THE DEGREE OF $\mathbb{Q}$-FANO THREEFOLDS

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1. Introduction

In this paper a $\mathbb{Q}$-*Fano variety* is a normal projective variety X with at worst $\mathbb{Q}$-factorial terminal singularities such that $-K_X$ is ample and $\operatorname{Pic} X$ is of rank one. Fano varieties with terminal singularities form in important class because, according to the minimal model program, every variety of negative Kodaira dimension should be birationally equivalent to a fibration $Y \to Z$ whose general fibre Y_η belong to this class. Moreover, in the case $\dim Z = 0$, $Y_\eta = Y$ is of Picard number one, i.e., Y is a $\mathbb{Q}$-Fano.

In dimension 2 the only $\mathbb{Q}$-Fano variety is the projective plane $\mathbb{P}^2$. In dimension 3 $\mathbb{Q}$-Fanos are bounded in the moduli sense by the following result of Kawamata:

(1.1) Theorem ([1]). *There exist positive integers r and d such that for an arbitrary $\mathbb{Q}$-Fano threefold X we have $-K_X^3 \le d$ and rK_X is Cartier.*

Since the Weil divisor $-K_X$ gives a natural polarization of a $\mathbb{Q}$-Fano variety X, the rational number $-K_X^3$ is a very important invariant. It is called *the degree* of X. In this paper we find a sharp bound for $-K_X^3$:

(1.2) Theorem. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein. Then $-K_X^3 \le 125/2$ and the equality holds if and only if X is isomorphic to the weighted projective space $\mathbb{P}(1, 1, 1, 2)$.*

Note that in the Gorenstein case we have the estimate $-K_X^3 \le 64$ by the classification of Iskovskikh and Mori-Mukai and by Namikawa's result [2].

The idea of the proof is as follows. In Sections 4 and 5 using Riemann-Roch formula for Weil divisors [3] and Kawamata's estimates [1] we produce a short list of possibilities for singularities of $\mathbb{Q}$-Fanos of degree $\ge 125/2$. Here, to check a finite (but very huge) number of Diophantine conditions, we use a computer program (cf. [4]). In

The author was partially supported by grants CRDF-RUM, No. 1-2692-MO-05 and RFBR, No. 05-01-00353-a.

Section 6 we exclude all these possibilities except for $\mathbb{P}(1,1,1,2)$ by applying some birational transformations described in Section 3. The techniques used on this step is a very common in birational geometry (see [5], [6], [7]). It goes back to Fano-Iskovskikh "double projection method". The present paper is a logical continuation of our previous papers [8], [9] where we studied effective bounds of degree for sertain singular Fano threefolds.

Acknowledgements. The work was carried out at Max-Planck-Institut für Mathematik, Bonn in 2006. The author would like to thank the institute for the support and hospitality.

2. Preliminaries

Throughout this paper, we work over the complex number field $\mathbb{C}$.

(2.1) By $\operatorname{Cl} X$ we denote the Weil divisor class group of a normal variety X (modulo linear equivalence). There is a natural embedding $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$. Let X be a Fano variety with at worst log terminal singularities. It is well-known that both $\operatorname{Pic} X$ and $\operatorname{Cl} X$ are finitely generated and $\operatorname{Pic} X$ is torsion free (see e.g. [10, §2.1]). Moreover, numerical equivalence of $\mathbb{Q}$-Cartier divisors coincides with $\mathbb{Q}$-linear one. Therefore one can define the following numbers:

$$\begin{aligned} qF(X) &:= \max\{q \mid -K_X \sim_{\mathbb{Q}} qH, \quad H \in \operatorname{Pic} X\}, \\ q\mathbb{Q}(X) &:= \max\{q \mid -K_X \sim_{\mathbb{Q}} qL, \quad L \in \operatorname{Cl} X\}, \\ qW(X) &:= \max\{q \mid -K_X \sim qL, \quad L \in \operatorname{Cl} X\}. \end{aligned}$$

By the above, all of them are positive, $q\mathbb{Q}(X), qW(X) \in \mathbb{Z}$, and $qF(X) \in \mathbb{Q}$. If X is smooth all these numbers coincide with the *Fano index* of X. In general, we obviously have $q\mathbb{Q}(X) \geq qF(X)$ and $q\mathbb{Q}(X) \geq qW(X)$.

(2.1.1) Proposition (see e.g. [10, §2.1]**).** $qF(X) \leq \dim X + 1$.

The index $qW(X)$ was considered in [4]. In particular, it was proved that $qW(X) \leq 19$ for any $\mathbb{Q}$-Fano threefold.

(2.2) Terminal singularities Let (X, P) be a three-dimensional terminal singularity. It follows from the classification that there is a one-parameter deformation $\mathfrak{X} \to \Delta \ni 0$ over a small disk $\Delta \subset \mathbb{C}$ such that the central fibre $\mathfrak{X}_0$ is isomorphic to X and the generic fibre $\mathfrak{X}_\lambda$ has only cyclic quotient singularities $P_{\lambda,k}$ (see, e.g., [3]). Thus, to every theefold X with terminal singularities, one can associate a collection $\mathbf{B} = \{(r_{P,k}, b_{P,k})\}$, where $P_{\lambda,k} \in \mathfrak{X}_\lambda$ is a singularity of type

$\frac{1}{r_{P,k}}(1, b_{P,k}, -b_{P,k})$. This collection is uniquely determined by X and called the *basket* of singularities of X. By abuse of notation, we also will write $\mathbf{B} = (r_{P,k})$ instead of $\mathbf{B} = \{(r_{P,k}, b_{P,k})\}$. The index of P is the least common multiple of indices of points $P_{\lambda,k}$.

(2.2.1) Lemma ([11, Corollary 5.2]). *Let (X, P) be a three-dimensional terminal singularity of index r and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. There is an integer, i such that $D \sim iK_X$ near P. In particular, rD is Cartier.*

(2.2.2) Corollary. *Let X be a Fano threefold with terminal singularities and let r be the Gorenstein index of X. Then*

(i) $\gcd(r, qW(X)) = 1$,
(ii) $qF(X)r = q\mathbb{Q}(X)$,
(iii) $qW(X) \le q\mathbb{Q}(X) \le 4r$.

(2.2.3) Let (X, P) be a three-dimensional terminal singularity of index r and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. By Lemma (2.2.1) there is an integer i such that $0 \le i < r$ and $D \sim iK_X$ near P. Deforming D with (X, P) we obtain Weil divisors D_λ on X_λ. Thus we have a collection of numbers i_k such that $0 \le i_k < r_k$ and $D_\lambda \sim i_k K_{X_\lambda}$ near $P_{\lambda,k}$.

(2.3) Riemann-Roch formula [3]. Let X be a threefold with terminal singularities and let D be a Weil $\mathbb{Q}$-Cartier divisor on X. Then

$$\chi(D) = \frac{1}{12} D \cdot (D - K_X) \cdot (2D - K_X) + \frac{1}{12} D \cdot c_2 + \sum_{P \in \mathbf{B}} c_P(D) + \chi(\mathcal{O}_X), \tag{2.3.1}$$

where

$$c_P(D) = -i_P \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_P - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

(2.4) Now let X be a Fano threefold with terminal singularities, let $q := q\mathbb{Q}(X)$, and let L be an ample Weil $\mathbb{Q}$-Cartier divisor on X such that $-K_X \sim_{\mathbb{Q}} qL$. By (2.3.1) we have

$$\chi(tL) = 1 + \frac{t(q+t)(q+2t)}{12} L^3 + \frac{tL \cdot c_2}{12} + \sum_{P \in \mathbf{B}} c_P(tL), \tag{2.4.1}$$

$$c_P(tL) = -i_{P,t} \frac{r_P^2 - 1}{12 r_P} + \sum_{j=1}^{i_{P,t} - 1} \frac{\overline{b_P j}(r_P - \overline{b_P j})}{2 r_P}.$$

If $q > 2$, then $\chi(-L) = 0$. Using this equality we obtain (see [4])

$$L^3 = \frac{12}{(q-1)(q-2)} \left(1 - \frac{L \cdot c_2}{12} + \sum_{P \in B} c_P(-L) \right). \tag{2.4.2}$$

(2.5) In the above notation, applying (2.3.1), Serre duality and Kawamata-Viehweg vanishing to $D = K_X$ we get the following important equality (see, e.g., [3]):

$$24 = -K_X \cdot c_2 + \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right). \tag{2.5.1}$$

Similarly, for $D = -K_X$ we have $H^i(X, -K_X) = 0$ for $i > 0$ and

$$c_P(-K_X) = \frac{r_P^2 - 1}{12 r_P} - \frac{b_P(r - b_P)}{2 r_P}.$$

(see [5, §2]). Combining this with (2.5.1) we obtain

$$\dim|-K_X| = -\frac{1}{2} K_X^3 + 2 - \sum_{P \in \mathbf{B}} \frac{b_P(r_P - b_P)}{2 r_P}. \tag{2.5.2}$$

In particular,

$$\dim|-K_X| \le -\frac{1}{2} K_X^3 + 2 - \frac{1}{2} \sum_{P \in \mathbf{B}} \left(1 - \frac{1}{r_P} \right) \le -\frac{1}{2} K_X^3 + 2. \tag{2.5.3}$$

(2.5.4) Theorem ([1], [12]). *In the above notation,* $-K_X \cdot c_2(X) \ge 0$.

As a corollary we have ([5, §2]):

$$\dim|-K_X| \ge -\frac{1}{2} K_X^3 - 2. \tag{2.5.5}$$

(2.5.6) Proposition ([5, §2]]). *Let X be a $\mathbb{Q}$-Fano threefold. If $\dim|-K_X| \ge 2$, then the linear system $|-K_X|$ has no base components and is not composed of a pencil. (In particular, a general element of $|-K_X|$ is reduced and irreducible.)*

(2.6) Now let X be a $\mathbb{Q}$-Fano threefold, let $q := q\mathbb{Q}(X)$, and let L be an ample Weil divisor on X that generates the group $\operatorname{Cl} X/\operatorname{Tors}$. Let $\mathcal{E}$ be the double dual to Ω^1_X. If $\mathcal{E}$ is not semistable, there is a maximal destabilizing subsheaf $\mathcal{F} \subset \mathcal{E}$. Clearly, $c_1(\mathcal{F}) \equiv -pL$ for some $p \in \mathbb{Z}$. Put $t := p/q$, so that $c_1(\mathcal{F}) \equiv tK_X$. According to [1] there are the following possibilities:

(2.6.1) $\mathcal{E}$ is semistable. Then $-K_X^3 \le -3K_X \cdot c_2(X)$.

(2.6.2) $\mathcal{E}$ is not semistable and $\operatorname{rk}\mathcal{F} = 2$. Then $q \geq 2$, $\quad 0 < t < 2/3$, and

$$t(4-3t)(-K_X^3) \leq -4K_X \cdot c_2(X).$$

(2.6.3) $\mathcal{E}$ is not semistable, $\operatorname{rk}\mathcal{F} = 1$, and $(\mathcal{E}/\mathcal{F})^{**}$ is semistable. Then $q \geq 4$, $\quad 0 < t < 1/3$, and

$$(1-t)(1+3t)(-K_X^3) \leq -4K_X \cdot c_2(X).$$

(2.6.4) $\mathcal{E}$ is not semistable, $\operatorname{rk}\mathcal{F} = 1$, and $(\mathcal{E}/\mathcal{F})^{**}$ is not semistable. Then again $q \geq 4$ and $0 < t < 1/3$. There exists an unstable reflexive sheaf $\mathcal{F} \subsetneq \mathcal{G} \subsetneq \mathcal{E}$. Write $c_1(\mathcal{G}/\mathcal{F}) \equiv -p'L$, $p' \in \mathbb{Z}$ and put $u := p'/q$, so that $c_1(\mathcal{G}/\mathcal{F}) \equiv uK_X$. Then $t < u < 1-t-u$ and

$$\Bigl(tu + (t+u)(1-t-u)\Bigr)(-K_X^3) \leq -K_X \cdot c_2(X),$$

(2.7) Corollary. *If $q\mathbb{Q}(X) = 1$, then $\mathcal{E}$ is semistable. If $q\mathbb{Q}(X) \leq 3$, then either $\mathcal{E}$ is semistable or we are in case* (2.6.2).

3. Two birational constructions

(3.1) Let X be a $\mathbb{Q}$-Fano threefold. Throughout this paper we assume that the linear system $|-K_X|$ is non-empty, has no fixed components, and is not composed of a pencil. Then a general member $H \in |-K_X|$ is irreducible. By (2.5.5) and (2.5.6) this holds automatically when $-K_X^3 \geq 8$. Let $q := q\mathbb{Q}(X)$ and L be the ample Weil divisor that generates the group $\operatorname{Cl} X/\operatorname{Tors}$. Thus we have $-K_X \equiv qL$. Put $\mathcal{H} := |-K_X|$. Let $H \in \mathcal{H}$ be a general member.

(3.2) Assume there is a diagram (Sarkisov link of type I or II)

(3.2.1)
$$\begin{array}{ccc} \tilde{X} & \overset{\chi}{\dashrightarrow} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & & Z \end{array}$$

where $\tilde{X}$ and Y have only $\mathbb{Q}$-factorial terminal singularities, $\rho(\tilde{X}) = \rho(Y) = 2$, g is a Mori extremal divisorial contraction, $\tilde{X} \dashrightarrow Y$ is a sequence of log flips, and f is a Mori extremal contraction (either divisorial or fibre type). Thus one of the following holds: a) $\dim Z = 1$ and f is a $\mathbb{Q}$-del Pezzo fibration, b) $\dim Z = 2$ and f is a $\mathbb{Q}$-conic bundle, or c) $\dim Z = 3$, f is a divisorial contraction, and Z is a $\mathbb{Q}$-Fano. Let E be the g-exceptional divisor. We assume that the composition $f \circ \chi \circ g^{-1}$ is not an isomorphism. For a divisor D on X, everywhere below $\tilde{D}$ and D_Y denote strict birational transforms of D on $\tilde{X}$ and Y, respectively. We also assume that the discrepancy $\alpha := a(E, X, \mathcal{H})$ is non-positive, i.e.,

(3.2.2) $$0 \sim f^*(K_X + \mathcal{H}) = K_{\tilde{X}} + \tilde{\mathcal{H}} + \alpha E, \quad \alpha \in \mathbb{Z}, \quad \alpha \geq 0.$$

By the above we have

(3.2.3) $$\dim |-K_{\tilde{X}}| \geq \dim \tilde{\mathcal{H}} = \dim |-K_X|.$$

(3.3) Similarly,

$$0 \sim_{\mathbb{Q}} g^*(K_X + qL) \sim_{\mathbb{Q}} K_{\tilde{X}} + q\tilde{L} + \beta E, \qquad \beta \geq 0.$$

Therefore,

(3.3.1) $$K_Y + qL_Y + \beta E_Y \sim_{\mathbb{Q}} 0.$$

If $q\mathbb{Q}(X) = qW(X)$, then $K_X + qL \sim 0$ and β is an integer $\geq \alpha$.

Let $F = f^{-1}(\mathrm{pt})$ be a general fibre. Recall that F is either $\mathbb{P}^1$ or a smooth del Pezzo surface. Restricting (3.3.1) to F we get

(3.3.2) $$K_F + qL_Y|_F + \beta E_Y|_F \sim 0.$$

Here $-K_F$, $L_Y|_F$, and $E_Y|_F$ are proportional nef Cartier divisors. Moreover, $-K_F$ and $E_Y|_F$ are ample.

(3.4) We will use construction (3.2.1) in the following two situationa:

(3.4.1) (see [6], [7]). Let $P \in X$ be a singularity of index r. Take g to be a divisorial blowup of P such that the discrepancy of the exceptional divisor E is equal to $1/r$. Assume that the divisor $-K_{\tilde{X}}$ is nef, big and the linear system $|-nK_{\tilde{X}}|$ does not contract any divisors. Then the transformation in (3.2.1) is so-called "two rays game". If $-K_{\tilde{X}}$ is ample, then $f \circ \chi$ is a composition of steps of the K-MMP. Otherwise, $f \circ \chi$ is a composition of a single flop followed by steps of the K-MMP. It is easy to see also that $f \circ \chi$ is an $-E$-MMP.

(3.4.2) (see [5]). The pair $(X, \mathcal{H})$ is not canonical. Let c be the canonical threeshold of $(X, \mathcal{H})$. Then $0 < c < 1$. Take g to be an extremal divisorial $K_X + c\mathcal{H}$-crepant blowup. In this situation, $\alpha > 0$ and $f \circ \chi$ is an $K + c\mathcal{H}$-MMP. In particular, f is an extremal $K_X + c\mathcal{H}$-negative contraction. The conditions of (3.2) are satisfied by [5].

(3.5) Properties of construction (3.2).

(3.5.1) Claim. *E_Y is not contracted by f.*

Proof. Assume the converse, i.e., $\dim f(E_Y) < \min(2, \dim Z)$. If f is birational, this implies that the map $f \circ \chi \circ g^{-1} \colon X \dashrightarrow Z$ is an isomorphism in codimension one. Since both X and Z are Fano threefolds, this implies that $f \circ \chi \circ g^{-1}$ is in fact an isomorphism. This contradicts our assumptions. If $\dim Z \le 2$, then E_Y is a pull-back of an ample Weil divisor on Z. But then nE_Y is movable for some $n > 0$. Again we derive a contradiction. □

(3.5.2) Claim. *For some n, $m > 0$ there is a decomposition $-nK_{\tilde{X}} \sim m\tilde{\mathcal{H}} + M$, where $|M|$ is a base point free linear system. In particular, $|-nK_{\tilde{X}}|$ has no fixed components.*

Proof. By (3.2.2), for some $0 < c \le 1$, we have $K_{\tilde{X}} + c\tilde{\mathcal{H}} = g^*(K_X + cH)$. Hence we can take n, $m > 0$ so that $|-nK_{\tilde{X}} - m\tilde{\mathcal{H}}|$ is base point free. □

(3.5.3) Lemma ([13]). *If f is a $\mathbb{Q}$-conic bundle, then Z is a del Pezzo surface with at worst Du Val singularities of type A_n and $\rho(Z) = 1$. Moreover, there is a natural embedding $f^* \colon \operatorname{Cl} Z \to \operatorname{Cl} Y$.*

Proof. The assertion about the base is an immediate consequence of the main result of [13] and the fact that Z is uniruled. The last statement is obvious because both Y and Z have only isolated singularities and $\operatorname{Pic}(Y/Z) \simeq \mathbb{Z}$. □

(3.5.4) Remark. (i) *In the above notation the generic fibre of f is a smooth rational curve. The locus $\Lambda := \{z \in Z \mid f \text{ is smooth over } z\}$ is a closed subset of codimension ≥ 1 in Z. The union of one-dimensional components of Λ is called the* discriminant curve.

(ii) *The classification of del Pezzo surfaces Z with Du Val singularities and $\rho(Z) = 1$ is well-known. In particular, we always have $K_Z^2 \le 9$ and $K_Z^2 \ne 7$. Moreover,*

- (i) *if $K_Z^2 = 9$, then $Z \simeq \mathbb{P}^2$;*
- (ii) *if $K_Z^2 = 8$, then $Z \simeq \mathbb{P}(1, 1, 2)$;*
- (iii) *if $K_Z^2 \le 6$, then on Z there is a rational curve C such that $-K_Z \cdot C = 1$.*

(3.5.5) Lemma. *Notation and assumptions as in* (3.2). *Assume additionally that $q\mathbb{Q}(X) \ge 4$ and f is not birational. Then $L_Y = f^*\Xi$ for some* (*integral*) *Weil divisor on Z. Moreover,* $\dim |\Xi| = \dim |L|$ *and the class of Ξ generates the group* $\operatorname{Cl} Z/\operatorname{Tors}$.

Proof. Since $q\mathbb{Q}(X) \geq 4$, relation (3.3.2) implies $L_Y|_F = 0$. Since f is a Mori contraction and Y is normal, $L_Y = f^*\Xi$, where $\Xi := f(L_Y)$. The rest follows by the fact that the group $\operatorname{Cl} Y/\operatorname{Tors}$ is generated by L_Y and E_Y. □

(3.5.6) Lemma. *Assume that $(X, |-K_X|)$ is not canonical and we are applying construction* (3.2)*. Further, assume that* $\dim Z = 2$ *and* $\alpha > 0$*. Then one of the following holds:*

(i) $\mathcal{H}_Y$ *is f-ample. Then the discriminant curve of f is empty.*
(ii) $\mathcal{H}_Y$ *is not f-ample. Then $q\mathbb{Q}(X) \geq 7$. Moreover, the equality holds only if $Z \simeq \mathbb{P}^2$ and* $\dim|-K_X| = 35$.

Proof. First we assume that $\mathcal{H}_Y$ is f-ample. By (3.2.2) and Claim (3.5.1) E_Y and general elements of $\mathcal{H}_Y$ are sections of f. Hence f is smooth outside of a finite number of degenerate fibres.

Now we assume that $\mathcal{H}_Y$ is not f-ample. Then $\mathcal{H}_Y = f^*\mathcal{M}$, where $\mathcal{M}$ is a linear system without fixed components. Let Ξ be an ample Weil divisor that generates $\operatorname{Cl} Z/\operatorname{Tors}$. We can write $\mathcal{M} \sim_{\mathbb{Q}} a\Xi$ and $-K_Z \sim_{\mathbb{Q}} q'\Xi$, where $q' := q\mathbb{Q}(Z)$, $a \in \mathbb{Z}$. Clearly, $q\mathbb{Q}(X) \geq a$.

By our assumption and by Reid's Riemann-Roch formula [3, (9.1)],

$$30 \leq \dim \mathcal{M} \leq \frac{1}{2}\mathcal{M} \cdot (\mathcal{M} - K_Z) + \sum c_P(\mathcal{M}) \leq \frac{a(a+q')}{2q'^2} K_Z^2.$$

Assume that $a \leq 7$. If $K_Z^2 \leq 6$, then $q' = K_Z^2$ by Remark (3.5.4). So, $60q' \leq a(a+q') \leq 49 + 7q'$, a contradiction. If $K_Z^2 = 8$, then $q' = 4$, so $120 \leq a(a+4) \leq 77$. Again we have a contradiction. Finally, let $K_Z^2 = 9$, i.e., $Z \simeq \mathbb{P}^2$. Then $q' = 3$, so $60 \leq a(a+3) \leq 70$. This inequality has only one solution: $a = 7$. But then $q\mathbb{Q}(X) \leq 7$. If $q\mathbb{Q}(X) = 7$, then $a = 7$, $\mathcal{M} = |\mathcal{O}_{\mathbb{P}^2}(7)|$, and $\dim \mathcal{M} = 35$. □

(3.5.7) Lemma. *Notation and assumptions as in* (3.2)*. Assume additionally that $q\mathbb{Q}(X) = 1$, Z is a surface, and the discriminant curve of f is empty. Then* $\dim|-K_X| < 30$.

Proof. Suppose $\dim|-K_X| \geq 30$. Let $\Gamma \subset Z$ is a smooth curve contained into the smooth locus of Z. Then $G := f^{-1}(\Gamma)$ is a smooth ruled surface over Γ. We claim that $\dim|-K_Y - G| \leq 0$. Indeed, otherwise $-K_Y \sim G + B$, where B is an integral effective divisor, $\dim|B| \geq 1$. Since $q\mathbb{Q}(X) = 1$, this gives a contradiction.

Now from (3.2.3) and from the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-K_Y - G) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \mathcal{O}_G(-K_Y) \longrightarrow 0$$

we get $h^0(\mathcal{O}_G(-K_Y)) \geq h^0(\mathcal{O}_Y(-K_Y)) - 1 \geq 30$. It is easy to see that

$$(-K_Y|_G)^2 = (-K_G + G|_G)^2 = K_G^2 - 2K_G \cdot G|_G = 8 - 8p_a(\Gamma) + 4\Gamma^2.$$

By Claim (3.5.2) the linear system $|-nK_Y|$ has no fixed components. Therefore we can take Γ so that $|-nK_Y|_G|$ has at worst isolated base points (in particular, it is nef). Moreover, $|-nK_Y|_G|$ is base point free for sufficiently large n. If $-K_Y|_G$ is ample, it is well-known that $h^0(\mathcal{O}_G(-K_Y)) \leq (-K_Y|_G)^2 + 2$ (see, e.g., [14]). If $-K_Y|_G$ is not ample, we obtain the above inequality by applying the same arguments to $\bar{G}$, where $\bar{G}$ is the image of G under the birational contraction given by $|-nK_Y|_G|$. In both cases we have

$$8 - 8p_a(\Gamma) + 4\Gamma^2 = (-K_Y|_G)^2 \geq h^0(\mathcal{O}_G(-K_Y)) - 2 \geq 28.$$

This gives us

$$\Gamma^2 \geq 2p_a(\Gamma) + 5 = K_Z \cdot \Gamma + \Gamma^2 + 7, \qquad -K_Z \cdot \Gamma \geq 7.$$

If $K_Z^2 < 8$, then we can take Γ to be a general member of $-K_Z$ and derive a contradiction. If $K_Z^2 = 8$ or 9, then we can take $\Gamma \in |-\frac{1}{2}K_Z|$, or $|-\frac{1}{3}K_Z|$, respectively. □

(3.5.8) Lemma. *If* $\dim Z = 1$ *and* $\dim|-K_X| \geq 30$*, then* $q\mathbb{Q}(X) \geq 3$.

Proof. Let F_1, F_2, F_3 be general fibres. Then from the exact sequence

$$0 \longrightarrow \mathcal{O}_Y\left(-K_Y - \sum F_i\right) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \bigoplus \mathcal{O}_{F_i}(-K_{F_i}) \longrightarrow 0$$

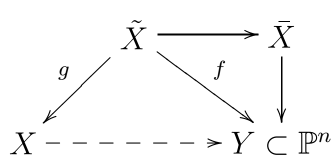
we obtain

$$h^0(-K_Y - \sum F_i) \geq h^0(-K_Y) - \sum h^0(-K_{F_i}).$$

Since F_i are smooth del Pezzo surfaces, $h^0(-K_{F_i}) = K_F^2 + 1 \leq 10$. Hence, $h^0(-K_Y - \sum F_i) > 0$ by (2.5.5) and we have a decomposition $-K_Y \sim \sum F_i + G$, where G is effective. Since F_i is movable, this gives us that $q\mathbb{Q}(X) \geq 3$. □

(3.6) Case: $(X, |-K_X|)$ **is canonical.**

(3.6.1) Consider the case when $(X, |-K_X| = \mathcal{H})$ is canonical. According to [5] there is the following diagram



where $g\colon (\tilde{X}, \tilde{\mathcal{H}}) \to (X, \mathcal{H})$ is a terminal modification of $(X, \mathcal{H})$, $n := \dim|-K_X|$, the morphism f is given by the (base point free) linear system $\tilde{\mathcal{H}}$, $\dim Y = 2$ or 3, and $\tilde{X} \to \bar{X} \to Y$ is the Stein factorization. We have

$$K_{\tilde{X}} + \tilde{\mathcal{H}} = g^*(K + \mathcal{H}) \sim 0.$$

Since $(\tilde{X}, \tilde{\mathcal{H}})$ is terminal, a general member $\tilde{H} \in \tilde{\mathcal{H}}$ is a smooth K3 surface. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}) \longrightarrow \mathcal{O}_{\tilde{H}}(-K_{\tilde{X}}) \longrightarrow 0$$

one can see that the restriction $f|_{\tilde{H}}$ is given by a complete linear system.

(3.6.2) Lemma. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that $(X, |-K_X| = \mathcal{H})$ is canonical and the image of the map given by $|-K_X|$ is a surface. If* $\dim|-K_X| \geq 6$, *then* $2q\mathbb{Q}(X) \geq \dim|-K_X| - 1$.

Proof. We use notation of (3.6.1). By our assumption $f(\tilde{H})$ is a curve. Thus $|-K_{\tilde{X}}|_{\tilde{H}}|$ is a base point free elliptic pencil on $\tilde{H}$ and $f(\tilde{H}) \subset \mathbb{P}^n$ is a rational normal curve of degree $n-1$. Hence $Y \subset \mathbb{P}^n$ is a surface of degree $n-1$. Let M be a hyperplane section of Y. It is well-known that in this situation one of the following halds (recall that $n \geq 6$):

(i) Y is a rational scroll, $Y \simeq \mathbb{F}_e$, $M \sim \Sigma + al$, where Σ and l are the minimal section and a fibre of $\mathbb{F}_e$, respectively, and a is an integer such that $a \geq e+1$, $n-1 = 2a - e$.

(ii) Y is a cone over a rational normal curve of degree $n-1$, $M \sim (n-1)l$, where l is a generator of the cone.

In case (i), $\tilde{\mathcal{H}} \sim f^*\Sigma + af^*l$. Here $|f^*l|$ is a linear system without fixed components and $f^*\Sigma$ is an effective divisor. So, $2q\mathbb{Q}(X) \geq 2a \geq n-1$. In case (ii) we have $\tilde{\mathcal{H}} \sim f^*(n-1)l$. Let $o \in Y$ be the vertex of the cone and let G be the closure of f^*l over $Y \setminus \{o\}$. Then G is an integral Weil divisor and $\tilde{H} \sim_{\mathbb{Q}} (n-1)G + T$, where T is effective. Clearly, g does not contract any component of G. This implies $q\mathbb{Q}(X) \geq n-1$. □

Now assume that $\dim Y = 3$.

(3.6.3) Lemma (cf. [8, Corollary 1.8]**).** *Let X be a $\mathbb{Q}$-Fano threefold. Assume that $(X, |-K_X| = \mathcal{H})$ is canonical and the image of the map given by $|-K_X|$ is three-dimensional. Then* $\dim|-K_X| \leq 37$. *If moreover* $q\mathbb{Q}(X) = 1$, *then* $\dim|-K_X| \leq 13$.

Proof. By the construction, $\bar{Y}$ is a Fano threefold with canonical Gorenstein singularities and $\bar{Y} \to Y \subset \mathbb{P}^N$ is the anticanonical map (see [5]). We have $\dim|-K_X| \leq \dim|-K_{\bar{Y}}| \leq 38$ by the main result of [8]. Moreover, if $\dim|-K_X| = 38$, then $\bar{Y}$ is isomorphic either $\mathbb{P}(3,1,1,1)$ or $\mathbb{P}(6,4,1,1)$. In particular, $\bar{Y}$ is a toric variety. Since $\tilde{X}$ is a terminal modification of $\bar{Y}$, it is also toric and so is X. By Lemma (3.6.4) below $\dim|-K_X| \le \dim|-K_{\bar{Y}}| \le 33$, a contradiction. If $q\mathbb{Q}(X) = 1$, then $-K_{\bar{Y}}$ cannot be decomposed into a sum of two movable divisors. According to [15], $\dim|-K_X| \le \dim|-K_{\bar{Y}}| \le 13$. □

(3.6.4) Lemma. *Let X be a toric $\mathbb{Q}$-Fano threefold. If $X \not\simeq \mathbb{P}^3$, then $-K_X^3 \le 125/2$ and $\dim|-K_X| \le 33$.*

Sketch of the proof. By considering cyclic covering tricks (cf. Proof of Proposition (5.3)) we reduce the question to the case $\operatorname{Cl} X \simeq \mathbb{Z}$. For toric varieties this preserves the property $\rho = 1$. Then X is a weighted projective space. Using the fact that X has only terminal singularities we get the following cases: $\mathbb{P}(1,1,1,2)$, $\mathbb{P}(1,1,2,3)$, $\mathbb{P}(1,2,3,5)$, $\mathbb{P}(1,3,4,5)$, $\mathbb{P}(2,3,5,7)$, $\mathbb{P}(3,4,5,7)$. The lemma follows. □

4. Case $q\mathbb{Q}(X) \le 3$

In this section we consider the case $q := q\mathbb{Q}(X) \le 3$.

(4.1) Proposition. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein, $q := q\mathbb{Q}(X) \le 3$ and $-K_X^3 \ge 125/2$. Then we have one of the following cases:*

(4.1.1) $q = 1$, $\mathbf{B} = (2)$, $-K_X^3 = 2g - 3/2$, $\dim|-K_X| = g + 1$, $32 \le g \le 35$;

(4.1.2) $q = 1$, $\mathbf{B} = (2,2)$, $-K_X^3 = 63$, $\dim|-K_X| = 33$;

(4.1.3) $q = 1$, $\mathbf{B} = (3)$, $-K_X^3 = 188/3$, $\dim|-K_X| = 33$;

(4.1.4) $q = 2$, $\mathbf{B} = (3)$, $L^3 = 25/3$, $\dim|L| = 9$, $\dim|-K_X| = 35$.

(4.2) Lemma. *In notation of Proposition (4.1) we have $-K_X \cdot c_2(X) \ge 125/8$ and $\sum_{P\in\mathbf{B}}(r_P - 1/r_P) \le 67/8$. In particular, $\sum r_P \le 10$.*

Proof. By Corollary (2.7) we have cases (2.6.1) or (2.6.2). Hence,

$$-K_X \cdot c_2(X) \ge \begin{cases} \dfrac{1}{3}(-K_X^3) \ge \dfrac{125}{6}, \\ \dfrac{1}{4}t(4-3t)(-K_X)^3 \ge \dfrac{1}{4q}\left(4 - \dfrac{3}{q}\right)\dfrac{125}{2} \ge \dfrac{125}{8}. \end{cases}$$

(In the second line we used that $t \geq 1/q \geq 1/3$ and the function $t(4-3t)$ is increasing for $t \leq 2/3$). In both cases we have $-K_X \cdot c_2(X) \geq 125/8$. Thus,

$$\sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) \leq 24 - \frac{125}{8} = \frac{67}{8}.$$

Hence $\mathbf{B}$ contains at most 5 points and $\sum r_P \leq \left\lfloor \frac{67}{8} + 5 \cdot \frac{1}{2} \right\rfloor \leq 10$. $\square$

(4.3) Proposition. *In notation of Proposition* (4.1) *we have* $\operatorname{Cl} X \simeq \mathbb{Z}$.

Proof. Let T be an s-torsion element in the Weil divisor class group. By Riemann-Roch (2.3.1), Kawamata-Viehweg vanishing theorem and Serre duality we have

$$\begin{aligned} 0 &= \chi(T) &&= 1 + \textstyle\sum_P c_P(T), \\ 0 &= \chi(K_X + T) &&= 1 + \tfrac{1}{12} K_X \cdot c_2(X) + \textstyle\sum_{P \in \mathbf{B}} c_P(K_X + T). \end{aligned}$$

Subtracting we get

$$0 = -\frac{1}{12} K_X \cdot c_2(X) + \sum_{P \in \mathbf{B}} (c_P(T) - c_P(K_X + T)).$$

Take $i_{T,P}$ so that $T \sim i_{T,P} K_X$ near $P \in \mathbf{B}$. Then $s i_{T,P} \equiv 0 \mod r_P$ and

$$0 = -\frac{1}{12} K_X \cdot c_2(X) + \frac{1}{12} \sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) - \sum_{P \in \mathbf{B}} \frac{\overline{b_P i_{T,P}}\ \left(r_P - \overline{b_P i_{T,P}} \right)}{2 r_P}.$$

Therefore,

$$2 = \sum_{P \in \mathbf{B}} \frac{\overline{b_P i_{T,P}}\ \left(r_P - \overline{b_P i_{T,P}} \right)}{2 r_P}.$$

If $i_{T,P} \not\equiv 0 \mod r_P$, we have

$$\frac{\overline{b_P i_{T,P}}\ \left(r_P - \overline{b_P i_{T,P}} \right)}{2 r_P} \leq \frac{r_P}{8}.$$

Combining the last two relations we get

$$\sum_{P \in \mathbf{B}'} r_P \geq 16,$$

where the sum runs over all $P \in \mathbf{B}$ such that $i_{T,P} \not\equiv 0 \mod r_P$. This contradicts Lemma (4.2). $\square$

Proof of Proposition (4.1). By Proposition (4.3) $q = q\mathbb{Q}(X) = qW(X)$. So, $\gcd(q, r_P) = 1$ for all $P \in \mathbf{B}$.

(4.4) Case $q = 3$. We will show that this case does not occur. By (2.4.2) we have

$$-K_X^3 = q^3L^3 = 162 + \frac{9}{2}K_X \cdot c_2(X) + 162\sum_{P\in\mathbf{B}} c_P(-L). \tag{4.4.1}$$

By Lemma (4.2) $-K_X \cdot c_2(X) \geq 125/8$ and $-K_X^3 \geq 125/2$ by our assumptions. Combining this we obtain $\sum c_P(-L) \geq -467/2592$.

Again by Lemma (4.2) we have $\sum(r_P - 1/r_P) \leq 67/8$. Assume that $r_P = 2$ for all $P \in \mathbf{B}$. Note that $c_P(L) = -1/8$ (because $-K_X \sim L$ near each P). Hence $\mathbf{B} = (2)$. Then $-K_X \cdot c_2(X) = 45/2$. By (4.4.1) we have $-K_X^3 = 81/2 < 125/2$, a contradiction.

Thus we assume that at least one on the r_P's is ≥ 3. Recall that $\sum r_P \leq 10$, $\sum(r_P - 1/r_P) \leq 67/8$ and $3 \nmid r_P$. This gives us the following possibilities for $\mathbf{B}$:

$$(4), (5), (7), (8), (2,4), (2,5), (2,7), (2,2,4), (2,2,5), (4,4), (2,2,2,4).$$

Take $0 \leq i_P < r_P$ so that $3i_P \equiv -1 \mod r_P$. Easy computations give us

r_P	2	4	5	7	8
i_P	1	1	3	2	5
c_P	$-1/8$	$-5/16$	$-1/5$	$-2/7, -3/7, -5/7$	$-5/32$

In all cases except for $\mathbf{B} = (8)$ we get a contradiction with $\sum c_P(-L) \geq -467/2592$. Consider the case $\mathbf{B} = (8)$. Then by (4.4.1) we have

$$-K_X^3 = 162 - \frac{9}{2}\cdot\frac{129}{8} - 162\frac{5}{32} = \frac{513}{8}.$$

Then by (2.5.2)

$$\dim|-K_X| = 2 + \frac{513}{16} - \frac{b_P(8-b_P)}{16} = 34 + \frac{1 - b_P(8-b_P)}{16}.$$

This number cannot be an integer, a contradiction.

(4.5) Case $q = 1$. By (2.6.1) we have

$$\sum_{P\in\mathbf{B}}\left(r_P - \frac{1}{r_P}\right) = 24 + K_X\cdot c_2(X) \leq 24 + \frac{1}{2}K_X^3 \leq 24 - \frac{125}{6} = \frac{19}{6}.$$

This gives the following possibilities: $\mathbf{B} = (2)$, (3), or $(2,2)$.

If $\mathbf{B} = (2,2)$, then $-K_X \cdot c_2(X) = 21$ and $-K_X^3 \le 63$. On the other hand, $-K_X^3 \in \frac{1}{2}\mathbb{Z}$ (see [4, Lemma 1.2]). Hence $-K_X^3 = 63$ or $125/2$. Further, by (2.5.2)

$$\dim |-K_X| = -\frac{1}{2}K_X^3 + \frac{3}{2}.$$

Since this number should be an integer, the only possibility is $-K_X^3 = 63$ and $\dim |-K_X| = 33$.

If $\mathbf{B} = (2)$, then $-K_X \cdot c_2(X) = 45/2$ and by (2.5.2)

$$\dim |-K_X| = -\frac{1}{2}K_X^3 + \frac{7}{4}.$$

Put $g := \dim |-K_X| - 1$. Then $-K_X^3 = 2g - 3/2$. We have

$$125/2 \le -K_X^3 = 2g - 3/2 \le 74 - \frac{9}{2}.$$

Hence $32 \le g \le 35$ and $-K_X^3 \in \{125/2, 129/2, 133/2, 137/2\}$.

Assume that $\mathbf{B} = (3)$. Then $-K_X \cdot c_2(X) = 64/3$ and $-K_X^3 \le 64$. As above,

$$\dim |-K_X| = -\frac{1}{2}K_X^3 + \frac{5}{3}.$$

We get only one possibility: $-K_X^3 = 188/3$ and $\dim |-K_X| = 33$.

(4.6) Case $q = 2$**.** If $\mathcal{E}$ is semistable, then as above by (2.6.1) $\mathbf{B} = (3)$. Otherwise we are in case (2.6.2) and as in the proof of Lemma (4.2) we have

$$\sum_{P \in \mathbf{B}} \left(r_P - \frac{1}{r_P} \right) = 24 + K_X \cdot c_2(X) \le 24 + \frac{5}{16}K_X^3 \le \frac{143}{32}.$$

Since $\gcd(r_P, q) = 1$, again we get the same possibility $\mathbf{B} = (3)$.

Then $-K_X \cdot c_2(X) = 64/3$ and $L \cdot c_2(X) = 32/3$. Hence

$$5/4(-K_X^3) \le t(4-3t)(-K_X^3) \le 4 \cdot 64/3.$$

Thus $125/2 \le -K_X^3 \le 1024/15$ and $125/16 \le L^3 \le 128/15$. Since $3L^3 \in \mathbb{Z}$ (see [4, Lemma 1.2]), we have $L^3 = 8$ or $25/3$. As above the case $L^3 = 8$ is impossible by (2.5.2). Thus $L^3 = 25/3$. Then one can easily compute $h^0(L)$ and $h^0(-K_X)$ by (2.4.1).

□

5. Case $q\mathbb{Q}(X) \geq 4$

(5.1) Proposition *Let X be a $\mathbb{Q}$-Fano threefold. Assume that X is not Gorenstein, $-K_X^3 \geq 125/2$, and $q := qW(X) = q\mathbb{Q}(X) \geq 4$. Then we have one of the following cases:*

(5.1.1) $q = 4$, $\mathbf{B} = (5)$, $-K_X^3 = 384/5$, $\dim |L| = 3$, $\dim |2L| = 10$, $\dim |-K_X| = 40$;

(5.1.2) $q = 4$, $\mathbf{B} = (5, 5)$, $-K_X^3 = 64$, $\dim |L| = 2$, $\dim |2L| = 8$, $\dim |-K_X| = 33$;

(5.1.3) $q = 5$, $\mathbf{B} = (2)$, $-K_X^3 = 125/2$, $\dim |L| = 2$, $\dim |2L| = 6$, $\dim |-K_X| = 33$;

(5.1.4) $q = 5$, $\mathbf{B} = (2, 6)$, $-K_X^3 = 250/3$, $\dim |L| = 2$, $\dim |2L| = 7$, $\dim |-K_X| = 43$;

(5.1.5) $q = 5$, $\mathbf{B} = (7)$, $-K_X^3 = 500/7$, $\dim |L| = 2$, $\dim |2L| = 6$, $\dim |-K_X| = 37$;

(5.1.6) $q = 5$, $\mathbf{B} = (2, 2, 3, 6)$, $-K_X^3 = 125/2$, $\dim |L| = 1$, $\dim |2L| = 5$, $\dim |-K_X| = 32$;

(5.1.7) $q = 6$, $\mathbf{B} = (5, 7)$, $-K_X^3 = 2592/35$, $\dim |L| = 1$, $\dim |2L| = 4$, $\dim |-K_X| = 38$;

(5.1.8) $q = 7$, $\mathbf{B} = (3, 9)$, $-K_X^3 = 686/9$, $\dim |L| = 1$, $\dim |2L| = 3$, $\dim |-K_X| = 39$;

(5.1.9) $q = 7$, $\mathbf{B} = (2, 10)$, $-K_X^3 = 343/5$, $\dim |L| = 1$, $\dim |2L| = 3$, $\dim |3L| = 6$, $\dim |-K_X| = 35$.

Proof. Let L be a Weil divisor such that $-K_X \sim qL$. Since $qW(X) = q\mathbb{Q}(X)$, the group $\operatorname{Cl} X/\operatorname{Tors}$ is generated by L. To get our cases we run a computer program. Below is the description of our algorithm.

1) By (2.5.1) and Theorem (2.5.4) we have $\sum_{P \in \mathbf{B}}(1 - 1/r_P) \leq 24$. Hence there is only a finite (but very huge) number of possibilities for the basket $\mathbf{B}$. In each case we know $-K_X \cdot c_2(X)$ from (2.5.1). Let $r := \operatorname{lcm}(\{r_P\})$ be the Gorenstein index of X.

2) By Corollary (2.2.2) $q \leq 4r$ and $\gcd(q, r) = 1$. Hence we have only a finite number of possibilities for the index q.

3) In each case we compute L^3 and $-K_X^3 = q^3 L^3$ by formula (2.4.2) and check the condition $-K_X^3 \geq 125/2$. Here, for $D = -L$, the number i_P is uniquely determined by conditions $q i_P \equiv 1 \mod r_P$ and $0 \leq i_P < r_P$.

4) Next we check Kawamata's inequalities (2.6), i.e., we check that at least one of inequalities (2.6.1) – (2.6.4) holds. In case (2.6.2) we use the fact that the function $t(4-3t)$ is increasing for $t<2/3$. Since $t \geq 1/q$, we have $\frac{1}{q}(4-\frac{3}{q}) \leq t(4-3t)$ and

$$\frac{1}{q}\left(4-\frac{3}{q}\right)(-K_X^3) \leq -4K_X \cdot c_2(X).$$

Similarly, in cases (2.6.3) and (2.6.4) we have, respectively,

$$\left(1-\frac{1}{q}\right)\left(1+\frac{3}{q}\right)(-K_X^3) \leq -4K_X \cdot c_2(X),$$

$$\frac{1}{q}\left(2-\frac{3}{q}\right)(-K_X^3) \leq -K_X \cdot c_2(X).$$

5) Finally, by the Kawamata-Viehweg vanishing theorem we have $\chi(tL) = h^0(tL) = 0$ for $-q < t < 0$. We check this condition by using (2.4.1).

At the end we get possibilities (5.1.1)–(5.1.9). □

(5.2) Corollary (cf. [4, Remark 2.14]**).** *Let X be a $\mathbb{Q}$-Fano threefold. If $qW(X) = q\mathbb{Q}(X)$, then $-K_X^3 \leq 250/3$.*

Now we show that the condition $qW(X) = q\mathbb{Q}(X)$ in Proposition (5.1) is satisfied automatically.

(5.3) Proposition. *Let X be a $\mathbb{Q}$-Fano threefold. Assume that $q := q\mathbb{Q}(X) > 3$ and $-K_X^3 > 45$. Then $\operatorname{Cl} X \simeq \mathbb{Z}$.*

Proof. Assume that the torsion part of $\operatorname{Cl} X$ is non-trivial for some X satisfying the conditions of Proposition (5.1). Take X so that $q\mathbb{Q}(X)$ is maximal. Write $K_X + qL \sim_{\mathbb{Q}} 0$, where L is an (ample) integral Weil divisor. Since $\operatorname{Cl} X$ is finitely generated and by cyclic covering trick [3, (3.6)], there is a finite étale in codimension one cover $\pi\colon X' \to X$ such that $\operatorname{Cl} X'$ torsion free. Here $K_{X'} + qL' \sim 0$, where $L' := \pi^*L$. Note that X' has only terminal singularities. Hence X' is a Fano threefold with terminal singularities with $qW(X') \geq q$. (It is possible however that X' is not $\mathbb{Q}$-factorial and $\rho(X') > 1$). Denote $n := \deg \pi$. Clearly, $-K_{X'}^3 = -nK_X^3 \geq -2K_X^3$. Hence $\dim|-K_{X'}| \geq -K_X^2 - 2 > 43$. Let $\sigma\colon X'' \to X'$ be a $\mathbb{Q}$-factorialization. (If X' is $\mathbb{Q}$-factorial, we take $X'' = X'$). Run K-MMP on X'': $\upsilon\colon X'' \dashrightarrow Y$. At the end we get a Mori-Fano fibre space $f\colon Y \to Z$. Let $L'' := \sigma^{-1}(L')$ and $L_Y := \upsilon_* L''$. Then $-K_Y \sim qL_Y$. If $\dim Z > 0$, then for a general fibre $F := f^{-1}(o)$, $o \in Z$ we have $-K_F \sim qL_Y|_F$. This is impossible if $q > 3$.

In the case $\dim Z = 0$, Y is a Fano with $\rho(Y) = 1$ and $qW(Y) \geq q$. By our assumption of maximality of $q = q\mathbb{Q}(X)$ we have $q\mathbb{Q}(Y) =$

$qW(Y) = q$. Hence, $-K_Y^3 \le 250/3$ by Corollary (5.2). By (2.5.3) we have $\dim|-K_Y| \le 43$. Using (2.5.5) we obtain

$$43 \ge \dim|-K_Y| \ge \dim|-K_{X''}| \ge -\frac{1}{2}K_{X''}^3 - 2 \ge -K_X^3 - 2.$$

Thus $-K_X^3 \le 45$, a contradiction. □

6. Proof of the main theorem

(6.1) To construct a Sarkisov link such as in (3.2.1), we need the following result basically due to Ambro and Kawachi.

(6.1.1) Proposition (cf. [6, Th. 4.1]**).** *Let X be a Fano threefold with terminal singularities, and let S be an ample Cartier divisor proportional to $-K_X$. Then the linear system $|S|$ is non-empty and a general member of $|S|$ is a reduced irreducible normal surface whose singularities are at worst log terminal of type T. Moreover, assume that $K_X^2 \cdot S > 1$ and $qF(X) \ge 1/2$. Then a general $S \in |S|$ does not pass through non-Gorenstein points (and has at worst Du Val singularities).*

Proof. According to [16] the pair (X, S) is plt for a general $S \in |S|$. Then singularities of S are of type T by [17]. Note that the restriction map $H^0(\mathcal{O}_X(S)) \to H^0(\mathcal{O}_S(S))$ is surjective. Let $P \in \operatorname{Bs}|S|$ be a non-Gorenstein point of X. Then $P \in S$ is a log terminal non-Du Val singularity of type T.

Recall that Kawachi's invariant of a normal surface singularity (S, P) is defined as $\delta_P := -(\Gamma - \Delta)^2$, where Δ is the codiscrepancy divisor of (S, P) on the minimal resolution $\hat{S} \to S$ and Γ is the fundamental cycle on $\hat{S}$ (see [18]). If (S, P) is a rational singularity, then $\delta_P = \Gamma^2 - \Delta^2 + 4$. Hence in our case Kawachi's invariant δ_P is integral (because $\Delta^2 \in \mathbb{Z}$, see [17]). On the other hand, $0 < \delta_P < 2$. Thus $\delta_P = 1$. Now we apply the main result of [18] to the linear system $|S|_S| = |K_S - K_X|_S|$. It follows that there is a curve C on S passing through P and such that $-K_X \cdot C < 1/2$. Since $qF(X) \ge 1/2$, this is impossible. □

(6.1.2) Proposition. *In notation of Proposition* (6.1.1) *assume additionally that $(2K_X + S)^2 \cdot S \ge 5$ and $-(2K_X + S)$ is an ample divisor which is divisible in* $\operatorname{Cl} X/\operatorname{Tors}$. *Then the linear system $|-K_X|$ has only isolated base points.*

Proof. Denote the restriction $-K_X|_S$ by D. Since S does not pass through non-Gorenstein points, D is Cartier. By the Kawamata-Viehweg vanishing the map

$$H^0(\mathcal{O}_X(-K_X)) \longrightarrow H^0(\mathcal{O}_S(D))$$

is surjective. Thus it is sufficient to show that the linear system $|D|$ is base point free. By the adjunction formula $D = K_S - (2K_X + S)|_S$. Let $\mu\colon \hat{S} \to S$ be the minimal resolution. Since S has at worst Du Val singularities, $K_{\hat{S}} = \mu^* K_S$. Thus we can write $\mu^* D = K_{\hat{S}} + M$, where $M = \mu^*(-(2K_X + S)|_S)$ is nef. It is easy to see that $M^2 = (2K_X + S)^2 \cdot S \geq 5$ by our assumption. Suppose that the linear system $|\mu^* D| = |K_{\hat{S}} + M|$ has a base point P. By the main theorem of [19] there is an effective divisor E on $\hat{S}$ passing through P such that either $M \cdot E = 0$, $E^2 = -1$ or $M \cdot E = 1$, $E^2 = 0$. In the former case E is contracted my μ and we get a contradiction by the genus formula. In the latter case we have $-(2K_X + S) \cdot \mu(E) = 1$. This is impossible because $-(2K_X + S)$ is divisible in $\operatorname{Cl} X/\mathrm{Tors}$ and $\mu(E)$ is contained in the Gorenstein locus of X. □

Since $qF(X) = q/r$, we have the following

(6.1.3) Corollary. *Let X be a $\mathbb{Q}$-Fano threefold, let $q := q\mathbb{Q}(X)$, and let r be the Gorenstein index of X. Assume that $-K_X^3 > q/r = qF(X)$, $2q - r \geq 2$, and $(-K_X^3)(2q-r)^2 r \geq 5q^3$. Then the linear system $|-K_X|$ has only isolated base points.*

Proof. Let L be the Weil divisor such that $-K_X \sim_{\mathbb{Q}} qL$. Take $S = rL$ and apply Proposition (6.1.2). □

Now we are in position to prove Theorem (1.2).

(6.2) Main assumption. Let X be a $\mathbb{Q}$-Fano threefold. We assume that $-K_X^3 \geq 125/2$. Then X is such as in Propositions (4.1) or (5.1). In particular, $\dim|-K_X| \geq 32$. By Propositions (4.3) and (5.3) we also have $\operatorname{Cl} X \simeq \mathbb{Z}$. We divide cases of (4.1) or (5.1) in four groups and treat these groups separately (see (6.3), (6.4) (6.5), (6.6)).

(6.2.1) Proposition. *Notation and assumptions as in* (6.2). *If there exists a Sarkisov link* (3.2.1) *with birational f, then $-K_Z^3 \geq 125/2$ except possibly for the following case*

- $\dim|-K_Z| = \dim|-K_X| = 32$.

Proof. Assume the converse. Then Z is a $\mathbb{Q}$-Fano with $\dim|-K_Z| \geq \dim|-K_X| \geq 32$ and $-K_Z^3 < 125/2$. By (2.5.3)

$$\dim|-K_Z| + \frac{1}{2}\sum_{P\in \mathbf{B}_Z}\left(1 - \frac{1}{r_P}\right) \leq -\frac{1}{2}K_Z^3 + 2 < \frac{133}{4}. \tag{6.2.2}$$

Therefore, $\dim|-K_Z| = 32$ or 33. Moreover, if $\dim|-K_Z| = 33$, then we have $r_P = 1$ for all $P \in \mathbf{B}_Z$, i.e., Z is Gorenstein (and factorial). In particular, $q\mathbb{Q}(Z) = qF(Z) = qW(Z)$ and $q\mathbb{Q}(Z)^3$ divides $-K_Z^3$. By Riemann-Roch, $-K_Z^3 = 62$. Therefore, $q\mathbb{Q}(Z) = 1$. But then $-K_Z$ cannot be decomposed into a sum of movable divisors. We derive a contradiction by [15]. $\square$

(6.3) Case (5.1.3)

(6.3.1) Proposition (see [20]**).** *In case* (5.1.3)*,* $X \simeq \mathbb{P}(1,1,1,2)$.

Proof. Let $S \in |2L|$ be a general member. Then S is Cartier and by Proposition (6.1.1) X is has at worst Du Val singularities. By the adjunction formula S is a del Pezzo surface of degree 9. It follows that S is smooth and $S \simeq \mathbb{P}^2$ (see Remark (3.5.4)). The restriction map $H^0(X, \mathcal{O}_X(S)) \to H^0(S, \mathcal{O}_S(S))$ is surjective. Hence the linear system $|S|$ is base point free and determines a morphism $\varphi\colon X \to \mathbb{P}^6$. We have $(\deg\varphi)(\deg\varphi(X)) = S^3 = 4$. So φ is birational and $\varphi(X) \subset \mathbb{P}^6$ is a variety of degree 4. A general hyperplane section $\varphi(S) \subset \varphi(X)$ is a Veronese surface. It is well-known that in this situation $\varphi(X)$ is a cone over $\varphi(S)$, i.e., $X \simeq \varphi(X) \simeq \mathbb{P}(1,1,1,2)$. $\square$

(6.4) Cases (4.1.4), (5.1.1), (5.1.2), (5.1.4), (5.1.5), (5.1.6), (5.1.8), (5.1.9). We apply construction (3.4.1). Let r be the Gorenstein index of X. First we construct a birational extremal extraction $g\colon \tilde{X} \to X$ such that $\tilde{X}$ has only terminal singularities and the exceptional divisor E of g has discrepancy $1/r$.

(6.4.1) Claim. *Either*

(i) *There is a cyclic quotient singularity* $P \in X$ *of type* $\frac{1}{r}(b, -b, 1)$*, where* $\gcd(r, b) = 1$*, or*
(ii) *we are in case* (5.1.2) *and there is a point* $P \in X$ *of type* $cA/5$ *of axial weight* 2.

Proof. Note that in all cases there is a basket point $P \in \mathbf{B}$ of index r. If this point is unique, it corresponds to a cyclic quotient singularity of X. The point $P \in \mathbf{B}$ of index r is not unique only in case (5.1.2). Then $r = 5$ and there are two points P_1, $P_2 \in \mathbf{B}$ of index 5. They correspond either two cyclic quotient singularities of X or a point $P \in X$ of type $cA/5$. $\square$

In case (i) the weighted blowup of $P \in X$ with weights $\frac{1}{r}(b, r-b, 1)$ gives us a desired contraction g. Similarly, in case (ii) a suitable weighted blowup gives us a desired contraction g (see [21]).

Further, $r\mathcal{H}$ is the linear system of Cartier divisors. Hence we can write $g^*\mathcal{H} = \tilde{\mathcal{H}} + \delta E$, where $\delta \geq 1/r$. Thus,

$$(6.4.2) \qquad -K_{\tilde{X}} \sim_{\mathbb{Q}} g^*(-K_X) - \tfrac{1}{r}E \sim_{\mathbb{Q}} \tilde{\mathcal{H}} + (\delta - \tfrac{1}{r})E.$$

By Corollary (6.1.3) the linear system $\tilde{\mathcal{H}}$ has only isolated base points outside of E. Therefore, $-K_{\tilde{X}}$ is nef.

If $g(E)$ is a cyclic quotient singularity, then $E \simeq \mathbb{P}(b, r-b, 1)$, $E|_E \sim \mathcal{O}_{\mathbb{P}(b,r-b,1)}(-r)$, and $E^3 = r^2/b(r-b)$. Therefore,

$$-K_{\tilde{X}}^3 = -K_X^3 - \tfrac{1}{r^3}E^3 \geq \frac{125}{2} - \frac{r^2}{b(r-b)} > 0.$$

This shows that $-K_{\tilde{X}}$ is big. Similar computations shows that this fact also holds in case (6.4.1), (ii).

Let C be a curve such that $-K_{\tilde{X}} \cdot C = 0$. By (3.3.1) we have $q\tilde{L} \cdot C + \beta E \cdot C = 0$. By (6.4.2) $E \cdot C > 0$. Hence $\tilde{L} \cdot C < 0$. Since $\dim |L| > 0$, there is at most a finite number of such curves. Thus the linear system $|-nK_{\tilde{X}}|$ does not contract any divisors.

(6.4.3) Consider diagram (3.2.1). Since $K_X + qL \sim 0$, the constant β in (3.3) is a non-negative integer. We can write

$$K_{\tilde{X}} = g^*K_X + \frac{1}{r}E, \qquad \tilde{L} = g^*L - \delta E,$$

where $\delta \in \mathbb{Q}$, $\delta > 0$. Since rL is Cartier (see Lemma (2.2.1)), $\delta = k/r$ for some $k \in \mathbb{Z}$, $k > 0$. Therefore,

$$\beta = -\frac{1}{r} + q\delta = \frac{qk-1}{r}$$

and the value of β is bounded from below as follows:

case	(4.1.4)	(5.1.1)(5.1.2)(5.1.8)	(5.1.4)(5.1.6)	(5.1.5)(5.1.9)
β	≥ 1	≥ 3	≥ 4	≥ 2

(6.4.4) First we assume that $\dim Z = \dim X$. Then f is a divisorial contraction and Z is a $\mathbb{Q}$-Fano threefold. By (3.3.1) we have $K_Z + qL_Z + \beta E_Z \sim_{\mathbb{Q}} 0$, where E_Z and L_Z are effective non-zero divisors. Hence, $q\mathbb{Q}(Z) \geq q + \beta > 4$. In particular, Z is not Gorenstein (see Corollary (2.2.2)).

Assume that $-K_Z^3 < 125/2$. By Proposition (6.2.1) $\dim |-K_X| = \dim |-K_Z| = 32$. Hence X is of type (5.1.6). By (6.2.2) $\dim |-K_Z| \geq 60$ and by (3.3.1) $q\mathbb{Q}(Z) \geq 9$. On the other hand, $\operatorname{discrep}(Z) \geq$

discrep($\tilde{X}$) $\geq 1/5$. Therefore the Gorenstein index of Z is at most 5 (see [21]). By Proposition (5.3) $\operatorname{Cl} Z \simeq \mathbb{Z}$. Let L' be the ample generator of $\operatorname{Cl} Z \simeq \mathbb{Z}$, let $r' \leq 5$ be the Gorenstein index of Z, and let $S \in |r'L'|$ a general member. Then S be the ample generator of $\operatorname{Pic} Z$. By Proposition (6.1.1) S has at worst Du Val singularities. By the adjunction formula $K_S = (r' - q\mathbb{Q}(Z))L'|_S$. Since $L'|_S$ is a Cartier divisor, S is a del Pezzo surface with $qF(S) \geq q\mathbb{Q}(Z) - r' \geq 4$. This is impossible (see (3.5.4)). Thus $-K_Z^3 \geq 125/2$ and Z is such as in (5.1).

Now we consider possibilities for X case by case. In cases (5.1.4), (5.1.6), (5.1.8), and (5.1.9) we have $q\mathbb{Q}(Z) \geq 9$, a contradiction. In cases (5.1.1), (5.1.2), and (5.1.5) we have $q\mathbb{Q}(Z) = 7$. Hence Z is such as in (5.1.8) or (5.1.9). Then $q + \beta = 7$. By (3.3.1) L_Z and E_Z are linear equivalent and they are generators of $\operatorname{Cl} Z$. On the other hand, $\dim |L| \geq 2 > \dim |L_Z| = 1$, a contradiction.

In case (4.1.4) $\tilde{X}$ is of Gorenstein index 2. Hence, discrep($\tilde{X}$) $= 1/2$. On the other hand, $f \circ \chi$ is a composition of a flop and steps of the K-MMP. Therefore, discrep(Z) $\geq 1/2$. This is possible only if Z of type (5.1.3). But then $35 = \dim |-K_X| > \dim |-K_Z| = 33$, a contradiction.

(6.4.5) Thus we may assume that $\dim Z < \dim X$. Let $M \in |2L|$ be a general member. Note that by (6.4.3) $q+\beta \geq 3$ and $q+\beta = 3$ only in case (4.1.4). By (3.3.2) L_Y can be f-horizontal only in case (4.1.4) and if Z is a curve. By Lemma (3.5.8) we have a contradiction. Hence L_Y is f-vertical. As in Lemma (3.5.5) we have $L_Y = f^*\Xi$ for some integral Weil divisor Ξ on Z, $\dim |\Xi| = \dim |L| \geq 1$, and Ξ is a generator of $\operatorname{Cl} Z/\mathrm{Tors}$.

(6.4.6) Assume that Z is a surface. From (3.3.2) we get $\beta \leq 2$. By (6.4.3) this is possible only in cases (4.1.4), (5.1.5) or (5.1.9). If $K_Z^2 < 8$, we have $\dim |\Xi| = 0$, a contradiction. Hence Z is either $\mathbb{P}^2$ or $\mathbb{P}(1,1,2)$. Consider the case $Z \simeq \mathbb{P}(1,1,2)$. Then $\dim |\Xi| = 1$ and we are in case (5.1.9). Let $M \in |3L|$ be a general member. We can write $K_Y + 2M_Y + L_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. This shows that M_Y is f-vertical. Thus $M_Y \sim 3L_Y = 3f^*\Xi$ and $\dim |M_Y| = \dim |3\Xi| = 4$, a contradiction.

Consider the case $Z \simeq \mathbb{P}^2$. Then $\dim |\Xi| = 2$ and we are in case (5.1.5). Let $M \in |2L|$ be a general member. We can write $K_Y + 2M_Y + L_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. This shows that $\gamma = \beta = 2$ and M_Y is f-vertical. Thus $M_Y \sim 2L_Y = f^*\Xi$ and $\dim |M_Y| = \dim |2\Xi| = 5$, a contradiction.

(6.4.7) Assume that Z is a curve. Then $Z \simeq \mathbb{P}^1$. Since $L_Y = f^*\Xi$ is not divisible in $\operatorname{Cl} Y$, $\dim |\Xi| \leq 1$. So we are in cases (5.1.6), (5.1.8), or (5.1.9). Moreover, since $\dim|L| > 0$, $\dim|\Xi| = 1$. Case (5.1.6) is impossible because then $\beta \geq 4$. Let $M \in |2L|$ be a general member. We can write $K_Y + 3M_Y + L_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. This shows that M_Y is f-vertical. Thus $M_Y \sim 2L_Y = 2f^*\Xi$ and $\dim|M_Y| = \dim|2\Xi| = 2$, a contradiction.

Now we consider case (5.1.7).

(6.5) Case (5.1.7). By Lemmas (3.6.2) and (3.6.3) the pair $(X, |-K_X|)$ is not canonical. Thus we apply the construction (3.2.1) in case (3.4.2). Then in (3.2.2) we have $\alpha > 0$. Assume that $\dim Z = 3$. Since $\alpha > 0$, and by Proposition (2.5.6) we have $\dim|-K_Z| > \dim|-K_X| = 38$. Then by Proposition (6.2.1) $-K_Z^3 \geq 125/2$. Hence Z is $\mathbb{Q}$-Fano such as in Proposition (5.1). Moreover, by (3.3.1) we have $q\mathbb{Q}(Z) \geq q\mathbb{Q}(X) + \beta = 6 + \beta$. This implies that $E_Z \sim L_Z$ is a generator of $\operatorname{Cl} Z$, $q\mathbb{Q}(Z) = 7$, and $\beta = 1$. So, the variety Z is of type (5.1.8). Obviously, $\dim|2L_Z| \geq \dim|2L|$. This contradicts Proposition (5.1).

Thus $\dim Z = 1$ or 2. If Z is a surface, then by Lemma (3.5.5) $Z \simeq \mathbb{P}(1,1,2)$. Let $M \in |2L|$ be a general member. We can write $K_Y + 3M_Y + \gamma E_Y \sim 0$, where $\gamma > 0$. Restricting to a general fibre we obtain that M_Y is f-vertical. Thus, $M_Y \sim 2L_Y = 2f^*\Xi$ and $\dim|M_Y| = \dim|2\Xi| \leq 3$, a contradiction.

Finally we consider cases when $q\mathbb{Q}(X) = 1$.

(6.6) Cases (4.1.1), (4.1.2), (4.1.3). By Lemmas (3.6.2) and (3.6.3) the pair $(X, |-K_X|)$ is not canonical. Thus we may apply construction (3.2) under assumptions (3.4.2).

Then in (3.2.2) we have $\alpha > 0$. Assume that $\dim Z = 3$. Similar to (6.5) $\dim|-K_Z| > \dim|-K_X|$ and $-K_Z^3 \geq 125/2$. Hence Z is $\mathbb{Q}$-Fano such as in Proposition (5.1) or (4.1) with $q\mathbb{Q}(Z) > 1$. By (6.3), (6.4), and (6.5) Z is of type (5.1.3) and $Z \simeq \mathbb{P}(1,1,1,2)$. Then $\dim|-K_X| < \dim|-K_Z| = 33$, so X is of type (4.1.1) and $\dim \mathcal{H}_Z \geq 32$. Easy computations show that $\mathcal{H}_Z \sim \mathcal{O}_{\mathbb{P}(1,1,1,2)}(n)$, with $n \geq 5$. On the other hand, $-K_Z \sim \mathcal{H}_Z + \alpha E_Z$, where $\alpha > 0$, a contradiction.

Therefore, $1 \leq \dim Z \leq 2$. If Z is a curve, we have a contradiction by Lemma (3.5.8). Thus Z is a surface. Then by Lemma (3.5.6) the fibration f has no discriminant curve. Hence by Lemma (3.5.7) we have $\dim|-K_X| < 30$, a contradiction.

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