THE DEGREE OF $\mathbb{Q}$-FANO THREEFOLDS

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1. Introduction

In this paper a $\mathbb{Q}$-*Fano variety* is a normal projective variety X with at worst $\mathbb{Q}$-factorial terminal singularities such that $-K_X$ is ample and $\operatorname{Pic} X$ is of rank one. Fano varieties with terminal singularities form in important class because, according to the minimal model program, every variety of negative Kodaira dimension should be birationally equivalent to a fibration $Y \to Z$ whose general fibre Y_η belong to this class. Moreover, in the case $\dim Z = 0$, $Y_\eta = Y$ is of Picard number one, i.e., Y is a $\mathbb{Q}$-Fano.

In dimension 2 the only $\mathbb{Q}$-Fano variety is the projective plane $\mathbb{P}^2$. In dimension 3 $\mathbb{Q}$-Fanos are bounded in the moduli sense by the following result of Kawamata:

(1.1) Theorem ([1]). *There exist positive integers r and d such that for an arbitrary $\mathbb{Q}$-Fano threefold X we have $-K_X^3 \le d$ and rK_X is Cartier.*

Since the Weil divisor $-K_X$ gives a natural polarization of a $\mathbb{Q}$-Fano variety X, the rational number $-K_X^3$ is a very important invariant. It is called *the degree* of X. In this paper we find a sharp bound for $-K_X^3$:

(1.2) Theorem. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein. Then $-K_X^3 \le 125/2$ and the equality holds if and only if X is isomorphic to the weighted projective space $\mathbb{P}(1,1,1,2)$.*

Note that in the Gorenstein case we have the estimate $-K_X^3 \le 64$ by the classification of Iskovskikh and Mori-Mukai and by Namikawa's result [2].

The idea of the proof is as follows. In Sections 4 and 5 using Riemann-Roch formula for Weil divisors [3] and Kawamata's estimates [1] we produce a short list of possibilities for singularities of $\mathbb{Q}$-Fanos of degree $\ge 125/2$. Here, to check a finite (but very huge) number of Diophantine conditions, we use a computer program (cf. [4]). In

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Section 6 we exclude all these possibilities except for $\mathbb{P}(1,1,1,2)$ by applying some birational transformations described in Section 3. The techniques used on this step is a very common in birational geometry (see [5], [6], [7]). It goes back to Fano-Iskovskikh "double projection method". The present paper is a logical continuation of our previous papers [8], [9] where we studied effective bounds of degree for sertain singular Fano threefolds.

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2. Preliminaries

Throughout this paper, we work over the complex number field $\mathbb{C}$.

(2.1) By $\operatorname{Cl} X$ we denote the Weil divisor class group of a normal variety X (modulo linear equivalence). There is a natural embedding $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$. Let X be a Fano variety with at worst log terminal singularities. It is well-known that both $\operatorname{Pic} X$ and $\operatorname{Cl} X$ are finitely generated and $\operatorname{Pic} X$ is torsion free (see e.g. [10, §2.1]). Moreover, numerical equivalence of $\mathbb{Q}$-Cartier divisors coincides with $\mathbb{Q}$-linear one. Therefore one can define the following numbers:

$$\begin{aligned} qF(X) &:= \max\{q \mid -K_X \sim_{\mathbb{Q}} qH, \quad H \in \operatorname{Pic} X\}, \\ q\mathbb{Q}(X) &:= \max\{q \mid -K_X \sim_{\mathbb{Q}} qL, \quad L \in \operatorname{Cl} X\}, \\ qW(X) &:= \max\{q \mid -K_X \sim qL, \quad L \in \operatorname{Cl} X\}. \end{aligned}$$

By the above, all of them are positive, $q\mathbb{Q}(X), qW(X) \in \mathbb{Z}$, and $qF(X) \in \mathbb{Q}$. If X is smooth all these numbers coincide with the *Fano index* of X. In general, we obviously have $q\mathbb{Q}(X) \geq qF(X)$ and $q\mathbb{Q}(X) \geq qW(X)$.

(2.1.1) Proposition (see e.g. [10, §2.1]**).** $qF(X) \leq \dim X + 1$.

The index $qW(X)$ was considered in [4]. In particular, it was proved that $qW(X) \leq 19$ for any $\mathbb{Q}$-Fano threefold.

(2.2) Terminal singularities Let (X, P) be a three-dimensional terminal singularity. It follows from the classification that there is a one-parameter deformation $\mathfrak{X} \to \mathit{\Delta} \ni 0$ over a small disk $\mathit{\Delta} \subset \mathbb{C}$ such that the central fibre $\mathfrak{X}_0$ is isomorphic to X and the generic fibre $\mathfrak{X}_\lambda$ has only cyclic quotient singularities $P_{\lambda,k}$ (see, e.g., [3]). Thus, to every theefold X with terminal singularities, one can associate a collection $\mathbf{B} = \{(r_{P,k}, b_{P,k})\}$, where $P_{\lambda,k} \in \mathfrak{X}_\lambda$ is a singularity of type

$\frac{1}{r_{P,k}}(1, b_{P,k}, -b_{P,k})$. This collection is uniquely determined by X and called the *basket* of singularities of X. By abuse of notation, we also will write $\mathbf{B} = (r_{P,k})$ instead of $\mathbf{B} = \{(r_{P,k}, b_{P,k})\}$. The index of P is the least common multiple of indices of points $P_{\lambda,k}$.

(2.2.1) Lemma ([11, Corollary 5.2]**).** *Let (X, P) be a three-dimensional terminal singularity of index r and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. There is an integer, i such that $D \sim iK_X$ near P. In particular, rD is Cartier.*

(2.2.2) Corollary. *Let X be a Fano threefold with terminal singularities and let r be the Gorenstein index of X. Then*

(i) $\gcd(r, qW(X)) = 1$,
(ii) $qF(X)r = q\mathbb{Q}(X)$,
(iii) $qW(X) \le q\mathbb{Q}(X) \le 4r$.

(2.2.3) Let (X, P) be a three-dimensional terminal singularity of index r and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. By Lemma (2.2.1) there is an integer i such that $0 \le i < r$ and $D \sim iK_X$ near P. Deforming D with (X, P) we obtain Weil divisors D_λ on X_λ. Thus we have a collection of numbers i_k such that $0 \le i_k < r_k$ and $D_\lambda \sim i_k K_{X_\lambda}$ near $P_{\lambda,k}$.

(2.3) Riemann-Roch formula [3]**.** Let X be a threefold with terminal singularities and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. Then

$$\chi(D) = \frac{1}{12} D \cdot (D - K_X) \cdot (2D - K_X) + \frac{1}{12} D \cdot c_2 + \sum_{P \in \mathbf{B}} c_P(D) + \chi(\mathcal{O}_X), \tag{2.3.1}$$

where

$$c_P(D) = -i_P \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_P - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

(2.4) Now let X be a Fano threefold with terminal singularities, let $q := q\mathbb{Q}(X)$, and let L be an ample Weil $\mathbb{Q}$-Cartier divisor on X such that $-K_X \sim_{\mathbb{Q}} qL$. By (2.3.1) we have

$$\chi(tL) = 1 + \frac{t(q+t)(q+2t)}{12} L^3 + \frac{tL \cdot c_2}{12} + \sum_{P \in \mathbf{B}} c_P(tL), \tag{2.4.1}$$

$$c_P(tL) = -i_{P,t} \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_{P,t} - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

If $q > 2$, then $\chi(-L) = 0$. Using this equality we obtain (see [4])

$$L^3 = \frac{12}{(q-1)(q-2)} \left(1 - \frac{L \cdot c_2}{12} + \sum_{P \in B} c_P(-L) \right). \tag{2.4.2}$$

(2.5) In the above notation, applying (2.3.1), Serre duality and Kawamata-Viehweg vanishing to $D = K_X$ we get the following important equality (see, e.g., [3]):

$$24 = -K_X \cdot c_2 + \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right). \tag{2.5.1}$$

Similarly, for $D = -K_X$ we have $H^i(X, -K_X) = 0$ for $i > 0$ and

$$c_P(-K_X) = \frac{r_P^2 - 1}{12 r_P} - \frac{b_P(r - b_P)}{2 r_P}.$$

(see [5, §2]). Combining this with (2.5.1) we obtain

$$\dim |-K_X| = -\frac{1}{2} K_X^3 + 2 - \sum_{P \in \mathbf{B}} \frac{b_P(r_P - b_P)}{2 r_P}. \tag{2.5.2}$$

In particular,

$$\dim |-K_X| \le -\frac{1}{2} K_X^3 + 2 - \frac{1}{2} \sum_{P \in \mathbf{B}} \left(1 - \frac{1}{r_P} \right) \le -\frac{1}{2} K_X^3 + 2. \tag{2.5.3}$$

(2.5.4) Theorem ([1], [12]). *In the above notation, $-K_X \cdot c_2(X) \ge 0$.*

As a corollary we have ([5, §2]):

$$\dim |-K_X| \ge -\frac{1}{2} K_X^3 - 2. \tag{2.5.5}$$

(2.5.6) Proposition ([5, §2]]). *Let X be a $\mathbb{Q}$-Fano threefold. If $\dim |-K_X| \ge 2$, then the linear system $|-K_X|$ has no base components and is not composed of a pencil. (In particular, a general element of $|-K_X|$ is reduced and irreducible.)*

(2.6) Now let X be a $\mathbb{Q}$-Fano threefold, let $q := q\mathbb{Q}(X)$, and let L be an ample Weil divisor on X that generates the group $\operatorname{Cl} X/\mathrm{Tors}$. Let $\mathcal{E}$ be the double dual to Ω_X^1. If $\mathcal{E}$ is not semistable, there is a maximal destabilizing subsheaf $\mathcal{F} \subset \mathcal{E}$. Clearly, $c_1(\mathcal{F}) \equiv -pL$ for some $p \in \mathbb{Z}$. Put $t := p/q$, so that $c_1(\mathcal{F}) \equiv tK_X$. According to [1] there are the following possibilities:

(2.6.1) $\mathcal{E}$ is semistable. Then $-K_X^3 \le -3K_X \cdot c_2(X)$.

(2.6.2) $\mathcal{E}$ is not semistable and $\operatorname{rk}\mathcal{F} = 2$**.** Then $q \geq 2$, $0 < t < 2/3$, and

$$t(4-3t)(-K_X^3) \leq -4K_X \cdot c_2(X).$$

(2.6.3) $\mathcal{E}$ is not semistable, $\operatorname{rk}\mathcal{F} = 1$**, and** $(\mathcal{E}/\mathcal{F})^{**}$ **is semistable.** Then $q \geq 4$, $0 < t < 1/3$, and

$$(1-t)(1+3t)(-K_X^3) \leq -4K_X \cdot c_2(X).$$

(2.6.4) $\mathcal{E}$ is not semistable, $\operatorname{rk}\mathcal{F} = 1$**, and** $(\mathcal{E}/\mathcal{F})^{**}$ **is not semistable.** Then again $q \geq 4$ and $0 < t < 1/3$. There exists an unstable reflexive sheaf $\mathcal{F} \subsetneq \mathcal{G} \subsetneq \mathcal{E}$. Write $c_1(\mathcal{G}/\mathcal{F}) \equiv -p'L$, $p' \in \mathbb{Z}$ and put $u := p'/q$, so that $c_1(\mathcal{G}/\mathcal{F}) \equiv uK_X$. Then $t < u < 1-t-u$ and

$$\Bigl(tu + (t+u)(1-t-u)\Bigr)(-K_X^3) \leq -K_X \cdot c_2(X),$$

(2.7) Corollary. *If $q\mathbb{Q}(X) = 1$, then $\mathcal{E}$ is semistable. If $q\mathbb{Q}(X) \leq 3$, then either $\mathcal{E}$ is semistable or we are in case* (2.6.2).

3. Two birational constructions

(3.1) Let X be a $\mathbb{Q}$-Fano threefold. Throughout this paper we assume that the linear system $|-K_X|$ is non-empty, has no fixed components, and is not composed of a pencil. Then a general member $H \in |-K_X|$ is irreducible. By (2.5.5) and (2.5.6) this holds automatically when $-K_X^3 \geq 8$. Let $q := q\mathbb{Q}(X)$ and L be the ample Weil divisor that generates the group $\operatorname{Cl} X/\mathrm{Tors}$. Thus we have $-K_X \equiv qL$. Put $\mathcal{H} := |-K_X|$. Let $H \in \mathcal{H}$ be a general member.

(3.2) Assume there is a diagram (Sarkisov link of type I or II)

(3.2.1)
$$\begin{array}{ccc} \tilde{X} & \overset{\chi}{\dashrightarrow} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & & Z \end{array}$$

where $\tilde{X}$ and Y have only $\mathbb{Q}$-factorial terminal singularities, $\rho(\tilde{X}) = \rho(Y) = 2$, g is a Mori extremal divisorial contraction, $\tilde{X} \dashrightarrow Y$ is a sequence of log flips, and f is a Mori extremal contraction (either divisorial or fibre type). Thus one of the following holds: a) $\dim Z = 1$ and f is a $\mathbb{Q}$-del Pezzo fibration, b) $\dim Z = 2$ and f is a $\mathbb{Q}$-conic bundle, or c) $\dim Z = 3$, f is a divisorial contraction, and Z is a $\mathbb{Q}$-Fano. Let E be the g-exceptional divisor. We assume that the composition $f \circ \chi \circ g^{-1}$ is not an isomorphism. For a divisor D on X, everywhere below $\tilde{D}$ and D_Y denote strict birational transforms of D on $\tilde{X}$ and Y, respectively. We also assume that the discrepancy $\alpha := a(E, X, \mathcal{H})$ is non-positive, i.e.,

$$(3.2.2) \qquad 0 \sim f^*(K_X + \mathcal{H}) = K_{\tilde{X}} + \tilde{\mathcal{H}} + \alpha E, \quad \alpha \in \mathbb{Z}, \quad \alpha \geq 0.$$

By the above we have

$$(3.2.3) \qquad \dim |-K_{\tilde{X}}| \geq \dim \tilde{\mathcal{H}} = \dim |-K_X|.$$

(3.3) Similarly,

$$0 \sim_{\mathbb{Q}} g^*(K_X + qL) \sim_{\mathbb{Q}} K_{\tilde{X}} + q\tilde{L} + \beta E, \qquad \beta \geq 0.$$

Therefore,

$$(3.3.1) \qquad K_Y + qL_Y + \beta E_Y \sim_{\mathbb{Q}} 0.$$

If $q\mathbb{Q}(X) = qW(X)$, then $K_X + qL \sim 0$ and β is an integer $\geq \alpha$.

Let $F = f^{-1}(\mathrm{pt})$ be a general fibre. Recall that F is either $\mathbb{P}^1$ or a smooth del Pezzo surface. Restricting (3.3.1) to F we get

$$(3.3.2) \qquad K_F + qL_Y|_F + \beta E_Y|_F \sim 0.$$

Here $-K_F$, $L_Y|_F$, and $E_Y|_F$ are proportional nef Cartier divisors. Moreover, $-K_F$ and $E_Y|_F$ are ample.

(3.4) We will use construction (3.2.1) in the following two situationa:

(3.4.1) (see [6], [7]). Let $P \in X$ be a singularity of index r. Take g to be a divisorial blowup of P such that the discrepancy of the exceptional divisor E is equal to $1/r$. Assume that the divisor $-K_{\tilde{X}}$ is nef, big and the linear system $|-nK_{\tilde{X}}|$ does not contract any divisors. Then the transformation in (3.2.1) is so-called "two rays game". If $-K_{\tilde{X}}$ is ample, then $f \circ \chi$ is a composition of steps of the K-MMP. Otherwise, $f \circ \chi$ is a composition of a single flop followed by steps of the K-MMP. It is easy to see also that $f \circ \chi$ is an $-E$-MMP.

(3.4.2) (see [5]). The pair $(X, \mathcal{H})$ is not canonical. Let c be the canonical threeshold of $(X, \mathcal{H})$. Then $0 < c < 1$. Take g to be an extremal divisorial $K_X + c\mathcal{H}$-crepant blowup. In this situation, $\alpha > 0$ and $f \circ \chi$ is an $K + c\mathcal{H}$-MMP. In particular, f is an extremal $K_X + c\mathcal{H}$-negative contraction. The conditions of (3.2) are satisfied by [5].

(3.5) Properties of construction (3.2).

(3.5.1) Claim. *E_Y is not contracted by f.*

Proof. Assume the converse, i.e., $\dim f(E_Y) < \min(2, \dim Z)$. If f is birational, this implies that the map $f \circ \chi \circ g^{-1} \colon X \dashrightarrow Z$ is an isomorphism in codimension one. Since both X and Z are Fano threefolds, this implies that $f \circ \chi \circ g^{-1}$ is in fact an isomorphism. This contradicts our assumptions. If $\dim Z \le 2$, then E_Y is a pull-back of an ample Weil divisor on Z. But then nE_Y is movable for some $n > 0$. Again we derive a contradiction. □

(3.5.2) Claim. *For some n, $m > 0$ there is a decomposition $-nK_{\tilde{X}} \sim m\tilde{\mathcal{H}} + M$, where $|M|$ is a base point free linear system. In particular, $|-nK_{\tilde{X}}|$ has no fixed components.*

Proof. By (3.2.2), for some $0 < c \le 1$, we have $K_{\tilde{X}} + c\tilde{\mathcal{H}} = g^*(K_X + cH)$. Hence we can take n, $m > 0$ so that $|-nK_{\tilde{X}} - m\tilde{\mathcal{H}}|$ is base point free. □

(3.5.3) Lemma ([13])**.** *If f is a $\mathbb{Q}$-conic bundle, then Z is a del Pezzo surface with at worst Du Val singularities of type A_n and $\rho(Z) = 1$. Moreover, there is a natural embedding $f^* \colon \operatorname{Cl} Z \to \operatorname{Cl} Y$.*

Proof. The assertion about the base is an immediate consequence of the main result of [13] and the fact that Z is uniruled. The last statement is obvious because both Y and Z have only isolated singularities and $\operatorname{Pic}(Y/Z) \simeq \mathbb{Z}$. □

(3.5.4) Remark. (i) *In the above notation the generic fibre of f is a smooth rational curve. The locus $\Lambda := \{z \in Z \mid f$ is smooth over $z\}$ is a closed subset of codimension ≥ 1 in Z. The union of one-dimensional components of Λ is called the* discriminant curve.

(ii) *The classification of del Pezzo surfaces Z with Du Val singularities and $\rho(Z) = 1$ is well-known. In particular, we always have $K_Z^2 \le 9$ and $K_Z^2 \ne 7$. Moreover,*

- (i) *if $K_Z^2 = 9$, then $Z \simeq \mathbb{P}^2$;*
- (ii) *if $K_Z^2 = 8$, then $Z \simeq \mathbb{P}(1, 1, 2)$;*
- (iii) *if $K_Z^2 \le 6$, then on Z there is a rational curve C such that $-K_Z \cdot C = 1$.*

(3.5.5) Lemma. *Notation and assumptions as in* (3.2)*. Assume additionally that $q\mathbb{Q}(X) \ge 4$ and f is not birational. Then $L_Y = f^*\Xi$ for some* (*integral*) *Weil divisor on Z. Moreover,* $\dim |\Xi| = \dim |L|$ *and the class of Ξ generates the group* $\operatorname{Cl} Z/\operatorname{Tors}$*.*

Proof. Since $q\mathbb{Q}(X) \geq 4$, relation (3.3.2) implies $L_Y|_F = 0$. Since f is a Mori contraction and Y is normal, $L_Y = f^*\Xi$, where $\Xi := f(L_Y)$. The rest follows by the fact that the group $\operatorname{Cl} Y/\operatorname{Tors}$ is generated by L_Y and E_Y. $\square$

(3.5.6) Lemma. *Assume that $(X, |-K_X|)$ is not canonical and we are applying construction* (3.2). *Further, assume that* $\dim Z = 2$ *and* $\alpha > 0$. *Then one of the following holds:*

(i) *$\mathcal{H}_Y$ is f-ample. Then the discriminant curve of f is empty.*
(ii) *$\mathcal{H}_Y$ is not f-ample. Then $q\mathbb{Q}(X) \geq 7$. Moreover, the equality holds only if $Z \simeq \mathbb{P}^2$ and* $\dim |-K_X| = 35$.

Proof. First we assume that $\mathcal{H}_Y$ is f-ample. By (3.2.2) and Claim (3.5.1) E_Y and general elements of $\mathcal{H}_Y$ are sections of f. Hence f is smooth outside of a finite number of degenerate fibres.

Now we assume that $\mathcal{H}_Y$ is not f-ample. Then $\mathcal{H}_Y = f^*\mathcal{M}$, where $\mathcal{M}$ is a linear system without fixed components. Let Ξ be an ample Weil divisor that generates $\operatorname{Cl} Z/\operatorname{Tors}$. We can write $\mathcal{M} \sim_{\mathbb{Q}} a\Xi$ and $-K_Z \sim_{\mathbb{Q}} q'\Xi$, where $q' := q\mathbb{Q}(Z)$, $a \in \mathbb{Z}$. Clearly, $q\mathbb{Q}(X) \geq a$.

By our assumption and by Reid's Riemann-Roch formula [3, (9.1)],

$$30 \leq \dim \mathcal{M} \leq \frac{1}{2}\mathcal{M} \cdot (\mathcal{M} - K_Z) + \sum c_P(\mathcal{M}) \leq \frac{a(a+q')}{2q'^2} K_Z^2.$$

Assume that $a \leq 7$. If $K_Z^2 \leq 6$, then $q' = K_Z^2$ by Remark (3.5.4). So, $60q' \leq a(a+q') \leq 49 + 7q'$, a contradiction. If $K_Z^2 = 8$, then $q' = 4$, so $120 \leq a(a+4) \leq 77$. Again we have a contradiction. Finally, let $K_Z^2 = 9$, i.e., $Z \simeq \mathbb{P}^2$. Then $q' = 3$, so $60 \leq a(a+3) \leq 70$. This inequality has only one solution: $a = 7$. But then $q\mathbb{Q}(X) \leq 7$. If $q\mathbb{Q}(X) = 7$, then $a = 7$, $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^2}(7)|$, and $\dim \mathcal{M} = 35$. $\square$

(3.5.7) Lemma. *Notation and assumptions as in* (3.2). *Assume additionally that $q\mathbb{Q}(X) = 1$, Z is a surface, and the discriminant curve of f is empty. Then* $\dim |-K_X| < 30$.

Proof. Suppose $\dim |-K_X| \geq 30$. Let $\Gamma \subset Z$ is a smooth curve contained into the smooth locus of Z. Then $G := f^{-1}(\Gamma)$ is a smooth ruled surface over Γ. We claim that $\dim |-K_Y - G| \leq 0$. Indeed, otherwise $-K_Y \sim G + B$, where B is an integral effective divisor, $\dim |B| \geq 1$. Since $q\mathbb{Q}(X) = 1$, this gives a contradiction.

Now from (3.2.3) and from the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-K_Y - G) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \mathcal{O}_G(-K_Y) \longrightarrow 0$$

we get $h^0(\mathcal{O}_G(-K_Y)) \ge h^0(\mathcal{O}_Y(-K_Y)) - 1 \ge 30$. It is easy to see that

$$(-K_Y|_G)^2 = (-K_G + G|_G)^2 = K_G^2 - 2K_G \cdot G|_G = 8 - 8p_a(\Gamma) + 4\Gamma^2.$$

By Claim (3.5.2) the linear system $|-nK_Y|$ has no fixed components. Therefore we can take Γ so that $|-nK_Y|_G|$ has at worst isolated base points (in particular, it is nef). Moreover, $|-nK_Y|_G|$ is base point free for sufficiently large n. If $-K_Y|_G$ is ample, it is well-known that $h^0(\mathcal{O}_G(-K_Y)) \le (-K_Y|_G)^2 + 2$ (see, e.g., [14]). If $-K_Y|_G$ is not ample, we obtain the above inequality by applying the same arguments to $\bar{G}$, where $\bar{G}$ is the image of G under the birational contraction given by $|-nK_Y|_G|$. In both cases we have

$$8 - 8p_a(\Gamma) + 4\Gamma^2 = (-K_Y|_G)^2 \ge h^0(\mathcal{O}_G(-K_Y)) - 2 \ge 28.$$

This gives us

$$\Gamma^2 \ge 2p_a(\Gamma) + 5 = K_Z \cdot \Gamma + \Gamma^2 + 7, \qquad -K_Z \cdot \Gamma \ge 7.$$

If $K_Z^2 < 8$, then we can take Γ to be a general member of $-K_Z$ and derive a contradiction. If $K_Z^2 = 8$ or 9, then we can take $\Gamma \in |-\frac{1}{2}K_Z|$, or $|-\frac{1}{3}K_Z|$, respectively. □

(3.5.8) Lemma. *If* $\dim Z = 1$ *and* $\dim|-K_X| \ge 30$*, then* $q\mathbb{Q}(X) \ge 3$.

Proof. Let F_1, F_2, F_3 be general fibres. Then from the exact sequence

$$0 \longrightarrow \mathcal{O}_Y\left(-K_Y - \sum F_i\right) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \bigoplus \mathcal{O}_{F_i}(-K_{F_i}) \longrightarrow 0$$

we obtain

$$h^0(-K_Y - \sum F_i) \ge h^0(-K_Y) - \sum h^0(-K_{F_i}).$$

Since F_i are smooth del Pezzo surfaces, $h^0(-K_{F_i}) = K_F^2 + 1 \le 10$. Hence, $h^0(-K_Y - \sum F_i) > 0$ by (2.5.5) and we have a decomposition $-K_Y \sim \sum F_i + G$, where G is effective. Since F_i is movable, this gives us that $q\mathbb{Q}(X) \ge 3$. □

(3.6) Case: $(X, |-K_X|)$ is canonical.

(3.6.1) Consider the case when $(X, |-K_X| = \mathcal{H})$ is canonical. According to [5] there is the following diagram

$$\begin{tikzcd} & \tilde{X} \arrow[r] \arrow[dl, "g"'] \arrow[dr, "f"] & \bar{X} \arrow[d] \\ X \arrow[rr, dashed] & & Y \subset \mathbb{P}^n \end{tikzcd}$$

where $g\colon (\tilde{X}, \tilde{\mathcal{H}}) \to (X, \mathcal{H})$ is a terminal modification of $(X, \mathcal{H})$, $n := \dim|-K_X|$, the morphism f is given by the (base point free) linear system $\tilde{\mathcal{H}}$, $\dim Y = 2$ or 3, and $\tilde{X} \to \bar{X} \to Y$ is the Stein factorization. We have

$$K_{\tilde{X}} + \tilde{\mathcal{H}} = g^*(K + \mathcal{H}) \sim 0.$$

Since $(\tilde{X}, \tilde{\mathcal{H}})$ is terminal, a general member $\tilde{H} \in \tilde{\mathcal{H}}$ is a smooth K3 surface. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}) \longrightarrow \mathcal{O}_{\tilde{H}}(-K_{\tilde{X}}) \longrightarrow 0$$

one can see that the restriction $f|_{\tilde{H}}$ is given by a complete linear system.

(3.6.2) Lemma. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that $(X, |-K_X| = \mathcal{H})$ is canonical and the image of the map given by $|-K_X|$ is a surface. If* $\dim |-K_X| \geq 6$, *then* $2q\mathbb{Q}(X) \geq \dim |-K_X| - 1$.

Proof. We use notation of (3.6.1). By our assumption $f(\tilde{H})$ is a curve. Thus $|-K_{\tilde{X}}|_{\tilde{H}}|$ is a base point free elliptic pencil on $\tilde{H}$ and $f(\tilde{H}) \subset \mathbb{P}^n$ is a rational normal curve of degree $n-1$. Hence $Y \subset \mathbb{P}^n$ is a surface of degree $n-1$. Let M be a hyperplane section of Y. It is well-known that in this situation one of the following halds (recall that $n \geq 6$):

(i) Y is a rational scroll, $Y \simeq \mathbb{F}_e$, $M \sim \Sigma + al$, where Σ and l are the minimal section and a fibre of $\mathbb{F}_e$, respectively, and a is an integer such that $a \geq e+1$, $n-1 = 2a-e$.

(ii) Y is a cone over a rational normal curve of degree $n-1$, $M \sim (n-1)l$, where l is a generator of the cone.

In case (i), $\tilde{\mathcal{H}} \sim f^*\Sigma + af^*l$. Here $|f^*l|$ is a linear system without fixed components and $f^*\Sigma$ is an effective divisor. So, $2q\mathbb{Q}(X) \geq 2a \geq n-1$. In case (ii) we have $\tilde{\mathcal{H}} \sim f^*(n-1)l$. Let $o \in Y$ be the vertex of the cone and let G be the closure of f^*l over $Y \setminus \{o\}$. Then G is an integral Weil divisor and $\tilde{H} \sim_{\mathbb{Q}} (n-1)G + T$, where T is effective. Clearly, g does not contract any component of G. This implies $q\mathbb{Q}(X) \geq n-1$. □

Now assume that $\dim Y = 3$.

(3.6.3) Lemma (cf. [8, Corollary 1.8]**).** *Let X be a $\mathbb{Q}$-Fano threefold. Assume that $(X, |-K_X| = \mathcal{H})$ is canonical and the image of the map given by $|-K_X|$ is three-dimensional. Then* $\dim |-K_X| \leq 37$. *If moreover* $q\mathbb{Q}(X) = 1$, *then* $\dim |-K_X| \leq 13$.

Proof. By the construction, $\bar{Y}$ is a Fano threefold with canonical Gorenstein singularities and $\bar{Y} \to Y \subset \mathbb{P}^N$ is the anticanonical map (see [5]). We have $\dim |-K_X| \leq \dim |-K_{\bar{Y}}| \leq 38$ by the main result of [8]. Moreover, if $\dim |-K_X| = 38$, then $\bar{Y}$ is isomorphic either $\mathbb{P}(3,1,1,1)$ or $\mathbb{P}(6,4,1,1)$. In particular, $\bar{Y}$ is a toric variety. Since $\tilde{X}$ is a terminal modification of $\bar{Y}$, it is also toric and so is X. By Lemma (3.6.4) below $\dim|-K_X| \le \dim|-K_{\bar{Y}}| \le 33$, a contradiction. If $q\mathbb{Q}(X) = 1$, then $-K_{\bar{Y}}$ cannot be decomposed into a sum of two movable divisors. According to [15], $\dim|-K_X| \le \dim|-K_{\bar{Y}}| \le 13$. □

(3.6.4) Lemma. *Let X be a toric $\mathbb{Q}$-Fano threefold. If $X \not\simeq \mathbb{P}^3$, then $-K_X^3 \le 125/2$ and $\dim|-K_X| \le 33$.*

Sketch of the proof. By considering cyclic covering tricks (cf. Proof of Proposition (5.3)) we reduce the question to the case $\operatorname{Cl} X \simeq \mathbb{Z}$. For toric varieties this preserves the property $\rho = 1$. Then X is a weighted projective space. Using the fact that X has only terminal singularities we get the following cases: $\mathbb{P}(1,1,1,2)$, $\mathbb{P}(1,1,2,3)$, $\mathbb{P}(1,2,3,5)$, $\mathbb{P}(1,3,4,5)$, $\mathbb{P}(2,3,5,7)$, $\mathbb{P}(3,4,5,7)$. The lemma follows. □

4. Case $q\mathbb{Q}(X) \le 3$

In this section we consider the case $q := q\mathbb{Q}(X) \le 3$.

(4.1) Proposition. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein, $q := q\mathbb{Q}(X) \le 3$ and $-K_X^3 \ge 125/2$. Then we have one of the following cases:*

(4.1.1) $q = 1$, $\mathbf{B} = (2)$, $-K_X^3 = 2g - 3/2$, $\dim|-K_X| = g + 1$, $32 \le g \le 35$;

(4.1.2) $q = 1$, $\mathbf{B} = (2,2)$, $-K_X^3 = 63$, $\dim|-K_X| = 33$;

(4.1.3) $q = 1$, $\mathbf{B} = (3)$, $-K_X^3 = 188/3$, $\dim|-K_X| = 33$;

(4.1.4) $q = 2$, $\mathbf{B} = (3)$, $L^3 = 25/3$, $\dim|L| = 9$, $\dim|-K_X| = 35$.

(4.2) Lemma. *In notation of Proposition (4.1) we have $-K_X \cdot c_2(X) \ge 125/8$ and $\sum_{P\in\mathbf{B}}(r_P - 1/r_P) \le 67/8$. In particular, $\sum r_P \le 10$.*

Proof. By Corollary (2.7) we have cases (2.6.1) or (2.6.2). Hence,

$$-K_X \cdot c_2(X) \ge \begin{cases} \dfrac{1}{3}(-K_X^3) \ge \dfrac{125}{6}, \\ \dfrac{1}{4}t(4-3t)(-K_X)^3 \ge \dfrac{1}{4q}\left(4 - \dfrac{3}{q}\right)\dfrac{125}{2} \ge \dfrac{125}{8}. \end{cases}$$

(In the second line we used that $t \geq 1/q \geq 1/3$ and the function $t(4-3t)$ is increasing for $t \leq 2/3$). In both cases we have $-K_X \cdot c_2(X) \geq 125/8$. Thus,

$$\sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) \leq 24 - \frac{125}{8} = \frac{67}{8}.$$

Hence $\mathbf{B}$ contains at most 5 points and $\sum r_P \leq \left\lfloor \frac{67}{8} + 5 \cdot \frac{1}{2} \right\rfloor \leq 10$. $\square$

(4.3) Proposition. *In notation of Proposition* (4.1) *we have* $\operatorname{Cl} X \simeq \mathbb{Z}$.

Proof. Let T be an s-torsion element in the Weil divisor class group. By Riemann-Roch (2.3.1), Kawamata-Viehweg vanishing theorem and Serre duality we have

$$\begin{aligned} 0 &= \chi(T) &&= 1 + \textstyle\sum_P c_P(T), \\ 0 &= \chi(K_X + T) &&= 1 + \tfrac{1}{12} K_X \cdot c_2(X) + \textstyle\sum_{P \in \mathbf{B}} c_P(K_X + T). \end{aligned}$$

Subtracting we get

$$0 = -\frac{1}{12} K_X \cdot c_2(X) + \sum_{P \in \mathbf{B}} (c_P(T) - c_P(K_X + T)).$$

Take $i_{T,P}$ so that $T \sim i_{T,P} K_X$ near $P \in \mathbf{B}$. Then $s i_{T,P} \equiv 0 \mod r_P$ and

$$0 = -\frac{1}{12} K_X \cdot c_2(X) + \frac{1}{12} \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) - \sum_{P \in \mathbf{B}} \frac{\overline{b_P i_{T,P}} \, \left(r_P - \overline{b_P i_{T,P}} \right)}{2 r_P}.$$

Therefore,

$$2 = \sum_{P \in \mathbf{B}} \frac{\overline{b_P i_{T,P}} \, \left(r_P - \overline{b_P i_{T,P}} \right)}{2 r_P}.$$

If $i_{T,P} \not\equiv 0 \mod r_P$, we have

$$\frac{\overline{b_P i_{T,P}} \, \left(r_P - \overline{b_P i_{T,P}} \right)}{2 r_P} \leq \frac{r_P}{8}.$$

Combining the last two relations we get

$$\sum_{P \in \mathbf{B}'} r_P \geq 16,$$

where the sum runs over all $P \in \mathbf{B}$ such that $i_{T,P} \not\equiv 0 \mod r_P$. This contradicts Lemma (4.2). $\square$

Proof of Proposition (4.1). By Proposition (4.3) $q = q\mathbb{Q}(X) = qW(X)$. So, $\gcd(q, r_P) = 1$ for all $P \in \mathbf{B}$.

(4.4) Case $q = 3$. We will show that this case does not occur. By (2.4.2) we have

$$(4.4.1) \qquad -K_X^3 = q^3L^3 = 162 + \frac{9}{2}K_X \cdot c_2(X) + 162 \sum_{P \in \mathbf{B}} c_P(-L).$$

By Lemma (4.2) $-K_X \cdot c_2(X) \geq 125/8$ and $-K_X^3 \geq 125/2$ by our assumptions. Combining this we obtain $\sum c_P(-L) \geq -467/2592$.

Again by Lemma (4.2) we have $\sum(r_P - 1/r_P) \leq 67/8$. Assume that $r_P = 2$ for all $P \in \mathbf{B}$. Note that $c_P(L) = -1/8$ (because $-K_X \sim L$ near each P). Hence $\mathbf{B} = (2)$. Then $-K_X \cdot c_2(X) = 45/2$. By (4.4.1) we have $-K_X^3 = 81/2 < 125/2$, a contradiction.

Thus we assume that at least one on the r_P's is ≥ 3. Recall that $\sum r_P \leq 10$, $\sum(r_P - 1/r_P) \leq 67/8$ and $3 \nmid r_P$. This gives us the following possibilities for $\mathbf{B}$:

$$(4), (5), (7), (8), (2, 4), (2, 5), (2, 7), (2, 2, 4), (2, 2, 5), (4, 4), (2, 2, 2, 4).$$

Take $0 \leq i_P < r_P$ so that $3i_P \equiv -1 \mod r_P$. Easy computations give us

r_P	2	4	5	7	8
i_P	1	1	3	2	5
c_P	$-1/8$	$-5/16$	$-1/5$	$-2/7, -3/7, -5/7$	$-5/32$

In all cases except for $\mathbf{B} = (8)$ we get a contradiction with $\sum c_P(-L) \geq -467/2592$. Consider the case $\mathbf{B} = (8)$. Then by (4.4.1) we have

$$-K_X^3 = 162 - \frac{9}{2} \cdot \frac{129}{8} - 162\frac{5}{32} = \frac{513}{8}.$$

Then by (2.5.2)

$$\dim|-K_X| = 2 + \frac{513}{16} - \frac{b_P(8 - b_P)}{16} = 34 + \frac{1 - b_P(8 - b_P)}{16}.$$

This number cannot be an integer, a contradiction.

(4.5) Case $q = 1$. By (2.6.1) we have

$$\sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P}\right) = 24 + K_X \cdot c_2(X) \leq 24 + \frac{1}{2}K_X^3 \leq 24 - \frac{125}{6} = \frac{19}{6}.$$

This gives the following possibilities: $\mathbf{B} = (2)$, (3), or $(2,2)$.

If $\mathbf{B} = (2,2)$, then $-K_X \cdot c_2(X) = 21$ and $-K_X^3 \le 63$. On the other hand, $-K_X^3 \in \frac{1}{2}\mathbb{Z}$ (see [4, Lemma 1.2]). Hence $-K_X^3 = 63$ or $125/2$. Further, by (2.5.2)

$$\dim|-K_X| = -\frac{1}{2}K_X^3 + \frac{3}{2}.$$

Since this number should be an integer, the only possibility is $-K_X^3 = 63$ and $\dim|-K_X| = 33$.

If $\mathbf{B} = (2)$, then $-K_X \cdot c_2(X) = 45/2$ and by (2.5.2)

$$\dim|-K_X| = -\frac{1}{2}K_X^3 + \frac{7}{4}.$$

Put $g := \dim|-K_X| - 1$. Then $-K_X^3 = 2g - 3/2$. We have

$$125/2 \le -K_X^3 = 2g - 3/2 \le 74 - \frac{9}{2}.$$

Hence $32 \le g \le 35$ and $-K_X^3 \in \{125/2, 129/2, 133/2, 137/2\}$.

Assume that $\mathbf{B} = (3)$. Then $-K_X \cdot c_2(X) = 64/3$ and $-K_X^3 \le 64$. As above,

$$\dim|-K_X| = -\frac{1}{2}K_X^3 + \frac{5}{3}.$$

We get only one possibility: $-K_X^3 = 188/3$ and $\dim|-K_X| = 33$.

(4.6) Case $q = 2$. If $\mathcal{E}$ is semistable, then as above by (2.6.1) $\mathbf{B} = (3)$. Otherwise we are in case (2.6.2) and as in the proof of Lemma (4.2) we have

$$\sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) = 24 + K_X \cdot c_2(X) \le 24 + \frac{5}{16}K_X^3 \le \frac{143}{32}.$$

Since $\gcd(r_P, q) = 1$, again we get the same possibility $\mathbf{B} = (3)$.

Then $-K_X \cdot c_2(X) = 64/3$ and $L \cdot c_2(X) = 32/3$. Hence

$$5/4(-K_X^3) \le t(4-3t)(-K_X^3) \le 4 \cdot 64/3.$$

Thus $125/2 \le -K_X^3 \le 1024/15$ and $125/16 \le L^3 \le 128/15$. Since $3L^3 \in \mathbb{Z}$ (see [4, Lemma 1.2]), we have $L^3 = 8$ or $25/3$. As above the case $L^3 = 8$ is impossible by (2.5.2). Thus $L^3 = 25/3$. Then one can easily compute $h^0(L)$ and $h^0(-K_X)$ by (2.4.1).

□

5. Case $q\mathbb{Q}(X) \geq 4$

(5.1) Proposition *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein, $-K_X^3 \geq 125/2$, and $q := qW(X) = q\mathbb{Q}(X) \geq 4$. Then we have one of the following cases:*

(5.1.1) $q = 4$, $\mathbf{B} = (5)$, $-K_X^3 = 384/5$, $\dim |L| = 3$, $\dim |2L| = 10$, $\dim |-K_X| = 40$;

(5.1.2) $q = 4$, $\mathbf{B} = (5, 5)$, $-K_X^3 = 64$, $\dim |L| = 2$, $\dim |2L| = 8$, $\dim |-K_X| = 33$;

(5.1.3) $q = 5$, $\mathbf{B} = (2)$, $-K_X^3 = 125/2$, $\dim |L| = 2$, $\dim |2L| = 6$, $\dim |-K_X| = 33$;

(5.1.4) $q = 5$, $\mathbf{B} = (2, 6)$, $-K_X^3 = 250/3$, $\dim |L| = 2$, $\dim |2L| = 7$, $\dim |-K_X| = 43$;

(5.1.5) $q = 5$, $\mathbf{B} = (7)$, $-K_X^3 = 500/7$, $\dim |L| = 2$, $\dim |2L| = 6$, $\dim |-K_X| = 37$;

(5.1.6) $q = 5$, $\mathbf{B} = (2, 2, 3, 6)$, $-K_X^3 = 125/2$, $\dim |L| = 1$, $\dim |2L| = 5$, $\dim |-K_X| = 32$;

(5.1.7) $q = 6$, $\mathbf{B} = (5, 7)$, $-K_X^3 = 2592/35$, $\dim |L| = 1$, $\dim |2L| = 4$, $\dim |-K_X| = 38$;

(5.1.8) $q = 7$, $\mathbf{B} = (3, 9)$, $-K_X^3 = 686/9$, $\dim |L| = 1$, $\dim |2L| = 3$, $\dim |-K_X| = 39$;

(5.1.9) $q = 7$, $\mathbf{B} = (2, 10)$, $-K_X^3 = 343/5$, $\dim |L| = 1$, $\dim |2L| = 3$, $\dim |3L| = 6$, $\dim |-K_X| = 35$.

Proof. Let L be a Weil divisor such that $-K_X \sim qL$. Since $qW(X) = q\mathbb{Q}(X)$, the group $\operatorname{Cl} X/\operatorname{Tors}$ is generated by L. To get our cases we run a computer program. Below is the description of our algorithm.

1) By (2.5.1) and Theorem (2.5.4) we have $\sum_{P \in \mathbf{B}}(1 - 1/r_P) \leq 24$. Hence there is only a finite (but very huge) number of possibilities for the basket $\mathbf{B}$. In each case we know $-K_X \cdot c_2(X)$ from (2.5.1). Let $r := \operatorname{lcm}(\{r_P\})$ be the Gorenstein index of X.

2) By Corollary (2.2.2) $q \leq 4r$ and $\gcd(q, r) = 1$. Hence we have only a finite number of possibilities for the index q.

3) In each case we compute L^3 and $-K_X^3 = q^3 L^3$ by formula (2.4.2) and check the condition $-K_X^3 \geq 125/2$. Here, for $D = -L$, the number i_P is uniquely determined by conditions $q i_P \equiv 1 \mod r_P$ and $0 \leq i_P < r_P$.

4) Next we check Kawamata's inequalities (2.6), i.e., we check that at least one of inequalities (2.6.1) – (2.6.4) holds. In case (2.6.2) we use the fact that the function $t(4-3t)$ is increasing for $t < 2/3$. Since $t \geq 1/q$, we have $\frac{1}{q}(4-\frac{3}{q}) \leq t(4-3t)$ and

$$\frac{1}{q}\left(4-\frac{3}{q}\right)\left(-K_X^3\right) \leq -4K_X \cdot c_2(X).$$

Similarly, in cases (2.6.3) and (2.6.4) we have, respectively,

$$\left(1-\frac{1}{q}\right)\left(1+\frac{3}{q}\right)\left(-K_X^3\right) \leq -4K_X \cdot c_2(X),$$

$$\frac{1}{q}\left(2-\frac{3}{q}\right)\left(-K_X^3\right) \leq -K_X \cdot c_2(X).$$

5) Finally, by the Kawamata-Viehweg vanishing theorem we have $\chi(tL) = h^0(tL) = 0$ for $-q < t < 0$. We check this condition by using (2.4.1).

At the end we get possibilities (5.1.1)–(5.1.9). $\square$

(5.2) Corollary (cf. [4, Remark 2.14]**).** *Let X be a $\mathbb{Q}$-Fano threefold. If $qW(X) = q\mathbb{Q}(X)$, then $-K_X^3 \leq 250/3$.*

Now we show that the condition $qW(X) = q\mathbb{Q}(X)$ in Proposition (5.1) is satisfied automatically.

(5.3) Proposition. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that $q := q\mathbb{Q}(X) > 3$ and $-K_X^3 > 45$. Then $\operatorname{Cl} X \simeq \mathbb{Z}$.*

Proof. Assume that the torsion part of $\operatorname{Cl} X$ is non-trivial for some X satisfying the conditions of Proposition (5.1). Take X so that $q\mathbb{Q}(X)$ is maximal. Write $K_X + qL \sim_{\mathbb{Q}} 0$, where L is an (ample) integral Weil divisor. Since $\operatorname{Cl} X$ is finitely generated and by cyclic covering trick [3, (3.6)], there is a finite étale in codimension one cover $\pi\colon X' \to X$ such that $\operatorname{Cl} X'$ torsion free. Here $K_{X'} + qL' \sim 0$, where $L' := \pi^*L$. Note that X' has only terminal singularities. Hence X' is a Fano threefold with terminal singularities with $qW(X') \geq q$. (It is possible however that X' is not $\mathbb{Q}$-factorial and $\rho(X') > 1$). Denote $n := \deg \pi$. Clearly, $-K_{X'}^3 = -nK_X^3 \geq -2K_X^3$. Hence $\dim |-K_{X'}| \geq -K_X^2 - 2 > 43$. Let $\sigma\colon X'' \to X'$ be a $\mathbb{Q}$-factorialization. (If X' is $\mathbb{Q}$-factorial, we take $X'' = X'$). Run K-MMP on X'': $\upsilon\colon X'' \dashrightarrow Y$. At the end we get a Mori-Fano fibre space $f\colon Y \to Z$. Let $L'' := \sigma^{-1}(L')$ and $L_Y := \upsilon_* L''$. Then $-K_Y \sim qL_Y$. If $\dim Z > 0$, then for a general fibre $F := f^{-1}(o)$, $o \in Z$ we have $-K_F \sim qL_Y|_F$. This is impossible if $q > 3$.

In the case $\dim Z = 0$, Y is a Fano with $\rho(Y) = 1$ and $qW(Y) \geq q$. By our assumption of maximality of $q = q\mathbb{Q}(X)$ we have $q\mathbb{Q}(Y) =$

$qW(Y) = q$. Hence, $-K_Y^3 \leq 250/3$ by Corollary (5.2). By (2.5.3) we have $\dim|-K_Y| \leq 43$. Using (2.5.5) we obtain

$$43 \geq \dim|-K_Y| \geq \dim|-K_{X''}| \geq -\frac{1}{2}K_{X''}^3 - 2 \geq -K_X^3 - 2.$$

Thus $-K_X^3 \leq 45$, a contradiction. $\square$

6. Proof of the main theorem

(6.1) To construct a Sarkisov link such as in (3.2.1), we need the following result basically due to Ambro and Kawachi.

(6.1.1) Proposition (cf. [6, Th. 4.1]**).** *Let X be a Fano threefold with terminal singularities, and let S be an ample Cartier divisor proportional to $-K_X$. Then the linear system $|S|$ is non-empty and a general member of $|S|$ is a reduced irreducible normal surface whose singularities are at worst log terminal of type T. Moreover, assume that $K_X^2 \cdot S > 1$ and $qF(X) \geq 1/2$. Then a general $S \in |S|$ does not pass through non-Gorenstein points* (*and has at worst Du Val singularities*).

Proof. According to [16] the pair (X, S) is plt for a general $S \in |S|$. Then singularities of S are of type T by [17]. Note that the restriction map $H^0(\mathcal{O}_X(S)) \to H^0(\mathcal{O}_S(S))$ is surjective. Let $P \in \operatorname{Bs}|S|$ be a non-Gorenstein point of X. Then $P \in S$ is a log terminal non-Du Val singularity of type T.

Recall that Kawachi's invariant of a normal surface singularity (S, P) is defined as $\delta_P := -(\Gamma - \Delta)^2$, where Δ is the codiscrepancy divisor of (S, P) on the minimal resolution $\hat{S} \to S$ and Γ is the fundamental cycle on $\hat{S}$ (see [18]). If (S, P) is a rational singularity, then $\delta_P = \Gamma^2 - \Delta^2 + 4$. Hence in our case Kawachi's invariant δ_P is integral (because $\Delta^2 \in \mathbb{Z}$, see [17]). On the other hand, $0 < \delta_P < 2$. Thus $\delta_P = 1$. Now we apply the main result of [18] to the linear system $|S|_S| = |K_S - K_X|_S|$. It follows that there is a curve C on S passing through P and such that $-K_X \cdot C < 1/2$. Since $qF(X) \geq 1/2$, this is impossible. $\square$

(6.1.2) Proposition. *In notation of Proposition* (6.1.1) *assume additionally that* $(2K_X + S)^2 \cdot S \geq 5$ *and* $-(2K_X + S)$ *is an ample divisor which is divisible in* $\operatorname{Cl} X/\operatorname{Tors}$. *Then the linear system $|-K_X|$ has only isolated base points.*

Proof. Denote the restriction $-K_X|_S$ by D. Since S does not pass through non-Gorenstein points, D is Cartier. By the Kawamata-Viehweg vanishing the map

$$H^0(\mathcal{O}_X(-K_X)) \longrightarrow H^0(\mathcal{O}_S(D))$$

is surjective. Thus it is sufficient to show that the linear system $|D|$ is base point free. By the adjunction formula $D = K_S - (2K_X + S)|_S$. Let $\mu\colon \hat{S} \to S$ be the minimal resolution. Since S has at worst Du Val singularities, $K_{\hat{S}} = \mu^* K_S$. Thus we can write $\mu^* D = K_{\hat{S}} + M$, where $M = \mu^*(-(2K_X + S)|_S)$ is nef. It is easy to see that $M^2 = (2K_X + S)^2 \cdot S \geq 5$ by our assumption. Suppose that the linear system $|\mu^* D| = |K_{\hat{S}} + M|$ has a base point P. By the main theorem of [19] there is an effective divisor E on $\hat{S}$ passing through P such that either $M \cdot E = 0$, $E^2 = -1$ or $M \cdot E = 1$, $E^2 = 0$. In the former case E is contracted my μ and we get a contradiction by the genus formula. In the latter case we have $-(2K_X + S) \cdot \mu(E) = 1$. This is impossible because $-(2K_X + S)$ is divisible in $\operatorname{Cl} X/\operatorname{Tors}$ and $\mu(E)$ is contained in the Gorenstein locus of X. $\square$

Since $qF(X) = q/r$, we have the following

(6.1.3) Corollary. *Let X be a $\mathbb{Q}$-Fano threefold, let $q := q\mathbb{Q}(X)$, and let r be the Gorenstein index of X. Assume that $-K_X^3 > q/r = qF(X)$, $2q - r \geq 2$, and $(-K_X^3)(2q-r)^2 r \geq 5q^3$. Then the linear system $|-K_X|$ has only isolated base points.*

Proof. Let L be the Weil divisor such that $-K_X \sim_{\mathbb{Q}} qL$. Take $S = rL$ and apply Proposition (6.1.2). $\square$

Now we are in position to prove Theorem (1.2).

(6.2) Main assumption. Let X be a $\mathbb{Q}$-Fano threefold. We assume that $-K_X^3 \geq 125/2$. Then X is such as in Propositions (4.1) or (5.1). In particular, $\dim |-K_X| \geq 32$. By Propositions (4.3) and (5.3) we also have $\operatorname{Cl} X \simeq \mathbb{Z}$. We divide cases of (4.1) or (5.1) in four groups and treat these groups separately (see (6.3), (6.4) (6.5), (6.6)).

(6.2.1) Proposition. *Notation and assumptions as in* (6.2)*. If there exists a Sarkisov link* (3.2.1) *with birational f, then $-K_Z^3 \geq 125/2$ except possibly for the following case*

- $\dim |-K_Z| = \dim |-K_X| = 32$.

Proof. Assume the converse. Then Z is a $\mathbb{Q}$-Fano with $\dim |-K_Z| \geq \dim |-K_X| \geq 32$ and $-K_Z^3 < 125/2$. By (2.5.3)

$$\dim |-K_Z| + \frac{1}{2} \sum_{P \in \mathbf{B}_Z} \left(1 - \frac{1}{r_P}\right) \leq -\frac{1}{2} K_Z^3 + 2 < \frac{133}{4}. \tag{6.2.2}$$

Therefore, $\dim |-K_Z| = 32$ or 33. Moreover, if $\dim |-K_Z| = 33$, then we have $r_P = 1$ for all $P \in \mathbf{B}_Z$, i.e., Z is Gorenstein (and factorial). In particular, $q\mathbb{Q}(Z) = qF(Z) = qW(Z)$ and $q\mathbb{Q}(Z)^3$ divides $-K_Z^3$. By Riemann-Roch, $-K_Z^3 = 62$. Therefore, $q\mathbb{Q}(Z) = 1$. But then $-K_Z$ cannot be decomposed into a sum of movable divisors. We derive a contradiction by [15]. $\square$

(6.3) Case (5.1.3)

(6.3.1) Proposition (see [20]**).** *In case* (5.1.3), $X \simeq \mathbb{P}(1,1,1,2)$.

Proof. Let $S \in |2L|$ be a general member. Then S is Cartier and by Proposition (6.1.1) X is has at worst Du Val singularities. By the adjunction formula S is a del Pezzo surface of degree 9. It follows that S is smooth and $S \simeq \mathbb{P}^2$ (see Remark (3.5.4)). The restriction map $H^0(X, \mathcal{O}_X(S)) \to H^0(S, \mathcal{O}_S(S))$ is surjective. Hence the linear system $|S|$ is base point free and determines a morphism $\varphi\colon X \to \mathbb{P}^6$. We have $(\deg\varphi)(\deg\varphi(X)) = S^3 = 4$. So φ is birational and $\varphi(X) \subset \mathbb{P}^6$ is a variety of degree 4. A general hyperplane section $\varphi(S) \subset \varphi(X)$ is a Veronese surface. It is well-known that in this situation $\varphi(X)$ is a cone over $\varphi(S)$, i.e., $X \simeq \varphi(X) \simeq \mathbb{P}(1,1,1,2)$. $\square$

(6.4) Cases (4.1.4), (5.1.1), (5.1.2), (5.1.4), (5.1.5), (5.1.6), (5.1.8), (5.1.9). We apply construction (3.4.1). Let r be the Gorenstein index of X. First we construct a birational extremal extraction $g\colon \tilde{X} \to X$ such that $\tilde{X}$ has only terminal singularities and the exceptional divisor E of g has discrepancy $1/r$.

(6.4.1) Claim. *Either*

(i) *There is a cyclic quotient singularity $P \in X$ of type $\frac{1}{r}(b, -b, 1)$, where* $\gcd(r, b) = 1$, *or*

(ii) *we are in case* (5.1.2) *and there is a point $P \in X$ of type $cA/5$ of axial weight* 2.

Proof. Note that in all cases there is a basket point $P \in \mathbf{B}$ of index r. If this point is unique, it corresponds to a cyclic quotient singularity of X. The point $P \in \mathbf{B}$ of index r is not unique only in case (5.1.2). Then $r = 5$ and there are two points P_1, $P_2 \in \mathbf{B}$ of index 5. They correspond either two cyclic quotient singularities of X or a point $P \in X$ of type $cA/5$. $\square$

In case (i) the weighted blowup of $P \in X$ with weights $\frac{1}{r}(b, r - b, 1)$ gives us a desired contraction g. Similarly, in case (ii) a suitable weighted blowup gives us a desired contraction g (see [21]).

Further, $r\mathcal{H}$ is the linear system of Cartier divisors. Hence we can write $g^*\mathcal{H} = \tilde{\mathcal{H}} + \delta E$, where $\delta \geq 1/r$. Thus,

$$(6.4.2) \qquad -K_{\tilde{X}} \sim_{\mathbb{Q}} g^*(-K_X) - \tfrac{1}{r}E \sim_{\mathbb{Q}} \tilde{\mathcal{H}} + (\delta - \tfrac{1}{r})E.$$

By Corollary (6.1.3) the linear system $\tilde{\mathcal{H}}$ has only isolated base points outside of E. Therefore, $-K_{\tilde{X}}$ is nef.

If $g(E)$ is a cyclic quotient singularity, then $E \simeq \mathbb{P}(b, r-b, 1)$, $E|_E \sim \mathcal{O}_{\mathbb{P}(b,r-b,1)}(-r)$, and $E^3 = r^2/b(r-b)$. Therefore,

$$-K_{\tilde{X}}^3 = -K_X^3 - \tfrac{1}{r^3}E^3 \geq \frac{125}{2} - \frac{r^2}{b(r-b)} > 0.$$

This shows that $-K_{\tilde{X}}$ is big. Similar computations shows that this fact also holds in case (6.4.1), (ii).

Let C be a curve such that $-K_{\tilde{X}} \cdot C = 0$. By (3.3.1) we have $q\tilde{L} \cdot C + \beta E \cdot C = 0$. By (6.4.2) $E \cdot C > 0$. Hence $\tilde{L} \cdot C < 0$. Since $\dim |L| > 0$, there is at most a finite number of such curves. Thus the linear system $|-nK_{\tilde{X}}|$ does not contract any divisors.

(6.4.3) Consider diagram (3.2.1). Since $K_X + qL \sim 0$, the constant β in (3.3) is a non-negative integer. We can write

$$K_{\tilde{X}} = g^*K_X + \frac{1}{r}E, \qquad \tilde{L} = g^*L - \delta E,$$

where $\delta \in \mathbb{Q}$, $\delta > 0$. Since rL is Cartier (see Lemma (2.2.1)), $\delta = k/r$ for some $k \in \mathbb{Z}$, $k > 0$. Therefore,

$$\beta = -\frac{1}{r} + q\delta = \frac{qk-1}{r}$$

and the value of β is bounded from below as follows:

case	(4.1.4)	(5.1.1)(5.1.2)(5.1.8)	(5.1.4)(5.1.6)	(5.1.5)(5.1.9)
β	≥ 1	≥ 3	≥ 4	≥ 2

(6.4.4) First we assume that $\dim Z = \dim X$. Then f is a divisorial contraction and Z is a $\mathbb{Q}$-Fano threefold. By (3.3.1) we have $K_Z + qL_Z + \beta E_Z \sim_{\mathbb{Q}} 0$, where E_Z and L_Z are effective non-zero divisors. Hence, $q\mathbb{Q}(Z) \geq q + \beta > 4$. In particular, Z is not Gorenstein (see Corollary (2.2.2)).

Assume that $-K_Z^3 < 125/2$. By Proposition (6.2.1) $\dim|-K_X| = \dim|-K_Z| = 32$. Hence X is of type (5.1.6). By (6.2.2) $\dim|-K_Z| \geq 60$ and by (3.3.1) $q\mathbb{Q}(Z) \geq 9$. On the other hand, discrep$(Z) \geq$

discrep$(\tilde{X}) \geq 1/5$. Therefore the Gorenstein index of Z is at most 5 (see [21]). By Proposition (5.3) $\operatorname{Cl} Z \simeq \mathbb{Z}$. Let L' be the ample generator of $\operatorname{Cl} Z \simeq \mathbb{Z}$, let $r' \leq 5$ be the Gorenstein index of Z, and let $S \in |r'L'|$ a general member. Then S be the ample generator of $\operatorname{Pic} Z$. By Proposition (6.1.1) S has at worst Du Val singularities. By the adjunction formula $K_S = (r' - q\mathbb{Q}(Z))L'|_S$. Since $L'|_S$ is a Cartier divisor, S is a del Pezzo surface with $qF(S) \geq q\mathbb{Q}(Z) - r' \geq 4$. This is impossible (see (3.5.4)). Thus $-K_Z^3 \geq 125/2$ and Z is such as in (5.1).

Now we consider possibilities for X case by case. In cases (5.1.4), (5.1.6), (5.1.8), and (5.1.9) we have $q\mathbb{Q}(Z) \geq 9$, a contradiction. In cases (5.1.1), (5.1.2), and (5.1.5) we have $q\mathbb{Q}(Z) = 7$. Hence Z is such as in (5.1.8) or (5.1.9). Then $q + \beta = 7$. By (3.3.1) L_Z and E_Z are linear equivalent and they are generators of $\operatorname{Cl} Z$. On the other hand, $\dim |L| \geq 2 > \dim |L_Z| = 1$, a contradiction.

In case (4.1.4) $\tilde{X}$ is of Gorenstein index 2. Hence, discrep$(\tilde{X}) = 1/2$. On the other hand, $f \circ \chi$ is a composition of a flop and steps of the K-MMP. Therefore, discrep$(Z) \geq 1/2$. This is possible only if Z of type (5.1.3). But then $35 = \dim |-K_X| > \dim |-K_Z| = 33$, a contradiction.

(6.4.5) Thus we may assume that $\dim Z < \dim X$. Let $M \in |2L|$ be a general member. Note that by (6.4.3) $q+\beta \geq 3$ and $q+\beta = 3$ only in case (4.1.4). By (3.3.2) L_Y can be f-horizontal only in case (4.1.4) and if Z is a curve. By Lemma (3.5.8) we have a contradiction. Hence L_Y is f-vertical. As in Lemma (3.5.5) we have $L_Y = f^*\Xi$ for some integral Weil divisor Ξ on Z, $\dim |\Xi| = \dim |L| \geq 1$, and Ξ is a generator of $\operatorname{Cl} Z/\mathrm{Tors}$.

(6.4.6) Assume that Z is a surface. From (3.3.2) we get $\beta \leq 2$. By (6.4.3) this is possible only in cases (4.1.4), (5.1.5) or (5.1.9). If $K_Z^2 < 8$, we have $\dim |\Xi| = 0$, a contradiction. Hence Z is either $\mathbb{P}^2$ or $\mathbb{P}(1,1,2)$. Consider the case $Z \simeq \mathbb{P}(1,1,2)$. Then $\dim |\Xi| = 1$ and we are in case (5.1.9). Let $M \in |3L|$ be a general member. We can write $K_Y + 2M_Y + L_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. This shows that M_Y is f-vertical. Thus $M_Y \sim 3L_Y = 3f^*\Xi$ and $\dim |M_Y| = \dim |3\Xi| = 4$, a contradiction.

Consider the case $Z \simeq \mathbb{P}^2$. Then $\dim |\Xi| = 2$ and we are in case (5.1.5). Let $M \in |2L|$ be a general member. We can write $K_Y + 2M_Y + L_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. This shows that $\gamma = \beta = 2$ and M_Y is f-vertical. Thus $M_Y \sim 2L_Y = f^*\Xi$ and $\dim |M_Y| = \dim |2\Xi| = 5$, a contradiction.

(6.4.7) Assume that Z is a curve. Then $Z \simeq \mathbb{P}^1$. Since $L_Y = f^*\Xi$ is not divisible in $\operatorname{Cl} Y$, $\dim |\Xi| \leq 1$. So we are in cases (5.1.6), (5.1.8), or (5.1.9). Moreover, since $\dim|L| > 0$, $\dim|\Xi| = 1$. Case (5.1.6) is impossible because then $\beta \geq 4$. Let $M \in |2L|$ be a general member. We can write $K_Y + 3M_Y + L_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. This shows that M_Y is f-vertical. Thus $M_Y \sim 2L_Y = 2f^*\Xi$ and $\dim|M_Y| = \dim|2\Xi| = 2$, a contradiction.

Now we consider case (5.1.7).

(6.5) Case (5.1.7). By Lemmas (3.6.2) and (3.6.3) the pair $(X, |-K_X|)$ is not canonical. Thus we apply the construction (3.2.1) in case (3.4.2). Then in (3.2.2) we have $\alpha > 0$. Assume that $\dim Z = 3$. Since $\alpha > 0$, and by Proposition (2.5.6) we have $\dim|-K_Z| > \dim|-K_X| = 38$. Then by Proposition (6.2.1) $-K_Z^3 \geq 125/2$. Hence Z is $\mathbb{Q}$-Fano such as in Proposition (5.1). Moreover, by (3.3.1) we have $q\mathbb{Q}(Z) \geq q\mathbb{Q}(X) + \beta = 6 + \beta$. This implies that $E_Z \sim L_Z$ is a generator of $\operatorname{Cl} Z$, $q\mathbb{Q}(Z) = 7$, and $\beta = 1$. So, the variety Z is of type (5.1.8). Obviously, $\dim|2L_Z| \geq \dim|2L|$. This contradicts Proposition (5.1).

Thus $\dim Z = 1$ or 2. If Z is a surface, then by Lemma (3.5.5) $Z \simeq \mathbb{P}(1,1,2)$. Let $M \in |2L|$ be a general member. We can write $K_Y + 3M_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. Restricting to a general fibre we obtain that M_Y is f-vertical. Thus, $M_Y \sim 2L_Y = 2f^*\Xi$ and $\dim|M_Y| = \dim|2\Xi| \leq 3$, a contradiction.

Finally we consider cases when $q\mathbb{Q}(X) = 1$.

(6.6) Cases (4.1.1), (4.1.2), (4.1.3). By Lemmas (3.6.2) and (3.6.3) the pair $(X, |-K_X|)$ is not canonical. Thus we may apply construction (3.2) under assumptions (3.4.2).

Then in (3.2.2) we have $\alpha > 0$. Assume that $\dim Z = 3$. Similar to (6.5) $\dim|-K_Z| > \dim|-K_X|$ and $-K_Z^3 \geq 125/2$. Hence Z is $\mathbb{Q}$-Fano such as in Proposition (5.1) or (4.1) with $q\mathbb{Q}(Z) > 1$. By (6.3), (6.4), and (6.5) Z is of type (5.1.3) and $Z \simeq \mathbb{P}(1,1,1,2)$. Then $\dim|-K_X| < \dim|-K_Z| = 33$, so X is of type (4.1.1) and $\dim \mathcal{H}_Z \geq 32$. Easy computations show that $\mathcal{H}_Z \sim \mathcal{O}_{\mathbb{P}(1,1,1,2)}(n)$, with $n \geq 5$. On the other hand, $-K_Z \sim \mathcal{H}_Z + \alpha E_Z$, where $\alpha > 0$, a contradiction.

Therefore, $1 \leq \dim Z \leq 2$. If Z is a curve, we have a contradiction by Lemma (3.5.8). Thus Z is a surface. Then by Lemma (3.5.6) the fibration f has no discriminant curve. Hence by Lemma (3.5.7) we have $\dim|-K_X| < 30$, a contradiction.

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