ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 10.

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In memory of
Professor D.A.Mit'kin,
one of the best pupils
of Professor N.M. Korobov.

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§10.0. Foreword.

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§10.1 Test the equality $-\nu^5(\nu+\alpha)^5E_4 = A^*_{\alpha,0}(z;-\nu-\alpha)A^*_{\alpha,0}(z;\nu)$.

We had established in [12] that the equality

$$-\nu^5(\nu+\alpha)^5E_4 = A^*_{\alpha,0}(z;-\nu-\alpha)A^*_{\alpha,0}(z;\nu) \tag{1}$$

is equivalent to the following system of equalities

$$\begin{cases} S^{**}_{\alpha,0}(\nu)S^{**}_{\alpha,0}(-\nu-\alpha) = -\nu^5(\nu+\alpha)^5E_4, \\ S^{**}_{\alpha,0}(\nu)V^{**}_{\alpha,0}(-\nu-\alpha) + V^{**}_{\alpha,0}(\nu)S^{**}_{\alpha,0}(-\nu-\alpha) = 0E_4 \\ V^{**}_{\alpha,0}(z;\nu)V^{**}_{\alpha,0}(z;-\nu-\alpha) = 0E_4, \end{cases} \tag{2}$$

where $S^{**}_{\alpha,0}(\nu)$ and $V^{**}_{\alpha,0}(\nu)$ are 4×4-matrices, elements of which respectively

$$s^{**}_{\alpha,0,i,k}(\nu) \text{ and } v^{**}_{\alpha,0,i,k}(\nu)$$

with $\{i,k\}\subset\{1,\,2,\,3,\,4\}$ are pointed in the §9.4 of [12] (here i denotes the number of row and k denotes the number of column). Let

$$b^{**}_{\alpha,0,i,k}(\nu) = \sum_{j=1}^{4} s^{**}_{\alpha,i,j}(\nu)s^{**}_{\alpha,j,k}(-\nu-\alpha), \tag{3}$$

where $\{i,\,k\}\subset\{1,\,2,\,3,\,4\}$.

In view of (111) – (131) in [12],

(4) $$b^{**}_{\alpha,0,2,1}(\nu) = b^{**}_{\alpha,0,3,1}(\nu) = b^{**}_{\alpha,0,3,2}(\nu) =$$
$$b^{**}_{\alpha,0,4,1}(\nu) = b^{**}_{\alpha,0,4,2}(\nu) = b^{**}_{\alpha,0,4,3}(\nu) = 0,$$

(5) $$b^{**}_{\alpha,0,1,1}(\nu) = b^{**}_{\alpha,0,2,2}(\nu) =$$
$$b^{**}_{\alpha,0,3,3}(\nu) = b^{**}_{\alpha,0,4,4}(\nu) = -\nu^5(\nu+\alpha)^5,$$

(6) $$b^{**}_{\alpha,0,1,2}(\nu) = b^{**}_{\alpha,0,2,3}(\nu) = b^{**}_{\alpha,0,3,4}(\nu) =$$
$$s^{**}_{\alpha,0,1,1}(\nu)s^{**}_{\alpha,0,1,2}(-\nu-\alpha) + s^{**}_{\alpha,0,1,2}(\nu)s^{**}_{\alpha,0,2,2}(-\nu-\alpha) =$$
$$\nu^3(\nu+\alpha)^2(-2(\nu+\alpha)^2(-2\nu-\alpha)(-\nu)) + (-2\nu^2(2\nu+\alpha)(\nu+\alpha))(-(\nu+\alpha)^3\nu^2) =$$
$$-2\nu^4(\nu+\alpha)^4(2\nu+\alpha) + 2\nu^4(\nu+\alpha)^4(2\nu+\alpha) = 0,$$

(7) $$b^{**}_{\alpha,0,1,3}(\nu) = b^{**}_{\alpha,0,2,4}(\nu) = s^{**}_{\alpha,0,1,1}(\nu)s^{**}_{\alpha,0,1,3}(-\nu-\alpha) +$$
$$s^{**}_{\alpha,0,1,2}(\nu)s^{**}_{\alpha,0,2,3}(-\nu-\alpha) + s^{**}_{\alpha,0,1,3}(\nu)s^{**}_{\alpha,0,3,3}(-\nu-\alpha) =$$
$$\nu^3(\nu+\alpha)^2(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))(-2(\nu+\alpha)^2(2\nu+\alpha)\nu+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha))(-(\nu+\alpha)^3\nu^2) =$$
$$\nu^3(\nu+\alpha)^3(2\nu+\alpha)(-(4\nu+\alpha) + 4(2\nu+\alpha-(4\nu+3\alpha)) = 0,$$

(8) $$b^{**}_{\alpha,0,1,4}(\nu) = s^{**}_{\alpha,0,1,1}(\nu)s^{**}_{\alpha,0,1,4}(-\nu-\alpha) +$$
$$s^{**}_{\alpha,0,1,2}(\nu)s^{**}_{\alpha,0,2,4}(-\nu-\alpha) + s^{**}_{\alpha,0,1,3}(\nu)s^{**}_{\alpha,0,3,4}(-\nu-\alpha)+$$
$$s^{**}_{\alpha,0,1,4}(\nu)s^{**}_{\alpha,0,4,4}(-\nu-\alpha) =$$
$$\nu^3(\nu+\alpha)^2(-2(2\nu+\alpha)(3\nu+\alpha))+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))+$$
$$(\nu(2\nu+\alpha)(4\nu+3\alpha))(-2(\nu+\alpha)^2(2\nu+\alpha)\nu)+$$
$$(-2(2\nu+\alpha)(3\nu+2\alpha))(-(\nu+\alpha)^3\nu^2)) =$$
$$2\nu^2(\nu+\alpha)^2(2\nu+\alpha)\times$$
$$(-3\nu^2-\alpha\nu-2\alpha(2\nu+\alpha)+3\nu^2+5\alpha\nu+2\alpha^2) = 0.$$

It follows from (5) – (8) that

(9) $$S^{**}_{\alpha,0}(z;\nu)S^{**}_{\alpha,0}(z;-\nu-\alpha) = S^{**}_{\alpha,0}(z;-\nu-\alpha)S^{**}_{\alpha,0}(z;\nu) =$$
$$-\nu^3(\nu+\alpha)^3E_4.$$

Let

$$c^{**}_{\alpha,0,i,k}(\nu) = \left(\sum_{j=1}^{4} s^{**}_{\alpha,i,j}(\nu) v^{**}_{\alpha,j,k}(-\nu-\alpha)\right) + \tag{10}$$

$$+\sum_{j=1}^{4} v^{**}_{\alpha,i,j}(\nu) s^{**}_{\alpha,j,k}(-\nu-\alpha),$$

where $\{i,\, k\} \subset \{1,\, 2,\, 3,\, 4\}$. In view of (111) – (131) in [12],

$$c^{**}_{\alpha,0,1,1}(\nu) = \nu^3(\nu+\alpha)^2 \times \tag{11}$$

$$(-\nu^2(2\nu+\alpha)(8(\nu+\alpha)^2 - (6+5\alpha)(\nu+\alpha) + 4\alpha))+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))\times$$
$$(-(\nu+\alpha)\nu^2(2\nu+\alpha)(6(\nu+\alpha)^2 - 4(\nu+\alpha)(1+\alpha) + 3\alpha))+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha)\times$$
$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2 - (2+3\alpha)(\nu+\alpha) + 2\alpha))+$$
$$(-2(2\nu+\alpha)(3\nu+2\alpha))(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))+$$
$$(-(\nu+\alpha)^3\nu^2)(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha) =$$
$$(2\nu+\alpha)(\nu+\alpha)^2\nu^2 c^{*}_{\alpha,0,1,1}(\nu),$$

where

$$c^{*}_{\alpha,0,1,1}(\nu) = \tag{12}$$

$$-\nu^3(8(\nu+\alpha)^2 - (6+5\alpha)(\nu+\alpha) + 4\alpha)+$$
$$(2\nu^2(2\nu+\alpha)(6(\nu+\alpha)^2 - 4(\nu+\alpha)(1+\alpha) + 3\alpha)+$$
$$(-\nu(2\nu+\alpha)(4\nu+3\alpha)(4(\nu+\alpha)^2 - (2+3\alpha)(\nu+\alpha) + 2\alpha))+$$
$$2(2\nu+\alpha)(3\nu+2\alpha)(\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha)+$$
$$(-(\nu+\alpha)^3(8\nu^2+(6+5\alpha)\nu+4\alpha)) =$$
$$\sum_{k=0}^{5} c^{\vee}_{0,1,1,k}(\nu).$$

Clearly,

$$c^{\vee}_{0,1,1,5}(\nu) = 2(2-2) = 0, c^{\vee}_{0,1,1,4}(\nu) = -3\nu + 8\nu + 4 - 5\nu - 4 = 0,$$

$$\deg_{\alpha}(c^{*}_{\alpha,0,1,1}(\nu)) \le 3, \text{ if } c^{*}_{\alpha,0,1,1}(\nu) \neq 0,$$

and

$$c^{*}_{0,0,1,1}(\nu) = \nu^4\times$$
$$(-8\nu + 6 + 24\nu - 16 - 32\nu + 16 + 24\nu - 8\nu - 6) = 0,$$
$$c^{*}_{-\nu,0,1,1}(\nu) = 4\nu^4 - 6\nu^4 + 2\nu^4 = 0,$$
$$c^{*}_{-2\nu,0,1,1}(\nu) = (-\nu^3+\nu^3)(8\nu^2 + \nu(-10\nu+6) - 8\nu) = 0,$$
$$c^{*}_{\nu,0,1,1}(\nu) = \nu^4\times$$

$$(-22\nu+8+96\nu-30-210\nu+42+240\nu+60-104\nu-80)=0.$$

Therefore

$$c^{**}_{\alpha,0,1,1}(\nu)=0. \tag{13}$$

In view of (111) – (131) in [12],

$$c^{**}_{\alpha,0,2,1}(\nu)=\nu^3(\nu+\alpha)^2\times \tag{14}$$

$$(-\nu^2(\nu+\alpha)(2\nu+\alpha)(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))\times$$
$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha)(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))+$$
$$(-(\nu+\alpha)^3\nu^2)(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))=$$
$$\nu^3(\nu+\alpha)^3(2\nu+\alpha)c^{*}_{\alpha,0,2,1}(\nu),$$

where

$$c^{*}_{\alpha,0,2,1}(\nu)= \tag{15}$$

$$-\nu^2(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha)+$$
$$2\nu(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$(-(4\nu+3\alpha)(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))+$$
$$(\nu+\alpha)^2(6\nu^2+4(1+\alpha)\nu+3\alpha)=$$
$$\sum_{k=0}^{3}c^{\vee}_{0,2,1,k}(\nu)\alpha^k,$$

Clearly,

$$c^{\vee}_{0,2,1,3}(\nu)=2\nu-6\nu-3+4\nu+3=0,$$
$$\deg_\alpha(c^{*}_{\alpha,0,2,1}(\nu))\le 2,\ \text{ if } c^{*}_{\alpha,0,2,1}(\nu)\ne 0,$$
$$c^{*}_{0,0,2,1}(\nu)=\nu^3(-6\nu+4+16\nu-8-16\nu+6\nu+4)=0,$$
$$c^{*}_{-\nu,0,1,1}(\nu)=3\nu^3-4\nu^3+\nu^3=0,$$
$$c^{*}_{-2\nu,0,1,1}(\nu)=(-\nu^2+\nu^2)(6\nu^2+4\nu(1-2\nu)-6\nu)=0.$$

Therefore

$$c^{**}_{\alpha,0,2,1}(\nu)=0. \tag{16}$$

In view of (111) – (131) in [12],

$$c^{**}_{\alpha,0,3,1}(\nu)=\nu^3(\nu+\alpha)^2\times \tag{17}$$

$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))+$$
$$(-(\nu+\alpha)^3\nu^2)\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)=$$

$$\nu^4(\nu+\alpha)^4(2\nu+\alpha)c^*_{\alpha,0,3,1}(\nu),$$

where

(18) $$c^*_{\alpha,0,3,1}(\nu) = -\nu(4(\nu+\alpha)^2 - (2+3\alpha)(\nu+\alpha)+2\alpha)+$$

$$2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha) - (\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha) =$$

$$\sum_{k=0}^{2} c^{\vee}_{0,3,1,k}(\nu)\alpha^k,$$

Clearly,

$$c^{\vee}_{0,3,1,2}(\nu) = -\nu+4\nu+2-3\nu-2 = 0,$$

$$\deg_\alpha(c^*_{\alpha,0,2,1}(\nu)) \le 1, \text{ if } c^*_{\alpha,0,2,1}(\nu) \neq 0,$$

$$c^*_{0,0,3,1}(\nu) = \nu^2(-4\nu+2+8\nu-4\nu-2) = 0, c^*_{-\nu,0,3,1}(\nu) = 2\nu^2-2\nu^2 = 0.$$

Therefore

(19) $$c^{**}_{\alpha,0,3,1}(\nu) = 0.$$

In view of (111) – (131) in [12],

(20) $$c^{**}_{\alpha,0,4,1}(\nu) = \nu^3(\nu+\alpha)^2 \times$$

$$(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))+$$

$$(-(\nu+\alpha)^3\nu^2)(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha)) =$$

$$(2\nu^2+2\alpha\nu+\alpha)\nu^5(\nu+\alpha)^5(2\nu+\alpha)(-1+1) = 0.$$

In view of (111) – (131) in [12],

(21) $$c^{**}_{\alpha,0,1,2}(\nu) = \nu^3(\nu+\alpha)^2 \times$$

$$(2\nu(2\nu+\alpha)(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2))+$$

$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))\times$$

$$\nu(\nu+\alpha)(2\nu+\alpha)(-8(\nu+\alpha)^2+(8+\alpha)(nu+\alpha)-6\alpha+3\alpha^2)+$$

$$\nu(2\nu+\alpha)(4\nu+3\alpha)\times$$

$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2+(2+\alpha)(\nu+\alpha)-2\alpha+\alpha^2)+$$

$$(-2(2\nu+\alpha)(3\nu+2\alpha))\times$$

$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$

$$(-(\nu+\alpha)^3\nu^2)2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)+$$

$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha) =$$

$$\nu(2\nu+\alpha)(\nu+\alpha)^2c^*_{\alpha,0,1,2}(\nu),$$

where

(22) $$c^*_{\alpha,0,1,2}(\nu) = (2\nu^3(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)) +$$

$$(-2\nu^2(2\nu+\alpha)(-8(\nu+\alpha)^2+(8+\alpha)(\nu+\alpha)-6\alpha+3\alpha^2))+$$

$$2(4\nu+3\alpha)(2\nu+\alpha)\nu(-3(\nu+\alpha)^2+(2+\alpha)(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(-2(3\nu+2\alpha)(\nu+\alpha)(2\nu+\alpha))(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2))+$$
$$(-2(\nu+\alpha)^2\nu(-5\nu^2-6\nu-4\alpha+2\alpha^2))+$$
$$(-2(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha))=$$
$$\sum_{k=0}^{4} c^{\vee}_{0,1,2,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,1,2,4}(\nu)=-6\nu+20\nu+8-4\nu-10\nu-8=0,$$
$$\deg_\alpha(c^*_{\alpha,0,1,2}(\nu))\le 3,\ \text{if } c^*_{\alpha,0,1,1}(\nu)\ne 0,$$

and

$$c^*_{0,0,1,2}(\nu)=\nu^4\times$$
$$(-10\nu+12+32\nu-32-48\nu+32+48\nu+10\nu+12-32\nu-24)=0,$$
$$c^*_{-\nu,0,1,2}(\nu)=\nu^4(4\nu+8-6\nu-12+4+2\nu)=0,$$
$$c^*_{-2\nu,0,1,2}(\nu)=(2\nu^2-2\nu^2)(-5\nu^2-6\nu+8\nu+8\nu^2)=0,$$
$$c^*_{\nu,0,1,2}(\nu)=\nu^4\times$$
$$\nu^4(-36\nu+16+162\nu-60-378\nu+84+540\nu+120+24\nu+80-312\nu-240)=0$$

Therefore

$$c^{**}_{\alpha,0,1,2}(\nu)=0. \tag{23}$$

In view of (111) – (131) in [12],

$$c^{**}_{\alpha,0,2,2}(\nu)=\nu^3(\nu+\alpha)^2\times \tag{24}$$
$$\nu(\nu+\alpha)(2\nu+\alpha)(-8(\nu+\alpha)^2+(8+\alpha)(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))\times$$
$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2+(2+\alpha)(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha)\times$$
$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(-(\nu+\alpha)^3\nu^2)(-\nu(\nu+\alpha)(2\nu+\alpha))(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2)+$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))=$$
$$\nu^2(2\nu+\alpha)(\nu+\alpha)^3c^*_{\alpha,0,2,2}(\nu),$$

where

(25)
$$c^*_{\alpha,0,2,2}(\nu)=\nu^2(-8(\nu+\alpha)^2+(8+\alpha)(nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-4\nu(2\nu+\alpha))(-3(\nu+\alpha)^2+(2+\alpha)(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(4\nu+3\alpha)(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$\nu(\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2)+$$

$$(2(\nu+\alpha)(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha)) =$$

$$\sum_{k=0}^{3} c^{\vee}_{0,2,2,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,2,2,3}(\nu) = 4\nu - 15\nu - 6 + 3\nu + 8\nu + 6 = 0,$$
$$\deg_\alpha(c^*_{\alpha,0,2,2}(\nu)) \le 2, \text{ if } c^*_{\alpha,0,2,2}(\nu) \ne 0,$$

and

$$c^*_{0,0,2,2}(\nu) = \nu^3 \times$$
$$(-8\nu + 8 + 24\nu - 16 - 32\nu - 8\nu - 8 + 24nu + 16) = 0,$$
$$c^*_{-\nu,0,2,2}(\nu) = \nu^3(3\nu - 6 - 4\nu + 8 + \nu - 2) = 0,$$
$$c^*_{-2\nu,0,2,2}(\nu) = (\nu^2 - \nu^2)(6\nu^2 + 4\nu) = 0.$$

Therefore

(26) $$c^{**}_{\alpha,0,2,2}(\nu) = 0.$$

In view of (111) – (131) in [12], 1

(27) $$c^{**}_{\alpha,0,3,2}(\nu) = \nu^3(\nu+\alpha)^2 \times$$
$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2 + (2+\alpha)(\nu+\alpha) - 2\alpha + \alpha^2) +$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))\times$$
$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2 + 3\alpha(\nu+\alpha) - 2\alpha + \alpha^2) +$$
$$(-(\nu+\alpha)^3\nu^2)\times$$
$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2 - 2\nu - \alpha\nu - 2\alpha + \alpha^2) +$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2))\times$$
$$\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2 + (2+3\alpha)\nu + 2\alpha) =$$

$$\nu^2(2\nu+\alpha)(\nu+\alpha)^4 c^*_{\alpha,0,2,2}(\nu),$$

where

(28) $$c^*_{\alpha,0,3,2}(\nu) =$$
$$2\nu^2(-3(\nu+\alpha)^2 + (2+\alpha)(\nu+\alpha) - 2\alpha + \alpha^2) +$$
$$(-2\nu(2\nu+\alpha))(-4(\nu+\alpha)^2 + 3\alpha(\nu+\alpha) - 2\alpha + \alpha^2) +$$
$$(-2\nu^2(-3\nu^2 - 2\nu - \alpha\nu - 2\alpha + \alpha^2)) +$$
$$(-2\nu(2\nu+\alpha))(4\nu^2 + (2+3\alpha)\nu + 2\alpha) =$$

$$\sum_{k=0}^{2} c^{\vee}_{0,3,2,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,3,2,2}(\nu) = \nu(-2\nu + 10\nu + 4 - 2\nu - 6\nu - 4) = 0,$$

$$\deg_\alpha(c^*_{\alpha,0,2,2}(\nu)) \le 1, \text{ if } c^*_{\alpha,0,3,2}(\nu) \ne 0,$$

and

$$c^*_{0,0,3,2}(\nu) = \nu^3(-6\nu + 4 + 16\nu + 6\nu + 4 - 16\nu - 8) = 0,$$
$$c^*_{-\nu,0,3,2}(\nu) = \nu^3(2\nu + 4 - 2\nu - 4 + 2\nu - 2\nu) = 0.$$

Therefore

(29) $$c^{**}_{\alpha,0,3,2}(\nu) = 0.$$

In view of (111) – (131) in [12], 1

(30) $$c^{**}_{\alpha,0,4,2}(\nu) = \nu^3(\nu+\alpha)^2 \times$$
$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2 + 3\alpha(\nu+\alpha) - 2\alpha + \alpha^2)+$$
$$(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2 - 3\alpha\nu - 2\alpha + \alpha^2))(-(\nu+\alpha)^3\nu^2)+$$
$$(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2 + 2\alpha\nu + \alpha))\times$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)) =$$
$$\nu^4(2\nu+\alpha)(\nu+\alpha)^4 c^*_{\alpha,0,4,2}(\nu),$$

where

(31) $$c^*_{\alpha,0,4,2}(\nu) =$$
$$(\nu+\alpha)(-4(\nu+\alpha)^2 + 3\alpha(\nu+\alpha) - 2\alpha + \alpha^2)+$$
$$\nu(-4\nu^2 - 3\alpha\nu - 2\alpha + \alpha^2) + 2(2\nu+\alpha)(2\nu^2 + 2\alpha\nu + \alpha) =$$
$$\sum_{k=0}^{2} c^\vee_{0,4,2,k}(\nu)\alpha^k.$$

Clearly,

$$c^\vee_{0,4,2,2}(\nu) = \nu(-5\nu - 2 + \nu + 4\nu + 2) = 0,$$
$$\deg_\alpha(c^*_{\alpha,0,4,2}(\nu)) \le 1, \text{ if } c^*_{\alpha,0,4,2}(\nu) \ne 0,$$

and

$$c^*_{0,0,4,2}(\nu) = \nu^3(-4 - 4 + 8) = 0, c^*_{-\nu,0,4,2}(\nu) = \nu^2(2-2) = 0.$$

Therefore

(32) $$c^{**}_{\alpha,0,4,2}(\nu) = 0.$$

In view of (111) – (131) in [12],

(33) $$c^{**}_{\alpha,0,4,3}(\nu) = \nu^3(\nu+\alpha)^2 \times$$
$$(2\nu+\alpha)(4(\nu+\alpha)^2 + (6 - 15\alpha)(\nu+\alpha) - 4\alpha + 8\alpha^2)+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))\times$$
$$(\nu+\alpha)(2\nu+\alpha)(2(\nu+\alpha)^2 + (4 - 10\alpha)(\nu+\alpha) - 3\alpha + 6\alpha^2)+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha)\times$$
$$(\nu+\alpha)^2(2\nu+\alpha)((2 - 5\alpha)(\nu+\alpha) - 2\alpha + 4\alpha^2)+$$

$$(-2(2\nu+\alpha)(3\nu+2\alpha))\times$$
$$(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)+$$
$$(-(\nu+\alpha)^3\nu^2)(-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2)+$$
$$2(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)+$$
$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha)=$$
$$(2\nu+\alpha)(\nu+\alpha)^2c^*_{\alpha,0,1,3}(\nu),$$

where

(34)
$$c^*_{\alpha,0,1,3}(\nu)=$$
$$\nu^3(4(\nu+\alpha)^2+(6-15\alpha)(\nu+\alpha)-4\alpha+8\alpha^2)+$$
$$-2\nu^2(2\nu+\alpha)(2(\nu+\alpha)^2+(4-10\alpha)(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$\nu(4\nu+3\alpha)(2\nu+\alpha)((2-5\alpha)(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$-2(3\nu+2\alpha)(\nu+\alpha)(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)+$$
$$(\nu+\alpha)\nu^2(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2)+$$
$$(-4\nu(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2))+$$
$$-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha)=$$
$$\sum_{k=0}^{4}c^\vee_{0,1,3,k}(\nu)\alpha^k.$$

Clearly,
$$c^\vee_{0,1,3,4}(\nu)=-3\nu+16\nu+4-8\nu-5\nu-4=0,$$
$$deg_\alpha(c^*_{\alpha,0,1,3}(\nu))\le 3,\ \text{if } c^*_{\alpha,0,1,3}(\nu)\ne 0,$$

and
$$c^*_{0,0,1,3}(\nu)=\nu^4\times$$
$$(4\nu+6-8\nu-4+16+24\nu+4\nu-6+40\nu+48-64\nu-48)=0,$$
$$c^*_{-\nu,0,1,3}(\nu)=\nu^4(8\nu+4-12\nu-6+4\nu+2)=0,$$
$$c^*_{-2\nu,0,1,3}(\nu)=(2\nu^2-2\nu^2)(6\nu^2+2\nu)=0,$$
$$c^*_{\nu,0,1,3}(\nu)=\nu^4\times$$
$$(-6\nu+8+36\nu-30-126\nu+42+360\nu+60+54\nu-20+72\nu+240-390\nu-300)=$$

Therefore

(35)
$$c^{**}_{\alpha,0,1,3}(\nu)=0.$$

In view of (111) – (131) in [12],

(36)
$$c^{**}_{\alpha,0,2,3}(\nu)=\nu^3(\nu+\alpha)^2\times$$
$$(\nu+\alpha)(2\nu+\alpha)(2(\nu+\alpha)^2+(4-10\alpha)(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))(\nu+\alpha)^2(2\nu+\alpha)((2-5\alpha)(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$\nu(2\nu+\alpha)(4\nu+3\alpha)(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)+$$

$$(-(\nu+\alpha)^3\nu^2)\nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)+$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)(-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))+$$
$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))=$$
$$\nu(2\nu+\alpha)(\nu+\alpha)^3 c^*_{\alpha,0,2,3}(\nu),$$

where

$$c^*_{\alpha,0,2,3}(\nu) = \tag{37}$$
$$\nu^2(2(\nu+\alpha)^2+(4-10\alpha)(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$-2\nu(2\nu+\alpha)((2-5\alpha)(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$(2\nu+\alpha)(4\nu+3\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)+$$
$$-\nu^2(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)+$$
$$2\nu(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2)+$$
$$(4\nu+\alpha)(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha)=$$
$$\sum_{k=0}^{3} c^{\vee}_{0,2,3,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,2,3,3}(\nu) = 2\nu-12\nu-3+6\nu+4\nu+3=0,$$
$$\deg_\alpha(c^*_{\alpha,0,2,3}(\nu)) \le 2, \text{ if } c^*_{\alpha,0,2,3}(\nu)\neq 0,$$

and

$$c^*_{0,0,2,3}(\nu)=\nu^3\times$$
$$(2\nu+4-8-16\nu-2\nu+4-32\nu-32+48\nu+32)=0,$$
$$c^*_{-\nu,0,2,3}(\nu)=\nu^3(6\nu+3-8\nu-4+2\nu+1+2\nu+1-8\nu-4+6\nu+3)=0,$$
$$c^*_{-2\nu,0,2,3}(\nu)=(\nu^2-\nu^2)(6\nu^2+2\nu)=0.$$

Therefore

$$c^{**}_{\alpha,0,2,3}(\nu)=0. \tag{38}$$

In view of (111) – (131) in [12],

$$c^{**}_{\alpha,0,3,3}(\nu)=\nu^3(\nu+\alpha)^2\times \tag{39}$$
$$(\nu+\alpha)^2(2\nu+\alpha)((2-5\alpha)(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)+$$
$$(-(\nu+\alpha)^3\nu^2)(-\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))+$$
$$2(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)+$$
$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)=$$
$$\nu^2(2\nu+\alpha)(\nu+\alpha)^3 c^*_{\alpha,0,3,3}(\nu),$$

where

$$c^*_{\alpha,0,3,3}(\nu) = \tag{40}$$

$$\nu(\nu+\alpha)((2-5\alpha)(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$(-2(2\nu+\alpha)(\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2))+$$
$$(\nu^2(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))+$$
$$(-4\nu(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2))+$$
$$(-(2\nu+\alpha)(4\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)) =$$
$$\sum_{k=0}^{3} c^{\vee}_{0,3,3,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,3,3,3}(\nu) = -\nu+8\nu+2-4\nu-3\nu-2 = 0,$$
$$\deg_\alpha(c^*_{\alpha,0,3,3}(\nu)) \le 2, \text{ if } c^*_{\alpha,0,3,3}(\nu) \neq 0,$$

and

$$c^*_{0,0,3,3}(\nu) = \nu^3(2+8\nu-2+24\nu+16-32\nu-16) = 0,$$
$$c^*_{-\nu,0,3,3}(\nu) = \nu^3(-\nu+4\nu-3\nu) = 0, c^*_{-2\nu,0,3,3}(\nu) = \nu^3(-6\nu+6\nu) = 0.$$

Therefore

$$c^{**}_{\alpha,0,3,3}(\nu) = 0. \tag{41}$$

In view of (111) – (131) in [12],

$$c^{**}_{\alpha,0,4,3}(\nu) = \nu^3(\nu+\alpha)^2 \times \tag{42}$$

$$(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)+$$
$$(-(\nu+\alpha)^3\nu^2)\nu^3(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2)+$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))+$$
$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha)) =$$
$$\nu^3(2\nu+\alpha)(\nu+\alpha)^3c^*_{\alpha,0,4,3}(\nu),$$

where

$$c^*_{\alpha,0,4,3}(\nu) = (\nu+\alpha)^2(-2(\nu+\alpha)^2-\alpha+2\alpha^2) + \tag{43}$$

$$(-\nu^2(-2\nu^2-\alpha+2\alpha^2)) + 2\nu(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2)+$$
$$(4\nu+\alpha)(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha) = \sum_{k=0}^{3} c^{\vee}_{0,4,3,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,3,4,3}(\nu) = -4\nu-1+2\nu+2\nu+1 = 0,$$
$$\deg_\alpha(c^*_{\alpha,0,4,3}(\nu)) \le 2, \text{ if } c^*_{\alpha,0,4,3}(\nu) \neq 0,$$

and

$$c^*_{0,0,4,3}(\nu) = \nu^4(-2+2-16+16) = 0,$$

$$c^*_{-\nu,0,3,3}(\nu) = \nu^3(-1+4-3) = 0,$$

$$c^*_{-2\nu,0,3,3}(\nu) = (\nu^3 - \nu^3)(6\nu+2) = 0.$$

Therefore

$$c^{**}_{\alpha,0,4,3}(\nu) = 0. \tag{44}$$

In view of (111) – (131) in [12],

$$\begin{gathered} c^{**}_{\alpha,0,1,4}(\nu) = 2\nu^3(\nu+\alpha)^2(2\nu+\alpha)(3\nu+\alpha) + \\ (-2\nu^2(2\nu+\alpha)(\nu+\alpha))(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha) + \\ \nu(2\nu+\alpha)(4\nu+3\alpha)2\nu(2\nu+\alpha)(\nu+\alpha)^2 + \\ (-2(2\nu+\alpha)(3\nu+2\alpha))(-(\nu+\alpha)^3\alpha(2\nu+\alpha)) + \\ (-2(2\nu+\alpha)(3\nu+\alpha))(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha) + \\ (-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2) + \\ (-2\nu(2\nu+\alpha)(\nu+\alpha)^2)(-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2)) + \\ (-\nu^2(\nu+\alpha)^3)2(2\nu+\alpha)(3\nu+2\alpha) = \\ (2\nu+\alpha)(\nu+\alpha)^2 c^*_{\alpha,0,1,4}(\nu), \end{gathered} \tag{45}$$

where

$$\begin{gathered} c^{**}_{\alpha,0,1,4}(\nu) = 2\nu^3(3\nu+\alpha) + (-2\nu^2(2\nu+\alpha)(4\nu+\alpha)) + \\ 2\nu^2(2\nu+\alpha)(4\nu+3\alpha) + 2(2\nu+\alpha)(3\nu+2\alpha)(\nu+\alpha)\alpha + \\ (-2(2\nu+\alpha)(3\nu+\alpha))(8\nu^2+(6+5\alpha)\nu+4\alpha) + \\ (-2(4\nu+\alpha))(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)) + \\ 2\nu(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2) + \\ (-2\nu^2(\nu+\alpha)(3\nu+2\alpha)) = \sum_{k=0}^{4} c^{\vee}_{0,4,1,k}(\nu)\alpha^k. \end{gathered} \tag{46}$$

Clearly,

$$c^{\vee}_{0,4,1,4}(\nu) = 4-4 = 0, \deg_\alpha(c^*_{\alpha,0,4,1}(\nu)) \le 3, \text{ if } c^*_{\alpha,0,1,4}(\nu) \neq 0,$$

and

$$c^*_{0,0,1,4}(\nu) = \nu^3(6\nu - 16\nu + 16\nu - 96\nu - 72 + 80\nu + 96 + 16\nu - 24 - 6\nu) = 0,$$

$$c^*_{-\nu,0,1,4}(\nu) = \nu^3(4\nu - 6\nu + 2\nu - 12\nu - 8 + 18\nu + 12 - 6\nu - 4) = 0$$

$$c^*_{\nu,0,1,4}(\nu) =$$

$$\nu^3(8\nu - 30\nu + 42\nu + 60\nu - 312\nu - 240 + 90\nu + 300 + 162\nu - 60 - 20\nu) = 0,$$

$$c^*_{-2\nu,0,1,4}(\nu) = \nu^4(2-2) = 0.$$

Therefore

$$c^{**}_{\alpha,0,1,4}(\nu) = 0. \tag{47}$$

In view of (111) – (131) in [12],

(48) $$c^{**}_{\alpha,0,2,4}(\nu) = \nu^3(\nu+\alpha)^2(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha)+$$

$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))2\nu(2\nu+\alpha)(\nu+\alpha)^2+$$

$$\nu(2\nu+\alpha)(4\nu+3\alpha)(-(\nu+\alpha)^3\alpha(2\nu+\alpha))+$$

$$(-2(2\nu+\alpha)(3\nu+\alpha))(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))+$$

$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))\times$$

$$(-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))+$$

$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2)\nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)+$$

$$(-\nu^2(\nu+\alpha)^3)(-\nu(2\nu+\alpha)(4\nu+3\alpha)) = \nu(2\nu+\alpha)(\nu+\alpha)^2c^{*}_{\alpha,0,2,4}(\nu),$$

where

(49) $$c^{**}_{\alpha,0,2,4}(\nu) = \nu^2(\nu+\alpha)(4\nu+\alpha)+$$

$$(-4\nu^2(2\nu+\alpha)(\nu+\alpha)) + (-(4\nu+3\alpha)(\nu+\alpha)\alpha(2\nu+\alpha))+$$

$$2(2\nu+\alpha)(3\nu+\alpha))(6\nu^2+4(1+\alpha)\nu+3\alpha)+$$

$$(2\nu+\alpha)(4\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2)+$$

$$(-2\nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2))+$$

$$\nu^2(\nu+\alpha)(4\nu+3\alpha) = \sum_{k=0}^{4} c^{\vee}_{0,2,4,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,4,2,4}(\nu) = -3+3 = 0,\ \deg_\alpha(c^{*}_{\alpha,0,2,4}(\nu)) \le 3,\ \text{ if } c^{*}_{\alpha,0,2,4}(\nu) \ne 0,$$

and

$$c^{*}_{0,0,2,4}(\nu) = \nu^3(4\nu-8\nu+72\nu+48-64\nu-64-8\nu+16+4\nu) = 0,$$

$$c^{*}_{-\nu,0,2,4}(\nu) = \nu^3(8\nu+4-12\nu-6+4\nu+2) = 0,$$

$$c^{*}_{\nu,0,2,4}(\nu) = \nu^3(10\nu-24\nu-42\nu+240\nu+168-90\nu-210-108\nu+42+14\nu = 0,$$

$$c^{*}_{-2\nu,0,2,4}(\nu) = \nu^4(-2+2) = 0.$$

Therefore

(50) $$c^{**}_{\alpha,0,2,4}(\nu) = 0.$$

In view of (111) – (131) in [12],

(51) $$c^{**}_{\alpha,0,3,4}(\nu) = 2\nu^3(\nu+\alpha)^2\nu(2\nu+\alpha)(\nu+\alpha)^2+$$

$$(-2\nu^2(2\nu+\alpha)(\nu+\alpha))(-(\nu+\alpha)^3\alpha(2\nu+\alpha))+$$

$$(-2(2\nu+\alpha)(3\nu+\alpha))\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)+$$

$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))\times$$

$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)+$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2))(-\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))+$$
$$(-\nu^2(\nu+\alpha)^3)2\nu^2(2\nu+\alpha)(\nu+\alpha)=\nu^2(2\nu+\alpha)(\nu+\alpha)^2c^*_{\alpha,0,3,4}(\nu),$$

where

(52)
$$c^*_{\alpha,0,3,4}(\nu)=2\nu^2(\nu+\alpha)^2+2(\nu+\alpha)^2\alpha(2\nu+\alpha)+$$
$$(-2(2\nu+\alpha)(3\nu+\alpha))(4\nu^2+(2+3\alpha)\nu+2\alpha)+$$
$$(-2(4\nu+\alpha))(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)+$$
$$2\nu(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2)+$$
$$(-2\nu^2(\nu+\alpha)^2)=\sum_{k=0}^{4}c^{\vee}_{0,3,4,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,3,4,4}(\nu)=2-2=0,\ \deg_\alpha(c^*_{\alpha,0,3,4}(\nu))\le 3,\ \text{ if } c^*_{\alpha,0,3,4}(\nu)\ne 0,$$

and

$$c^*_{0,0,3,4}(\nu)=\nu^3(2\nu-48\nu-24+48\nu+32-8-2\nu)=0,$$
$$c^*_{-\nu,0,3,4}(\nu)=\nu^3(-4\nu+6\nu-2\nu)=0,$$
$$c^*_{\nu,0,3,4}(\nu)=\nu^3(8\nu+24\nu-168\nu-96+90\nu+120+54\nu-24-8\nu)=0,$$
$$c^*_{-2\nu,0,3,4}(\nu)=\nu^4(2-2)=0.$$

Therefore

(53)
$$c^{**}_{\alpha,0,3,4}(\nu)=0.$$

In view of (111) – (131) in [12],

(54)
$$c^{**}_{\alpha,0,4,4}(\nu)=\nu^3(\nu+\alpha)^2(-(\nu+\alpha)^3\alpha(2\nu+\alpha))+$$
$$(-2(2\nu+\alpha)(3\nu+\alpha))(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))+$$
$$(-(\nu+\alpha)(2\nu+\alpha)(4\nu+\alpha))\times$$
$$(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))+$$
$$(-2\nu(2\nu+\alpha)(\nu+\alpha)^2))\nu^3(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2)+$$
$$(-\nu^2(\nu+\alpha)^3)(-\nu^3\alpha(2\nu+\alpha))=\nu^3(2\nu+\alpha)(\nu+\alpha)^2c^*_{\alpha,0,4,4}(\nu),$$

where

(55)
$$c^*_{\alpha,0,4,4}(\nu)=-(\nu+\alpha)^3\alpha+$$
$$2(2\nu+\alpha)(3\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha)+$$
$$(4\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2)+$$
$$(-2\nu(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2))+$$

$$\nu^2(\nu+\alpha)\alpha = \sum_{k=0}^{4} c^{\vee}_{0,4,4,k}(\nu)\alpha^k.$$

Clearly,

$$c^{\vee}_{0,4,4,4}(\nu) = -1+1 = 0,\ \deg_\alpha(c^*_{\alpha,0,4,4}(\nu)) \le 3,\ \text{ if } c^*_{\alpha,0,4,4}(\nu) \ne 0,$$

and

$$c^*_{0,0,4,4}(\nu) = \nu^4(24-32+8) = 0,$$
$$c^*_{-\nu,0,4,4}(\nu) = \nu^3(-4+6-2) = 0,$$
$$c^*_{\nu,0,4,4}(\nu) = \nu^3(-8\nu+96\nu+24-90\nu-30+6+2\nu) = 0,$$
$$c^*_{-2\nu,0,2,4}(\nu) = \nu^4(-2+2) = 0.$$

Therefore

(56) $$c^{**}_{\alpha,0,4,4}(\nu) = 0.$$

It follows from (11) – (56) that

(57) $$S^{**}_{\alpha,0}(z;\nu)V^{**}_{\alpha,0}(z;-\nu-\alpha) +$$
$$V^{**}_{\alpha,0}(z;\nu)S^{**}_{\alpha,0}(z;-\nu-\alpha) = 0E_4.$$

Let

(58) $$d^{**}_{\alpha,0,i,k}(\nu) = \sum_{j=1}^{4} v^{**}_{\alpha,i,j}(\nu)v^{**}_{\alpha,j,k}(-\nu-\alpha),$$

where $\{i,\ k\} \subset \{1,\ 2,\ 3,\ 4\}$. In view of (111) – (131) in [12],

(59) $$d^{**}_{\alpha,0,1,1}(\nu) =$$
$$(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha)\times$$
$$(-\nu^2(2\nu+\alpha)(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha))+$$
$$2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)\times$$
$$(-\nu^2(\nu+\alpha)(2\nu+\alpha)(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$(-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\times$$
$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$2(2\nu+\alpha)(3\nu+2\alpha)\times$$
$$(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha)) =$$
$$\nu^2(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,1,1}(\nu),$$

where

(60) $$d^*_{\alpha,0,1,1}(\nu) =$$
$$(8\nu^2+(6+5\alpha)\nu+4\alpha)\times$$
$$(-(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha))+$$

$$2(-5\nu^2-6\nu-4\alpha+2\alpha^2)\times$$
$$(-(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$(-(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\times$$
$$(-(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$2(3\nu+2\alpha)\times$$
$$(-(\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha))=$$
$$\sum_{k=0}^{4} d^{\vee}_{0,1,1,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,1,1,4}(\nu)=-8+8=0,$$
$$\deg_\alpha(d^*_{\alpha,0,1,1}(\nu))\leq 3,\ \text{if } d^*_{\alpha,0,1,1}(\nu)\neq 0,$$

and

$$d^*_{0,0,1,1}(\nu)=-\nu^2(8\nu+6)(8\nu-6)+2\nu^2(5\nu+6)(6\nu-4)+$$
$$\nu^2(4\nu-6)(4\nu-2)-12\nu^2=$$
$$\nu^2(-64\nu^2+36+60\nu^2+32\nu-48+16\nu^2-32\nu+12\nu^2)=0,$$
$$d^*_{-\nu,0,1,1}(\nu)=-4\nu(3\nu^2+2\nu)+$$
$$(-3\nu^2-2\nu)(-6\nu)+(-3\nu^2-2\nu)(2\nu)=0,$$
$$d^*_{\nu,0,1,1}(\nu)=-\nu^2((13\nu+10)(22\nu-8)+2(3\nu+10)(16\nu-5))+$$
$$\nu^2((27\nu-10)(10\nu-2)-20(4\nu^2+\nu)=\nu^4(-286+96+270-80)+$$
$$\nu^3(-220+104+320-30-54-100-20))+\nu^2(80-100+20)=0,$$
$$d^*_{-2\nu,0,1,1}(\nu)=\nu^2(-4(u+1)^2+2(3\nu+2)(2\nu+2))+$$
$$\nu^2(-(6\nu+2)(2\nu+2)+4(\nu^2+\nu))=4\nu^2(\nu+1)(-\nu-1+3\nu+2-3\nu-1+\nu)=0.$$

Therefore

(61) $$d^{**}_{\alpha,0,1,1}(\nu)=0.$$

In view of (111) – (131) in [12],

(62) $$d^{**}_{\alpha,0,2,1}(\nu)=$$

$$(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))\times$$

$$(-\nu^2(2\nu+\alpha)(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha))+$$
$$(-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))\times$$
$$(-\nu^2(\nu+\alpha)(2\nu+\alpha)(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$\nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)\times$$
$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$(-\nu(2\nu+\alpha)(4\nu+3\alpha))\times$$

$$(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha)) =$$
$$\nu^3(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,2,1}(\nu),$$

where

(63)
$$d^*_{\alpha,0,2,1}(\nu) =$$

$$(6\nu^2+4(1+\alpha)\nu+3\alpha)\times$$
$$(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha))+$$
$$(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2)\times$$
$$(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha)+$$
$$(-(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2))\times$$
$$(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$((4\nu+3\alpha))\times$$
$$((\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha)) =$$
$$\sum_{k=0}^{4} d^{\vee}_{0,2,1,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,2,1,4}(\nu) = 6-6 = 0,$$
$$\deg_\alpha(d^*_{\alpha,0,2,1}(\nu)) \le 3, \text{ if } d^*_{\alpha,0,2,1}(\nu) \ne 0,$$

and

$$d^*_{0,0,2,1}(\nu) = \nu^2((6\nu+4)(8\nu-6)-8(\nu+1)(6\nu-4))-$$
$$\nu^2(2\nu-4)(4\nu-2)+8\nu^2 = \nu^4(48-48-8+8)+$$
$$\nu^3(-4-16+16+4)+\nu^2(-24+32-48) = 0,$$
$$d^*_{-\nu,0,2,1}(\nu) = \nu^2(2\nu+1)(-4+6-2) = 0$$
$$d^*_{\nu,0,2,1}(\nu) = \nu^2((10\nu+7)(22\nu-8)-(6\nu+14)(16\nu-5))-$$
$$\nu^2((18\nu-7)(10\nu-2)+14(4\nu^2+\nu) = \nu^4(220-96-180+56)+$$
$$\nu^3(74-194+106+14)+\nu^2(-56+70-14) = 0,$$
$$d^*_{-2\nu,0,2,1}(\nu) = \nu^2(4((\nu+1)^2-4(3\nu+2)(\nu+1))+$$
$$\nu^2(4(3\nu+1)(\nu+1)-4\nu^3(\nu+1) = 4\nu^2(\nu+1-3\nu-2+3\nu+1-\nu) = 0.$$

Therefore

(64)
$$d^{**}_{\alpha,0,2,1}(\nu) = 0.$$

In view of (111) – (131) in [12],

(65)
$$d^{**}_{\alpha,0,3,1}(\nu) =$$

$$\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)\times$$
$$(-\nu^2(2\nu+\alpha)(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha))+$$

$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$(-\nu^2(\nu+\alpha)(2\nu+\alpha)(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$(-\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$2\nu^2(2\nu+\alpha)(\nu+\alpha)\times$$
$$(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha))=$$
$$\nu^4(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,3,1}(\nu),$$

where

$$d^*_{\alpha,0,3,1}(\nu)= \tag{66}$$
$$(4\nu^2+(2+3\alpha)\nu+2\alpha)\times$$
$$(-(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha))+$$
$$2(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$(-(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$(-(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$(-(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$2(-(\nu+\alpha)^2(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha))=$$
$$\sum_{k=0}^{4} d^\vee_{0,3,1,k}(\nu)\alpha^k.$$

Clearly,

$$d^\vee_{0,3,1,4}(\nu)=-12+16-4=0,$$
$$\deg_\alpha(d^*_{\alpha,0,3,1}(\nu))\le 3,\ \text{ if } d^*_{\alpha,0,3,1}(\nu)\ne 0,$$

and

$$d^*_{0,0,3,1}(\nu)=$$
$$\nu^2((4\nu+2)(-8\nu+6)+2(-3\nu-2)(-6\nu+4))-8\nu+4-4\nu^2)=$$
$$\nu^4(-32+36-4)+\nu^3(8-8)+\nu^2(12-16+4)=0,$$
$$d^*_{-\nu,0,3,1}(\nu)=4\nu^3-6\nu^3+2\nu^3=0,$$
$$d^*_{\nu,0,3,1}(\nu)=$$
$$\nu^2((7\nu+4)(-22\nu+8)+(-6\nu-8)(-16\nu+5))+$$
$$\nu^2((-9\nu+4)(-10\nu+2)+8(-4\nu^2-\nu)=\nu^4(-154+96+90-32)+$$
$$\nu^3(-88+56+128-30-18-40-8)+nu^2(32-40+8)=0,$$
$$d^*_{-2\nu,0,3,1}(\nu)=$$
$$\nu^2(-4(\nu+1)^2+(2\nu+4)(2\nu+2)+$$
$$\nu^2(-6\nu-2\nu)(2\nu+2)+(4\nu^2+4\nu))=$$
$$4\nu^2(\nu+1)(-\nu-1+\nu+2-3\nu-1+\nu)=0.$$

Therefore

$$d^{**}_{\alpha,0,3,1}(\nu) = 0. \tag{67}$$

In view of (111) – (131) in [12],

$$d^{**}_{\alpha,0,4,1}(\nu) = \tag{68}$$

$$(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))\times$$
$$(-\nu^2(2\nu+\alpha)(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha)))+$$
$$(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))\times$$
$$(-\nu^2(\nu+\alpha)(2\nu+\alpha)(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$\nu^3(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2)\times$$
$$(-\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha)))+$$
$$(-\nu^3\alpha(2\nu+\alpha))\times$$
$$(-\nu^2(\nu+\alpha)^3(2\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha)) =$$
$$\nu^5(2\nu+\alpha)^2(\nu+\alpha)^2 d^{*}_{\alpha,0,4,1}(\nu),$$

where

$$d^{*}_{\alpha,0,4,1}(\nu) = \tag{69}$$

$$(2\nu^2+2\alpha\nu+\alpha))\times$$
$$(8(\nu+\alpha)^2-(6+5\alpha)(\nu+\alpha)+4\alpha)))+$$
$$(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))\times$$
$$(6(\nu+\alpha)^2-4(1+\alpha)(\nu+\alpha)+3\alpha))+$$
$$(-2\nu^2-\alpha+2\alpha^2)\times$$
$$(-(4(\nu+\alpha)^2-(2+3\alpha)(\nu+\alpha)+2\alpha))+$$
$$(\alpha)((\nu+\alpha)(2(\nu+\alpha)^2-2\alpha(\nu+\alpha)+\alpha)) =$$
$$\sum_{k=0}^{4} d^{\vee}_{0,4,1,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,4,1,4}(\nu) = 2-2 = 0,$$
$$\deg_\alpha(d^{*}_{\alpha,0,4,1}(\nu)) \leq 3,\ \text{if } d^{*}_{\alpha,0,4,1}(\nu) \neq 0,$$

and

$$d^{*}_{0,0,4,1}(\nu) = 2\nu^3(8\nu-6)+(-4\nu^3(6\nu-4)+2\nu^3(4\nu-2) =$$
$$\nu^4(16-24+8)+\nu^3(-12+16-4) = 0,$$
$$d^{*}_{-\nu,0,3,1}(\nu) = 4\nu^2-6\nu^2+2\nu^2 = 0$$
$$d^{*}_{\nu,0,4,1}(\nu) = \nu^2((4\nu+1)(22\nu-8)+(-6\nu-2)(16\nu-5))+$$
$$\nu^2((10\nu-2)+8\nu^2+2\nu)) = \nu^4(88-96+8)+$$
$$\nu^3(-10-2+10+2)+\nu^2(-8+10-2) = 0,$$

$$d^*_{-2\nu,0,4,1}(\nu) = 4\nu^2(\nu+1)^2 + \nu^2(6\nu+4)(-2\nu-2)+$$
$$\nu^2(6\nu+2)(2\nu+2) + 2\nu^2(-2\nu^2-2\nu) =$$
$$\nu^2(\nu+1)(4\nu+4-12\nu-8+12\nu+4-4\nu) = 0.$$

Therefore

$$d^{**}_{\alpha,0,4,1}(\nu) = 0. \tag{70}$$

In view of (111) – (131) in [12],

$$d^{**}_{\alpha,0,1,2}(\nu) = \tag{71}$$

$$(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha)\times$$
$$2\nu(2\nu+\alpha)(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)\times$$
$$\nu(\nu+\alpha)(2\nu+\alpha)(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\times$$
$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$2(2\nu+\alpha)(3\nu+2\alpha)\times$$
$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2) =$$
$$\nu(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,1,2}(\nu),$$

where

$$d^*_{\alpha,0,1,2}(\nu) = \tag{72}$$

$$2(8\nu^2+(6+5\alpha)\nu+4\alpha)\times$$
$$(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$2(-5\nu^2-6\nu-4\alpha+2\alpha^2)\times$$
$$(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\times$$
$$2(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$2(3\nu+2\alpha)\times$$
$$(\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2) =$$
$$\sum_{k=0}^{4} d^{\vee}_{0,1,2,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,1,2,4}(\nu) = -8+8 = 0,$$
$$\deg_\alpha(d^*_{\alpha,0,1,2}(\nu)) \le 3, \text{ if } d^*_{\alpha,0,1,2}(\nu) \neq 0,$$

and

$$d^*_{0,0,1,2}(\nu) = \nu^2(2(8\nu+6)(-5\nu+6)+2(-5\nu-6)(-8\nu+8))+$$

$$\nu^2(-2(4\nu-6)(-3\nu+2)-24\nu^2) = \nu^4(-80+80+24-24)+$$
$$\nu^3(36+16-36-16)+\nu^2(72-96+24) = 0,$$
$$d^*_{-\nu,0,1,2}(\nu) =$$
$$\nu^2(6\nu+4)(2\nu+4)+(-6\nu-4)(3\nu+6)+(3\nu+2)(2\nu+4)) =$$
$$\nu^2(\nu+2)(12\nu+8-18\nu-12+6\nu+4) = 0,$$
$$d^*_{\nu,0,1,2}(\nu) =$$
$$2\nu^2((13\nu+10)(-18\nu+8)+(-3\nu-10)(-27\nu+10))+$$
$$2\nu^2((-27\nu+10)(-9\nu+2)+10\nu^2(-9\nu^2-2\nu) =$$
$$2\nu^4(-234+81+243-90)+2\nu^3(-76+240-90-54-20)+2\nu^2(80-100+20) = 0,$$
$$d^*_{-2\nu,0,1,2}(\nu) =$$
$$2\nu^2((-2\nu-2)(3\nu+2)+(3\nu+2)(6\nu+4))+$$
$$2\nu^2((-6\nu-2)(3\nu+2)+6\nu^2+4\nu) =$$
$$2\nu^4(3\nu+2)(-2\nu-2+6\nu+4-6\nu-2+2\nu) = 0.$$

Therefore

$$d^{**}_{\alpha,0,1,2}(\nu) = 0. \tag{73}$$

In view of (111) – (131) in [12],

$$d^{**}_{\alpha,0,2,2}(\nu) = \tag{74}$$
$$(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))\times$$
$$2\nu(2\nu+\alpha)(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$(-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))\times$$
$$\nu(\nu+\alpha)(2\nu+\alpha)(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$\nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)\times$$
$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(-\nu(2\nu+\alpha)(4\nu+3\alpha))\times$$
$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2) =$$
$$\nu^2(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,2,2}(\nu),$$

where

$$d^*_{\alpha,0,2,2}(\nu) = \tag{75}$$
$$(-(6\nu^2+4(1+\alpha)\nu+3\alpha))\times$$
$$2(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$(-(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))\times$$
$$(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)\times$$

$$2(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(-(4\nu+3\alpha))\times$$
$$(\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)=$$
$$\sum_{k=0}^{4} d^{\vee}_{0,2,2,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,2,2,4}(\nu)=12-12=0,$$
$$\deg_\alpha(d^*_{\alpha,0,2,2}(\nu))\le 3,\ \text{ if } d^*_{\alpha,0,2,1}(\nu)\neq 0,$$

and

$$d^*_{0,0,2,2}(\nu)=\nu^2(-2(6\nu+4)(-5\nu+6)+8(\nu+1)(-8\nu+8))+$$
$$\nu^2(2\nu-4)(-6\nu+4)+16\nu^4=\nu^4(60-64-12+16)+$$
$$\nu^3(-32+32)+\nu^2(-48+64-16)=0,$$
$$d^*_{-\nu,0,2,2}(\nu)=\nu^2(-2(2\nu+1))(2\nu+4)+(4\nu+2)(3\nu+6)+$$
$$\nu^2(-2\nu-1)(2\nu+4)=\nu^2(2\nu+1)(-4\nu-8+6\nu+12-2\nu-4)=0,$$
$$d^*_{\nu,0,2,2}(\nu)=$$
$$2\nu^2((-10\nu-7)(-18\nu+8)+(3\nu+7)(-27\nu+10)+$$
$$2\nu^2((18\nu-7)(-9\nu+2)+(-14\nu^2)(-9\nu^2-2\nu))=$$
$$2\nu^4(180-81-162+63)+2\nu^3(46-159+99+14)+2\nu^2(-56+70-14)=0,$$
$$d^*_{-2\nu,0,2,2}(\nu)=$$
$$2\nu^2((2\nu+2)(3\nu+2)-(3\nu+2)(6\nu+4))+$$
$$2\nu^2((6\nu+2)(3\nu+2)-(6\nu^2+4\nu))=$$
$$4\nu^2(3\nu+2)(\nu+1-3\nu-2+3\nu+1-\nu)=0.$$

Therefore

(76) $$d^*_{\alpha,0,2,2}(\nu)=0.$$

In view of (111) – (131) in [12],

(77) $$d^{**}_{\alpha,0,3,2}(\nu)=$$

$$\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)\times$$
$$2\nu(2\nu+\alpha)(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$\nu(\nu+\alpha)(2\nu+\alpha)(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$2\nu^2(2\nu+\alpha)(\nu+\alpha)\times$$

$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)=$$
$$\nu^3(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,3,1}(\nu),$$

where

(78)
$$d^*_{\alpha,0,3,2}(\nu)=$$
$$(4\nu^2+(2+3\alpha)\nu+2\alpha)\times$$
$$2(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$2(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$2(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$2(\nu+\alpha)^2(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)=$$
$$\sum_{k=0}^{4} d^{\vee}_{0,3,2,k}(\nu)\alpha^k.$$

Clearly,
$$d^{\vee}_{0,3,1,4}(\nu)=-8+8=0,$$
$$\deg_\alpha(d^*_{\alpha,0,3,2}(\nu))\le 3,\ \text{if}\ d^*_{\alpha,0,3,2}(\nu)\neq 0,$$

and
$$d^*_{0,0,3,2}(\nu)=\nu^2((4\nu+2)(-10\nu+12)+(-6\nu-4)(-8\nu+8))+$$
$$-12\nu^3+8\nu^2-8\nu^4=\nu^4(-40+48-8)+\nu^3(28-16-12)\nu^2(24-32+8)=0,$$
$$d^*_{-\nu,0,3,2}(\nu)=2\nu^3(\nu+2)-6\nu^3(\nu+2)+4\nu^3(\nu+2)=0,$$
$$d^*_{\nu,0,3,2}(\nu)=\nu^2((14\nu+8)(-18\nu+8)+(-6\nu-8)(-27\nu+10))+$$
$$\nu^2((-9\nu+4\nu)(-18\nu+4)+8\nu^2(-9\nu^2-2\nu)=\nu^4(-252+162+162-72)+$$
$$\nu^3(-32+156-108-16)+\nu^2(64-80+16)=0,$$
$$d^*_{-2\nu,0,3,2}(\nu)=\nu^2((-2\nu-2)(6\nu+4)+(6\nu+4)(6\nu+4))+$$
$$\nu^2(-(6\nu+2)(6\nu+4)+2(6\nu^2+4\nu))=$$
$$\nu^2(6\nu+4)(-2\nu-2+6\nu+4-6\nu-2+2\nu)=0.$$

Therefore

(79)
$$d^{**}_{\alpha,0,3,2}(\nu)=0.$$

In view of (111) – (131) in [12],

(80)
$$d^{**}_{\alpha,0,4,2}(\nu)=$$
$$(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))\times$$
$$2\nu(2\nu+\alpha)(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))\times$$

$$\nu(\nu+\alpha)(2\nu+\alpha)(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$\nu^3(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2)\times$$
$$2\nu(\nu+\alpha)^2(2\nu+\alpha)(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(-\nu^3\alpha(2\nu+\alpha))\times$$
$$\nu(\nu+\alpha)^3(2\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)=$$
$$\nu^4(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,4,2}(\nu),$$

where

(81)
$$d^*_{\alpha,0,4,2}(\nu)=$$
$$(-(2\nu^2+2\alpha\nu+\alpha))\times$$
$$2(-5(\nu+\alpha)^2+6(\nu+\alpha)-4\alpha+2\alpha^2)+$$
$$(-(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))\times$$
$$(-8(\nu+\alpha)^2+8(\nu+\alpha)+\alpha(\nu+\alpha)-6\alpha+3\alpha^2)+$$
$$(-2\nu^2-\alpha+2\alpha^2)\times$$
$$2(-3(\nu+\alpha)^2+2(\nu+\alpha)+\alpha(\nu+\alpha)-2\alpha+\alpha^2)+$$
$$(-\alpha(\nu+\alpha)(-4(\nu+\alpha)^2+3\alpha(\nu+\alpha)-2\alpha+\alpha^2)=$$
$$\sum_{k=0}^{4} d^{\vee}_{0,4,2,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,4,2,4}(\nu)=4-4=0,$$
$$\deg_\alpha(d^*_{\alpha,0,4,2}(\nu))\le 3,\ \text{ if } d^*_{\alpha,0,4,2}(\nu)\ne 0,$$

and

$$d^*_{0,0,4,2}(\nu)=\nu^3(20\nu-12-32\nu+32+12\nu)=0,$$
$$d^*_{-\nu,0,4,2}(\nu)=\nu^3(4\nu+8-6\nu-12+2\nu+4)=0,$$
$$d^*_{\nu,0,4,2}(\nu)=\nu^2((-8\nu-2)(-18\nu+8)+(6\nu+2)(-27\nu+10))+$$
$$\nu^2(18\nu-4+18\nu^2+4\nu)=\nu^4(144-162+18)+$$
$$\nu^3(-28+6+18+4)+\nu^2(-16+20-4)=0,$$
$$d^*_{-2\nu,0,4,2}(\nu)=\nu^2((2\nu+2)(6\nu+4)+(-6\nu-4)(6\nu+4)+$$
$$\nu^2((6\nu+2)(6\nu+4)-2(6\nu^2+4\nu))=\nu^2(6\nu+4)(2nu+2-6\nu-4+6\nu+2-2\nu)=0.$$

Therefore

(82)
$$d^{**}_{\alpha,0,4,2}(\nu)=0.$$

In view of (111) – (131) in [12],

(83)
$$d^{**}_{\alpha,0,1,3}(\nu)=$$
$$(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha)\times$$

$$(2\nu+\alpha)(4(\nu+\alpha)^2+6(\nu+\alpha)-15(\nu+\alpha)\alpha-4\alpha+8\alpha^2)+$$
$$2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)\times$$
$$(\nu+\alpha)(2\nu+\alpha)(2(\nu+\alpha)^2+4(\nu+\alpha)-10\alpha(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$(-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\times$$
$$(\nu+\alpha)^2(2\nu+\alpha)(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$2(2\nu+\alpha)(3\nu+2\alpha)\times$$
$$(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)=$$
$$(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,1,3}(\nu),$$

where

(84)
$$d^*_{\alpha,0,1,3}(\nu)=$$
$$(8\nu^2+(6+5\alpha)\nu+4\alpha)\times$$
$$(4(\nu+\alpha)^2+6(\nu+\alpha)-15(\nu+\alpha)\alpha-4\alpha+8\alpha^2)+$$
$$2(-5\nu^2-6\nu-4\alpha+2\alpha^2)\times$$
$$(2(\nu+\alpha)^2+4(\nu+\alpha)-10\alpha(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$(-(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\times$$
$$(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$2(3\nu+2\alpha)\times$$
$$(\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)=$$
$$\sum_{k=0}^{4} d^{\vee}_{0,1,3,k}(\nu)\alpha^k.$$

Clearly,
$$d^{\vee}_{0,1,3,4}(\nu)=-8+8=0,$$
$$\deg_\alpha(d^*_{\alpha,0,1,3}(\nu))\le 3,\ \text{if } d^*_{\alpha,0,1,3}(\nu)\ne 0,$$

and
$$d^*_{0,0,1,3}(\nu)=\nu^2((8\nu+6)(4\nu+6)+2(-5\nu-6)(2\nu+4))+$$
$$\nu^2(-2(4\nu-6)-12\nu^2)=$$
$$\nu^4(32-20-12)+\nu^3(-24+40-24+8)+\nu^2(36-48+12)=0,$$
$$d^*_{-\nu,0,1,3}(\nu)=$$
$$\nu^2((3\nu+2)(8\nu+4)+(-6\nu-4)(6\nu+3)+(3\nu+2)(4\nu+2))=$$
$$\nu^2(3\nu+2)(8\nu+4-12\nu-6+4\nu+2)=0,$$
$$d^*_{\nu,0,1,3}(\nu)=$$
$$\nu^2((13\nu+10)(-6\nu+8)+(-6\nu-20)(-6\nu+5))+$$
$$\nu^2((-27\nu+10)(-6\nu+2)-120\nu^2-20\nu)=\nu^4(-78+36+162-120)+$$
$$\nu^3(44+90-114-20)+\nu^2(80-100+20)=0,$$

$$d^*_{-2\nu,0,1,3}(\nu) =$$

$$\nu^2((-2\nu-2)(6\nu+2)+(6\nu+4)(6\nu+2))+$$

$$\nu^2((-6\nu-2)(6\nu+2)+12\nu^2+4\nu) =$$

$$\nu^2(6\nu+2)(-2\nu-2+6\nu+4-6nu-2+2\nu) = 0.$$

Therefore

(85) $$d^*_{\alpha,0,1,3}(\nu) = 0.$$

In view of (111) – (131) in [12],

(86) $$d^{**}_{\alpha,0,2,3}(\nu) =$$

$$(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2+4(1+\alpha)\nu+3\alpha))\times$$

$$(2\nu+\alpha)(4(\nu+\alpha)^2+6(\nu+\alpha)-15(\nu+\alpha)\alpha-4\alpha+8\alpha^2)+$$

$$(-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))\times$$

$$(\nu+\alpha)(2\nu+\alpha)(2(\nu+\alpha)^2+4(\nu+\alpha)-10\alpha(\nu+\alpha)-3\alpha+6\alpha^2)+$$

$$\nu(2\nu+\alpha)(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)\times$$

$$(\nu+\alpha)^2(2\nu+\alpha)(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$

$$(-\nu(2\nu+\alpha)(4\nu+3\alpha))\times$$

$$(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2) =$$

$$\nu(\nu+\alpha)^2(2\nu+\alpha)^2 d^*_{\alpha,0,2,3}(\nu),$$

where

(87) $$d^*_{\alpha,0,2,3}(\nu) =$$

$$(-(6\nu^2+4(1+\alpha)\nu+3\alpha))\times$$

$$(4(\nu+\alpha)^2+6(\nu+\alpha)-15(\nu+\alpha)\alpha-4\alpha+8\alpha^2)+$$

$$(-(-8\nu^2-8\nu-\alpha\nu-6\alpha+3\alpha^2))\times$$

$$(2(\nu+\alpha)^2+4(\nu+\alpha)-10\alpha(\nu+\alpha)-3\alpha+6\alpha^2)+$$

$$(2\nu^2-4\nu+10\alpha\nu-3\alpha+6\alpha^2)\times$$

$$(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$

$$(-(4\nu+3\alpha))\times$$

$$(\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2) =$$

$$\sum_{k=0}^{4} d^{\vee}_{0,2,3,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,2,3,4}(\nu) = -6+6 = 0,$$

$$\deg_\alpha(d^*_{\alpha,0,2,2}(\nu)) \leq 3, \text{ if } d^*_{\alpha,0,2,1}(\nu) \neq 0,$$

and

$$d^*_{0,0,2,3}(\nu) = \nu^2(-(6\nu+4)(4\nu+6)+8(\nu+1)(2\nu+4))+$$

$$\nu^2(2(2\nu-4))+8\nu^2) = \nu^4(-24+16+8)+\nu^3(-52+48+4)+\nu^2(-24+32-8) = 0,$$

$$d^*_{-\nu,0,2,3}(\nu) = \nu^2((-2\nu-1)(8\nu+4)+(4\nu+2)(6\nu+3))+$$
$$\nu^2((-2\nu-1)(4\nu+2)) = \nu^2(2\nu+1)^2(-4+6-2) = 0,$$
$$d^*_{\nu,0,2,3}(\nu) = \nu^2((-10\nu-7)(-6\nu+8)+(6\nu+14)(-6\nu+5))+$$
$$\nu^2((18\nu-7)(-6\nu+2)-14(-6\nu^2-\nu)) = \nu^4(60-36-108+84)+$$
$$\nu^3(-38-54+78+14)+\nu^2(-56+70-14) = 0,$$
$$d^*_{-2\nu,0,2,3}(\nu) = \nu^2((2\nu+2)(6\nu+2)+(-6\nu-4)(6\nu+2))+$$
$$\nu^2((6\nu+2)(6\nu+2)-2(6\nu^2+2\nu)) = \nu^2(6\nu+2)(2\nu+2-6\nu-4+6\nu+2-2\nu) = 0.$$

Therefore

(88) $$d^*_{\alpha,0,2,3}(\nu) = 0.$$

In view of (111) – (131) in [12],

(89) $$d^{**}_{\alpha,0,3,3}(\nu) =$$
$$\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2+(2+3\alpha)\nu+2\alpha)\times$$
$$(2\nu+\alpha)(4(\nu+\alpha)^2+6(\nu+\alpha)-15(\nu+\alpha)\alpha-4\alpha+8\alpha^2)+$$
$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$(\nu+\alpha)(2\nu+\alpha)(2(\nu+\alpha)^2+4(\nu+\alpha)-10\alpha(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$(-\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$(\nu+\alpha)^2(2\nu+\alpha)(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$2\nu^2(2\nu+\alpha)(\nu+\alpha)\times$$
$$(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2) =$$
$$\nu^2(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,3,3}(\nu),$$

where

(90) $$d^*_{\alpha,0,3,3}(\nu) =$$
$$(4\nu^2+(2+3\alpha)\nu+2\alpha)\times$$
$$(4(\nu+\alpha)^2+6(\nu+\alpha)-15(\nu+\alpha)\alpha-4\alpha+8\alpha^2)+$$
$$2(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$(2(\nu+\alpha)^2+4(\nu+\alpha)-10\alpha(\nu+\alpha)-3\alpha+6\alpha^2)+$$
$$(-(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$2(\nu+\alpha)^2(-2(\nu+\alpha)^2-\alpha+2\alpha^2) =$$
$$\sum_{k=0}^{4} d^{\vee}_{0,3,3,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,3,3,4}(\nu) = -4 + 4 = 0,$$

$$\deg_\alpha(d^*_{\alpha,0,3,2}(\nu)) \leq 3, \text{ if } d^*_{\alpha,0,3,3}(\nu) \neq 0,$$

and

$$d^*_{0,0,3,3}(\nu) = \nu^2((4\nu+2)(4\nu+6) + (-6\nu-4)(2\nu+4))+$$
$$4\nu^2 - 4\nu^4 = \nu^4(16-12-4) + \nu^3(32-32) + \nu^2(12-16+4) = 0,$$
$$d^*_{-nu,0,3,3}(\nu) = \nu^2(8\nu^2 + 4\nu - 12\nu^2 - 6\nu + 4\nu^2 + 2\nu) = 0,$$
$$d^*_{\nu,0,3,3}(\nu) = \nu^2((7\nu+4)(-6\nu+8) + (-6\nu-8)(-6\nu+5))+$$
$$\nu^2((-9\nu+4)(-6\nu+2) - 48\nu^2 - 8\nu) = \nu^4(-42+36+54-48)+$$
$$\nu^3(32+18-42-8) + \nu^2(32-40+8) = 0,$$
$$d^*_{-2\nu,0,3,3}(\nu) = \nu^2((-2\nu-2)(6\nu+2) + (6\nu+4)(6\nu+2))+$$
$$\nu^2(-(6\nu+2)(6\nu+2) + 2\nu(6\nu+2)) =$$
$$\nu^2(6\nu+2)(-2\nu-2+6\nu+4-6\nu-2+2\nu) = 0.$$

Therefore

$$d^*_{\alpha,0,3,3}(\nu) = 0. \tag{91}$$

In view of (111) – (131) in [12],

$$d^{**}_{\alpha,0,4,3}(\nu) = \tag{92}$$

$$(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))\times$$
$$(2\nu+\alpha)(4(\nu+\alpha)^2 + 6(\nu+\alpha) - 15(\nu+\alpha)\alpha - 4\alpha + 8\alpha^2)+$$
$$(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2 - 3\alpha\nu - 2\alpha + \alpha^2))\times$$
$$(\nu+\alpha)(2\nu+\alpha)(2(\nu+\alpha)^2 + 4(\nu+\alpha) - 10\alpha(\nu+\alpha) - 3\alpha + 6\alpha^2)+$$
$$\nu^3(2\nu+\alpha)(-2\nu^2 - \alpha + 2\alpha^2)\times$$
$$(\nu+\alpha)^2(2\nu+\alpha)(2(\nu+\alpha) - 5\alpha(\nu+\alpha) - 2\alpha + 4\alpha^2)+$$
$$(-\nu^3\alpha(2\nu+\alpha))\times$$
$$(\nu+\alpha)^3(2\nu+\alpha)(-2(\nu+\alpha)^2 - \alpha + 2\alpha^2) =$$
$$\nu^3(2\nu+\alpha)^2(\nu+\alpha)^2 d^*_{\alpha,0,4,3}(\nu),$$

where

$$d^*_{\alpha,0,4,3}(\nu) = \tag{93}$$

$$(-(2\nu^2+2\alpha\nu+\alpha))\times$$
$$(4(\nu+\alpha)^2 + 6(\nu+\alpha) - 15(\nu+\alpha)\alpha - 4\alpha + 8\alpha^2)+$$
$$(-(-4\nu^2 - 3\alpha\nu - 2\alpha + \alpha^2))\times$$
$$(2(\nu+\alpha)^2 + 4(\nu+\alpha) - 10\alpha(\nu+\alpha) - 3\alpha + 6\alpha^2)+$$
$$(-2\nu^2 - \alpha + 2\alpha^2)\times$$

$$(2(\nu+\alpha)-5\alpha(\nu+\alpha)-2\alpha+4\alpha^2)+$$
$$(-\alpha(\nu+\alpha)(-2(\nu+\alpha)^2-\alpha+2\alpha^2)=$$
$$\sum_{k=0}^{4} d^{\vee}_{0,4,3,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,4,3,4}(\nu)=2-2=0,$$
$$\deg_\alpha(d^*_{\alpha,0,4,3}(\nu))\le 3,\ \text{ if } d^*_{\alpha,0,4,3}(\nu)\ne 0,$$

and

$$d^*_{0,0,4,3}(\nu)=\nu^2(-8\nu^2-12\nu+8\nu^2+16\nu-4\nu)=0,$$
$$d^*_{-\nu,0,4,3}(\nu)=\nu^2(8\nu+4-12\nu-6+4\nu+2)=0,$$
$$d^*_{\nu,0,4,3}(\nu)=\nu^2((-4\nu-1)(-6\nu+8)+(6\nu+2)(-6\nu+5))+$$
$$\nu^2(6\nu-2+12\nu^2+2\nu)=\nu^4(24-36+12)+\nu^3(-26+18+8)+\nu^2(-8+10-2)=0,$$
$$d^*_{-2\nu,0,4,2}(\nu)=\nu^2((2\nu+2)(6\nu+2)+(-6\nu-4)(6\nu+2)+$$
$$\nu^2((6\nu+2)(6\nu+2)-2(6\nu^2+2\nu))=$$
$$\nu^2(6\nu+2)(2\nu+2-6\nu-4+6\nu+2-2\nu)=0.$$

Therefore

$$d^{**}_{\alpha,0,4,3}(\nu)=0. \tag{94}$$

In view of (111) – (131) in [12],

$$d^{**}_{\alpha,0,1,4}(\nu)= \tag{95}$$
$$(\nu+\alpha)^2(2\nu+\alpha)(8\nu^2+(6+5\alpha)\nu+4\alpha)(-2(2\nu+\alpha)(-3\nu-\alpha))+$$
$$2(\nu+\alpha)(2\nu+\alpha)(-5\nu^2-6\nu-4\alpha+2\alpha^2)(-(\nu\alpha)(2\nu+\alpha)(-4\nu-\alpha))+$$
$$(-(2\nu+\alpha)(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))2\nu(2\nu+\alpha)(\nu+\alpha)^2+$$
$$2(2\nu+\alpha)(3\nu+2\alpha)(-(\nu+\alpha)^3\alpha(2\nu+\alpha))=(2\nu+\alpha)^2(\nu+\alpha)^2d^*_{\alpha,0,1,4}(\nu),$$

where

$$d^*_{\alpha,0,1,4}(\nu)= \tag{96}$$
$$(8\nu^2+(6+5\alpha)\nu+4\alpha)(-2(-3\nu-\alpha))+$$
$$2(-5\nu^2-6\nu-4\alpha+2\alpha^2)(-(-4\nu-\alpha))+$$
$$2(-(4\nu^2-6\nu+15\nu\alpha-4\alpha+8\alpha^2))\nu(2\nu+\alpha)(\nu+\alpha)^2+$$
$$2(3\nu+2\alpha)(-(\nu+\alpha)\alpha)=\sum_{k=0}^{3} d^{\vee}_{0,1,3,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,1,4,4}(\nu)=4-4=0,$$
$$\deg_\alpha(d^*_{\alpha,0,1,4}(\nu))\le 2,\ \text{ if } d^*_{\alpha,0,1,4}(\nu)\ne 0,$$

and

$$d^*_{0,0,1,4}(\nu) = \nu^2(48\nu + 36 - 40\nu - 48 - 8\nu + 12) = 0,$$
$$d^*_{-\nu,0,1,4}(\nu) = \nu^2(12\nu + 8 - 18\nu - 12 + 6\nu + 4) = 0,$$
$$d^*_{\nu,0,1,4}(\nu) =$$
$$\nu^2(104\nu + 80 - 30\nu - 100 - 54\nu + 20 - 20\nu) = 0.$$

Therefore

(97) $$d^*_{\alpha,0,1,4}(\nu) = 0.$$

In view of (111) – (131) in [12],

(98) $$d^{**}_{\alpha,0,2,4}(\nu) =$$
$$(-\nu(\nu+\alpha)^2(2\nu+\alpha)(6\nu^2 + 4(1+\alpha)\nu + 3\alpha))(-2(2\nu+\alpha)(-3\nu-\alpha))+$$
$$(-\nu(\nu+\alpha)(2\nu+\alpha)(-8\nu^2 - 8\nu - \alpha\nu - 6\alpha + 3\alpha^2))\times$$
$$(-(\nu\alpha)(2\nu+\alpha)(-4\nu-\alpha))+$$
$$2\nu(2\nu+\alpha)(2\nu^2 - 4\nu + 10\alpha\nu - 3\alpha + 6\alpha^2)\nu(2\nu+\alpha)(\nu+\alpha)^2+$$
$$(-\nu(2\nu+\alpha)(4\nu+3\alpha))(-(\nu+\alpha)^3\alpha(2\nu+\alpha)) = \nu(\nu+\alpha)^2(2\nu+\alpha)^2 d^*_{\alpha,0,2,4}(\nu),$$

where

(99) $$d^*_{\alpha,0,2,4}(\nu) =$$
$$(-(6\nu^2 + 4(1+\alpha)\nu + 3\alpha))(-2(-3\nu-\alpha))+$$
$$(-(-8\nu^2 - 8\nu - \alpha\nu - 6\alpha + 3\alpha^2))(-(-4\nu-\alpha))+$$
$$2\nu(2\nu^2 - 4\nu + 10\alpha\nu - 3\alpha + 6\alpha^2)+$$
$$(-(4\nu+3\alpha))(-(\nu+\alpha)\alpha) = \sum_{k=0}^{3} d^{\vee}_{0,2,4,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,2,4,3}(\nu) = -3 + 3 = 0,$$
$$\deg_\alpha(d^*_{\alpha,0,2,4}(\nu)) \le 2, \text{ if } d^*_{\alpha,0,2,4}(\nu) \ne 0,$$

and

$$d^*_{0,0,2,4}(\nu) = \nu^2(-36\nu - 24 + 32\nu + 32 + 4\nu - 8) = 0,$$
$$d^*_{-\nu,0,2,4}(\nu) = \nu^2(-8\nu - 4 + 12\nu + 6 - 4\nu - 2) = 0,$$
$$d^*_{\nu,0,2,4}(\nu) = \nu^2(-80\nu - 56 + 30\nu + 70 + 36\nu - 14 + 14\nu = 0.$$

Therefore

(100) $$d^*_{\alpha,0,2,4}(\nu) = 0.$$

In view of (111) – (131) in [12],

(101) $$d^{**}_{\alpha,0,3,4}(\nu) =$$
$$\nu^2(\nu+\alpha)^2(2\nu+\alpha)(4\nu^2 + (2+3\alpha)\nu + 2\alpha)\times$$

$$(-2(2\nu+\alpha)(-3\nu-\alpha))+$$
$$2\nu^2(\nu+\alpha)(2\nu+\alpha)(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)\times$$
$$(-(\nu\alpha)(2\nu+\alpha)(-4\nu-\alpha))+$$
$$(-\nu^2(2\nu+\alpha)(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))\times$$
$$2\nu(2\nu+\alpha)(\nu+\alpha)^2+$$
$$2\nu^2(2\nu+\alpha)(\nu+\alpha)(-(\nu+\alpha)^3\alpha(2\nu+\alpha))=\nu^2(2\nu+\alpha)^2(\nu+\alpha)^2d^*_{\alpha,0,3,4}(\nu),$$

where

(102)
$$d^*_{\alpha,0,3,4}(\nu)=$$
$$(4\nu^2+(2+3\alpha)\nu+2\alpha)(-2(-3\nu-\alpha))+$$
$$2(-3\nu^2-2\nu-\alpha\nu-2\alpha+\alpha^2)(-(-4\nu-\alpha))+$$
$$2\nu(-(-2\nu+5\alpha\nu-2\alpha+4\alpha^2))+$$
$$(-2(\nu+\alpha)^2\alpha)=\sum_{k=0}^{4}d^{\vee}_{0,3,3,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,3,4,4}(\nu)=-2+2=0,$$
$$\deg_\alpha(d^*_{\alpha,0,3,4}(\nu))\le 2,\ \text{ if } d^*_{\alpha,0,3,4}(\nu)\ne 0,$$

and

$$d^*_{0,0,3,4}(\nu)=\nu^2(24\nu+12-24\nu-16+4)=0,$$
$$d^*_{-\nu,0,3,4}(\nu)=\nu^2(4\nu-6\nu+2\nu)=0,$$
$$d^*_{\nu,0,3,4}(\nu)=\nu^2(56\nu+32-30\nu-40-18\nu+8-8\nu)=0.$$

Therefore

(103)
$$d^*_{\alpha,0,3,4}(\nu)=0.$$

In view of (111) – (131) in [12],

(104)
$$d^{**}_{\alpha,0,4,4}(\nu)=$$
$$(-\nu^3(\nu+\alpha)^2(2\nu+\alpha)(2\nu^2+2\alpha\nu+\alpha))(-2(2\nu+\alpha)(-3\nu-\alpha))+$$
$$(-\nu^3(\nu+\alpha)(2\nu+\alpha)(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))(-(\nu\alpha)(2\nu+\alpha)(-4\nu-\alpha))+$$
$$\nu^3(2\nu+\alpha)(-2\nu^2-\alpha+2\alpha^2)2\nu(2\nu+\alpha)(\nu+\alpha)^2+$$
$$(-\nu^3\alpha(2\nu+\alpha))(-(\nu+\alpha)^3\alpha(2\nu+\alpha))=$$
$$\nu^3(2\nu+\alpha)^2(\nu+\alpha)^2d^*_{\alpha,0,4,4}(\nu),$$

where

(105)
$$d^*_{\alpha,0,4,4}(\nu)=(-(2\nu^2+2\alpha\nu+\alpha))(-2(-3\nu-\alpha))+$$
$$(-(-4\nu^2-3\alpha\nu-2\alpha+\alpha^2))(-(-4\nu-\alpha))+$$

$$2\nu(-2\nu^2-\alpha+2\alpha^2)+(\nu+\alpha)\alpha^2=\sum_{k=0}^{3} d^{\vee}_{0,4,4,k}(\nu)\alpha^k.$$

Clearly,

$$d^{\vee}_{0,4,3,4}(\nu)=1-1=0, \deg_\alpha(d^*_{\alpha,0,4,4}(\nu))\le 2,\ \text{if}\ d^*_{\alpha,0,4,4}(\nu)\ne 0,$$

and

$$d^*_{0,0,4,4}(\nu)=\nu^3(-12+16-4)=0,$$
$$d^*_{-\nu,0,4,3}(\nu)=\nu^3(4-6+2)=0,$$
$$d^*_{\nu,0,4,3}(\nu)=\nu^2(-32\nu-8+30\nu+10-2+2\nu)=0.$$

Therefore

(106) $$d^*_{\alpha,0,4,4}(\nu)=0.$$

It follows from (59) – (106) that

(107) $$V^{**}_{\alpha,0}(z;\nu)V^{**}_{\alpha,0}(z;-\nu-\alpha)=0E_4.$$

So, (1) and (2) hold. It is interesing to compare the equality (1) and the equality

(108) $$-\nu^6E_4=A^*_0(z;-\nu)A^*_0(z;\nu),$$

which was established in [5] – [8]. In view of (70), (72), (74), (77) – (88) in [11] and (110), (111) – (131) in [12],

(109) $$a^*_{0,0,i,k}(z;\nu)=\nu^2a^*_{0,i,k}(z;\nu)$$

for $\{i,k\}\subset\{1, 2, 3, 4\}$. Therefore

(110) $$A^*_{0,0}(z;\nu)=\nu^2A^*_0(z;\nu).$$

Consequently, (108) follows from (1) with $\alpha=0$.

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