On convergence rates of harmonic maps near points of discontinuity

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Let M^m and N^n be Riemannian manifolds, with metrics given by $ds_M^2 = \gamma_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta$ and $ds_N^2 = g_{ij}(u)\, du^i\, du^j$. A map $f : M \longrightarrow N$ is said to be <u>harmonic</u> if it is stationary for Dirichlet's integral

$$(1) \qquad E(f) = \int_M |Df|^2 d\, vol_M \ ,$$

where $|Df|^2 = \gamma^{\alpha\beta}(x) g_{ij}(f(x)) D_\alpha f^i D_\beta f^j$, $D_\alpha := \partial/\partial x_\alpha$, and $d\, vol_M$ is the natural volume form $\sqrt{\det(\gamma_{\alpha\beta})}\, dx^1 \ldots dx^m$. For certain purposes, it is necessary to choose an isometric embedding of N^n into $\mathbb{R}^d$ and define the admissible class of functions $H^1(M,N)$ to be the subset of the Sobolev space $H^1(M,\mathbb{R}^d)$ (mappings whose first distributional derivatives are square-integrable) having values in N almost everywhere. Then the space $H^1(M,N)$ is independent of the choice of isometric embedding of N into $\mathbb{R}^d$. It should be noted that for $m \geq 3$, a mapping $f \in H^1(M,N)$ need not be (equal almost everywhere to) a continuous function, so that definitions given originally in terms of local coordinates on N need to be rewritten. For example, Dirichlet's integral (1) should be defined with the integrand $|Df|^2 = \gamma^{\alpha\beta}(x)\langle D_\alpha f, D_\beta f\rangle$, where $\langle\ ,\ \rangle$ is the inner product of $\mathbb{R}^d$. Similarly, one should define the Euler-Lagrange equations for the functional (1) by treating the condition $f(x) \in N$ as a constraint for mappings $f : M \longrightarrow \mathbb{R}^d$. One finds that $f : M \longrightarrow N$ is harmonic if and only if it is a weak solution of the elliptic system of equations

$$(2) \qquad \Delta_M f + \gamma^{\alpha\beta}(x) B(D_\alpha f, D_\beta f) = 0 .$$

Here, B is the second fundamental form of N in $\mathbb{R}^d$, and Δ_M is the geometric Laplace operator of M :

$$\Delta_M f := -D_\alpha(\sqrt{\gamma}\ \gamma^{\alpha\beta} D_\beta f)/\sqrt{\gamma} ,$$

where we have written $\gamma = \det(\gamma_{\alpha\beta})$.

Not only the admissible mappings, but even the solutions of equation (2) may fail to be continous. This was shown by Hildebrandt and Widman in [HW] with the example $f_o(x) = x/|x|$ as a mapping from the euclidean ball B^m in $\mathbb{R}^m$ to its boundary S^{m-1} , for $m \geq 3$. It was recently shown, moreover, that f_o minimizes E among maps having the same Dirichlet boundary values by Coron and Gulliver in [CG], following earlier results of Jäger-Kaul $(m \geq 7)$ and Brézis-Coron-Lieb $(m = 3)$.

It is no accident that the examples of discontinuous solutions have domains of dimension $m \geq 3$ and targets of dimension $n \geq 2$. In fact, if $n = 1$, then f is the solution of a single uniformly elliptic equation with a mild nonlinearity;

f must be as smooth as suggested by the equation itself ([G], including references on pp. 51-54). On the other hand, if $m = 2$, then a minimizing harmonic map is as regular as the target N by Morrey's theorem ([M], Theorem 1.10.4 (iii) and pp. 34-37). With $m = 1$, we have the solution of an ordinary differential equation, whose smoothness is well understood.

A harmonic mapping f_o from $\mathbb{R}^m$ into a manifold N^n is called a <u>homogeneous tangent map</u> if $f_o(\lambda x) = f_o(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^m$. It may be shown by elementary means that the restriction of a homogeneous tangent map to S^{m-1} is a harmonic mapping: $S^{m-1} \longrightarrow N^n$; since this restriction represents f_o faithfully, it is sometimes referred to as the homogeneous tangent map. If $0 \in M$ is an interior singularity of a minimizing harmonic map $f : M \longrightarrow N$, then we may consider the "blow-up limit" at 0 by defining $f_\lambda(x) = f(\lambda x)$ and letting $\lambda \longrightarrow 0^+$. Schoen and Uhlenbeck show that, modulo a small correction factor, $E(f_\lambda)$ is a monotone increasing function of λ ([SU], p. 313). It follows that $f_{\lambda(i)}$ converges weakly in $H^1(B^m,N)$ to some mapping f_o, for a sequence $\lambda(i) \longrightarrow 0$. They proceed to show much more:

<u>Theorem 1</u> ([SU]). <u>Every sequence tending to zero has a subsequence</u> $\lambda(i)$ <u>such that</u> $f_{\lambda(i)} \longrightarrow f_o$ <u>in the</u> H^1-<u>norm on some neighborhood</u> $B_r(0)$, <u>and uniformly on the annulus</u> $B_{2r}(0)\setminus B_r(0)$. <u>Moreover</u>, f_o <u>is a homogeneous tangent map and minimizes</u> E <u>for its boundary values</u>.

With Theorem 1, it becomes clear that the study of points of discontinuity of harmonic maps is conveniently divided into the study of homogeneous tangent maps (which are essentially harmonic maps from S^{m-1} to N), and the degree to which f is approximated by its homogeneous tangent maps. In the present report, we shall concern ourselves with the second of these questions.

Theorem 1 does not yet settle the important problem of the uniqueness of the homogeneous tangent map f_o . The first result in this direction was a theorem of Allard and Almgren concerning the analogous question for area-minimizing integral currents ([AA]); the corresponding analysis for harmonic maps was carried out by Simon ([S2], pp. 270-276).

Given a harmonic map $g : S^{m-1} \longrightarrow N$ (for example, the restriction of f_o), a vector field $\varphi : S^{m-1} \longrightarrow TN$ along g is called a <u>harmonic-Jacobi field</u> if φ is a weak solution of the linearized equation

$$(3) \qquad \Delta\varphi + 2B((D_\alpha\varphi)^T, D_\alpha g) + (D_\varphi B)(D_\alpha g, D_\alpha g) = 0 ,$$

where the euclidean Laplace operator is $\Delta = -D_\alpha D_\alpha$; a vector V is written in terms of its components V^T tangent to N and $V^\perp$ normal to N ; and $D_\varphi B$ is the covariant derivative of the second fundamental form B .

<u>Theorem 2</u> ([AA], cf. [S2]). <u>Let</u> $f \in H^1(M,N)$ <u>be a harmonic mapping which minimizes</u> E <u>on some neighborhood of</u> $0 \in M$. <u>Let</u> $f_o : \mathbb{R}^m \longrightarrow N$ <u>be the weak limit of some blowup sequence</u> $f_{\lambda(i)}$

with $\lambda(i) \longrightarrow 0^+$. Assume that f_o is smooth on S^{m-1} and satisfies the following integrability hypothesis: for some integer $k \geq 0$, there is a k-parameter family $F : \mathbb{R}^k \times S^{m-1} \longrightarrow N$ of harmonic maps such that $F(0,\cdot) = f_o$ and every harmonic-Jacobi field φ along f_o equals $\frac{d}{dt} F(tv,\cdot)(t=0)$ for some $v \in \mathbb{R}^k$. Then f_o is the unique homogeneous tangent map to f at 0, and for some $\alpha > 0$,

$$(4) \qquad \| f-f_o \|_{C^2(|x|=\rho)} + \rho \| D_\rho f \|_{C^1(|x|=\rho)} \leq \text{const.}\ \rho^\alpha .$$

The integrability hypothesis is used in an essential way in the proof of Theorem 2: roughly speaking, it allows one to replace f_o iteratively by another homogeneous tangent map which gives a better approximation to f at smaller radii (*cf*. Lemma II.6.4 of [S2]). It seems reasonable to conjecture that the integrability hypothesis always holds for generic metrics on N; but this remains unproven, and the only broad context in which it is known to hold is given in Theorem 4 below.

Simon later succeeded in proving the uniqueness of the homogeneous tangent map at a singularity without requiring the troublesome integrability hypothesis, a result we state as Theorem 3. Nonetheless, his conclusion was weaker than that of Theorem 2 as regards the rate of convergence to f_o.

Theorem 3 ([S1]; see also [S2], pp. 215-6, 240-1). *Let $f \in H^1(M,N)$ be a harmonic map which minimizes E on some neighborhood of $0 \in M$. Let $f_o : \mathbb{R}^m \longrightarrow N$ be the weak limit of some blowup sequence $f_{\lambda(i)}$ with $\lambda(i) \longrightarrow 0^+$. Assume that N is real-*

analytic, and that f_o *is smooth on* S^{m-1} *. Then* f_o *is the unique homogeneous tangent map to* f *at* 0 *, and as* $\rho \longrightarrow 0^+$,

$$(5) \qquad \| f-f_o \|_{C^2(|x|=\rho)} + \rho \| D_\rho f \|_{C^1(|x|=\rho)} \longrightarrow 0 \ .$$

Simon's proof of Theorem 3 involves a remarkable analysis of growth rates as $\rho = |x| \longrightarrow 0$, or as $t := -\log \rho \longrightarrow +\infty$. Roughly speaking, he shows that after f has become sufficiently C^1-close to f_o , any later sufficiently long t-interval consists of a possible initial interval of exponential decay, a possible middle period in which f is nearly constant, and a possible final interval of exponential growth (see [S2], p. 252). This holds with estimates independent of t , and the same estimates are valid for $D_t f = -\rho D_\rho f$. The final growth interval is readily arranged to be empty for f itself, but for $D_t f$, much more work is needed. In particular, one requires an estimate of the time in which a path of steepest descent in a Banach manifold will reach a critical point; such an estimate can hold only in the real-analytic context. The nature of these indirect arguments rules out any explicit estimate of the rate of convergence to f_o .

We prove that the stronger conclusion (4) of Theorem 2 always holds in the lowest dimensions for which singularities may occur:

Theorem 4 ([GW]). *Let* M *and* N *be manifolds of respective dimensions* $m = 3$ *and* $n = 2$ *. Let* $f : M \longrightarrow N$ *be a locally minimizing harmonic map near a singularity* $0 \in M$ *. Then* N *has the topological type of* S^2 *or* $\mathbb{RP}^2$ *; there is a unique homogeneous tangent mapping* $f_o : \mathbb{R}^3 \longrightarrow N$ *; and* f *converges*

<u>to</u> f_o <u>at a rate controlled by a positive power of</u> $|x|$, <u>as in inequality</u> (4).

The proof of Theorem 4 proceeds by showing that the integrability hypothesis of Theorem 2 is satisfied. The special character of the dimensions $m = 3$ and $n = 2$ is related to the special properties of conformal mappings. In general, the restriction of f_o to the sphere defines a <u>harmonic</u> mapping $g : S^{m-1} \longrightarrow N^n$; but if $m-1 = n = 2$, then g is automatically <u>conformal</u>, since the quadratic form $\langle g_z, g_z \rangle dz^2$ is holomorphic (and must therefore vanish identically). Since $0 \in M$ is a singularity, f_o and g must be nonconstant ([SU]). It follows that N and its universal covering space must be compact, so $N \cong S^2$ or $\mathbb{R}P^2$; we may assume $N \cong S^2$ with no loss of generality.

If $\varphi : S^2 \longrightarrow TN$ is a harmonic-Jacobi field, then we may rewrite equation (3) with respect to a conformal coordinate z on S^2 :

$$\varphi_{z\bar{z}} = B(\varphi_z^T, g_{\bar{z}}) + B(g_z, \varphi_{\bar{z}}^T) + (D_\varphi B)(g_z, g_{\bar{z}}) \ .$$

A straightforward computation reveals that the quadratic form $\langle \varphi_z, g_z \rangle dz^2$ is holomorphic, and therefore zero, which is to say that φ is a <u>conformal-Jacobi field</u> along g . We now introduce a conformal coordinate on $N^2 \cong S^2$, so that g is represented by a rational function $P(z)/Q(z)$. We may assume that $\deg P \leq \deg Q = \deg g =: d$. Since $\langle \varphi_z, g_z \rangle \equiv 0$, φ is represented by a meromorphic function; an analysis of the poles of g shows that this meromorphic function has the form $R(z)/Q(z)^2$ for some polynomial R. But P and Q are relatively prime; using the euclidean algorithm, we may find polynomials A and B with $\deg B < d$ and $\deg A \leq d$, so that $R = AQ - BP$. Therefore, by the quotient rule,

$$(6) \qquad \frac{R}{Q^2} = \frac{d}{dt}\frac{P+tA}{Q+tB}\ (t = 0)\ .$$

We may now show that g satisfies the integrability hypothesis of Theorem 2. In fact, we may define $g_{A,B}$ to be the conformal (and therefore harmonic) mapping represented by the rational function $(P+A)/(Q+B)$. As B ranges over the complex polynomials of degree at most $d-1$ and A over those of degree at most d, this forms a real $(4d+2)$-parameter family with $g_{0,0} = g$. Equation (6) shows that every harmonic-Jacobi field arises from this family.

The conclusion of Theorem 4 may fail in higher dimensions:

<u>Theorem 5</u> ([GW]). <u>There is a real-analytic manifold</u> N^3 <u>and a harmonic map</u> $f : B^3 \longrightarrow N^3$ <u>from the euclidean ball</u> B^3, <u>such that as</u> $|x| \longrightarrow 0$, $f(x) \longrightarrow f_0(x/|x|)$ <u>more slowly than any positive power of</u> $|x|$.

More generally, we may consider the manifold $N^m = S^{m-1} \times \mathbb{R}$ (for the special case $n = m$) with the $\mathbb{O}(m)$-invariant metric

(7) $$ds_N^2 = dt^2 + \Gamma(t)^2 \, ds_\Sigma^2(\omega) \,,$$

where ds_Σ^2 is the standard metric on S^{m-1}, and $\Gamma : \mathbb{R} \longrightarrow (0,\infty)$ is a smooth function. For example, certain metrics of the form (7), and locally every metric of this form, arise from hypersurfaces of revolution $z = \Lambda(t)$, $|y| = \Gamma(t)$ in $\mathbb{R}^{m+1}$, where $(y,z) \in \mathbb{R}^m \times \mathbb{R} = \mathbb{R}^{m+1}$ and $(\Lambda')^2 + (\Gamma')^2 = 1$. Suppose that $f : B^m \longrightarrow N^m$ is a mapping in the $\mathbb{O}(m)$-equivariant form

(8) $$f(\rho\omega) = (\omega, u(\rho)) \in S^{m-1} \times \mathbb{R} \cong N^m$$

where $x = \rho\omega$, $0 \leq \rho \leq 1$ and $\omega \in S^{m-1}$, gives spherical coordinates for B^m. Then the elliptic system (2) reduces to the ordinary differential equation

(9) $$\rho^{3-m} D_\rho(\rho^{m-1} D_\rho u) = (m-1)\Gamma(u)\Gamma'(u) \,.$$

In particular, any homogeneous tangent map $f_o : \mathbb{R}^m \longrightarrow N^m$ which is $\mathbb{O}(m)$-equivariant must be of the form $f_o(\rho\omega) = (\omega, t_o)$ where $\Gamma'(t_o) = 0$.

For the proof of Theorem 5, one may choose $u(\rho) = (C - 2\log\rho)^{-1/2}$ to construct $f : B^3 \longrightarrow N^3$ in the equivariant form (8), with any positive constant C. We observe that $\rho D_\rho u \equiv u^3$, so that equation (9) is satisfied with $m = 3$ and the metric corresponding to $\Gamma(t)^2 := 1 + t^4/4 + t^6/2$. Note that $t = 0$ is a <u>degenerate</u> critical point of Γ, and that the integrability hypothesis of Theorem 2 fails (as it must, since the conclusion of Theorem 2 does not hold). Namely,

$f_o(\rho\omega) = (\omega, 0)$ is a homogeneous tangent map, and the vector field D_t is a harmonic-Jacobi field along f_o, but any non-constant homogeneous tangent map must have the same image as f_o.

A more penetrating analysis of the ordinary differential equation (9) reveals that for an appropriate class of $\mathbb{O}(m)$-invariant manifolds N, every bounded harmonic map of the equivariant form (8) tends logarithmically to its homogeneous tangent map. In the statement of our last theorem, we shall use the notation $O_2(t^k)$ for any function η such that the ratios $\eta(t)/t^k$, $\eta'(t)/t^{k-1}$ and $\eta''(t)/t^{k-2}$ are all bounded as $t \longrightarrow 0$. Here and throughout, the dash represents d/dt.

Theorem 6. Let N^m be the manifold $S^{m-1} \times \mathbb{R}$ equipped with the $\mathbb{O}(m)$-invariant Riemannian metric (7). We assume that Γ is a smooth positive function for $-\infty < t < \infty$, having as its only critical point a degenerate local minimum of finite order $k + 2$ at $t = 0$, such that Γ^2 is convex on a neighborhood $(-t_o, t_o)$. Then any bounded $\mathbb{O}(m)$-equivariant harmonic map $f \in H^1(B^m, N^m)$ converges to the homogeneous tangent map $f_o(x) = (x/|x|, 0)$ at a rate proportional to $(-\log|x|)^{-1/k}$. More precisely, if

(10) $$\Gamma(t)^2 = a_o + a\, t^{k+2} + O_2(t^{k+3}) ,$$

for positive constants a and a_o, then

(11) $$d(f(x), f_o(x)) = (-A \log|x|)^{-1/k} + O(-\log|x|)^{-2/k}$$

where $A := (m-1)k(k+2)a/(2(m-2))$.

Remark. We may write the hypothesis on critical points of Γ in the form

(12) $\quad t\,\Gamma'(t) > 0$ for $t \neq 0$.

Note also that the convexity hypothesis, or the asymptotic formula (10), implies that

(13) $\quad (\Gamma^2)''(t) \geq 0$ for $-t_o \leq t \leq t_o$,

where $t_o > 0$. Since $t = 0$ is assumed to be a degenerate local minimum, we have k even and ≥ 2 .

Theorem 6 is the only new result in this report. Its proof begins with the observation that if a map $f \in H^1(B^m, N^m)$ in the equivariant form (8) is weakly harmonic, then $u(\rho)$ is a weak solution of the ordinary differential equation (9) ($\rho = |x|$ as above). The regularity theory for o.d.e.'s implies that u is smooth on $0 < \rho < 1$.

We introduce the geometrically scaled variable $\Theta = \log \rho$ (as in [S1], which uses the notation $t = -\Theta$). Write $\gamma(t) := (m-1)\Gamma(t)\Gamma'(t)$, and let $b := m-2 > 0$. Then for $-\infty < \Theta < 0$, we have

(14) $\quad u_{\Theta\Theta} + b\, u_\Theta = \gamma(u(\Theta))$.

Proposition 1. Suppose $b > 0$ *and that* $t\,\gamma(t) > 0$ *for* $t \neq 0$. *Then any nontrivial bounded solution* $u(\Theta)$ *of equation* (14) *on the interval* $(-\infty, 0)$ *is of one sign, is strictly monotone, and tends to zero as* $\Theta \longrightarrow -\infty$.

Proof. We first note that $\tilde{u}(\Theta) := -\tilde{u}(\Theta)$ satisfies an o.d.e. of the same form as (14), with the right-hand side $\tilde{\gamma}(t) := -\gamma(-t)$, which satisfies the hypothesis required of γ .

Now suppose, for contradiction, that $u(\Theta_1) > 0$ and $u_\Theta(\Theta_1) \leq 0$ at some $-\infty < \Theta_1 < 0$. Then on an interval $[\Theta_0, \Theta_1]$ we have $u(\Theta) \geq t_1 := u(\theta_1)/2$ and hence $\gamma(u(\Theta)) \geq \gamma_0 := \inf\{\gamma(t) : t_1 \leq t \leq \sup u\} > 0$. Then we may compare u to the solution v of $v_{\Theta\Theta} + b\,v_\Theta = \gamma_0$ satisfying $v(\Theta_1) = u(\Theta_1)$ and $v_\Theta(\Theta_1) = u_\Theta(\Theta_1)$: we find that

$$(15) \qquad u(\Theta) \geq v(\Theta) = u(\Theta_1) - \gamma_0(\Theta_1 - \Theta)/b + c_1(e^{b(\Theta_1-\Theta)} - 1) ,$$

where $c_1 := b^{-2}(\gamma_0 - b\,u_\Theta(\Theta_1)) \geq b^{-2}\gamma_0$. Since $e^x - 1 \geq x$, we find $u(\Theta) \geq u(\Theta_1) > t_1$ for all $\Theta \in [\Theta_0, \Theta_1]$. But this implies that the requirement $u(\Theta) \geq t_1$ continues to hold as the interval $[\Theta_0, \Theta_1]$ is extended to the left, and finally inequality (15) must hold for all $\Theta \in (-\infty, \Theta_1]$. But this implies that $u(\Theta)$ is unbounded, contrary to hypothesis. It follows also that $u(\Theta_1) < 0$ and $u_\Theta(\Theta_1) \geq 0$ cannot hold simultaneously, by reversing the sign of $u(\Theta)$. This shows that u_Θ has always the same sign as u .

We may now show that $u(\Theta_3) \neq 0$ for all Θ_3 in $(-\infty, 0)$. In fact, if $u(\Theta_3) = 0$, then, since we have assumed u is not the trivial solution, there holds $u_\Theta(\Theta_3) \neq 0$ by the uniqueness of solutions to the initial-value problem. For $\Theta_2 < \Theta_3$ and close to Θ_3, it would follow that $u(\Theta_2)$ and $u_\Theta(\Theta_2)$ have opposite signs, a contradiction.

We have shown that u and u_Θ are both either positive or negative on $(-\infty,0)$. In particular, $t_2 := \lim_{\Theta\to-\infty} u(\Theta)$ exists. Consider the case $u > 0$, without loss of generality. If $t_2 > 0$, then we may argue as above to show that inequality (15) holds on $(-\infty,\Theta_1]$, for any choice of Θ_1, where now $\gamma_0 := \inf\{\gamma(t) : t_2 \leq t \leq \sup u\}$. Since $u(\Theta)$ is bounded above, the exponential coefficient c_1 must be nonpositive, which is to say

$$u_\Theta(\Theta_1) \geq \gamma_o/b$$

for any Θ_1 in $(-\infty,0)$. Choosing any $\Theta_4 < 0$, we may integrate this last inequality to obtain

$$u(\Theta) \leq u(\Theta_4) - (\Theta_4-\Theta)\gamma_o/b$$

for all Θ in $(-\infty,\Theta_4]$. This contradicts the property $u(\Theta) > 0$; we conclude that $t_2 = \lim_{\Theta\to-\infty} u(\Theta) = 0$.

q.e.d.

Returning to the proof of Theorem 6, we may apply Proposition 1 to our solution $u(\rho)$ of equation (9), since inequality (12) holds for its right-hand side. Thus u and u_ρ have one sign on $0 < \rho < 1$, which we take to be positive with no loss of generality.

We may therefore define a function $\Phi(t)$ for $0 < t < \sup u$ by the relation

$$(16) \qquad u_\Theta = \rho\, u_\rho(\rho) =: \Phi(u(\rho)) \ .$$

Then equation (14) is immediately equivalent to the first-order o.d.e.

$$(17) \qquad (\Phi'(t) + m-2)\Phi(t) = \gamma(t) \ ,$$

which is singular whenever $\Phi(t) = 0$. Certain properties of the auxiliary problem (17) may be stated as a proposition. Note that the inequality (12) implies that $\gamma(t) > 0$ for $t > 0$, while inequality (13) implies that $\gamma'(t) \geq 0$ for $0 \leq t \leq t_o$.

<u>Proposition 2</u>. <u>Assume</u> $m > 2$. <u>Suppose that</u> $\gamma(t) > 0$ <u>and</u> $\gamma'(t) \geq 0$ <u>for</u> $0 < t \leq t_o$. <u>Then for each</u> $c \geq 0$, <u>there is a unique nonnegative solution, satisfying</u> $\Phi(0) = c$, <u>of the ordinary differential equation</u> (17) <u>for</u> $0 \leq t \leq t_o$. <u>In the case of singular initial data</u> $c = 0$, <u>this solution satisfies</u>

$$(18) \qquad (1-\beta(t))\gamma(t) \leq (m-2)\Phi(t) \leq \gamma(t) \ ,$$

<u>where we define</u>

$$\beta(t_1) := (m-2)^{-2}\sup\{\gamma'(t) : 0 \leq t \leq t_1\} \ .$$

Remark. Since, in the context of Theorem 6, $\beta(t) = O(t^k)$ for an even integer $k \geq 2$, inequality (18) is a strong statement of the behavior of u_Θ as $\Theta \longrightarrow -\infty$ (and hence $u \longrightarrow 0$ by Proposition 1).

Proof. We may transform the singularity of equation (17) by defining a new dependent variable $H := \Phi^2/2$. Then equation (17) becomes

$$(19) \qquad H'(t) + b\sqrt{2H(t)} = \gamma(t) ,$$

where $b = m-2$ as before (recall $b > 0$). Now if H_1 and H_2 are two nonnegative solutions of equation (19), with $H_2(t) \geq H_1(t)$, then

$$H_2' - H_1' = b[(2H_1)^{1/2} - (2H_2)^{1/2}] \leq 0 .$$

This shows that the absolute difference of solutions is nonincreasing, which implies uniqueness on $[0,t_o]$ for the initial-value problem.

Now consider the singular case $c = 0$. For $\alpha \geq 0$, we define the comparison function $h_\alpha(t) = \alpha^2\gamma(t)^2/(2b^2)$. Then $h_\alpha' = \alpha^2 b^{-2}\gamma\gamma'$, while $\gamma(t) - b(2h_\alpha(t))^{1/2} = (1-\alpha)\gamma(t)$. In particular, h_1 is a supersolution of equation (19) on $[0,t_o]$, since $\gamma' \geq 0$ by hypothesis. On any interval $0 \leq t \leq t_1$, where $t_1 \leq t_o$, we have $h_\alpha'(t) \leq \alpha^2\gamma(t)\beta(t_1)$, so that h_α will be a subsolution provided that $\alpha^2\beta(t_1) \leq 1-\alpha$; we choose $\alpha = 1-\beta(t_1)$ for simplicity. The resulting comparisons for the forward initial-value problem:

$$h_\alpha(t_1) \le H(t_1) \le h_1(t_1)$$

are equivalent to inequality (18).

q.e.d.

Returning once again to the proof of Theorem 6, we claim that the function Φ defined by the equation (16) in fact satisfies $\lim_{t\to 0+} \Phi(t) = 0$. Otherwise, $\Phi(t)$ solves the regular o.d.e. (17) and remains positive for small positive t , by Proposition 2. Specifically, we have $\Phi(t) \ge \varepsilon > 0$ for $0 \le t \le t_5$, while $0 < u(\Theta) \le t_5$ for $\Theta \in (-\infty,\Theta_5]$, which implies that

$$u(\Theta) \le u(\Theta_5) - \varepsilon(\Theta_5-\Theta)$$

for all $\Theta \le \Theta_5$, contradicting the conclusion of Proposition 1.

It remains to estimate our solution u of equation (9) in terms of inequality (18). As above, we write $\Theta = \log \rho$. Then equation (16) yields

$$\frac{d\Theta}{du} = \frac{1}{\Phi(u)} ,$$

since $u_\Theta > 0$ by Proposition 1. Choose $\rho_1 > 0$ such that $t_1 := u(\rho_1) < t_o$; then inequality (18) yields

$$(1-\beta(t_1))\gamma(u(\rho)) \le (m-2)\Phi(u(\rho)) \le \gamma(u(\rho))$$

for all $\rho \in [0,\rho_1]$. Now the asymptotic relation (10) for Γ implies that

$$\gamma(t) = a(m-1)(k+2)t^{k+1}/2 + O_1(t^{k+2}) .$$

From this follows the estimate $\beta(t) = O(t^k)$; recall that $k \geq 2$ by assumption. Writing $A := a(m-1)k(k+2)/(2m-4)$, we find that

$$\Phi(t) = A\ t^{k+1}/k + O(t^{k+2}) ,$$

and hence for $u < t_1$ that

$$\frac{d\Theta}{du} = (k/A)u^{-k-1}(1 + O(u)) .$$

An integration yields

$$\Theta = [C_1 - u^{-k}/A](1 + O(u)) ,$$

that is, for $\rho < \rho_1$:

$$\begin{aligned} u &= [C_2 - A\Theta]^{-1/k}(1 + O(u)) \\ &= [C_2 - A \log \rho]^{-1/k} + O(-\log \rho)^{-2/k} , \end{aligned}$$

where C_1 and C_2 are constants. On the other hand, the geodesic distance in N from $f_o(\rho\omega) = (\omega,0)$ to $f(\rho\omega) = (\omega,u(\rho))$ is exactly $u(\rho)$, so that this last relation is equivalent to the conclusion (11) of Theorem 6.

Adams and Simon have recently proved in [AS] that for any homogeneous tangent map $f_o : \mathbb{R}^m \longrightarrow N^n$ for which the integrability hypothesis of Theorem 2 fails, and which is smooth on $\mathbb{R}^m \setminus \{0\}$, there is a harmonic map f into N with an isolated singularity $x = x_o$, so that f converges to f_o at the rate $(-\log|x|)^K$. Their result includes Theorem 5 above. Their analysis is valid for a much broader class of elliptic systems, as in [S1]. This theory applies equally to the rate of convergence of a singular submanifold to its tangent cone, assuming that the tangent cone has only one singularity; as well as to other problems of geometric interest.

We would like to point out, in the context of Theorem 6, that it is not known whether a minimizing harmonic map for given $\mathbb{O}(m)$-equivariant boundary conditions is itself equivariant.

REFERENCES

[AS] Adams, D. and L. Simon: Rates of asymptotic convergence near isolated singularities of geometric extrema. Preprint.

[AA] Allard, W. and F. Almgren: On the radial behaviour of minimal surfaces and the uniqueness of their tangent cones. Annals of Math. 113, 215-265 (1981).

[CG] Coron, J.-M. and R. Gulliver: Minimizing p-harmonic maps into spheres. MPI Preprint 87-13.

[G] Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Annals of Math. Study no. 105, Princeton 1983.

[GW] Gulliver, R. and B. White: The rate of convergence of a harmonic map at a singular point. Math. Ann. (to appear).

[HW] Hildebrandt, S. and K.-Ø. Widman: Some regularity results for quasilinear elliptic systems of second order. Math. Z. 142, 67-86 (1975).

[M] Morrey, C.B.: Multiple Integrals in the Calculus of Variations. Berlin-Heidelberg-New York: Springer 1966.

[SU] Schoen, R. and K. Uhlenbeck: A regularity theory for harmonic maps. J. Differential Geom. 17, 307-335 (1982).

[S1] Simon, L.: Asymptotics for a class of non-linear evolution equations, with applications to geometric problems. Annals of Math. 118, 525-571 (1983).

[S2] Simon, L.: Isolated singularities of extrema of geometric variational problems. Springer Lecture Notes in Math. 1161, 206-277 (1985).