = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 16, 24\}$. More precisely, $W_{(i)}$ is a single two-sided cell if $i \in \{0, 1, 2, 4, 5, 7, 10, 13, 16, 24\}$, and is a union of two two-sided cells if $i \in \{3, 6, 9\}$.

1.11 We see that there exist some elements of the form w_I, $I \subset S$, in all the sets $W_{(i)}$ of $W_a(\widetilde{F}_4)$, $i \in \Lambda \setminus \{13\}$. By the results of [21] and by 1.7,(4),(7), at this stage we can find a representative set of the two-sided cells Ω of $W_a(\widetilde{F}_4)$ with $a(\Omega) \neq 6, 9$ as follows. e, s_4, w_{02}, w_{01}, w_{34}, w_{23}, w_{023}, w_{0124}, w_{0234}, $x = w_{234}s_1s_2s_3s_2s_1s_4s_3s_2s_3s_4$, w_{0123} and w_{1234}, where we have $x \in W_{(13)}$ by 1.9, 1.10, and by the fact $w_0x = s_1s_2s_3s_2s_1 \in W_{(1)}$.

§2. An algorithm with some related results.

Here and later, the notation W_a always stands for an affine Weyl group with S its Coxeter generator set. The main purpose of the present paper is to describe the left cells of the affine Weyl group W_a of type $\widetilde{F}_4$ by finding its l.c.r. set together with all its left cell graphs (or with the corresponding essential graphs). We need an algorithm to do this, which was designed in [21] and then improved in [24]. This algorithm is applicable to a certain family of crystallographic groups including all the Weyl groups and all the affine Weyl groups. In this section, we shall recall the algorithm and some related results in [21] and [24].

2.1 To each element $x \in W_a$, we associate a set $\Sigma(x)$ of all left cells Γ of W_a satisfying the condition that there is some element $y \in \Gamma$ with y—x, $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $a(y) = a(x)$. We have the following result.

Theorem [21, Theorem 2.1]. *If $x \underset{L}{\sim} y$ in W_a, then $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$.*

2.2 To each $x \in W_a$, we denote by $M(x)$ the set of all elements y such that there are a sequence of elements $x_0 = x, x_1, \cdots, x_r = y$ in W_a with $r \geq 0$, where for every i, $1 \leq i \leq r$, the conditions $x_{i-1}^{-1}x_i \in S$ and $\mathcal{R}(x_{i-1}) \overset{\not\supseteq}{\not\subseteq} \mathcal{R}(x_i)$ are satisfied.

2.3 A subset $K \subset W_a$ is said to be *distinguished* if $K \neq \emptyset$ and $x \underset{L}{\not\sim} y$ for any $x \neq y$ in K. The following are three processes on a non-empty set $P \subset W_a$ (see [24]).

(**A**) Find a largest possible subset Q from the set $\bigcup_{x \in P} M(x)$ with Q distinguished.

(**B**) To each $x \in P$, find elements $y \in W_a$ such that $y^{-1}x \in S$, $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$ and $a(y) = a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.

(**C**) To each $x \in P$, find elements $y \in W_a$ such that $y < x$, y—x, $\mathcal{R}(y) \supsetneqq \mathcal{R}(x)$ and $a(y) = a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.

A subset P of W_a is called **A**-*saturated* (resp. **B**-*saturated*, resp. **C**-*saturated*), if Process (**A**) (resp. (**B**), resp. (**C**)) on P can't produce any element z satisfying $z \underset{L}{\not\sim} x$ for all $x \in P$.

2.4 Say a set Σ of left cells of W_a to be *represented* by a set $K \subset W_a$ if Σ is the set of all left cells Γ of W_a with $\Gamma \cap K \neq \emptyset$. K is called a *representative set* for Σ, if K represents Σ with K distinguished.

By [21, Theorem 3.1] and [7, 2.3f], we see that a representative set of left cells (an l.c.r. set for short) of W_a in a two-sided cell Ω is exactly a distinguished subset of Ω which is **A**-, **B**- and **C**-saturated. So to get such a set, we may use the following **ALGORITHM** [24, 2.7].

(1) Find a non-empty subset P of Ω (Usually we take P to be distinguished for avoiding unnecessary complication whenever it is possible);

(2) Perform Processes (**A**), (**B**) and (**C**) alternately on P until the resulting distinguished set can't be further enlarged by these processes.

In general, Process (**A**) (resp. (**B**)) is easier to be performed than Process (**B**) (resp. (**C**)) in applying the algorithm. So we shall make the priority first to Process (**A**) and second to Process (**B**). In other words, in applying the algorithm, we always first perform Process (**A**); Process (**B**) is performed only when Process (**A**) alone can not make any further progress; finally Process (**C**) is performed when no progress can be made only by Processes (**A**) and (**B**).

In applying Algorithm 2.4, we need some results and terminologies. Note that the terminologies concerning graphs are adopted from [24] which differ from those in [21]. From 1.7, (3) and Theorem 2.1, we have the following result on a set $M(x)$.

Proposition 2.5 [24, Proposition 3.1]. *(1) For any $x \in W_a$, the set $M(x)$ is wholely contained in some right cell of W_a.*

(2) If $x \underset{\mathrm{L}}{\sim} y$ in W_a, then $M(x)$ and $M(y)$ represent the same set of left cells of W_a.

2.6 In a Coxeter system (W, S), a sequence of elements of the form

$$\underbrace{ys,\ yst,\ ysts,\ \dots}_{m-1 \text{ terms}} \tag{2.6.1}$$

is called *an $\{s,t\}$-string* (or just call it *a string*) if $s, t \in S$ and $y \in W$ satisfy the conditions that the order $o(st)$ of the product st is m and $\mathcal{R}(y) \cap \{s,t\} = \emptyset$.

It is easily seen that a string is wholely contained in some right cell of W. For any $x \in W_a$, we can re-define $M(x)$ to be the minimal set containing x, subject to the requirement: any string (regarded as a set) meeting $M(x)$ must be wholely contained in $M(x)$. Suppose that we are given two $\{s,t\}$-strings $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{m-1}$ with $o(st) = m$. We denote the integers $\mu(x_i, y_j)$ (see 1.1) by a_{ij} for $1 \leq i, j \leq m-1$. Then it is known that

Proposition 2.7 [8, 10.4]. *In the above setup, the following assertions hold.*

(a) When $m = 3$, we have $a_{12} = a_{21}$, $a_{11} = a_{22}$;

(b) When $m = 4$, we have $a_{12} = a_{21} = a_{23} = a_{32}$, $a_{11} = a_{33}$, $a_{13} = a_{31}$ and $a_{22} = a_{11} + a_{13}$.

We have the following result corresponding to this.

Proposition 2.8 [21, Proposition 4.6]. *Keep the setup of 2.6.*

(1) If $m = 3$, then

(a) $x_1 \underset{L}{\sim} y_1 \Longleftrightarrow x_2 \underset{L}{\sim} y_2$;

(b) $x_1 \underset{L}{\sim} y_2 \Longleftrightarrow x_2 \underset{L}{\sim} y_1$.

(2) If $m = 4$, then

(a) $x_1 \underset{L}{\sim} y_2 \Longleftrightarrow x_2 \underset{L}{\sim} y_1 \Longleftrightarrow x_2 \underset{L}{\sim} y_3 \Longleftrightarrow x_3 \underset{L}{\sim} y_2$;

(b) $x_1 \underset{L}{\sim} y_1 \Longleftrightarrow x_3 \underset{L}{\sim} y_3$;

(c) $x_1 \underset{L}{\sim} y_3 \Longleftrightarrow x_3 \underset{L}{\sim} y_1$;

(d) $x_2 \underset{L}{\sim} y_2 \Longleftrightarrow$ *either* $x_1 \underset{L}{\sim} y_1$ *or* $x_1 \underset{L}{\sim} y_3$

2.9 Two elements $x, y \in W_a$ form *a primitive pair*, if there exist two sequences of elements $x_0 = x, x_1, \cdots, x_r$ and $y_0 = y, y_1, \cdots, y_r$ in W_a such that the following conditions are satisfied.

(a) x_i—y_i for all i, $0 \leq i \leq r$.

(b) For every i, $1 \leq i \leq r$, there exist some $s_i, t_i \in S$ such that x_{i-1}, x_i (and also y_{i-1}, y_i)) are two neighboring terms in some $\{s_i, t_i\}$-string.

(c) Either $\mathcal{R}(x) \not\subseteq \mathcal{R}(y)$ and $\mathcal{R}(y_r) \not\subseteq \mathcal{R}(x_r)$, or $\mathcal{R}(y) \not\subseteq \mathcal{R}(x)$ and $\mathcal{R}(x_r) \not\subseteq \mathcal{R}(y_r)$ hold.

In this case, we have $x \underset{R}{\sim} y$ by Proposition 2.5.

Assume that x, x' (and also y, y') are two neighboring terms in some $\{s,t\}$- string with x—y and that at least one of x, y is a terminal term of the $\{s,t\}$-string containing it. Then by Proposition 2.7, we have x'—y'. In particular, it is always the case when $o(st) = 3$. Thus, if in (b), we have in addition that at least one of x_i, y_i is a terminal term of the $\{s_i, t_i\}$-string containing it for any i, $0 \leq i < r$, then we can replace condition (a) by the following weaker one in the definition of a primitive pair:

(a') x_0—y_0.

2.10 By a graph, we mean that a set of vertices M together with a set of edges, where each edge is a two-elements subset of M, and each vertex is labelled by a subset of S.

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two graphs with their vertex sets M and M'. They are said to be isomorphic, written $\mathfrak{M} \cong \mathfrak{M}'$, if there exists a bijective map η from the set M to the set M' satisfying the following two conditions.

(1) The labelling of w is the same as that of $\eta(w)$ for any $w \in M$.

(2) For $w, z \in M$, $\{w, z\}$ is an edge of $\mathfrak{M}$ if and only if $\{\eta(w), \eta(z)\}$ is an edge of $\mathfrak{M}'$.

This is an equivalence relation on graphs.

2.11 We define *a graph $\mathfrak{M}(x)$ associated to an element $x \in W_a$* as follows. Its vertex set is $M(x)$. Its edge set consists of all two-elements subsets $\{y, z\} \subset M(x)$ with y, z two neighboring terms of a string. Each vertex $y \in M(x)$ is labelled by the set $\mathcal{R}(y)$.

A left cell graph associated to an element $x \in W_a$, written $\mathfrak{M}_L(x)$, is by definition a graph, whose vertex set $M_L(x)$ consists of all left cells Γ of W_a with $\Gamma \cap M(x) \neq \emptyset$. Two vertices $\Gamma, \Gamma' \in M_L(x)$ are joined by an edge, if there are two elements $x \in M(x) \cap \Gamma$ and $x' \in M(x) \cap \Gamma'$ such that $\{x, x'\}$ is an edge of $\mathfrak{M}(x)$. Each vertex Γ of $\mathfrak{M}_L(x)$ is labelled by the common labelling of elements of $M(x) \cap \Gamma$ (This makes sense by [7, Proposition 2.4]). Clearly, the graph $\mathfrak{M}_L(x)$ is always connected.

2.12 A subgraph $\mathfrak{M}$ of $\mathfrak{M}(x)$ ($x \in W_a$) is said to be *essential*, if there is an isomorphism η from $\mathfrak{M}$ to $\mathfrak{M}_L(x)$ with $y \in \eta(y)$ for each vertex y of $\mathfrak{M}$.

It is easily seen that when a subgraph $\mathfrak{M}$ of $\mathfrak{M}(x)$ is essential, its vertex set must be distinguished. In particular, the graph $\mathfrak{M}(x)$ itself is essential if and only if its vertex set $M(x)$ is distinguished. But it should be careful that in general there does not always exist an essential subgraph in $\mathfrak{M}(x)$ (A counter-example could be found in the two-sided cell $W_{(3)}$ of the affine Weyl group $W_a(\widetilde{D}_4)$ or in $W_{(1)}$ of $W_a(\widetilde{A}_n)$, $n > 1$). However, we shall see that for any $x \in W_a(\widetilde{F}_4)$, there always exists some essential subgraph of $\mathfrak{M}(x)$ containing x as its vertex.

2.13 Let $\mathfrak{N}$ and $\mathfrak{N}'$ be two graphs with N and N' the corresponding vertex sets. We say that $\mathfrak{N}'$ is opposed to $\mathfrak{N}$ (up to isomorphism), written $\mathfrak{N}' = \mathfrak{N}^{\mathrm{op}}$, if there exists a bijective map ϕ from the set N to N' satisfying that for any $x, y \in N$,

(a) $\mathcal{R}(\phi(x)) = S \setminus \mathcal{R}(x)$;

(b) $\{x, y\}$ is an edge of $\mathfrak{N}$ if and only if $\{\phi(x), \phi(y)\}$ is an edge of $\mathfrak{N}'$.

Clearly, the relation of two graphs being opposed is mutual. So $\mathfrak{N} = (\mathfrak{N}^{\mathrm{op}})^{\mathrm{op}}$.

2.14 By *a path* in graph $\mathfrak{M}(x)$, we mean a sequence of vertices $z_0, z_1, \ldots, z_t$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathfrak{M}(x)$ for any i, $1 \leq i \leq t$. Two elements $x, x' \in W_a$ have the same *generalized τ-invariant*, if for any path $z_0 = x, z_1, \ldots, z_t$ in graph $\mathfrak{M}(x)$, there is a path $z'_0 = x', z'_1, \ldots, z'_t$ in $\mathfrak{M}(x')$ with $\mathcal{R}(z'_i) = \mathcal{R}(z_i)$ for every i, $0 \leq i \leq t$, and if the same condition holds when interchanging the roles of x with x'.

2.15 It may happen that for two elements $x, y \in W_a$ with $x \underset{\mathrm{L}}{\sim} y$, the graphs $\mathfrak{M}(x)$ and $\mathfrak{M}(y)$ are not isomorphic (take $x = s_0$ and $y = s_1 s_0$ in $W_a(\widetilde{C}_4)$ for example). But we have the following result.

Proposition [24, 3.10]. *(a) The elements in the same left cell of W_a have the same generalized τ-invariant.*

(b) If $x \underset{\mathrm{L}}{\sim} y$ in W_a, then the left cell graphs $\mathfrak{M}_L(x)$ and $\mathfrak{M}_L(y)$ are isomorphic.

§3. Some techniques in applying the algorithm.

3.1 We shall apply the algorithm to find an l.c.r. set, together with all left cell graphs or with the corresponding essential graphs, in each two-sided cell Ω of $W_a = W_a(\widetilde{F}_4)$. This has been done for all the two-sided cells Ω with $a(\Omega) \in \{3, 4, 5\}$ in my previous paper [21]. On the other hand, an l.c.r. set in the two-sided cells $W_{(i)}$, $i \in \{0, 1, 2, 24\}$

have been found before (see [5], [8], [18] and [19]). Thus actually we need only deal with the two-sided cells Ω of $a(\Omega) \in \{6, 7, 9, 10, 13, 16\}$.

We shall use the notation **i** for the simple reflection s_i $(0 \le i \le 4)$ in the subsequent discussion.

3.2 First we choose the starting set P of the algorithm. Write $y = w_{I_y} \cdot y'$ with $I_y = \mathcal{L}(y)$ for any $y \in W_a$. We prefer to (but not have to) choose the elements x for the set P to satisfy the following conditions.

(1) $a(x) - \ell(w_{I_x}) = \min\{a(y) - \ell(w_{I_y}) \mid y \in \Omega\}$.

(2) Let A be the set of all the elements y in Ω which satisfy condition (1) for y instead of x. Then $\ell(x') = \min\{\ell(y') \mid y \in A\}$.

Thus the elements of the form w_I, $I \subset S$, are the best candidates to be chosen into P whenever they are contained in Ω.

Each $W_{(i)}$ $(i \in \{7, 10, 16\})$ consists of a single two-sided cell, which contains a unique element of the form w_I, $I \subset S$, i.e. w_{0124}, w_{0234} and w_{0123}, respectively. Thus in dealing with the two-sided cells $W_{(i)}$, $i \in \{7, 10, 16\}$, we can take the starting set P of the algorithm to be $\{w_{0124}\}$, $\{w_{0234}\}$ and $\{w_{0123}\}$, respectively. The set $W_{(6)}$ (resp. $W_{(9)}$) consists of two two-sided cells. There are two elements of the form w_I in the set $W_{(6)}$ (resp. $W_{(9)}$), i.e. w_{012} and w_{0134} (resp. w_{234} and w_{123}). We don't know in advance whether or not these two elements are in the same two-sided cell. Thus in dealing with a two-sided cell in $W_{(6)}$ (resp. $W_{(9)}$), the starting set P of the algorithm will be taken as $\{w_{012}\}$ or $\{w_{0134}\}$ (resp. $\{w_{234}\}$ or $\{w_{123}\}$) rather than $\{w_{012}, w_{0134}\}$ (resp. $\{w_{234}, w_{123}\}$). Let $x = w_{0234}\mathbf{1232143234}$ and $y = w_{234}\mathbf{1232143234}$. We have $y \in W_{(13)}$ by 1.11. We can show that $\{x^{-1}, y^{-1}\}$ is a primitive pair (see 2.9). This implies $x \underset{L}{\sim} y$ and hence $x \in W_{(13)}$. So for the two-sided cell $W_{(13)}$, we shall take $\{x\}$ as the starting set P of the algorithm.

For any $z \in W_a$, we denote by $\Omega(z)$ (resp. $\Gamma(z)$) the two-sided cell (resp. the left cell) of W_a containing z.

3.3 In applying the algorithm, we shall first deal with the two-sided cell $W_{(16)}$, then $W_{(10)}$, $W_{(13)}$, $\Omega(w_{123})$, $W_{(9)} \setminus \Omega(w_{123})$, $W_{(7)}$, $\Omega(w_{012})$, $W_{(6)} \setminus \Omega(w_{012})$ in turn. The reason for taking such an order is to make it easier in performing processes (**B**) and (**C**), in particular in the determination of the a-values of the elements occurring in the intermediate steps of these two processes.

3.4 Now we introduce some techniques which we shall use in section 4: (1) to find an essential subgraph from a graph of the form $\mathfrak{M}(x)$; (2) to determine the a-value of an element; (3) to tell whether or not two sets of the form $M(x)$ represent the same set of left cells. We illustrate our methods by some examples.

3.4.1 To examine whether or not a graph $\mathfrak{M} = \mathfrak{M}(\alpha)$ ($\alpha \in W_a$) is essential, we should first consider the generalized τ-invariants of vertices of $\mathfrak{M}$. If they are all different, then $\mathfrak{M}$ is essential. If there are some pair of vertices of $\mathfrak{M}$, say x, y, having the same generalized τ-invariant, then we should further compare the set $\Sigma(x)$ with $\Sigma(y)$. If for all such pairs x, y, we have $\Sigma(x) \neq \Sigma(y)$, then $\mathfrak{M}$ is still essential. For example, let $y_0 = w_{0123} \cdot \mathbf{43234}$. Then $y_0 \in W_{(16)}$ by 1.7,(6),(7), 1.10 and by the alcove form of y_0. The graph $\mathfrak{M}(y_0)$ is isomorphic to $\mathfrak{M}_{18}$ with y_0 the vertex labelled by $\boxed{\mathbf{0234}}$ (here and later, the graphs denoted by $\mathfrak{M}_i$, $i \geq 1$, are displayed at the end of section 4). We want to show that the graph $\mathfrak{M}(y_0)$ itself is essential. By Proposition 2.15, (a), it is enough to show that $\boxed{\mathbf{012}}_1 \underset{L}{\not\sim} \boxed{\mathbf{012}}_2$ in $\mathfrak{M}(y_0)$, where by abuse of notations, we identify a vertex with its labelling in the graph $\mathfrak{M}(y_0)$, the numbers inside a box represent the corresponding elements in S, and the subscripts of boxes are used to distinguish the positions of the vertices with the same labelling (such an identification will be used quite often later, which will not cause any confusion in the context). Assume $\ell(\boxed{\mathbf{012}}_1) < \ell(\boxed{\mathbf{012}}_2)$ for the sake of definity. We may check that $\Gamma(\boxed{\mathbf{0124}}_1) \in \Sigma(\boxed{\mathbf{012}}_2) \setminus \Sigma(\boxed{\mathbf{012}}_1)$ and hence $\Sigma(\boxed{\mathbf{012}}_1) \neq \Sigma(\boxed{\mathbf{012}}_2)$. This implies $\boxed{\mathbf{012}}_1 \underset{L}{\not\sim} \boxed{\mathbf{012}}_2$ by Theorem 2.1.

3.4.2 In examining whether or not a graph $\mathfrak{M} = \mathfrak{M}(\alpha)$ is essential, it may happen that there are more than one pairs of vertices $\{x, y\}$ in $\mathfrak{M}$ having the same generalized τ-invariants. In such a case, it is not always necessary to check the equation $\Sigma(x) = \Sigma(y)$ for each pair $\{x, y\}$. We need only to check the equations on some pairs and then apply Proposition 2.8 to get the conclusion on the remains. For example, let $x = w_{0234} \cdot \mathbf{1232143234}$. Then $x \in W_{(13)}$ by 3.2. The graph $\mathfrak{M}(x)$ is isomorphic to the one in Fig. 1 with the vertex x labelled by $\boxed{234}_1$. We want to find an essential subgraph $\mathfrak{M}_e(x)$ in $\mathfrak{M}(x)$. Let $y = 0x$ and $z = 21 \cdot y$. Then $x \underset{L}{\sim} y \underset{L}{\sim} z$ by 3.2 and by the fact $z^{-1} \in M(y^{-1})$. We have $\mathfrak{M}(z) \cong \mathfrak{M}_6$. By Proposition 2.15, (b), we see that essential graphs $\mathfrak{M}_e(x)$ and $\mathfrak{M}_e(z)$ should be isomorphic whenever they exist. By comparing $\mathfrak{M}(x)$ with $\mathfrak{M}(z)$ and by Proposition 2.15, (a), we can take $\mathfrak{M}_e(x)$ to be a subgraph of $\mathfrak{M}(x)$ isomorphic to $\mathfrak{M}_{25}^{\mathrm{op}}$ (The graph $\mathfrak{M}_i^{\mathrm{op}}$ is obtained from $\mathfrak{M}_i$ by replacing the number set I in each box by $\{0, 1, 2, 3, 4\} \setminus I$ and with the subscripts of boxes unchanged whenever they are attached.

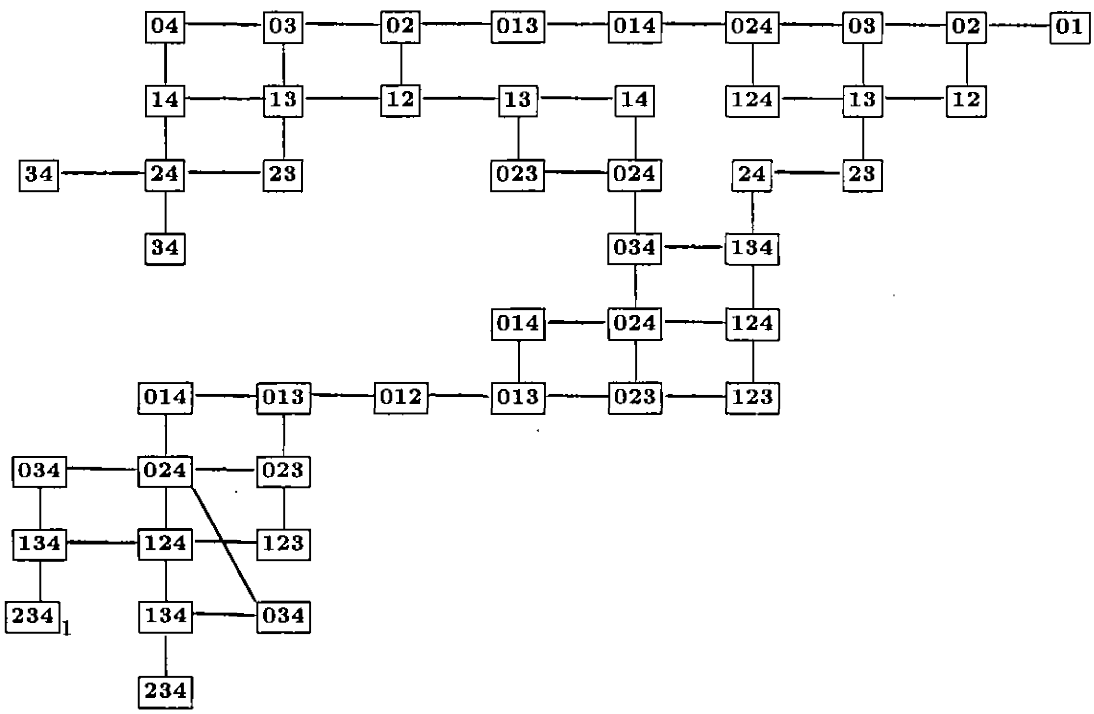


Fig. 1. $\mathfrak{M}(x)$

3.4.3 Now we shall deal with a more complicated example, where in addition of the above task, we shall do two more things. One is to determine the a-values of some elements in virtue of primitive pairs. The other is to conclude a pair of elements α, β having the relation $\alpha \underset{L}{\sim} \beta$ by observing that one is a left extension of the other with

$a(\alpha) = a(\beta)$. Let $z = w_{012} \cdot \mathbf{3421321}$. Then we claim $a(z) = 6$. For, let $y = z\mathbf{1}$, then y is the vertex labelled by $\boxed{2}_1$ in $\mathfrak{M}(w_{012}) \cong \mathfrak{M}_{17}$, and z is the vertex labelled by $\boxed{12}_1$ in $\mathfrak{M}(z)$ (see Fig. 2). So $\{y, z\}$ is a primitive pair and hence $a(z) = a(y) = a(w_{012}) = 6$. In the graph $\mathfrak{M}(z)$, the vertex labelled by $\boxed{01}_2$ is a left extension of that by $\boxed{01}_1$. On the other hand, let z' be the vertex labelled by $\boxed{12}_2$ in $\mathfrak{M}(z)$. Then we can show that $\Gamma(z' \cdot \mathbf{4}) \in \Sigma(z') \setminus \Sigma(z)$ and hence $z \underset{L}{\not\sim} z'$ by Theorem 2.1. Thus we have an essential subgraph $\mathfrak{M}_e(z)$ of $\mathfrak{M}(z)$ isomorphic to $\mathfrak{M}_{12}$.

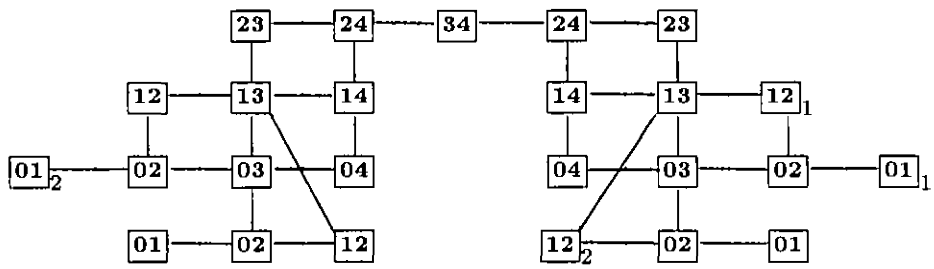


Fig. 2. $\mathfrak{M}(z)$

3.4.4 Let $x = w_{0234} \cdot \mathbf{1232143234}$. Then we have $x \in W_{(13)}$ by 3.2. The graph $\mathfrak{M}(x)$ is displayed in Fig. 1. Assume that an l.c.r. set of W_a in the two-sided cell $W_{(16)}$ has been found. Then we see that there is no graph in $W_{(16)}$ of the form $\mathfrak{M}(\alpha)$ which is isomorphic to $\mathfrak{M}(x)$.

This fact will help us in finding an l.c.r. set of the two-sided cell $W_{(13)}$. Let $y_0 = w_{0234} \cdot \mathbf{123214323101234323123 40}$. Then the graph $\mathfrak{M}(y_0)$ is isomorphic to $\mathfrak{M}_{23}$ with y_0 the vertex labelled by $\boxed{034}_1$. We claim $y_0 \in W_{(13)}$. For, we have $y_1 = y_0\mathbf{0} \in M(x)$ and that $\{y_0, y_1\}$ is a primitive pair. We want to find an essential subgraph $\mathfrak{M}_e(y_0)$ in $\mathfrak{M}(y_0)$. Let α, β be the vertices of $\mathfrak{M}(y_0)$ labelled by $\boxed{3}_1$ and $\boxed{3}_2$ respectively (see $\mathfrak{M}_{23}$). Then $\alpha = y_0 \cdot \mathbf{123}$ and $\beta = y_0 \cdot \mathbf{23123}$. We have $\mathbf{3210321} \cdot \alpha = \mathbf{20321} \cdot \beta$, denote it by y. Then y is a commen left extension of α and β. We have $\mathfrak{M}(y) \cong \mathfrak{M}(y_0)$. Now we want to determine the a-value of y. By 1.7,(6),(7), 1.10 and by the alcove form of y, we have $a(y) \in \{13, 16\}$. Let $y_1 = y\mathbf{32132}$ and $y_2 = y_1\mathbf{0}$. Then we have $y_1 \in M(y)$, $\mathfrak{M}(y_2) \cong \mathfrak{M}(x)$ and a primitive pair $\{y_1, y_2\}$. So by the above observation, we have

$a(y) = a(y_2) = 13$. This implies $\alpha \underset{L}{\sim} y \underset{L}{\sim} \beta$ by 1.7, (3). Therefore $\mathfrak{M}(y_0)$ contains an essential subgraph $\mathfrak{M}_e(y_0)$ isomorphic to $\mathfrak{M}_{22}$.

3.4.5 Suppose that we are given two isomorphic graphs, say $\mathfrak{N}_1$ and $\mathfrak{N}_2$. We want to examine whether or not their vertex sets N_1, N_2 represent the same left cell set of W_a. It is known that N_1 and N_2 represent the same left cell set if and only if there exist some vertices α_i of $\mathfrak{N}_i$ $(i = 1, 2)$ with $\alpha_1 \underset{L}{\sim} \alpha_2$ (hence α_1 and α_2 must have the same generalized τ-invariant). Thus the problem is reduced to checking whether or not the relation $\alpha_1 \underset{L}{\sim} \alpha_2$ holds for some $\alpha_i \in N_i$. To do this, we first choose vertices $\alpha_i \in N_i$, $i = 1, 2$, such that α_1 and α_2 have the same generalized τ-invariant. Thus $\mathfrak{N}_1$ and $\mathfrak{N}_2$ represent the same set of left cells if $\alpha_1 \underset{L}{\sim} \alpha_2$. For example, let $a_1 = w_{0234} \cdot \mathbf{12321432312012343}$, $a_2 = w_{0234} \cdot \mathbf{12321432341012342 31230}$ and $a_3 = w_{0234} \cdot \mathbf{1232143231201234323 41230}$. Then the graphs $\mathfrak{M}(a_i)$ $(i = 1, 2, 3)$ are all essential and isomorphic to $\mathfrak{M}_2$. We shall show that $a_i \in W_{(13)}$ for $i = 1, 2, 3$, that the left cell set represented by $M(a_1)$ is disjoint with that by $M(a_2)$ and that $M(a_2)$ and $M(a_3)$ represent the same set of left cells. Let $x_1 = a_1\mathbf{3}$, $y_0 = a_2\mathbf{4}$, $y = y_0\mathbf{0}$ and $y_1 = a_3\mathbf{4}$. Then we can check that $\{x_1, a_1\}$, $\{y_0, a_2\}$, $\{y, y_0\}$ and $\{y_1, a_3\}$ are all primitive pairs. We also see that $y_1 \in M(y_0)$ and $x_1, y \in \mathfrak{M}(x)$, where $x \in W_{(13)}$ is defined as in 3.2. This implies $a_i \in W_{(13)}$ for $i = 1, 2, 3$. Let b_j $(j = 1, 2, 3)$ be the vertex labelled by $\boxed{03}$ in $\mathfrak{M}(a_j)$ (see $\mathfrak{M}_2$). To show $M(a_1)$ and $M(a_2)$ representing disjoint left cell sets, it suffices to show $b_1 \underset{L}{\not\sim} b_2$. But this follows by Theorem 2.1 and by the fact $\Gamma(b_2 \cdot \mathbf{4}) \in \Sigma(b_2) \setminus \Sigma(b_1)$. Finally, we want to show that $M(a_2)$ and $M(a_3)$ represent the same set of left cells. It is enough to show $b_2 \underset{L}{\sim} b_3$. We have $\mathbf{3210321}b_2 = \mathbf{02321}b_3$, denote it by w. Then w is a commen left extension of b_2 and b_3. By 1.7, (6), 1.10 and the alcove form of w, we have $a(w) \in \{13, 16\}$. Let $w_0 = w\mathbf{4}$, $v = w_0\mathbf{32}$ and $v_0 = v\mathbf{0}$. Then it can be checked that both $\{w, w_0\}$ and $\{v, v_0\}$ are primitive pairs, that $w_0 \in M(v)$, and that $\mathfrak{M}(v_0)$ is isomorphic to $\mathfrak{M}(x)$ in Fig. 1. By the observation at the beginning of 3.4.4, this implies $a(w) = a(v_0) = 13$ and hence $b_2 \underset{L}{\sim} w \underset{L}{\sim} b_3$ by 1.7, (3). Our result follows.

Remark 3.5 (1) Note that the elements y in 3.4.4 and w in 3.4.5 are found in virtue of alcove forms and by Proposition 1.6, (3).

(2) Besides the above techniques, sometimes we use the property of distinguished involutions of W_a to determine whether or not two elements x, y in the same $\{s,t\}$-string ($o(st) = 4$) with $\mathcal{R}(x) = \mathcal{R}(y)$ satisfy the relation $x \underset{\mathrm{L}}{\sim} y$. An element $x \in W_a$ is called *distinguished*, if $\ell(x) - a(x) - 2\delta(x) = 0$, where $\delta(x) = \deg P_{e,x}$. It is known that any distinguished element of W_a is an involution and that each left cell of W_a contains a unique distinguished involution. Denote by $d(x)$ the distinguished involution in the left cell containing x. Suppose that $y \cdot s$, $y \cdot st$, $y \cdot sts$ is an $\{s,t\}$-string with $o(st) = 4$. Written $d(yst) = \alpha \cdot z \cdot \beta$ with α, β in the group generated by s, t and $s, t \notin \mathcal{L}(z) \cup \mathcal{R}(z)$. Then by [20, Proposition 5.12], we know that $ys \underset{\mathrm{L}}{\sim} ysts$ if and only if $\{sz, tz\} = \{zs, zt\}$. This result can be used, for example, to conclude that a graph isomorphic to $\mathfrak{M}_{15}$ is essential in the two-sided cell $\Omega(w_{0134})$.

§4. l.c.r. sets of the two-sided cells.

In the present section, we shall give an l.c.r. set, together with all the left cell graphs (or all the corresponding essential graphs) for each two-sided cell of $W_a = W_a(\widetilde{F}_4)$. Since the techniques applied are more or less similar to those in 3.4, we shall only give very brief arguments in the most cases.

4.1 Let $x = w_{0123}$, $y = x \cdot \mathbf{4323}$, $y_0 = y \cdot \mathbf{4}$, $y_1 = y_0 \cdot \mathbf{12340}$, $a_1 = y_1 \cdot \mathbf{1}$, $z = y_0 \cdot \mathbf{1232}$, $z_0 = z \cdot \mathbf{1}$, $w = y_1 \cdot \mathbf{2}$, $w_0 = w \cdot \mathbf{3}$, $y_2 = w \cdot \mathbf{1234}$, $a_2 = y_2 \cdot \mathbf{0}$, $w_1 = w_0 \cdot \mathbf{12340}$, $a_3 = w_1 \cdot \mathbf{1}$, $w_2 = w_1 \cdot \mathbf{231}$, $a_4 = w_2 \cdot \mathbf{0}$, $v = y_2 \cdot \mathbf{213}$, $v_0 = v \cdot \mathbf{4}$, $u = v \cdot \mathbf{0}$, $u_0 = u \cdot \mathbf{1}$, $h = z_0 \cdot \mathbf{43230123214323}$, $h_0 = h \cdot \mathbf{4}$, $j = u_0 \cdot \mathbf{210}$, $j_0 = j \cdot \mathbf{4}$, $u_1 = j \cdot \mathbf{34}$, $a_5 = u_1 \cdot \mathbf{3}$, $j_1 = j_0\mathbf{321243}$, $a_6 = j_1 \cdot \mathbf{0}$, $j_2 = j_0 \cdot \mathbf{321343}$, $a_7 = j_2 \cdot \mathbf{0}$, $k = u_0 \cdot \mathbf{21343210}$, $k_0 = k \cdot \mathbf{1}$, $k_1 = k_0 \cdot \mathbf{321304}$, $a_8 = k_1 \cdot \mathbf{3}$, $m = j_2 \cdot \mathbf{234}$, $b_1 = m \cdot \mathbf{1}$, $n = j_2 \cdot \mathbf{210}$, $n_0 = n \cdot \mathbf{1}$, $n_1 = n_0 \cdot \mathbf{324312343}$, $b_2 = n_1 \cdot \mathbf{1}$, $p = n_1 \cdot \mathbf{21}$ and $p_0 = p \cdot \mathbf{2}$. Let $I = \{x, y_0, z_0, w_0, v_0, u_0, h_0, j_0, k_0, n_0, p_0, a_i\ (1 \leq i \leq 8), b_l\ (l = 1,2)\}$. We see that each $\alpha \in I$ has some zero entries in its alcove form and satisfies $\mathcal{L}(\alpha) = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}$. This implies $I \subset W_{(16)}$ by 1.10. By the techniques of 3.4, we can show that all the graphs $\mathfrak{M}(\alpha)$ $(\alpha \in I)$ are essential. Thus we get the following table.

α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$	α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$
x	$\boxed{0123}$	$\mathfrak{M}_9^{op}$	a_1	$\boxed{0134}$	$\mathfrak{M}_2$
y_0	$\boxed{0234}$	$\mathfrak{M}_{18}$	a_2	$\boxed{0134}$	$\mathfrak{M}_2$
z_0	$\boxed{123}_1$	$\mathfrak{M}_6$	a_3	$\boxed{0134}$	$\mathfrak{M}_2$
w_0	$\boxed{0234}$	$\mathfrak{M}_{23}^{op}$	a_4	$\boxed{0134}$	$\mathfrak{M}_2$
v_0	$\boxed{134}_1$	$\mathfrak{M}_{26}$	a_5	$\boxed{0134}$	$\mathfrak{M}_2$
u_0	$\boxed{013}_1$	$\mathfrak{M}_{24}$	a_6	$\boxed{0134}$	$\mathfrak{M}_2$
h_0	$\boxed{234}$	$\mathfrak{M}_3$	a_7	$\boxed{0134}$	$\mathfrak{M}_2$
j_0	$\boxed{0124}$	$\mathfrak{M}_{18}^{op}$	a_8	$\boxed{0134}$	$\mathfrak{M}_2$
k_0	$\boxed{0124}$	$\mathfrak{M}_{23}$	b_1	$\boxed{134}$	$\mathfrak{M}_9$
n_0	$\boxed{0124}$	$\mathfrak{M}_{19}$	b_2	$\boxed{134}$	$\mathfrak{M}_9$
p_0	$\boxed{124}$	$\mathfrak{M}_{21}$			

We can show that the set $\bigcup_{\alpha\in I} M(\alpha)$ forms an l.c.r. set of $W_{(16)}$.

4.2 Now consider the two-sided cell $W_{(10)}$. Let $x = w_{0234}$, $y = x \cdot \mathbf{1232}$, $x_1 = x \cdot \mathbf{12340}$, $x_2 = x_1 \cdot \mathbf{21234213}$, $y_0 = y \cdot \mathbf{1}$, $a_1 = x_1 \cdot \mathbf{1}$, $b_1 = x_2 \cdot \mathbf{4}$, $y_1 = y_0 \cdot \mathbf{043}$, $z = y_0 \cdot \mathbf{014}$, $y_2 = y_0 \cdot \mathbf{01234}$, $w = y_2 \cdot \mathbf{323123}$, $a_2 = y_1 \cdot \mathbf{1}$, $z_0 = z \cdot \mathbf{2}$, $a_3 = y_2 \cdot \mathbf{3}$, $w_0 = w \cdot \mathbf{0}$, $w_1 = a_3 \cdot \mathbf{2310}$. $w_2 = w_0 \cdot \mathbf{431234}$, $v = w_2 \cdot \mathbf{21}$, $a_4 = w_1 \cdot \mathbf{4}$, $b_2 = w_2 \cdot \mathbf{1}$ and $v_0 = v \cdot \mathbf{2}$. Then $y, x_1, x_2 \in M(x)$, $y_1, y_2, z, w \in M(y_0)$ and $w_1, w_2, v \in M(w_0)$. We can check that $\{y, y_0\}$, $\{x_1, a_1\}$, $\{x_2, b_1\}$, $\{y_1, a_2\}$, $\{z, z_0\}$, $\{y_2, a_3\}$, $\{w, w_0\}$, $\{w_1, a_4\}$, $\{w_2, b_2\}$ and $\{v, v_0\}$ are all primitive pairs. Let $I = \{x, y_0, z_0, w_0, v_0, a_i\ (1 \le i \le 4), b_l\ (l = 1, 2)\}$. Then $I \subset W_{(10)}$. By the techniques of 3.4, we see that $\mathfrak{M}(\alpha)$ $(\alpha \in I)$ are all essential. So we get the following table.

α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$	α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$
x	$\boxed{0234}$	$\mathfrak{M}_{18}^{op}$	a_2	$\boxed{0134}$	$\mathfrak{M}_2$
y_0	$\boxed{123}_1$	$\mathfrak{M}_3$	a_3	$\boxed{0134}$	$\mathfrak{M}_2$
z_0	$\boxed{0124}$	$\mathfrak{M}_{23}$	a_4	$\boxed{0134}$	$\mathfrak{M}_2$
w_0	$\boxed{023}_1$	$\mathfrak{M}_{19}$	b_1	$\boxed{134}$	$\mathfrak{M}_9$
v_0	$\boxed{124}$	$\mathfrak{M}_{21}$	b_2	$\boxed{134}$	$\mathfrak{M}_9$
a_1	$\boxed{0134}$	$\mathfrak{M}_2$			

It can be shown that the set $\bigcup_{\alpha\in I} M(\alpha)$ forms an l.c.r. set of $W_{(10)}$.

4.3 Next consider the two-sided cell $W_{(13)}$. Let $x = w_{0234} \cdot \mathbf{1232143234}$, $x_0 = x \cdot \mathbf{0}$,

$x_1 = x\mathbf{12401234}$, $a_1 = x_1 \cdot \mathbf{3}$, $y = x_1\mathbf{32312342}$, $y_0 = y \cdot \mathbf{0}$, $a_2 = y_0\mathbf{4}$, $z = x_1 \cdot \mathbf{23123}$, $z_0 = z \cdot \mathbf{4}$, $z_1 = z_0\mathbf{1234031}$, $a_3 = z_1 \cdot \mathbf{3}$, $w = z_1\mathbf{42}$, $w_0 = w \cdot \mathbf{4}$, $w_1 = w \cdot \mathbf{34}$, $a_4 = w_1 \cdot \mathbf{3}$, $v = z_0\mathbf{1243}$, $v_0 = v\mathbf{1}$, $v_1 = v_0\mathbf{420}$, $a_5 = v_1 \cdot \mathbf{1}$, $u = w_1\mathbf{13210}$, $u_0 = u \cdot \mathbf{2}$, $u_1 = u_0 \cdot \mathbf{32432}$, $a_6 = u_1\mathbf{2}$, $h = x_0 \cdot \mathbf{1232}$, $h_0 = h \cdot \mathbf{1}$, $v_2 = v_1 \cdot \mathbf{21234234}$, $b_1 = v_2 \cdot \mathbf{1}$, $u_2 = u_1 \cdot \mathbf{1243}$, $b_2 = u_2 \cdot \mathbf{4}$, $j = u_2\mathbf{4121}$ and $j_0 = j \cdot \mathbf{2}$. Let $I = \{x, x_0, y_0, z_0, w_0, v_0, u_0, h_0, j_0, a_i\ (1 \le i \le 6), b_l\ (l = 1, 2)\}$. We shall show $I \subset W_{(13)}$. It is known already that $x \in W_{(13)}$ (see 3.2). We also have the relations $x_1, y, z \in M(x)$, $w, z_1, w_1, v, u \in M(z_0)$, $h \in M(x_0)$, $v_1, v_2 \in M(v_0)$ and $u_1, u_2, j \in M(u_0)$. On the other hand, $\{x, x_0\}$, $\{x_1, a_1\}$, $\{y, y_0\}$, $\{y_0, a_2\}$, $\{z, z_0\}$, $\{z_1, a_3\}$, $\{w, w_0\}$, $\{w_1, a_4\}$, $\{v, v_0\}$, $\{v_1, a_5\}$, $\{u, u_0\}$, $\{u_1, a_6\}$, $\{h, h_0\}$, $\{v_2, b_1\}$, $\{u_2, b_2\}$ and $\{j, j_0\}$ are all primitive pairs. So $I \subset W_{(13)}$. By the techniques of 3.4, we see that for $\alpha \in I$, the graph $\mathfrak{M}(\alpha)$ is essential if and only if $\alpha \in \{z_0, w_0, v_0, u_0, j_0, a_i\ (1 \le i \le 6), b_l\ (l = 1, 2)\}$. so for these α, we have $\mathfrak{M}_e(\alpha) = \mathfrak{M}(\alpha)$. On the other hand, we can find an essential subgraph $\mathfrak{M}_e(\beta)$ in each $\mathfrak{M}(\beta)$, $\beta \in \{x, x_0, y_0, h_0\}$. We get the following table.

α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$	α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$
x	$\boxed{\mathbf{234}}$	$\mathfrak{M}_{25}^{\mathrm{op}}$	a_1	$\boxed{\mathbf{0134}}$	$\mathfrak{M}_2$
x_0	$\boxed{\mathbf{0234}}$	$\mathfrak{M}_{22}^{\mathrm{op}}$	a_2	$\boxed{\mathbf{03}}$	$\mathfrak{M}_2$
y_0	$\boxed{\mathbf{034}}$	$\mathfrak{M}_{22}$	a_3	$\boxed{\mathbf{0134}}$	$\mathfrak{M}_2$
z_0	$\boxed{\mathbf{234}}$	$\mathfrak{M}_3$	a_4	$\boxed{\mathbf{0134}}$	$\mathfrak{M}_2$
w_0	$\boxed{\mathbf{0124}}$	$\mathfrak{M}_{23}$	a_5	$\boxed{\mathbf{0134}}$	$\mathfrak{M}_2$
v_0	$\boxed{\mathbf{23}}_1$	$\mathfrak{M}_{18}^{\mathrm{op}}$	a_6	$\boxed{\mathbf{03}}$	$\mathfrak{M}_2$
u_0	$\boxed{\mathbf{0124}}$	$\mathfrak{M}_{19}$	b_1	$\boxed{\mathbf{134}}$	$\mathfrak{M}_9$
h_0	$\boxed{\mathbf{123}}$	$\mathfrak{M}_{25}$	b_2	$\boxed{\mathbf{134}}$	$\mathfrak{M}_9$
j_0	$\boxed{\mathbf{124}}$	$\mathfrak{M}_{21}$			

Let $M_e(\alpha)$ be the vertex set of the chosen essential subgraph $\mathfrak{M}_e(\alpha)$ (such a notation will be used throughout the remaining part of the paper). Then we can show that the set $\bigcup_{\alpha \in I} M_e(\alpha)$ forms an l.c.r. set of $W_{(13)}$.

4.4 Let us consider the two-sided cell $\Omega(w_{123})$. Let $x = w_{123}$, $y = x \cdot \mathbf{014}$, $z = x \cdot \mathbf{432101}$, $y_0 = y \cdot \mathbf{2}$, $a_1 = y \cdot \mathbf{3}$, $z_0 = z \cdot \mathbf{2}$, $z_1 = z_0 \cdot \mathbf{32343}$, $a_2 = z_1\mathbf{4}$, $w = z_1 \cdot \mathbf{1213}$, $v = w \cdot \mathbf{21}$, $w_0 = w \cdot \mathbf{4}$ and $v_0 = v \cdot \mathbf{4}$. Then $y, z \in M(x)$ and $z_1, w, v \in M(z_0)$. We can check that $\{y, y_0\}$, $\{y, a_1\}$, $\{z, z_0\}$, $\{z_1, a_2\}$, $\{w, w_0\}$ and $\{v, v_0\}$ are all primitive pairs. Let $I = \{x, y_0, z_0, w_0, v_0, a_1, a_2\}$. Then $I \subset \Omega(w_{123})$. It can be shown that the graph $\mathfrak{M}(\alpha)$ is essential if $\alpha \in \{z_0, w_0, v_0, a_1, a_2\}$ and is not essential if $\alpha \in \{x, y_0\}$. By choosing some essential subgraphs from these graphs, we get the following table.

α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$	α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$
x	$\boxed{123}$	$\mathfrak{M}_7$	v_0	$\boxed{124}$	$\mathfrak{M}_{21}$
y_0	$\boxed{0124}$	$\mathfrak{M}_{22}$	a_1	$\boxed{0134}$	$\mathfrak{M}_2$
z_0	$\boxed{0124}$	$\mathfrak{M}_{19}$	a_2	$\boxed{03}$	$\mathfrak{M}_2$
w_0	$\boxed{134}$	$\mathfrak{M}_9$			

We can show that the union $\bigcup_{\alpha \in I} M_e(\alpha)$ forms an l.c.r. set of $\Omega(w_{123})$.

4.5 From the results in 4.4, we see that $w_{234} \notin \Omega(w_{123})$. This implies $\Omega(w_{234}) = W_{(9)} \setminus \Omega(w_{123})$ by 1.10. Now we consider the two-sided cell $\Omega(w_{234})$.

Let $x = w_{\mathbf{234}}$, $x_1 = x \cdot \mathbf{10213}$, $y = x_1 \cdot \mathbf{234}$, $z = y \cdot \mathbf{3231234}$, $a_1 = x_1 \cdot \mathbf{4}$, $y_0 = y \cdot \mathbf{3}$, $z_0 = z \cdot \mathbf{0}$, $z_1 = z_0\mathbf{24}$, $y_1 = y_0 \cdot \mathbf{2123}$, $a_2 = z_1\mathbf{2}$, $b_1 = y_1 \cdot \mathbf{4}$, $w = z_0 \cdot \mathbf{123421}$, $z_2 = z_0 \cdot \mathbf{2123}$, $z_3 = z_2 \cdot \mathbf{2134}$, $w_0 = w \cdot \mathbf{2}$, $b_2 = z_2 \cdot \mathbf{4}$ and $b_3 = z_3 \cdot \mathbf{3}$. Then $x_1, y, z \in M(x)$, $y_1 \in M(y_0)$ and $w, z_1, z_2, z_3 \in M(z_0)$. We can check that $\{x_1, a_1\}$, $\{y, y_0\}$, $\{z, z_0\}$, $\{z_1, a_2\}$, $\{y_1, b_1\}$, $\{w, w_0\}$, $\{z_2, b_2\}$ and $\{z_3, b_3\}$ are all primitive pairs. Let $I = \{x, y_0, z_0, w_0, a_i\ (i = 1, 2), b_l\ (1 \le l \le 3)\}$. Then $I \subset \Omega(w_{234})$. We can show that $\mathfrak{M}(\alpha)$ $(\alpha \in I)$ are all essential and hence get the following table.

α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$	α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$
x	$\boxed{234}$	$\mathfrak{M}_{16}$	a_2	$\boxed{03}$	$\mathfrak{M}_2$
y_0	$\boxed{034}_1$	$\mathfrak{M}_{13}$	b_1	$\boxed{134}$	$\mathfrak{M}_9$
z_0	$\boxed{034}_1$	$\mathfrak{M}_{15}$	b_2	$\boxed{134}$	$\mathfrak{M}_9$
w_0	$\boxed{124}$	$\mathfrak{M}_{21}$	b_3	$\boxed{134}$	$\mathfrak{M}_9$
a_1	$\boxed{0134}$	$\mathfrak{M}_2$			

The set $\bigcup_{\alpha \in I} M(\alpha)$ can be shown to form an l.c.r. set of $\Omega(w_{234})$.

4.6 Next consider the two-sided cell $W_{(7)}$. Let $x = w_{\mathbf{0124}}$, $y = x \cdot \mathbf{3}$, $z = x \cdot \mathbf{323431234}$, $w = x \cdot \mathbf{21}$, $y_0 = y \cdot \mathbf{4}$, $z_0 = z \cdot \mathbf{1}$ and $w_0 = w \cdot \mathbf{2}$. Then $y, z, w \in M(x)$. We can check that $\{y, y_0\}$, $\{z, z_0\}$ and $\{w, w_0\}$ are all primitive pairs. Let $I = \{x, y_0, z_0, w_0\}$. Then this implies $I \subset W_{(7)}$. It can be shown that all the graphs $\mathfrak{M}(\alpha)$ $(\alpha \in I)$ are essential. Hence we have the following table.

α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$	α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$
x	$\boxed{0124}$	$\mathfrak{M}_{19}$	z_0	$\boxed{134}$	$\mathfrak{M}_9$
y_0	$\boxed{0134}$	$\mathfrak{M}_2$	w_0	$\boxed{124}$	$\mathfrak{M}_{21}$

We can show that the set $\bigcup_{\alpha \in I} M(\alpha)$ forms an l.c.r. set of $W_{(7)}$.

4.7 Now consider the two-sided cell $\Omega(w_{012})$. Let $x = w_{012}$, $y = x \cdot \mathbf{342132}$, $z = x \cdot \mathbf{323432123}$, $y_0 = y \cdot \mathbf{3}$, $z_0 = z \cdot \mathbf{4}$, $w = y_0 \cdot \mathbf{14}$ and $w_0 = w \cdot \mathbf{2}$. Then $y, z \in M(x)$ and $w \in M(y_0)$. One can check that $\{y, y_0\}$, $\{z, z_0\}$ and $\{w, w_0\}$ are all primitive pairs. Let $I = \{x, y_0, z_0, w_0\}$. Then $I \subset \Omega(w_{012})$. We can show that a graph $\mathfrak{M}(\alpha)$ is essential if $\alpha \in \{z_0, w_0\}$, and is not essential if $\alpha \in \{x, y_0\}$. Hence by choosing an essential subgraph $\mathfrak{M}_e(\alpha)$ from $\mathfrak{M}(\alpha)$ for $\alpha \in I$, we get the following table.

α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$	α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$
x	$\boxed{012}$	$\mathfrak{M}_{17}$	z_0	$\boxed{134}$	$\mathfrak{M}_9$
y_0	$\boxed{23}$	$\mathfrak{M}_{12}$	w_0	$\boxed{124}$	$\mathfrak{M}_{21}$

It can be shown that the set $\bigcup_{\alpha \in I} M_e(\alpha)$ forms an l.c.r. set of $\Omega(w_{012})$.

4.8 Finally consider the two-sided cell $W_{(6)} \setminus \Omega(w_{012})$. By 1.10, it is equal to $\Omega(w_{0134})$ since $w_{0134} \notin \Omega(w_{012})$ by 4.7. Let $x = w_{0134}$, $y = x \cdot \mathbf{23}$, $y_0 = y \cdot \mathbf{2}$, $y_1 = y_0 \cdot \mathbf{1423}$, $b_1 = y_1 \cdot \mathbf{4}$, $y_2 = y_0 \cdot \mathbf{12432134}$, $b_2 = y_2 \cdot \mathbf{3}$, $z = y_1\mathbf{1421}$ and $z_0 = z \cdot \mathbf{2}$. Then $y \in M(x)$ and $y_1, y_2, z \in M(y_0)$. We can check that $\{y, y_0\}$, $\{y_1, b_1\}$, $\{y_2, b_2\}$ and $\{z, z_0\}$ are all primitive pairs. Let $I = \{x, y_0, z_0, b_1, b_2\}$. Then $I \subset \Omega(w_{0134})$. We can show that all $\mathfrak{M}(\alpha)$ $(\alpha \in I)$ are essential. So we have the following table.

α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$	α	position of α in $\mathfrak{M}(\alpha)$	isom. cls of $\mathfrak{M}(\alpha)$
x	$\boxed{0134}$	$\mathfrak{M}_2$	b_1	$\boxed{134}$	$\mathfrak{M}_9$
y_0	$\boxed{023}$	$\mathfrak{M}_{15}$	b_2	$\boxed{134}$	$\mathfrak{M}_9$
z_0	$\boxed{124}$	$\mathfrak{M}_{21}$			

We can show that the set $\bigcup_{\alpha \in I} M(\alpha)$ forms an l.c.r. set of $\Omega(w_{0134})$.

4.9 To be complete, we shall also give an l.c.r. set for each two-sided cell Ω with $a(\Omega) \leq 5$ or $a(\Omega) = 24$. These results are either obtained from my paper [21] (for the two-sided cells of a-values 3, 4, 5) or by a direct calculation (for the two-sided cells of a-values 0, 1, 2, 24). $W_{(0)}$ is the l.c.r. set of itself.

(1) Let $x = \mathbf{4} \in W_{(1)}$. Take an essential subgraph $\mathfrak{M}_e(x)$ in $\mathfrak{M}(x)$ such that $\mathfrak{M}_e(x)$ is isomorphic to $\mathfrak{M}_1$ with x the vertex labelled by $\boxed{\mathbf{4}}$. The set $M_e(x)$ forms an l.c.r. set of $W_{(1)}$.

(2) Let $x = \mathbf{24} \in W_{(2)}$. Take an essential subgraph $\mathfrak{M}_e(x)$ in $\mathfrak{M}(x)$ such that $\mathfrak{M}_e(x)$ is isomorphic to $\mathfrak{M}_4$ with x the vertex labelled by $\boxed{\mathbf{24}}$. The set $M_e(x)$ forms an l.c.r. set of $W_{(2)}$.

There are two two-sided cells with a-value 3: $\Omega(w_{01})$ and $\Omega(w_{34})$.

(3) Let $x = w_{01}$ and $y = w_{024}$. Then the graph $\mathfrak{M}(x)$ (resp. $\mathfrak{M}(y)$) is essential, isomorphic to $\mathfrak{M}_5$ (resp. $\mathfrak{M}_8$), and has x (resp. y) as its vertex labelled by $\boxed{\mathbf{01}}$ (resp. $\boxed{\mathbf{024}}$). The union $M(x) \bigcup M(y)$ forms an l.c.r. set of $\Omega(w_{01})$.

(4) Let $x = w_{34}$. Then the graph $\mathfrak{M}(x)$ is essential, isomorphic to $\mathfrak{M}_{20}$, and has x as its vertex labelled by $\boxed{\mathbf{34}}$. The set $M(x)$ forms an l.c.r. set of $\Omega(w_{34})$.

(5) Let $x = w_{034}$, $y = w_{014}$ and $z = w_{23}$ be in $W_{(4)}$. Then the graphs $\mathfrak{M}(x)$, $\mathfrak{M}(y)$ and $\mathfrak{M}(z)$ are all essential, isomorphic to $\mathfrak{M}_9$, $\mathfrak{M}_{21}$, $\mathfrak{M}_{10}$, respectively. The elements x, y, z are the vertices labelled by $\boxed{\mathbf{034}}$, $\boxed{\mathbf{014}}$ and $\boxed{\mathbf{23}}$ in $\mathfrak{M}_9$, $\mathfrak{M}_{21}$, $\mathfrak{M}_{10}$, respectively. The set $M(x) \bigcup M(y) \bigcup M(z)$ forms an l.c.r. set of $W_{(4)}$.

(6) Let $x = w_{023}$, $b_1 = x \cdot \mathbf{12343}$, $y_0 = x \cdot \mathbf{431232}$, $z_0 = b_1 \cdot \mathbf{21}$ and $b_2 = y_0 \cdot \mathbf{431}$. Then the graphs $\mathfrak{M}(b_1)$, $\mathfrak{M}(b_2)$ and $\mathfrak{M}(z_0)$ are essential but $\mathfrak{M}(x)$ and $\mathfrak{M}(y_0)$ are not. Let $I = \{x, y_0, z_0, b_1, b_2\}$. By choosing an essential subgraph $\mathfrak{M}_e(\alpha)$ from $\mathfrak{M}(\alpha)$ for $\alpha \in I$, we get the following table.

α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$	α	position of α in $\mathfrak{M}_e(\alpha)$	isom. cls of $\mathfrak{M}_e(\alpha)$
x	$\boxed{\mathbf{023}}$	$\mathfrak{M}_{14}$	b_1	$\boxed{\mathbf{134}}$	$\mathfrak{M}_9$
y_0	$\boxed{\mathbf{23}}$	$\mathfrak{M}_{11}$	b_2	$\boxed{\mathbf{134}}$	$\mathfrak{M}_9$
z_0	$\boxed{\mathbf{124}}$	$\mathfrak{M}_{21}$			

The set $\bigcup_{\alpha\in I} M_e(\alpha)$ forms an l.c.r. set of $W_{(5)}$.

4.10 According to the results of [18], [19], we know that there are $|W_0|$ left cells of W_a in the two-sided cell $W_{(24)}$ each of which is represented by a sign type over the symbol set $\{+,-\}$ (see [17] for the definition). It is known that there is a unique shortest element in each of such left cells and that the l.c.r. set consisting of such elements can be described explicitly as below.

$$M = \{w \in W_{(24)} \mid sw \notin W_{(24)} \quad \text{for any} \quad s \in \mathcal{L}(w)\} \qquad (\textit{loc. cit.}).$$

It can be shown that for any $x \in M$, the graph $\mathfrak{M}(x)$ is essential with $M(x) \subseteq M$. Thus it remains to find all the left cell graphs of $W_{(24)}$.

It is known that if X is a sign type over $\{+,-\}$ and if Y is obtained from X by transposing the symbols $+$ and $-$, then Y is also a sign type over $\{+,-\}$. Call Y the opposed sign type of X and denote it by X^{op}. For a given left cell graph $\mathfrak{M}$ in $W_{(24)}$, we can replace each vertex (represented by a sign type) of $\mathfrak{M}$ by its opposed sign type to get another left cell graph of $W_{(24)}$ opposed to $\mathfrak{M}$. Keeping this fact in mind, we can describe the left cell graphs of $W_{(24)}$ as below. Define the following sign types:

$$X_{01} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ -& &+&+& &- \\ +& &+&-& &- \\ -& &+&+& &+ \\ -& &-&+& &- \end{array} \quad X_{02} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &-&+& &- \\ +& &+&-& &- \\ -& &+&+& &+ \\ -& &-&+& &- \end{array} \quad X_{03} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &-&+& &- \\ +& &+&-& &+ \\ -& &+&+& &+ \\ -& &-&+& &- \end{array} \quad X_{04} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &+&+& &- \\ +& &+&-& &- \\ -& &+&+& &+ \\ -& &-&+& &- \end{array}$$

$$X_{05} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ -& &+&+& &- \\ +& &+&-& &+ \\ -& &+&+& &+ \\ -& &-&+& &- \end{array} \quad X_{06} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &+&+& &- \\ +& &+&+& &- \\ -& &+&+& &+ \\ -& &-&+& &- \end{array} \quad X_{07} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &+&+& &- \\ +& &+&-& &+ \\ -& &+&+& &+ \\ -& &-&+& &- \end{array} \quad X_{08} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ -& &+&+& &- \\ +& &+&+& &- \\ -& &+&+& &+ \\ -& &-&+& &- \end{array}$$

$$X_{11} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &-&+& &- \\ +& &+&-& &+ \\ -& &+&+& &+ \\ -& &-&+& &+ \end{array} \quad X_{12} = \begin{array}{cccccc} &-& & &+& \\ &+& & &+& \\ &+& & &-& \\ &-& & &+& \\ +& &-&+& &- \\ +& &+&-& &+ \\ -& &+&+& &+ \\ -& &-&+& &+ \end{array} \quad X_{13} = \begin{array}{cccccc} &+& & &+& \\ &+& & &+& \\ &+& & &-& \\ &+& & &+& \\ +& &-&+& &- \\ +& &+&-& &+ \\ -& &+&+& &+ \\ -& &-&+& &+ \end{array}$$

$$X_{21} = \begin{array}{ccccccc} & + & & & & + & \\ & + & & & & + & \\ & + & & & & + & \\ & + & & & & - & \\ - & & + & & + & & - \\ - & & + & & + & & + \\ + & & + & & + & & - \\ - & & + & & - & & - \end{array} \qquad X_{22} = \begin{array}{ccccccc} & + & & & & + & \\ & + & & & & + & \\ & - & & & & + & \\ & + & & & & - & \\ - & & + & & + & & - \\ - & & + & & + & & + \\ + & & + & & + & & - \\ - & & + & & - & & - \end{array} \qquad X_{31} = \begin{array}{ccccccc} & + & & & & - & \\ & + & & & & + & \\ & + & & & & + & \\ & - & & & & + & \\ - & & + & & - & & - \\ + & & - & & + & & - \\ + & & + & & + & & - \\ + & & + & & + & & - \end{array} \qquad X_{32} = \begin{array}{ccccccc} & + & & & & - & \\ & + & & & & + & \\ & - & & & & + & \\ & - & & & & + & \\ - & & + & & - & & - \\ + & & - & & + & & - \\ + & & + & & + & & - \\ + & & + & & + & & - \end{array}$$

$$X_4 = \begin{array}{ccccccc} & + & & & & + & \\ & + & & & & + & \\ & + & & & & + & \\ & + & & & & + & \\ + & & + & & + & & + \\ + & & + & & + & & + \\ + & & + & & + & & + \\ + & & + & & + & & + \end{array} \qquad X_5 = \begin{array}{ccccccc} & + & & & & + & \\ & + & & & & + & \\ & + & & & & + & \\ & + & & & & - & \\ - & & + & & + & & + \\ + & & + & & + & & + \\ + & & + & & + & & - \\ - & & + & & - & & - \end{array} \qquad X_6 = \begin{array}{ccccccc} & + & & & & + & \\ & + & & & & + & \\ & + & & & & + & \\ & + & & & & - & \\ - & & + & & + & & - \\ + & & + & & + & & + \\ + & & + & & + & & - \\ - & & + & & - & & - \end{array}$$

$$X_7 = \begin{array}{ccccccc} & - & & & & - & \\ & - & & & & - & \\ & - & & & & - & \\ & + & & & & - & \\ + & & - & & + & & - \\ - & & + & & - & & + \\ - & & - & & - & & + \\ - & & - & & - & & + \end{array} \qquad X_8 = \begin{array}{ccccccc} & - & & & & - & \\ & - & & & & - & \\ & - & & & & - & \\ & - & & & & - & \\ - & & - & & - & & - \\ - & & + & & - & & - \\ - & & - & & - & & + \\ - & & - & & - & & + \end{array} \qquad X_9 = \begin{array}{ccccccc} & - & & & & - & \\ & - & & & & - & \\ & - & & & & - & \\ & - & & & & - & \\ + & & - & & - & & - \\ - & & + & & - & & + \\ - & & - & & - & & + \\ - & & - & & - & & + \end{array}$$

Then $\mathfrak{M}_L(X_P)$, $\mathfrak{M}_L(X_P^{\mathrm{op}})$ with $P \in \{0h, 1i, 1j, 2k, m \mid 1 \leq h \leq 8, 1 \leq i \leq 3, 1 \leq j, k \leq 2, 4 \leq m \leq 9\}$, form a complete set of left cell graphs of $W_{(24)}$. The corresponding isomorphism classes of the graphs $\mathfrak{M}_L(X_P)$ are listed as below.

X	position of X in $\mathfrak{M}_L(X)$	isom. cls of $\mathfrak{M}_L(X)$	X	position of X in $\mathfrak{M}_L(X)$	isom. cls of $\mathfrak{M}_L(X)$
X_{0h} $(1 \leq h \leq 8)$	$\boxed{0134}$	$\mathfrak{M}_2$	X_5	$\boxed{0124}$	$\mathfrak{M}_{19}$
X_{1i} $(1 \leq i \leq 3)$	$\boxed{034}$	$\mathfrak{M}_9$	X_6	$\boxed{0124}_1$	$\mathfrak{M}_{24}$
X_{2j} $(1 \leq j \leq 2)$	$\boxed{0124}$	$\mathfrak{M}_{23}$	X_7	$\boxed{234}$	$\mathfrak{M}_3$
X_{3k} $(1 \leq k \leq 2)$	$\boxed{1}$	$\mathfrak{M}_{18}$	X_8	$\boxed{234}$	$\mathfrak{M}_6$
X_4	$\boxed{0}$	$\mathfrak{M}_{21}$	X_9	$\boxed{234}$	$\mathfrak{M}_{26}$

4.11 It is worth to mention that owing to the priority we made on the processes of the algorithm (see 2.4), all the elements so far we have got for our l.c.r. set of $W_a(\widetilde{F}_4)$ are only by Processes (**A**) and (**B**), none of them by Process (**C**).

4.12 Let Ω be a two-sided cell of $W_a = W_a(\widetilde{F}_4)$. We denote by $n(\Omega)$ the number of left cells of W_a in Ω, by $\mathbf{u}(\Omega)$ the corresponding unipotent class of the complex algebraic group G of type F_4 under the map in Theorem 1.8 (presented by its type with the notation as in [4, Chapter 13], also see 4.13), and by $A(\Omega) = C(u)/C(u)^{\circ}$ the component group of the centralizer of an element $u \in \mathbf{u}(\Omega)$, the latter makes sense since it is independent of the choice of u up to isomorphism. Then by the above results, Theorem 1.8 and the results in [4, Chapter 13], we have the following table.

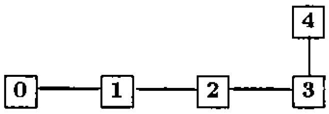
Ω	$n(\Omega)$	$\mathbf{u}(\Omega)$	$A(\Omega)$	Ω	$n(\Omega)$	$\mathbf{u}(\Omega)$	$A(\Omega)$
$W_{(0)}$	1	F_4	1	$\Omega(w_{0134})$	96	$\widetilde{A}_2+A_1$	1
$W_{(1)}$	5	$F_4(a_1)$	S_2	$W_{(7)}$	96	$A_2+\widetilde{A}_1$	1
$W_{(2)}$	14	$F_4(a_2)$	S_2	$\Omega(w_{123})$	168	A_2	S_2
$\Omega(w_{01})$	24	C_3	1	$\Omega(w_{234})$	192	$\widetilde{A}_2$	1
$\Omega(w_{34})$	24	B_3	1	$W_{(10)}$	288	$A_1+\widetilde{A}_1$	1
$W_{(4)}$	42	$F_4(a_3)$	S_4	$W_{(13)}$	432	$\widetilde{A}_1$	S_2
$W_{(5)}$	96	$C_3(a_1)$	S_2	$W_{(16)}$	576	A_1	1
$\Omega(w_{012})$	96	B_2	S_2	$W_{(24)}$	1152	1	1

Thus the total number of left cells in $W_a(\widetilde{F}_4)$ is 3302.

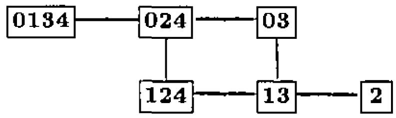
4.13 According to Bala-Carter Theorem, there is a bijective map between unipotent conjugacy classes of G and G-classes of pairs $(L, P_{L'})$, where L is a Levi subgroup of G and $P_{L'}$ is a distinguished parabolic subgroup of the semisimple part L' of L. The unipotent class corresponding to the pair $(L, P_{L'})$ contains the dense orbit of $P_{L'}$ on its unipotent radical (see [4, Theorem 5.9.6]). Now for a two-sided cell Ω of W_a, let $(L, P_{L'})$ be the pair associated to the unipotent conjugacy class $\mathbf{u}(\Omega)$ of G. Then the type of $\mathbf{u}(\Omega)$ listed in the above table just records such a correspondence. Let W_L be the Weyl group of L'. Then from the above table, we see that the number $n(\Omega)$ of left cells in Ω is equal to $|W_0|/|W_L|$ if and only if the corresponding component group $A(\Omega)$ is trivial. Note that such a phenomenon does not always occur in an irreducible affine Weyl group. By the existing datum, we see that it occurs in the affine Weyl groups of type $\widetilde{A}_n$, $n \geq 1$, of rank ≤ 4 with its type $\neq \widetilde{D}_4$, and in the two-sided cells Ω of $W_a(\widetilde{B}_\ell)$ ($\ell \geq 3$) or $W_a(\widetilde{C}_m)$ ($m \geq 2$) with $a(\Omega) \leq 4$, but not in the affine Weyl groups of types $\widetilde{D}_n$, $n \geq 4$, $\widetilde{E}_7$ and $\widetilde{E}_8$.

4.14 It has been shown in [25] that the Lusztig bijective map $\mathbf{u} \longrightarrow c(\mathbf{u})$ from the set of unipotent conjugacy classes of the complex algebraic group G of type F_4 to the set of two-sided cells of $W_a(\widetilde{F}_4)$ is order-preserving: $\mathbf{u}$ is contained in the closure of $\mathbf{u}'$ (in the variety of unipotent elements of G) if and only if $c(\mathbf{u}) \underset{LR}{\leq} c(\mathbf{u}')$ (see 1.8). For a two-sided cell Ω of W_a, let $T(\Omega)$ be the set of all subsets I of S such that $I = \mathcal{L}(w)$ for some $w \in \Omega$. Then we can find the following fact in the group $W_a(\widetilde{F}_4)$: two two-sided cells Ω, $\Omega' \neq \{e\}$ satisfy the relation $\Omega \underset{LR}{\leq} \Omega'$ if and only if $T(\Omega) \supseteq T(\Omega')$. This result may be expected to hold in any affine Weyl group.

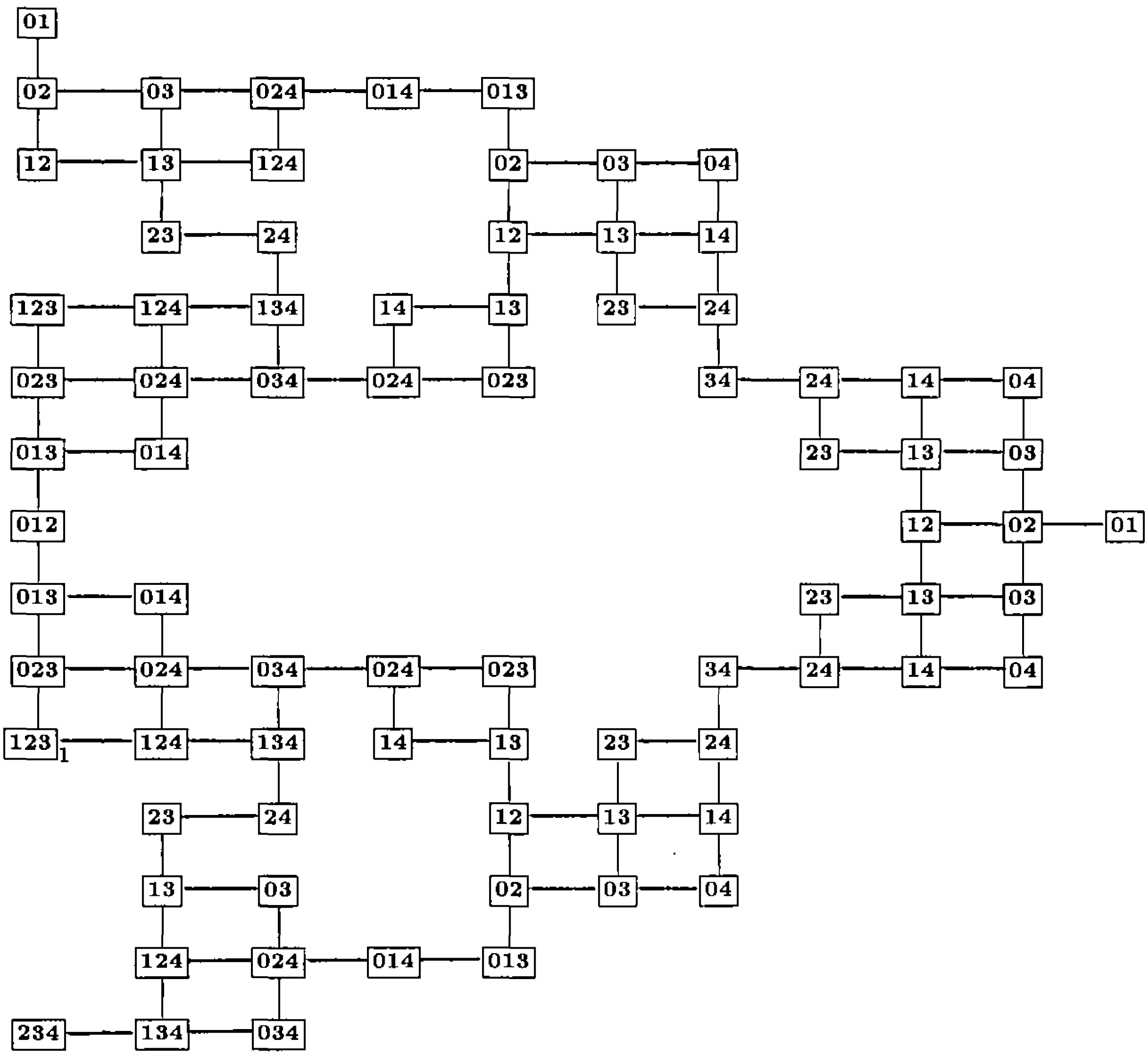
4.15 The following are the graphs $\mathfrak{M}_i$, $1 \leq i \leq 26$, mentioned in sections 3 and 4.



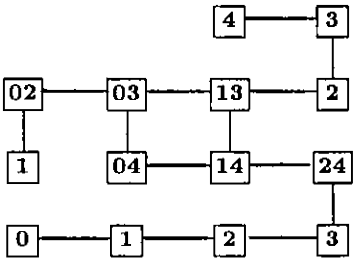
$\mathfrak{M}_1$



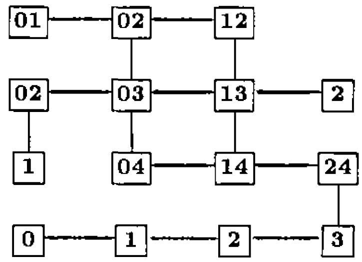
$\mathfrak{M}_2$



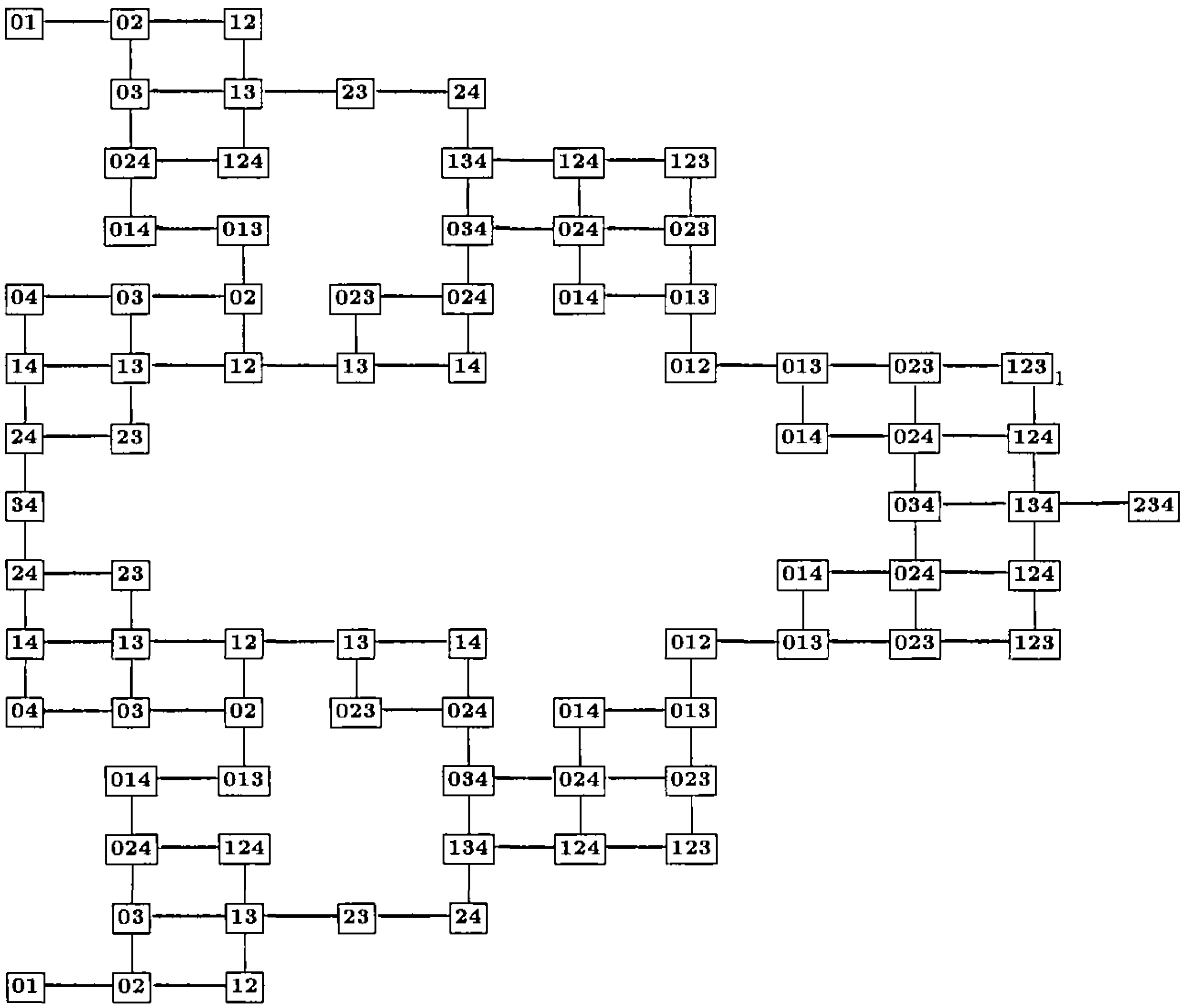
$\mathfrak{M}_3$



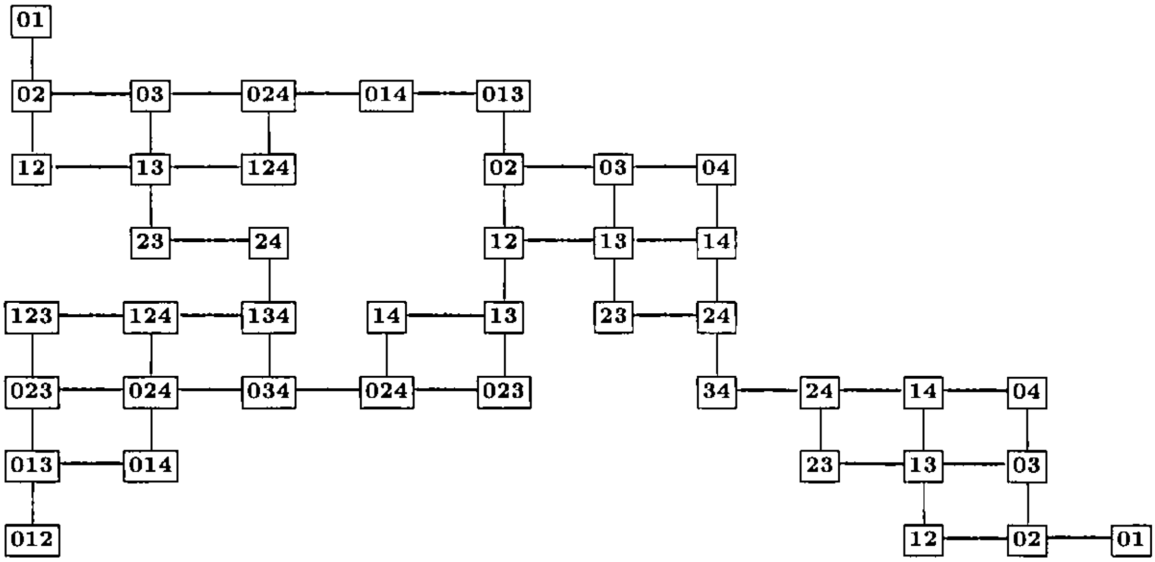
$\mathfrak{M}_4$



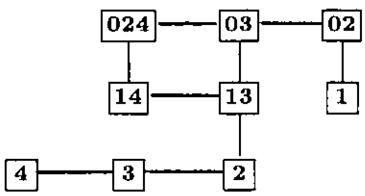
$\mathfrak{M}_5$



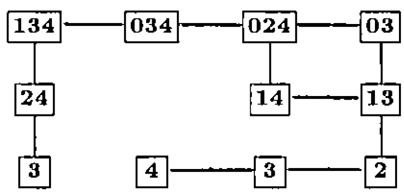
$\mathfrak{M}_6$



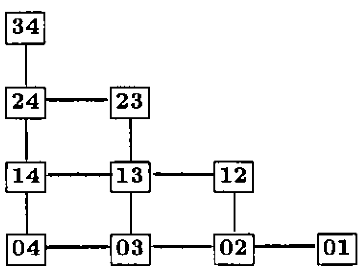
$\mathfrak{M}_7$



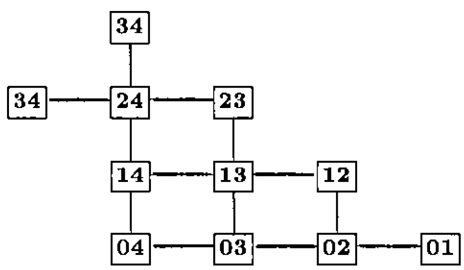
$\mathfrak{M}_8$



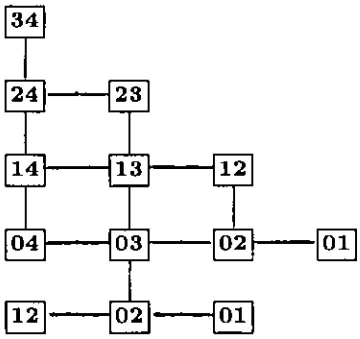
$\mathfrak{M}_9$



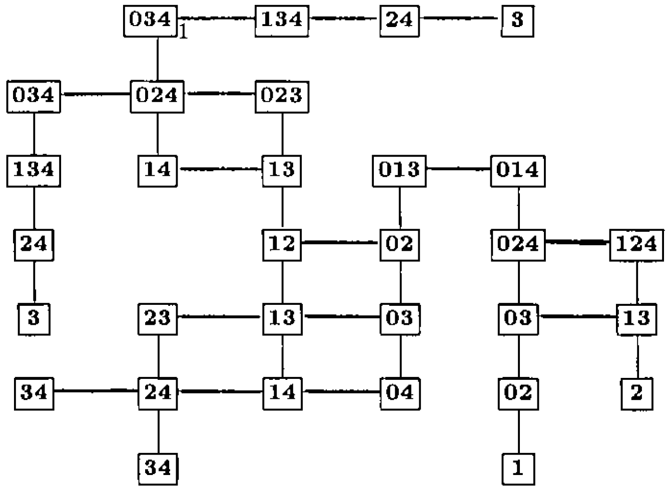
$\mathfrak{M}_{10}$



$\mathfrak{M}_{11}$



$\mathfrak{M}_{12}$



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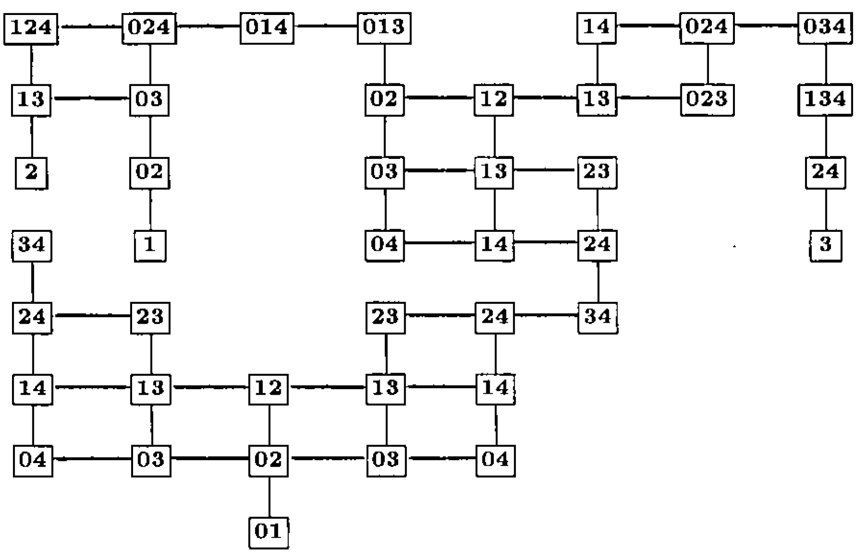
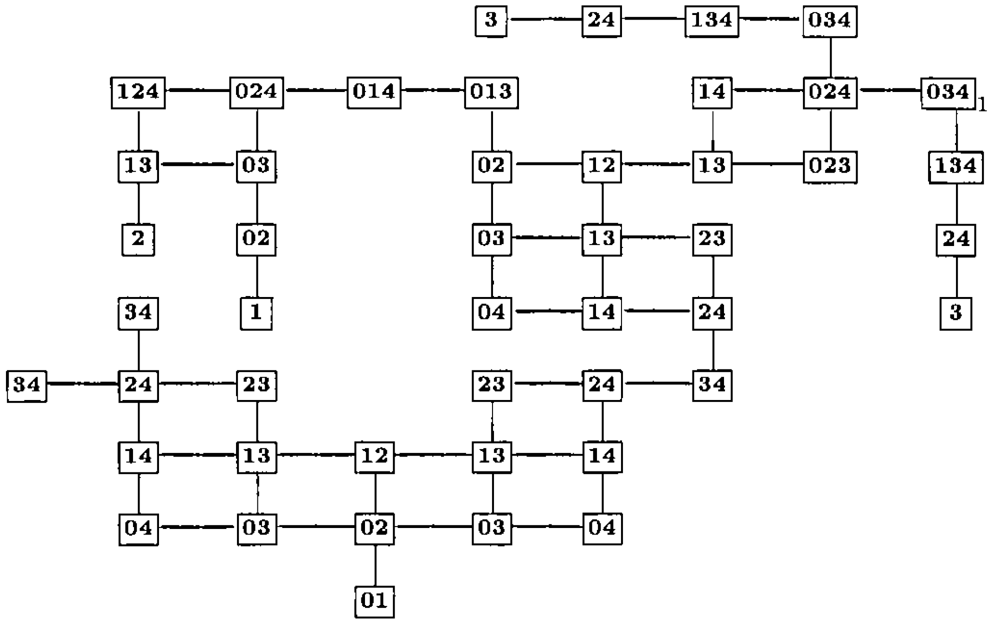
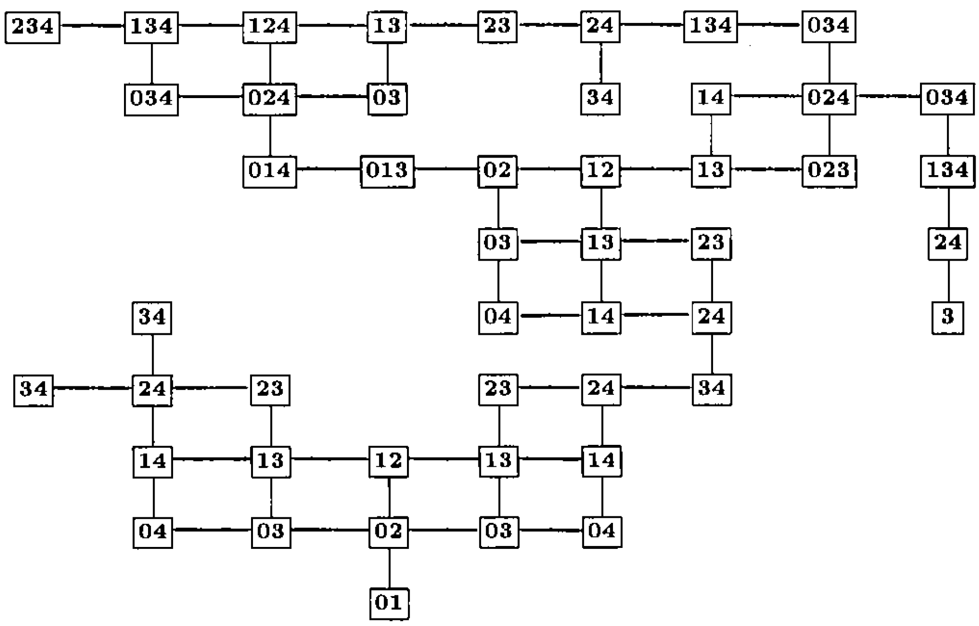


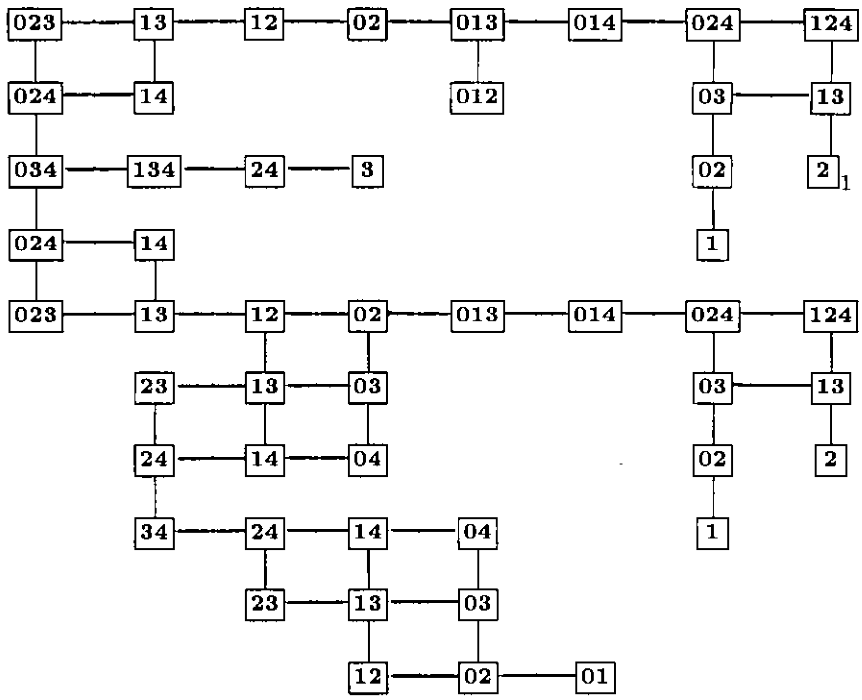
$\mathfrak{M}_{14}$



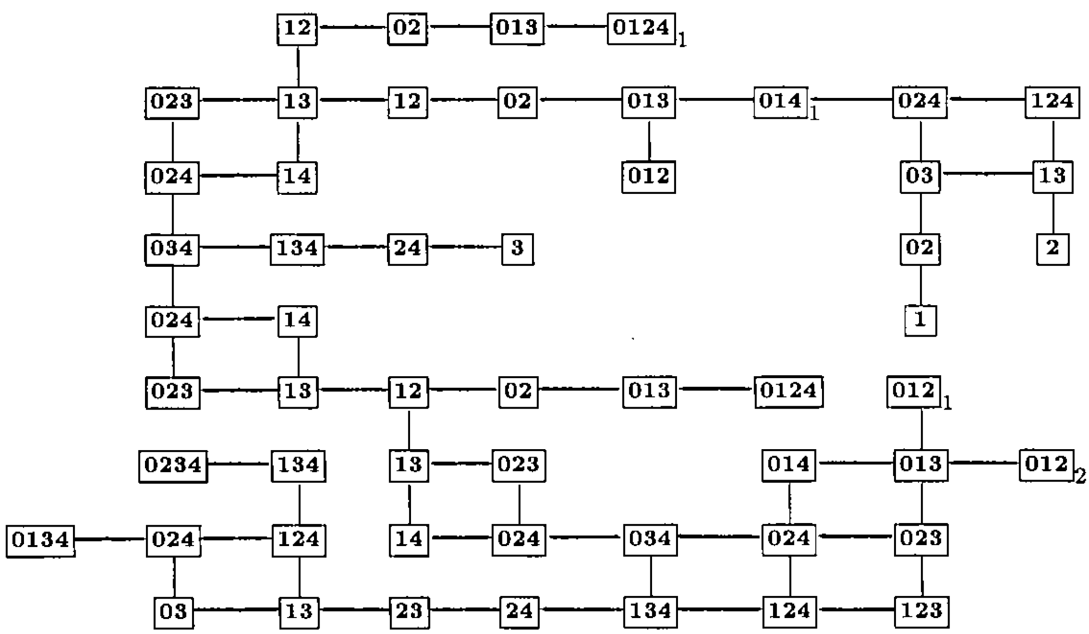
$\mathfrak{M}_{15}$



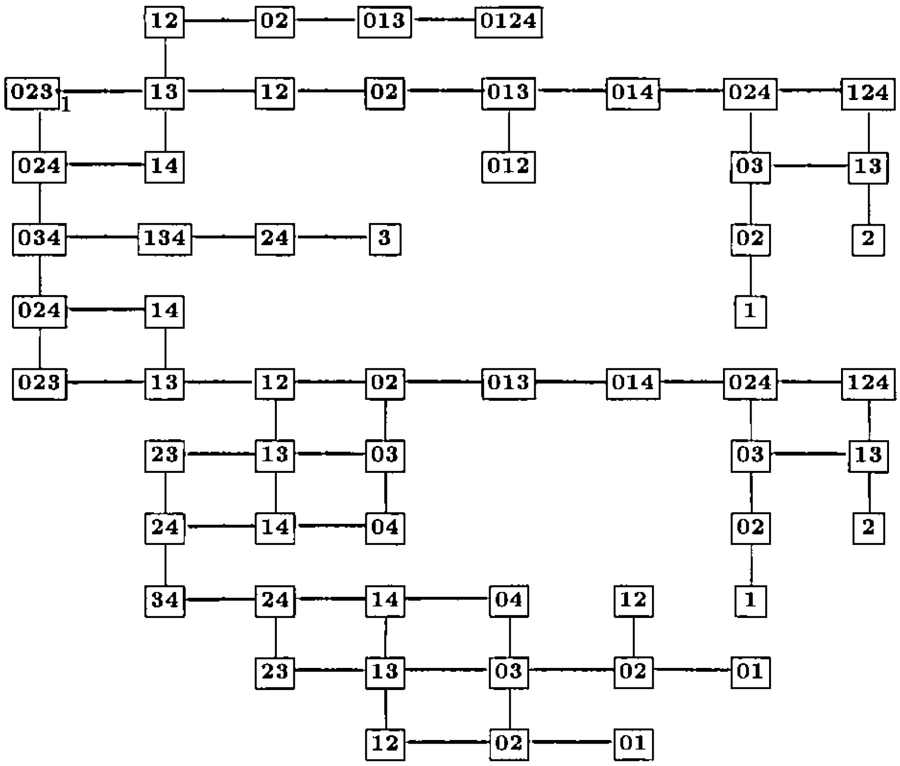
$\mathfrak{M}_{16}$



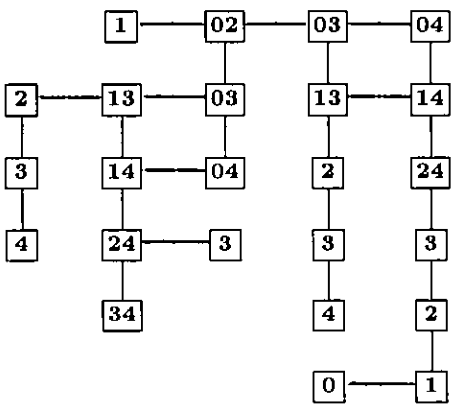
$\mathfrak{M}_{17}$



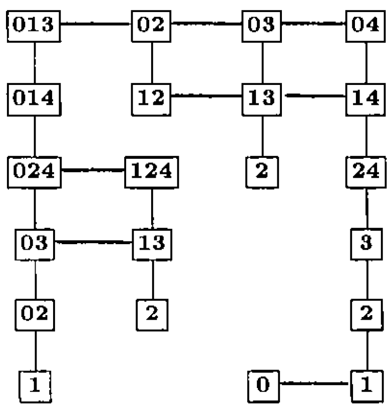
$\mathfrak{M}_{18}$



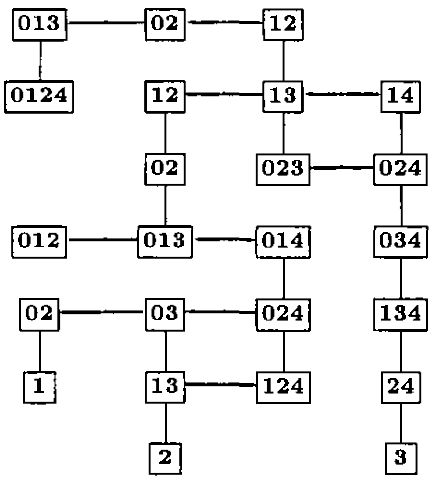
$\mathfrak{M}_{19}$



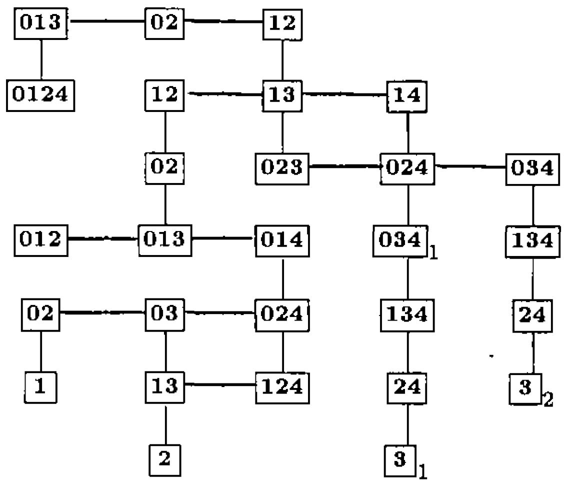
$\mathfrak{M}_{20}$



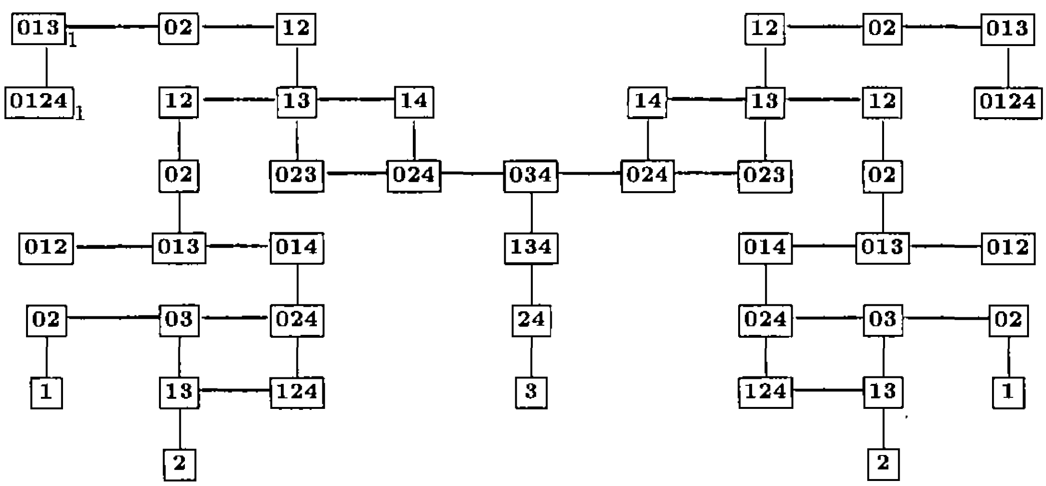
$\mathfrak{M}_{21}$



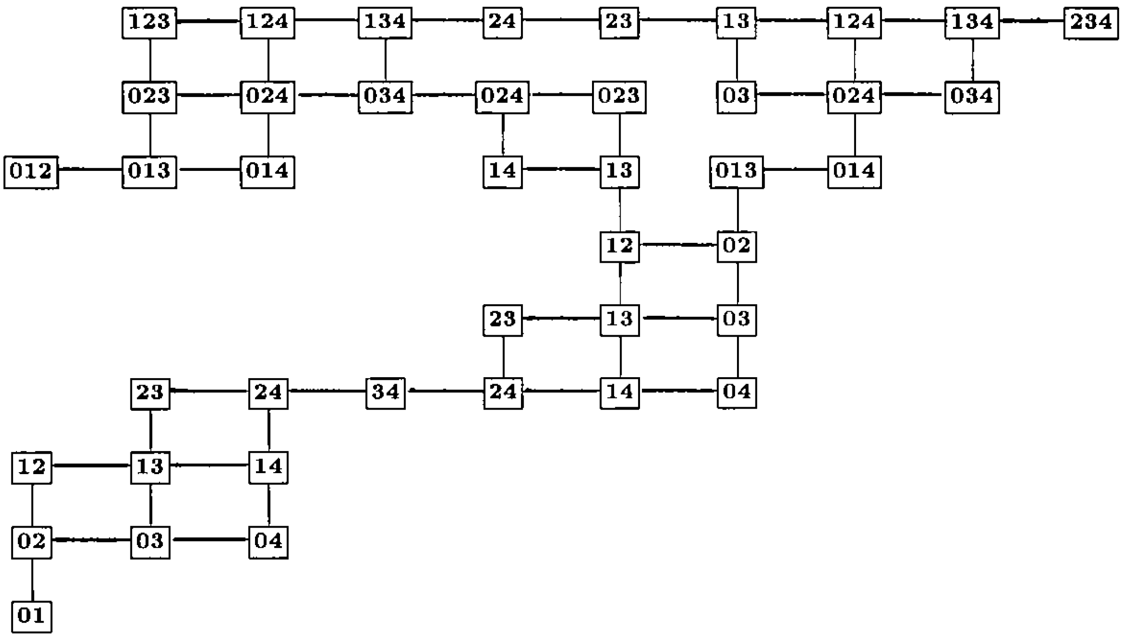
$\mathfrak{M}_{22}$



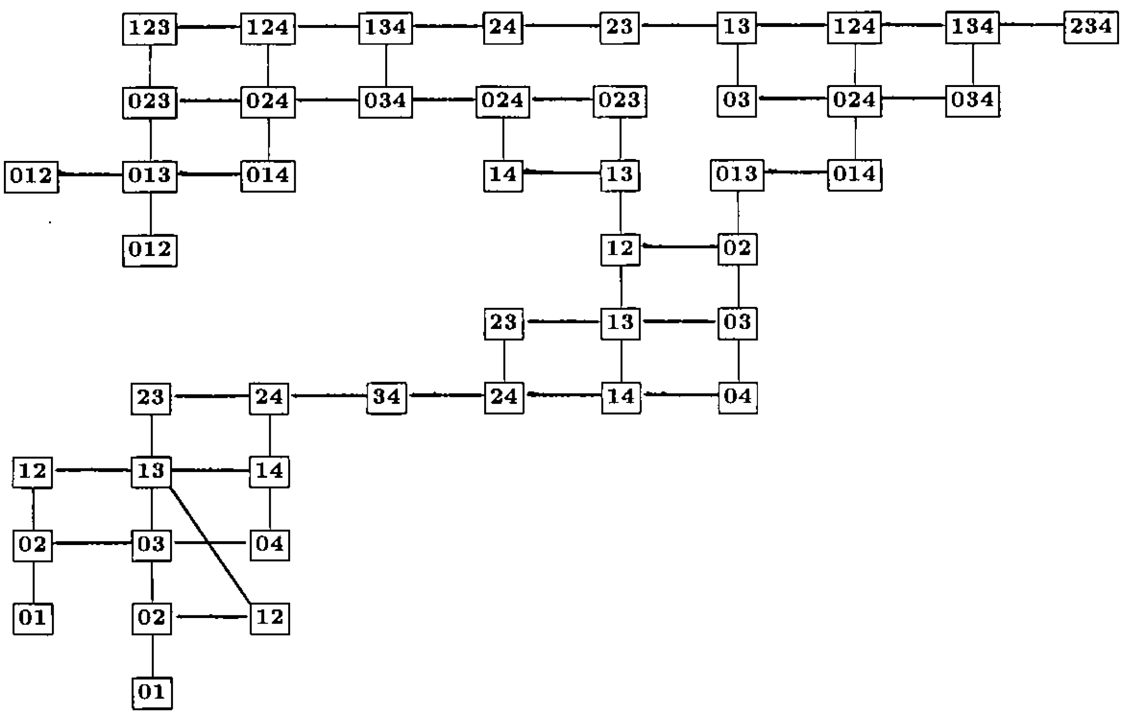
$\mathfrak{M}_{23}$



$\mathfrak{M}_{24}$



$\mathfrak{M}_{25}$



$\mathfrak{M}_{26}$

§5. On a conjecture.

5.1 I proposed the following conjecture in the paper [21, Conjecture 2.3].

Conjecture. *Let W be either a Weyl group or an affine Weyl group. For $x, y \in W$, $x \underset{L}{\sim} y$ if and only if $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$.*

5.2 In my paper [23], I verified this conjecture but with the following cases in $W_a(\widetilde{F}_4)$ excluded: $a(x) \in \Theta = \{6, 7, 9, 10, 13, 16\}$ and $\mathcal{R}(x) = \mathcal{R}(y) \in \{\{0, 1, 2\}, \{3, 4\}\}$. Now we can deal with these exceptional cases. By Theorem 2.1, we need only to show the direction " $\Longleftarrow$ ": if $\mathcal{R}(x) = \mathcal{R}(y)$ and $\Sigma(x) = \Sigma(y)$ then $x \underset{L}{\sim} y$. For the sake of definity, we may assume $\ell(x) \leq \ell(y)$ without loss of generality.

5.3 Let us assume $a(x) \in \Theta$, $\Sigma(x) = \Sigma(y)$ and $\mathcal{R}(x) = \mathcal{R}(y)$. By the condition $\Sigma(x) = \Sigma(y)$ (which is non-empty by our assumption), we have that $x \underset{LR}{\sim} y$ and that the elements x and y have the same generalized τ-invariant. We may assume that both x and y belong to the l.c.r. set of $\Omega(x)$ chosen in §4 by replacing x and y by some elements in $\Gamma(x)$ and $\Gamma(y)$ respectively if necessary. Hence we have $y \in M(x)$. We argue by contrary. Assume $x \underset{L}{\not\sim} y$. If $\mathcal{R}(x) = \mathcal{R}(y) = \{012\}$, then by the results of §4, we have $a(x) = a(y) = 16$, that the graph $\mathfrak{M}(x)$ is isomorphic to $\mathfrak{M}_{18}$ or $\mathfrak{M}_{26}$, and that x and y are the vertices labelled by $\boxed{012}_1$ and $\boxed{012}_2$ respectively in $\mathfrak{M}(x)$. It can be shown that $\Gamma(y \cdot 4) \in \Sigma(y) \setminus \Sigma(x)$. If $\mathcal{R}(x) = \mathcal{R}(y) = \{34\}$, then one of the following cases must occur.

(1) The graph $\mathfrak{M}(x)$ is isomorphic to $\mathfrak{M}_{15}$, and $x \in \Omega(w_{0134}) \cup \Omega(w_{234})$;

(2) The graph $\mathfrak{M}(x)$ is isomorphic to $\mathfrak{M}_{16}$, and $x \in \Omega(w_{234})$;

(3) The graph $\mathfrak{M}(x)$ is isomorphic to $\mathfrak{M}_{13}$, and $x \in \Omega(w_{234})$;

(4) The graph $\mathfrak{M}(x)$ is isomorphic to $\mathfrak{M}_{18}^{\mathrm{op}}$, and $a(x)$ is equal to 10, 13 or 16.

It can be shown that $\Gamma(y \cdot 1) \in \Sigma(y) \setminus \Sigma(x)$ in the cases (1), (3) and (4), and that $\Gamma(y \cdot 0) \in \Sigma(y) \setminus \Sigma(x)$ in the case (2). Thus in all the above cases, we have $\Sigma(x) \neq \Sigma(y)$, a contradiction. Therefore Conjecture 5.1 is verified without any exception.

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