EXPLICIT SHUFFLE RELATIONS AND A GENERALIZATION OF EULER'S DECOMPOSITION FORMULA

LI GUO AND BINGYONG XIE

Abstract. We give an explicit formula of the shuffle relation in a general framework that specializes to shuffle relations of multiple zeta values and multiple polylogarithms. As a consequence, we generalize the decomposition formula of Euler that expresses the product of two single (Riemann) zeta values as a sum of double zeta values to a formula that expresses the product of two multiple polylogarithm values as a sum of other multiple polylogarithm values.

1. Introduction

The decomposition formula of Euler is the equation

$$(1) \qquad \zeta(r)\zeta(s) = \sum_{k=0}^{s-1} \binom{r+k-1}{k} \zeta(r+k, s-k) + \sum_{k=0}^{r-1} \binom{s+k-1}{k} \zeta(s+k, r-k), \quad r, s \geqslant 2,$$

expressing the product of two Riemann zeta values as a sum of double zeta values. In this paper we generalize this formula in two directions, from the product of one variable functions to that of multiple variables and from multiple zeta values to multiple polylogarithms. In fact, we obtain our formula in a general setting of shuffle algebras and quasi-shuffle algebras in order to provide a natural framework to treat these special values uniformly and to connect our generalization with the extended double shuffle relations of multiple zeta values.

To motivate our generalization, we describe the relationship between Euler's formula and double shuffle relations of multiple zeta values. Multiple zeta values (MZVs) have been studied quite intensively since the early 1990s [21, 30] involving many areas of mathematics and physics, from mixed Tate motives [12, 29] and combinatorial number theory [3, 6, 22] to quantum field theory [9]. Especially interesting are the algebraic and linear relations among the MZVs. Because of the representations of an MZV as an iterated sum and as an iterated integral, the multiplication of two MZVs can be expressed in two ways as the sum of other MZVs, one way following the **quasi-shuffle (stuffle) relation** and the other way following the **shuffle relation**. The combination of these two relations (called the **double shuffle relations**) generates an extremely rich family of relations among MZVs. In fact, as a conjecture, all relations among MZVs can be derived from these relations and their degenerated forms, altogether called the **extended double shuffle relations** [24, 27]. A consequence of this conjecture is the irrationality of $\zeta(n)$ for all odd integers $n \geqslant 3$.

Naturally, determining all the extended double shuffle relations is challenging and the efforts have utilized a wide range of methods. One difficulty is that the shuffle relations have not been explicitly formulated in terms of the MZVs. For example, to determine the double shuffle relation from multiplying two Riemann zeta values $\zeta(r)$ and $\zeta(s)$, $r, s \geqslant 2$, one uses their sum representations and easily gets the quasi-shuffle relation

$$\zeta(r)\zeta(s) = \zeta(r,s) + \zeta(s,r) + \zeta(r+s). \tag{2}$$

On the other hand, to get their shuffle relation, one first uses their integral representations to express $\zeta(r)$ and $\zeta(s)$ as iterated integrals of dimensions r and s, respectively. One then uses the shuffle relation (or more concretely, repeated applications of the integration by parts formula) to express the product of these two iterated integrals as a sum of $\binom{r+s}{r}$ iterated integrals of dimension $r+s$. Finally, these last iterated integrals are translated back to MZVs and give the shuffle relation of $\zeta(r)\zeta(s)$. Explicitly, this shuffle relation is precisely the formula of Euler in Eq. (1). Then together with Eq. (2), we have the double shuffle relation obtained from $\zeta(r)$ and $\zeta(s)$.

In general, even though the computation of the shuffle relation can be performed recursively for any given pair of MZVs, an explicit formula is missing so far. As this example shows, such an explicit formula not only provides an effective way to evaluate the shuffle relation, but also is important in the theoretical study of MZVs, especially the double shuffle relations. There are several families of special values in addition to MZVs, such as the alternating Euler sums [2], the polylogarithms and multiple polylogarithms [3, 13], especially at roots of unity [27], where the double shuffle relations are also studied [5, 27, 33], but are less understood. Such an explicit formula for these values should also contribute their study.

In this paper, we prove an explicit formula in a general double shuffle framework. Consequently we obtain explicit shuffle formulas for the product of any two MZVs, alternating Euler sums and multiple polylogarithms, thereby generalizing Euler's formula. As a concrete example, we obtain, for integers $r_1, s_1 \geqslant 2$ and $s_2 \geqslant 1$,

$$\begin{aligned}\zeta(r_1)\,\zeta(s_1,s_2) = \sum_{\substack{t_1\geqslant 2,\, t_2,t_3\geqslant 1\\ t_1+t_2+t_3\\ = r_1+s_1+s_2}} &\left[\binom{t_1-1}{r_1-1}\binom{t_2-1}{s_2-t_3}\binom{t_3-1}{s_2-1} + \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-t_3}\right.\\ &\left. + \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\right]\zeta(t_1,t_2,t_3).\end{aligned} \tag{3}$$

As another instance, for integers $r_1, s_1 \geqslant 2$ and $r_2, s_2 \geqslant 1$, we have

$$\begin{aligned}&\zeta(r_1,r_2)\,\zeta(s_1,s_2) =\\ &\sum_{\substack{t_1\geqslant 2,\, t_2,t_3,t_4\geqslant 1\\ t_1+t_2+t_3+t_4=\\ r_1+r_2+s_1+s_2}} \left[\binom{t_1-1}{r_1-1}\binom{t_2-1}{r_2-1}\binom{t_3-1}{s_2-t_4}\binom{t_4-1}{s_2-1} + \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\binom{t_3-1}{r_2-t_4}\binom{t_4-1}{r_2-1}\right.\\ &\qquad + \binom{t_1-1}{r_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{s_2-t_4} + \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{r_2-t_4}\\ &\qquad \left. + \binom{t_1-1}{r_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{s_2-1} + \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{r_2-1}\right]\\ &\qquad \times\zeta(t_1,t_2,t_3,t_4).\end{aligned} \tag{4}$$

We hope this framework can be further extended to deal with other generalizations of multiple zeta values that have emerged recently, such as the multiple q-zeta values [7, 31, 32] and renormalized MZVs [19, 20, 26].

The organization of the paper is as follows. In Section 2, we first describe the algebraic formulation of double shuffle algebras and then state our main formula in two forms (Theorem 2.1 and Theorem 2.2). There we also give applications of the main formula to MZVs and other special values (Corollary 2.3 and Corollary 2.4), and illustrate its computations in low dimensions in Section 2.4. The proof of the main formula is quite long. So several lemmas are first proved in Section 3. Then these lemmas are applied in Section 4 to prove the main formula by induction. As an appendix, Section 5 includes a shuffle product formulation of the main formula.

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2. Statements of the main theorems

We first set up in Section 2.1 a framework of general double shuffles to give a uniform formulation of the double shuffle relations for multiple zeta values, alternating Euler sums and multiple polylogarithms. We then state in Section 2.2 our main formula in two variations in this framework. Applications of the main theorem to the aforementioned special values are presented in Section 2.3. Computations in low dimensions and examples are provided in Section 2.4.

2.1. The general double shuffle framework.

We formulate the framework to state our main theorems in Section 2.2. See Section 2.3 for the concrete cases that have been considered before [3, 13, 22, 27, 33].

We first introduce some notations. For any set Y, denote $M(Y)$ for the free monoid generated by Y. Let $\mathcal{H}(Y)$ be the free abelian group $\mathbb{Z}M(Y)$ with $M(Y)$ as a basis but without considering the product from the monoid $M(Y)$. When $\mathcal{H}(Y)$ is equipped with an associative multiplication $\circ$, we use $\mathcal{H}^\circ(Y)$ to denote the algebra $(\mathcal{H}(Y), \circ)$.

Let G be a given set. Define

$$\overline{G} = \{x_0\} \cup \{x_b \mid b \in G\}$$

to be a set of symbols indexed by $\{0\} \sqcup G$. Then the shuffle algebra [28, 25] generated by $\overline{G}$ is

$$\mathcal{H}^{ш}(\overline{G}) := \mathbb{Z}M(\overline{G}) \tag{5}$$

equipped with the shuffle product $ш$ that is defined recursively by

$$(a_1\mathfrak{a}) ш (b_1\mathfrak{b}) = a_1(\mathfrak{a} ш (b_1\mathfrak{b})) + b_1((a_1\mathfrak{a}) ш \mathfrak{b}), a_1, b_1 \in \overline{G}, \mathfrak{a}, \mathfrak{b} \in M(\overline{G})$$

with the convention that $1 ш \mathfrak{b} = \mathfrak{b} = \mathfrak{b} ш 1$ for $\mathfrak{b} \in M(\overline{G})$. Define the subalgebra

$$\mathcal{H}^{ш}_1(\overline{G}) := \mathbb{Z} \oplus \big(\oplus_{b\in G} \mathcal{H}^{ш}(\overline{G})x_b \big) \subseteq \mathcal{H}^{ш}(\overline{G}). \tag{6}$$

For the given set G, let $\widehat{G}$ be the set product

$$\widehat{G} := \mathbb{Z}_{\geqslant 1} \times G = \{w := \left[\begin{smallmatrix} s \\ b \end{smallmatrix}\right] \mid s \in \mathbb{Z}_{\geqslant 1}, b \in G\}.$$

We will denote the non-unit elements in the free monoid $M(\widehat{G})$ by vectors

$$\vec{\nu} := [\nu_1, \cdots, \nu_k] = \left[\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right] = \left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right].$$

Consider the free abelian group

$$\mathcal{H}(\widehat{G}) := \mathbb{Z}M(\widehat{G}) = \bigoplus_{\vec{\nu}\in \widehat{G}^k,\, k\geqslant 0} \mathbb{Z}\vec{\nu}, \quad \widehat{G}^0 = \{1\}.$$

As in the case of the shuffle algebra from MZVs, elements of $\mathcal{H}^{ш}_1(\overline{G})$ of the form

$$x_0^{s_1-1}x_{b_1}x_0^{s_2-1}x_{b_2}\cdots x_0^{s_k-1}x_{b_k}, \quad s_i \geqslant 1, b_i \in G, 1 \leqslant i \leqslant k, k \geqslant 1,$$

together with 1, form a basis of $\mathcal{H}^{ш}_1(\overline{G})$. Since $\mathcal{H}(\widehat{G})$ with the concatenation product is the free non-commutative algebra generated by $\widehat{G}$, there is a natural linear bijection

$$\rho : \mathcal{H}^{ш}_1(\overline{G}) \to \mathcal{H}^*(\widehat{G}), \quad x_0^{s_1-1}x_{b_1}\cdots x_0^{s_k-1}x_{b_k} \leftrightarrow \left[\begin{smallmatrix} s_1,\ s_2,\ \cdots,\ s_k \\ b_1,\ b_2,\ \cdots,\ b_k \end{smallmatrix}\right], \quad 1 \leftrightarrow 1. \tag{7}$$

Through ρ, the shuffle product ш on $\mathcal{H}^{ш}_1(\overline{G})$ defined a product on $\mathcal{H}(\widehat{G})$ by

$$\vec{\mu} ш_\rho \vec{\nu} := \rho(\rho^{-1}(\vec{\mu}) ш \rho^{-1}(\vec{\nu})), \quad \vec{\mu}, \vec{\nu} \in \mathcal{H}(\widehat{G}). \tag{8}$$

Following our notations, we use $\mathcal{H}^{ш_\rho}(\widehat{G})$ to denote this algebra.

Now assume that G is a multiplicative abelian group. Define $\widehat{G} = \mathbb{Z}_{>0} \times G$ to be the abelian semigroup with the component multiplication: $\left[\begin{smallmatrix} s_1 \\ z_1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} s_2 \\ z_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} s_1+s_2 \\ z_1z_2 \end{smallmatrix}\right]$. Then we define the quasi-shuffle algebra [23] on $\widehat{G}$ to be

$$\mathcal{H}^*(\widehat{G}) := \mathbb{Z}M(\widehat{G}) \tag{9}$$

where the multiplication $*$ is defined by the recursion

$$[\mu_1, \vec{\mu}']*[\nu_1, \vec{\nu}'] = [\mu_1, (\vec{\mu}'*[\nu_1, \vec{\nu}'])]+[\nu_1, [\mu_1, \vec{\mu}']*\vec{\nu}']+[(\mu_1\cdot\nu_1), \vec{\mu}'*\vec{\nu}'], \mu_1, \nu_1 \in \widehat{G}, \vec{\mu}', \vec{\nu}' \in M(\widehat{G})$$

with the initial condition that $1 * \vec{\nu} = \vec{\nu} = \vec{\nu} * 1$ for $\vec{\nu} \in M(\widehat{G})$. See [16, 18, 23] for its explicit description and its structure. We use $\mathcal{H}^*(\widehat{G})$ to denote the resulting commutative algebra $(\mathcal{H}(\widehat{G}), *)$.

We define a linear bijection

$$\theta : \mathcal{H}^*(\widehat{G}) \to \mathcal{H}^*(\widehat{G}), \quad \left[\begin{smallmatrix} s_1,\cdots,s_k \\ b_1,\cdots,b_k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} s_1,\, s_2,\cdots,\, s_k \\ \frac{1}{b_1},\, \frac{b_1}{b_2},\cdots,\, \frac{b_{k-1}}{b_k} \end{smallmatrix}\right] \tag{10}$$

whose inverse is given by

$$\theta^{-1} : \mathcal{H}^*(\widehat{G}) \to \mathcal{H}^*(\widehat{G}), \quad \left[\begin{smallmatrix} s_1,\cdots,s_k \\ z_1,\cdots,z_k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} s_1,\, s_2,\cdots,\, s_k \\ \frac{1}{z_1},\, \frac{1}{z_1z_2},\cdots,\, \frac{1}{z_1\cdots z_k} \end{smallmatrix}\right] \tag{11}$$

Note that the action of θ is defined by an action on the lower row of elements in $\mathcal{H}^*(\widehat{G})$ which is again denoted by θ:

$$\theta(b_1, \cdots, b_k) = (\frac{1}{b_1}, \frac{b_1}{b_2}, \cdots, \frac{b_{k-1}}{b_k}). \tag{12}$$

The composition of ρ and θ gives a natural bijection of abelian groups (but *not* as algebras)

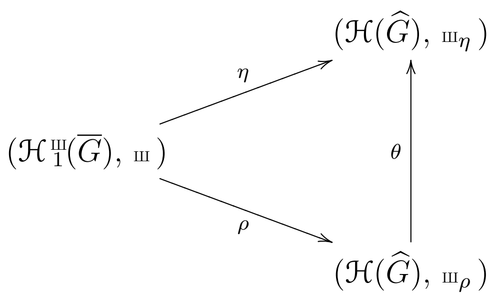
$$\eta : \mathcal{H}_1^{ш}(\overline{G}) \to \mathcal{H}^*(\widehat{G}), \quad x_0^{s_1-1}x_{b_1}\cdots x_0^{s_k-1}x_{b_k} \leftrightarrow \left[\begin{smallmatrix} s_1, & s_2, & \cdots, & s_k \\ \frac{1}{b_1}, & \frac{b_1}{b_2}, & \cdots, & \frac{b_{k-1}}{b_k} \end{smallmatrix} \right] \tag{13}$$

whose inverse is given by $\left[\begin{smallmatrix} s_1,\cdots,s_k \\ z_1,\cdots,z_k \end{smallmatrix} \right] \mapsto x_0^{s_1-1}x_{z_1^{-1}}x_0^{s_2-1}x_{(z_1z_2)^{-1}}\cdots x_0^{s_k-1}x_{(z_1\cdots z_k)^{-1}}$.

Through η, the shuffle product ш on $\mathcal{H}_1^{ш}(\overline{G})$ transports to a product $ш_\eta$ on $\mathcal{H}(\widehat{G})$, resulting a commutative algebra $\mathcal{H}^{ш_\eta}(\widehat{G}) = (\mathcal{H}(\widehat{G}), ш_\eta)$. More precisely, for $\vec{\mu}, \vec{\nu} \in \mathcal{H}(\widehat{G})$,

$$\vec{\mu} ш_\eta \vec{\nu} := \eta(\eta^{-1}(\vec{\mu}) ш \eta^{-1}(\vec{\nu})). \tag{14}$$

Then we have the following commutative diagram of commutative algebras:



The purpose of this paper is to give an explicit formula for $\vec{\mu} ш_\eta \vec{\nu}$ which naturally gives shuffle formulas for MZVs, MPVs and alternating Euler sums. However, as we will see later, for the proof of this formula, it is more convenient to work with the product $ш_\rho$ since it is more compatible with the module structure on $\mathcal{H}^*(\widehat{G})$. This approach also allows us to obtain a formula without requiring that G is a semigroup, further extending its potential of applications that will be discussed in a future work.

2.2. **The statement of the main theorem.** We first introduce some notations. For positive integers k and ℓ, denote $[k] = \{1, \cdots, k\}$ and $[k+1, k+\ell] = \{k+1, \cdots, k+\ell\}$. Define

$$\mathcal{I}_{k,\ell} = \left\{ (\varphi, \psi) \;\middle|\; \begin{array}{c} \varphi : [k] \to [k+\ell], \psi : [\ell] \to [k+\ell] \text{ are order preserving} \\ \text{injective maps and } \mathrm{im}(\varphi) \sqcup \mathrm{im}(\psi) = [k+\ell] \end{array} \right\} \tag{15}$$

In fact, in the definition it suffice to use one of the three conditions of the injectivity, the disjointness of $\mathrm{im}(\varphi)$ and $\mathrm{im}(\psi)$, or $\mathrm{im}(\varphi) \cup \mathrm{im}(\psi) = [k+\ell]$. We simply list them all for ease of application. Let $\vec{a} \in G^k$, $\vec{b} \in G^\ell$ and $(\varphi, \psi) \in \mathcal{I}_{k,\ell}$. We define $\vec{a} ш_{(\varphi,\psi)} \vec{b}$ to be the vector whose ith component is

$$(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i = \left\{ \begin{array}{ll} a_j & \text{if } i = \varphi(j) \\ b_j & \text{if } i = \psi(j) \end{array} \right. = a_{\varphi^{-1}(i)} b_{\psi^{-1}(i)}, \quad 1 \leqslant i \leqslant k+\ell, \tag{16}$$

with the convention that $a_\emptyset = b_\emptyset = 1$.

Let $\vec{r} = (r_1, \cdots, r_k) \in \mathbb{Z}_{\geqslant 1}^k$, $\vec{s} = (s_1, \cdots, s_\ell) \in \mathbb{Z}_{\geqslant 1}^\ell$ and $\vec{t} = (t_1, \cdots, t_{k+\ell}) \in \mathbb{Z}_{\geqslant 1}^{k+\ell}$ with $|\vec{r}| + |\vec{s}| = |\vec{t}|$. Here $|\vec{r}| = r_1 + \cdots + r_k$ and similarly for $|\vec{s}|$ and $|\vec{t}|$. Denote $R_i = r_1 + \cdots + r_i$ for $i \in [k]$, $S_i = s_1 + \cdots + s_i$ for $i \in [\ell]$ and $T_i = t_1 + \cdots + t_i$ for $i \in [k+\ell]$. For $i \in [k+\ell]$, define

$$h_{(\varphi,\psi),i} = h_{(\varphi,\psi),(\vec{r},\vec{s}),i} = \begin{cases} r_j & \text{if } i = \varphi(j) \\ s_j & \text{if } i = \psi(j) \end{cases} = r_{\varphi^{-1}(i)} s_{\psi^{-1}(i)}, \tag{17}$$

with the convention that $r_\emptyset = s_\emptyset = 1$.

With these notations, we define

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \begin{cases} \binom{t_i - 1}{h_{(\varphi,\psi),i} - 1} & \begin{array}{l}\text{if } i = 1 \text{ or} \\ \text{if } i-1, i \in \operatorname{im}(\varphi) \text{ or if } i-1, i \in \operatorname{im}(\psi),\end{array} \\ \binom{t_i - 1}{T_i - R_{|\varphi^{-1}([i])|} - S_{|\psi^{-1}([i])|}} = \binom{t_i - 1}{\sum_{j=1}^{i} t_j - \sum_{j=1}^{i} h_{(\varphi,\psi),j}} & \text{otherwise.} \end{cases} \tag{18}$$

Denote

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} := \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \prod_{j=1}^{k} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(\varphi(j)) \prod_{j=1}^{\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(\psi(j)). \tag{19}$$

Now we can state our main theorem.

Theorem 2.1. *Let G be a set and let $\mathcal{H}^{ш_\rho}(\widehat{G}) = (\mathcal{H}(\widehat{G}), ш_\rho)$ be as defined by Eq. (8). Then for $\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix} \right] \in \widehat{G}^k$ and $\left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix} \right] \in \widehat{G}^\ell$ in $\mathcal{H}^{ш_\rho}(\widehat{G})$, we have*

$$\begin{aligned} \left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix} \right] ш_\rho \left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix} \right] &= \sum_{\substack{(\varphi,\psi) \in \mathcal{I}_{k,\ell} \\ \vec{t} \in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}| + |\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{smallmatrix} \vec{t} \\ \vec{a}\, ш_{(\varphi,\psi)} \vec{b} \end{smallmatrix} \right] \\ &= \sum_{\substack{(\varphi,\psi) \in \mathcal{I}_{k,\ell} \\ \vec{t} \in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}| + |\vec{s}|}} \left(\prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) \right) \left[\begin{smallmatrix} \vec{t} \\ \vec{a}\, ш_{(\varphi,\psi)} \vec{b} \end{smallmatrix} \right], \end{aligned} \tag{20}$$

where $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ is given in Eq. (18) and $\vec{a}\, ш_{(\varphi,\psi)} \vec{b}$ is given in Eq. (16).

For the purpose of applications to MZVs and multiple polylogarithms, we give an equivalent form of Theorem 2.1 under the condition that G is an abelian group. For $\vec{w} \in G^k$ and $\vec{z} \in G^\ell$, we define

$$(\vec{w} \star_{(\varphi,\psi)} \vec{z})_i = \begin{cases} w_j & \text{if } i = \varphi(j) \text{ and either } i = 1 \text{ or } i-1 \in \operatorname{im}(\varphi), \\ z_j & \text{if } i = \psi(j) \text{ and either } i = 1 \text{ or } i-1 \in \operatorname{im}(\varphi), \\ \frac{w_1 \cdots w_j}{z_1 \cdots z_{i-j}} & \text{if } i = \varphi(j) \text{ and } i-1 \in \operatorname{im}(\psi), \\ \frac{z_1 \cdots z_j}{w_1 \cdots w_{i-j}} & \text{if } i = \psi(j) \text{ and } i-1 \in \operatorname{im}(\varphi). \end{cases} \tag{21}$$

Theorem 2.2. *Let G be an abelian group and let $\mathcal{H}^{\text{ш}_\eta}(\widehat{G}) = (\mathcal{H}(\widehat{G}), \text{ш}_\eta)$ be as defined by Eq. (14). Then for $\left[\begin{smallmatrix}\vec{r}\\ \vec{a}\end{smallmatrix}\right] \in \widehat{G}^k$ and $\left[\begin{smallmatrix}\vec{s}\\ \vec{b}\end{smallmatrix}\right] \in \widehat{G}^\ell$ in $\mathcal{H}^{\text{ш}_\eta}(\widehat{G})$, we have*

$$
\begin{aligned}
\left[\begin{smallmatrix}\vec{r}\\ \vec{w}\end{smallmatrix}\right] \text{ш}_\eta \left[\begin{smallmatrix}\vec{s}\\ \vec{z}\end{smallmatrix}\right] &= \sum_{\substack{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}\\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{smallmatrix}\vec{t}\\ \vec{w}\star_{(\varphi,\psi)}\vec{z}\end{smallmatrix}\right] \\
&= \sum_{\substack{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}\\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|}} \left(\prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\right) \left[\begin{smallmatrix}\vec{t}\\ \vec{w}\star_{(\varphi,\psi)}\vec{z}\end{smallmatrix}\right],
\end{aligned}
\tag{22}
$$

where $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ is given in Eq. (18) and $\vec{w}\star_{(\varphi,\psi)}\vec{z}$ is given in Eq. (21).

We will next give applications and examples of Theorem 2.2 in Section 2.3 and Section 2.4. Theorem 2.2 will be shown to follow from Theorem 2.1 in Section 4.1, and Theorem 2.1 will be proved in Section 4.2. Preparational lemmas will be given in Section 3.

2.3. **Applications.** In this section, Theorem 2.2 is specialized to give formulas for multiple zeta values, alternating Euler sums and multiple polylogarithms. We start with multiple polylogarithms and then specialize further to MZVs and alternating Euler sums. In Section 2.4 we demonstrate how to apply these formulas by computing examples in low dimensions.

2.3.1. *Multiple polylogarithms.* A **Multiple polylogarithm value (**MPV**)** [3, 13, 14] is defined by

$$
\operatorname{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) := \sum_{n_1>\cdots>n_k\geqslant 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}} \tag{23}
$$

where $|z_i| \leqslant 1$, $s_i \in \mathbb{Z}_{\geqslant 1}$, $1 \leqslant i \leqslant k$, and $(s_1, z_1) \neq (1,1)$. When $z_i = 1, 1 \leqslant i \leqslant k$, we obtain the multiple zeta values $\zeta(s_1,\cdots,s_k)$ that we will consider in Section 2.3.2. More generally, the special cases when z_i are roots of unity have been studied [3, 6, 14, 27] in connection with high cyclotomic theory, mixed motives and combinatorics, and have been found in the computations of Feynman diagrams [10].

In the notation of [3], we have

$$
\begin{aligned}
&\operatorname{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) = \lambda\left(\begin{smallmatrix} s_1,\cdots,s_k\\ b_1,\cdots,b_k\end{smallmatrix}\right), \quad \text{where}\\
&(b_1,\cdots,b_k) = \theta(z_1,\cdots,z_k) = (z_1^{-1},(z_1z_2)^{-1},\cdots,(z_1\cdots z_k)^{-1}).
\end{aligned}
\tag{24}
$$

Here θ is as defined in Eq. (12).

The product of two sums representing two MPVs is a $\mathbb{Z}$-linear combination of other such sums. This way the $\mathbb{Z}$-linear span of these values form an algebra which we denote by

$$
\mathbf{MPV} = \mathbb{Z}\{\operatorname{Li}_{s_1,\cdots,s_k}(z_1,\cdots,z_k) \mid s_i \in \mathbb{Z}_{\geqslant 1}, |z_i| \leqslant 1, (s_1,z_1)\neq(1,1)\}.
$$

In the framework of Section 2.1 and 2.2, let the abelian group G be $S^1 := \{z \in \mathbb{C}^\times \mid |z| = 1\}$, and consider the subalgebra

$$\mathcal{H}_0^*(\widehat{S}^1) := \mathbb{Z} \oplus \Big(\bigoplus_{\left[\begin{smallmatrix} s_1 \\ z_1 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]} \mathbb{Z}\left[\begin{smallmatrix} s_1, s_2, \cdots, s_k \\ z_1, z_2, \cdots, z_k \end{smallmatrix}\right] \Big) \subseteq \mathcal{H}^*(\widehat{S}^1).$$

Then $\mathcal{H}^*(\widehat{G})$ coincides with the quasi-shuffle (stuffle) algebra [14, 27] encoding MPVs, and the multiplication rule of two MPVs according to their sum representations in Eq. (23) follows from the fact that the linear map

$$\mathrm{Li}^* : \mathcal{H}_0^*(\widehat{S}^1) \to \mathbf{MPV}, \quad \left[\begin{smallmatrix} s_1, \cdots, s_k \\ z_1, \cdots, z_k \end{smallmatrix}\right] \mapsto \mathrm{Li}_{s_1, \cdots, s_k}(z_1, \cdots, z_k)$$

is an algebra homomorphism.

We also consider the shuffle algebra $\mathcal{H}^{ш}(\overline{S^1})$ and its subalgebras

$$\begin{aligned} \mathcal{H}_0^{ш}(\overline{S}^1) &:= \mathbb{Z} \oplus \big(\oplus_{a,b \in \{0\} \cup S^1, a \neq 1, b \neq 0}\, x_a \mathcal{H}^{ш}(\overline{S}^1) x_b \big) \\ &\subseteq \mathcal{H}_1^{ш}(\overline{S}^1) := \mathbb{Z} \oplus \big(\oplus_{b \in S^1} \mathcal{H}^{ш}(\overline{S}^1) x_b \big) \subseteq \mathcal{H}^{ш}(\overline{S}^1). \end{aligned}$$

They agree with the shuffle algebras [13, 27] encoding a MPV through its integral representation [3, 13, 27]

$$\mathrm{Li}_{s_1, \cdots, s_k}(z_1, \cdots, z_k) = \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{|\vec{s}|-1}} \frac{du_1}{f_1(u_1)} \cdots \frac{du_{|\vec{s}|}}{f_{|\vec{s}|}(u_{|\vec{s}|})}. \tag{25}$$

Here

$$f_j(u_j) = \begin{cases} (z_1 \cdots z_i)^{-1} - u_j = b_i - u_j & \text{if } j = s_1 + \cdots + s_i, 1 \leqslant i \leqslant k, \\ u_j & \text{otherwise.} \end{cases}$$

for the b_i in Eq. (24). Thus $\lambda\big(\begin{smallmatrix} s_1, \cdots, s_k \\ b_1, \cdots, b_k \end{smallmatrix} \big)$ has a simpler integration representation than that of $\mathrm{Li}_{s_1, \cdots, s_k}(z_1, \cdots, z_k)$. This is the fact that gives the simpler form of the shuffle formula in Theorem 2.1 in comparison with Theorem 2.2.

The multiplication rule of two MPVs according to their integral representations follows from the algebra homomorphism

$$\mathrm{Li}^{ш} : \mathcal{H}_0^{ш}(\overline{S}^1) \to \mathbf{MPV}, \quad x_0^{s_1-1} x_{b_1} \cdots x_0^{s_k-1} x_{b_k} \mapsto \lambda\big(\begin{smallmatrix} s_1, \cdots, s_k \\ b_1, \cdots, b_k \end{smallmatrix} \big) = \mathrm{Li}_{s_1, \cdots, s_k}(z_1, \cdots, z_k),$$

where $(z_1, \cdots, z_k) = \theta^{-1}(b_1, \cdots, b_k)$ is defined in Eq. (11). Therefore, applying $\mathrm{Li}^{ш}$ to the two sides of Eq. (22) we obtain

Corollary 2.3. *Let $\vec{r} \in \mathbb{Z}_{\geqslant 1}^k$ and $\vec{s} \in \mathbb{Z}_{\geqslant 1}^\ell$. Let $\vec{w} = (w_1, \cdots, w_k) \in (S^1)^k$ and $\vec{z} = (z_1, \cdots, z_\ell) \in (S^1)^\ell$ such that $\left[\begin{smallmatrix} r_1 \\ w_1 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} s_1 \\ z_1 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Then*

$$\mathrm{Li}_{\vec{r}}(\vec{w})\, \mathrm{Li}_{\vec{s}}(\vec{z}) = \sum_{\vec{t} \in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}| + |\vec{s}|} \sum_{(\varphi, \psi) \in \mathcal{I}_{k,\ell}} \Big(\prod_{i=1}^{k+\ell} c_{\vec{r}, \vec{s}}^{\vec{t}, (\varphi, \psi)}(i) \Big) \mathrm{Li}_{\vec{t}}(\vec{w} \star_{(\varphi, \psi)} \vec{z})$$

where $c_{\vec{r}, \vec{s}}^{\vec{t}, (\varphi, \psi)}(i)$ is given in Eq. (18) and $\vec{w} \star_{(\varphi, \psi)} \vec{z}$ is given in Eq. (21).

See Section 2.4 for examples in low dimensions. With the notation of $\lambda\left(\begin{smallmatrix}\vec{s}\\ \vec{b}\end{smallmatrix}\right)$, Corollary 2.3 has the form

$$\lambda\left(\begin{smallmatrix}\vec{r}\\ \vec{a}\end{smallmatrix}\right)\lambda\left(\begin{smallmatrix}\vec{s}\\ \vec{b}\end{smallmatrix}\right)=\sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|}\ \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}}\Big(\prod_{i=1}^{k+\ell}c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\Big)\lambda\left(\begin{smallmatrix}\vec{t}\\ \vec{a}\,ш_{(\varphi,\psi)}\vec{b}\end{smallmatrix}\right).$$

2.3.2. *Multiple zeta values and alternating Euler sums.* Taking $z_i=1, 1\leqslant i\leqslant r$, in a MPV defined in Eq. (23) and its integral representation in Eq. (25), we obtain the MZV and its integral representation:

$$\begin{aligned}\zeta(s_1,\cdots,s_k):&=\sum_{n_1>\cdots>n_k\geqslant 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}\\ &=\int_0^1\int_0^{u_1}\cdots\int_0^{u_{|\vec{s}|-1}}\frac{du_1}{f_1(u_1)}\cdots\frac{du_{|\vec{s}|}}{f_{|\vec{s}|}(u_{|\vec{s}|})}\end{aligned}$$

for integers $s_i\geqslant 1$ and $s_1>1$. Here

$$f_j(u_j)=\begin{cases}1-u_j & \text{if } j=s_1,s_1+s_2,\cdots,s_1+\cdots+s_k,\\ u_j & \text{otherwise.}\end{cases}$$

This is also the case when $G=\{1\}$ in our framework in Section 2.1 and 2.2. Then we can identify $\widehat{G}$ with $\mathbb{Z}_{\geqslant 1}$ and denote $\vec{\nu}=\left[\begin{smallmatrix}s_1,\cdots,s_k\\ z_1,\cdots,z_k\end{smallmatrix}\right]=\left[\begin{smallmatrix}s_1,\cdots,s_k\\ 1,\cdots,1\end{smallmatrix}\right]$ by $\mathfrak{z}=\mathfrak{z}_{s_1}\cdots\mathfrak{z}_{s_k}$. Then $\mathcal{H}^*(\widehat{G})$ coincides with the quasi-shuffle algebra $\mathcal{H}^*$ encoding MZVs [23, 24] through the identification $\mathfrak{z}_{s_1}\cdots\mathfrak{z}_{s_k}\leftrightarrow z_{s_1}\cdots z_{s_k}$. We will use $\mathfrak{z}_{s_1}\cdots\mathfrak{z}_{s_k}$ in place of $z_{s_1}\cdots z_{s_k}$ to avoid confusion with the vector $(z_1,\cdots,z_k)$ in $\vec{\nu}$. $\mathcal{H}^*$ contains the subalgebra

$$\mathcal{H}_0^*:=\mathbb{Z}\oplus\mathbb{Z}\big\{\mathfrak{z}_{s_1}\cdots\mathfrak{z}_{s_k}\ \big|\ s_i\geqslant 1, s_1>1, 1\leqslant i\leqslant k, k\geqslant 1\big\}.$$

Likewise the shuffle algebra $\mathcal{H}^{ш}(\widehat{G})$ when $G=\{1\}$ coincides with the shuffle algebra $\mathcal{H}^{ш}$ [22, 24] encoding MZVs, and there are subalgebras

$$\mathcal{H}_0^{ш}:=\mathbb{Z}\oplus x_0\mathcal{H}^{ш}x_1\subseteq\mathcal{H}_1^{ш}:=\mathbb{Z}\oplus\mathcal{H}^{ш}x_1\subseteq\mathcal{H}^{ш},$$

where $\mathcal{H}_1^{ш}$ coincides with $\mathcal{H}_1^{ш}(\widehat{G})$ defined in Eq. (6). The natural isomorphism $\eta:\mathcal{H}_1^{ш}\to\mathcal{H}^*$ of abelian groups in Eq. (13) restricts to an isomorphism of abelian groups

$$\eta:\mathcal{H}_0^{ш}\to\mathcal{H}_0^*,\quad x_0^{s_1-1}x_1\cdots x_0^{s_k-1}x_1\leftrightarrow\mathfrak{z}_{s_1}\cdots\mathfrak{z}_{s_k}.$$

With the notation $\mathfrak{z}'ш_\eta\mathfrak{z}'':=\eta(\eta^{-1}(\mathfrak{z}')ш\eta^{-1}(\mathfrak{z}''))$ from Eq. (14), the **double shuffle relation** of MZVs is simply the ideal generated by the set

$$\{\mathfrak{z}'ш_\eta\mathfrak{z}''-\mathfrak{z}'*\mathfrak{z}''\mid\mathfrak{z}',\mathfrak{z}''\in\mathcal{H}_0^*\}$$

and the **extended double shuffle relation** of MZVs [24] is the ideal generated by the set

$$\{\mathfrak{z}'ш_\eta\mathfrak{z}''-\mathfrak{z}'*\mathfrak{z}'',\ \mathfrak{z}_1ш_\eta\mathfrak{z}''-\mathfrak{z}_1*\mathfrak{z}''\mid\mathfrak{z}',\mathfrak{z}''\in\mathcal{H}_0^*\}.$$

While the product $\mathfrak{z}'*\mathfrak{z}''$ simply follows from the quasi-shuffle relation, the evaluation of $\mathfrak{z}'ш_\eta\mathfrak{z}''$ involves first pulling $\mathfrak{z}'$ and $\mathfrak{z}''$ back to $\mathcal{H}_0^{ш}$ by η, then expressing the shuffle product $\eta(\mathfrak{z}')ш\eta(\mathfrak{z}'')$ as a linear combination of words in $M(x_0,x_1)$, and then sending the result forward to $\mathcal{H}_0^*$ by η. While this process can be defined recursively (see Proposition 4.3), the explicit formula is found only in special cases, such as when $\mathfrak{z}'=\mathfrak{z}_r, \mathfrak{z}''=\mathfrak{z}_s$ are both of length one. As we have discussed in the Introduction, the explicit formula in this case is Euler's formula in Eq. (1). See the recent papers [1, 4] for its proofs and see [8, 31] for its q-analogs.

Our Theorem 2.2 provides an explicit formula for $ш_\eta$ and hence for the shuffle product of MZVs in the full generality.

Corollary 2.4. *Let $\vec{r}\in\mathbb{Z}_{\geqslant 1}^k$ and $\vec{s}\in\mathbb{Z}_{\geqslant 1}^\ell$ with $r_1, s_1\geqslant 2$. Then*

$$\zeta(\vec{r})\,\zeta(\vec{s}) = \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|} \Big(\sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}} \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\Big)\zeta(\vec{t})$$

where $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ is given in Eq. (18).

See Section 2.4 for its specialization to Euler's decomposition formula and other special cases.

Proof. Since $\zeta(\vec{r})=\mathrm{Li}_{\vec{r}}(\vec{w})$ and $\zeta(\vec{s})=\mathrm{Li}_{\vec{s}}(\vec{z})$ where the vectors $\vec{w}$ and $\vec{z}$ have 1 as the components, the vectors $\vec{w}\star_{(\varphi,\psi)}\vec{z}$ also have 1 as their components and thus are independent of the choice of $(\varphi,\psi)\in\mathfrak{I}_{k,\ell}$. Then the corollary follows Corollary 2.3. □

Between the case of MZVs and the case of MPVs, there is the case of **alternating Euler sums**, defined by

$$\zeta(s_1,\cdots,s_k;\sigma_1,\cdots,\sigma_k) := \sum_{n_1>\cdots>n_k\geqslant 1} \frac{\sigma_1^{n_1}\cdots\sigma_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}\,,$$

where $\sigma_i=\pm 1, 1\leqslant i\leqslant k$. This corresponds to the case when $G=\{\pm 1\}$ in our framework. More generally when G is the group of k-th roots of unity, we have the **multiple polylogarithms at roots of unity** [27]. We will not go into the details, but will give an example in Eq. (26) that generalizes Euler's formula.

2.4. **Examples.** We now consider some special cases of Theorem 2.2, Corollary 2.3 and Corollary 2.4.

2.4.1. *The case of $r=s=1$.* In this case $\vec{r}=r_1$ and $\vec{s}=s_1$ are positive integers, and $\vec{w}=w_1$ and $\vec{z}=z_1$ are in G. Let $\vec{t}=(t_1,t_2)\in\mathbb{Z}_{\geqslant 1}^2$ with $t_1+t_2=r_1+s_1$. If $(\varphi,\psi)\in\mathfrak{I}_{1,1}$, then either $\varphi(1)=1$ and $\psi(1)=2$, or $\psi(1)=1$ and $\varphi(1)=2$. If $\varphi(1)=1$ and $\psi(1)=2$, then by Eq. (18), we obtain

$$c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(1)=\binom{t_1-1}{r_1-1},\quad c_{r_1,s_1}^{\vec{t}(\varphi,\psi)}(2)=\binom{t_2-1}{t_1+t_2-r_1-s_1}=1$$

and thus

$$c_{r_1,s_1}^{\vec{t},(\varphi,\psi)} = c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(1)\, c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(2) = \binom{t_1-1}{r_1-1}.$$

By Eq. (21), we have

$$\vec{w}\star_{(\varphi,\psi)}\vec{z}=(w_1, z_1/w_1).$$

If $\psi(1)=1$ and $\varphi(1)=2$, then by Eq. (18), we obtain

$$c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(1)=\binom{t_1-1}{s_1-1},\quad c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(2)=\binom{t_2-1}{t_1+t_2-r_1-s_1}=1$$

and thus

$$c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}=c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(1)\,c_{r_1,s_1}^{\vec{t},(\varphi,\psi)}(2)=\binom{t_1-1}{s_1-1}.$$

By Eq. (21), we have $\vec{w}\star_{(\varphi,\psi)}\vec{z}=(z_1,w_1/z_1)$. Therefore,

$$\begin{aligned}\left[\begin{smallmatrix} r_1\\ w_1\end{smallmatrix}\right]ш_\eta\left[\begin{smallmatrix} s_1\\ z_1\end{smallmatrix}\right]&=\sum_{t_1,t_2\geqslant 1,t_1+t_2=r_1+s_1}\binom{t_1-1}{r_1-1}\left[\begin{smallmatrix} t_1,t_2\\ w_1,z_1/w_1\end{smallmatrix}\right]+\sum_{t_1,t_2\geqslant 1,t_1+t_2=r_1+s_1}\binom{t_1-1}{s_1-1}\left[\begin{smallmatrix} t_1,t_2\\ z_1,w_1/z_1\end{smallmatrix}\right]\\&=\sum_{t_1,t_2\geqslant 1,t_1+t_2=r_1+s_1}\binom{t_1-1}{t_1-r_1}\left[\begin{smallmatrix} t_1,t_2\\ w_1,z_1/w_1\end{smallmatrix}\right]+\sum_{t_1,t_2\geqslant 1,t_1+t_2=r_1+s_1}\binom{t_1-1}{t_1-s_1}\left[\begin{smallmatrix} t_1,t_2\\ z_1,w_1/z_1\end{smallmatrix}\right]\\&=\sum_{k=0}^{s_1-1}\binom{r_1+k-1}{k}\left[\begin{smallmatrix} r_1+k,s_1-k\\ w_1,z_1/w_1\end{smallmatrix}\right]+\sum_{k=0}^{r_1-1}\binom{s_1+k-1}{k}\left[\begin{smallmatrix} s_1+k,r_1-k\\ z_1,w_1/z_1\end{smallmatrix}\right]\end{aligned}$$

by a change of variables $k=t_1-r_1$ for the first sum and $k=t_1-s_1$ for the second sum. Then by Corollary 2.3, we obtain the following relation for double polylogarithms

$$\mathrm{Li}_{r_1}(w_1)\mathrm{Li}_{s_1}(z_1)=\sum_{k=0}^{s_1-1}\binom{r_1+k-1}{k}\mathrm{Li}_{r_1+k,s_1-k}(w_1,z_1/w_1)+\sum_{k=0}^{r_1-1}\binom{s_1+k-1}{k}\mathrm{Li}_{s_1+k,r_1-k}(z_1,w_1/z_1),$$

where $r_1,s_1\geqslant 1$, $w_1,z_1\in S^1$ and $(r_1,w_1)\neq(1,1)\neq(s_1,z_1)$. In the special case when $w_1=\pm1$ and $z_1=\pm1$, we have the following relation for alternating Euler sums

$$\begin{aligned}\zeta(r_1;w_1)\zeta(s_1;z_1)=&\sum_{k=0}^{s_1-1}\binom{r_1+k-1}{k}\zeta(r_1+k,s_1-k;w_1,z_1/w_1)\\&+\sum_{k=0}^{r_1-1}\binom{s_1+k-1}{k}\zeta(s_1+k,r_1-k;z_1,w_1/z_1),\end{aligned}\tag{26}$$

when $r_1,s_1\geqslant 1$ and $(r_1,w_1)\neq(1,1)\neq(s_1,z_1)$.

Further specializing, when $r_1,s_1\geqslant 2$ and $w_1=z_1=1$, we obtain the decomposition formula of Euler in Eq. (1).

2.4.2. *The case of* $r=1,s=2$. In this case $\left[\begin{smallmatrix}\vec{r}\\ \vec{w}\end{smallmatrix}\right]=\left[\begin{smallmatrix} r_1\\ w_1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\vec{s}\\ \vec{z}\end{smallmatrix}\right]=\left[\begin{smallmatrix} s_1,s_2\\ z_1,z_2\end{smallmatrix}\right]$. Let $\vec{t}=(t_1,t_2,t_3)\in\mathbb{Z}_{\geqslant 1}^3$ with $t_1+t_2+t_3=r_1+s_1+s_2$. There are 3 pairs (φ,ψ) in $\mathcal{I}_{1,2}$.

When $\varphi(1)=1$, $\psi(1)=2$ and $\psi(2)=3$, by Eq. (18), we have

$$c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}(1)=\binom{t_1-1}{r_1-1},\quad c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}(2)=\binom{t_2-1}{t_1+t_2-r_1-s_1}=\binom{t_2-1}{s_2-t_3},\quad c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}(3)=\binom{t_3-1}{s_2-1}$$

and thus

$$c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}=c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}(1)\,c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}(2)\,c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)}(3)=\binom{t_1-1}{r_1-1}\binom{t_2-1}{s_2-t_3}\binom{t_3-1}{s_2-1}.$$

By Eq. (21) we have

$$\vec{w}\star_{(\varphi,\psi)}\vec{z} = (w_1, z_1/w_1, z_2).$$

Similarly, when $\varphi(1)=2$, $\psi(1)=1$ and $\psi(2)=3$, we have

$$c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-t_3}, \quad \vec{w}\star(\varphi,\psi)\vec{z} = (z_1, w_1/z_1, z_1z_2/w_1),$$

and when $\varphi(1)=3$, $\psi(1)=1$ and $\psi(2)=2$, we have

$$c_{r_1,\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}, \quad \vec{w}\star(\varphi,\psi)\vec{z} = (z_1, z_2, w_1/(z_1z_2)).$$

Combining these computations with Corollary 2.3 we obtain, for $r_1, s_1, s_2 \geqslant 1$ and $(r_1, w_1) \neq (1,1) \neq (s_1, z_1)$,

$$\begin{aligned}\mathrm{Li}_{r_1}(w_1)\,\mathrm{Li}_{s_1,s_2}(z_1,z_2) = \sum_{\substack{t_1,t_2,t_3\geqslant 1\\ t_1+t_2+t_3\\ =r_1+s_1+s_2}} \Bigg[&\binom{t_1-1}{r_1-1}\binom{t_2-1}{s_2-t_3}\binom{t_3-1}{s_2-1}\mathrm{Li}_{(t_1,t_2,t_3)}(w_1, z_1/w_1, z_2)\\ &+\binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-t_3}\mathrm{Li}_{(t_1,t_2,t_3)}(z_1, w_1/z_1, z_1z_2/w_1)\\ &+\binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\mathrm{Li}_{(t_1,t_2,t_3)}(z_1, z_2, w_1/(z_1z_2))\Bigg].\end{aligned}$$

Taking $w_1 = z_1 = z_2 = 1$ (or by Corollary 2.4) we obtain the relation in Eq. (3) among MZVs.

2.4.3. *The case of $r = s = 2$.* In this case $\left[\begin{smallmatrix}\vec{r}\\ \vec{w}\end{smallmatrix}\right] = \left[\begin{smallmatrix}r_1,r_2\\ w_1,w_2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\vec{s}\\ \vec{z}\end{smallmatrix}\right] = \left[\begin{smallmatrix}s_1,s_2\\ z_1,z_2\end{smallmatrix}\right]$. Let $\vec{t} = (t_1,t_2,t_3,t_4) \in \mathbb{Z}^4_{\geqslant 1}$ with $t_1+t_2+t_3+t_4 = r_1+r_2+s_1+s_2$. Then there are $\binom{4}{2} = 6$ choices of $(\varphi,\psi) \in \mathfrak{I}_{2,2}$.

If $\varphi(1)=1$, $\varphi(2)=2$, $\psi(1)=3$ and $\psi(2)=4$, by Eq. (18), we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1) = \binom{t_1-1}{r_1-1}, \quad c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(2) = \binom{t_2-1}{r_2-1},$$

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(3) = \binom{t_3-1}{t_1+t_2+t_3-r_1-r_2-s_1} = \binom{t_3-1}{s_2-t_4}, \quad c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(4) = \binom{t_4-1}{s_2-1}$$

and thus

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1)\,c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(2)\,c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(3)\,c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(4) = \binom{t_1-1}{r_1-1}\binom{t_2-1}{r_2-1}\binom{t_3-1}{s_2-t_4}\binom{t_4-1}{s_2-1}.$$

Similarly, if $\varphi(1)=3$, $\varphi(2)=4$, $\psi(1)=1$ and $\psi(2)=2$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{s_1-1}\binom{t_2-1}{s_2-1}\binom{t_3-1}{r_2-t_4}\binom{t_4-1}{r_2-1}.$$

If $\varphi(1)=1$, $\varphi(2)=3$, $\psi(1)=2$ and $\psi(2)=4$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{r_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{s_2-t_4}.$$

If $\varphi(1)=2$, $\varphi(2)=4$, $\psi(1)=1$ and $\psi(2)=3$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{r_2-t_4}.$$

If $\varphi(1)=1$, $\varphi(2)=4$, $\psi(1)=2$ and $\psi(2)=3$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{s_2-1}.$$

If $\varphi(1)=2$, $\varphi(2)=3$, $\psi(1)=1$ and $\psi(2)=4$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \binom{t_1-1}{s_1-1}\binom{t_2-1}{t_1+t_2-r_1-s_1}\binom{t_3-1}{r_2-1}.$$

Then from Corollary 2.4, we obtain Eq. (4). We likewise obtain formulas for the products of double multiple polylogarithms and double alternating Euler sums.

3. Preparational lemmas

In this section we prove some properties of the coefficients $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}$ in our Theorem 2.1 and Theorem 2.2 in preparation for their proofs in the next section.

We recall some notations from Section 2.2. Let $k,\ell \geqslant 1$, $\vec{r}\in\mathbb{Z}_{\geqslant 1}^k$, $\vec{s}\in\mathbb{Z}_{\geqslant 1}^\ell$, $\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell}$ with $|\vec{t}|=|\vec{r}|+|\vec{s}|$ and $(\varphi,\psi)\in\mathcal{I}_{k,\ell}$ be given. For $1\leqslant i\leqslant k+\ell$, denote

$$h_{(\varphi,\psi),i} = h_{(\varphi,\psi),(\vec{r},\vec{s}),i} = \begin{cases} r_j & \text{if } i=\varphi(j), \\ s_j & \text{if } i=\psi(j). \end{cases} \tag{27}$$

We note that, if we define

$$\varepsilon_{\varphi,\psi}(i) = \begin{cases} 1 & \text{if} \quad i\in \operatorname{im}(\varphi), \\ -1 & \text{if} \quad i\in \operatorname{im}(\psi), \end{cases} \tag{28}$$

then Eq. (18) can be rewritten as

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \begin{cases} \binom{t_i-1}{h_{(\varphi,\psi),i}-1} & \begin{array}{l}\text{if } i=1 \\ \text{or if } i\geqslant 2 \text{ and } \varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1)=1,\end{array} \\ \binom{t_i-1}{\sum_{j=1}^{i} t_j-\sum_{j=1}^{i} h_{(\varphi,\psi),j}} & \text{if } i\geqslant 2 \text{ and } \varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1)=-1. \end{cases} \tag{29}$$

Also recall

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i).$$

For the inductive proof to work, we also include the case when one of k or ℓ (but not both) is zero which corresponds to the case when $\vec{\mu}$ or $\vec{\nu}\in\mathcal{H}_0^*(\widehat{G})$ is the empty word 1. We will use the convention that $\mathbb{Z}_{\geqslant 1}^0=\{\mathbf{e}\}$ and denote $|\mathbf{e}|=0$. When $k=0$, $\ell\geqslant 1$, we will also denote $\vec{r}=\mathbf{e}$, denote $\mathbf{f}:[k](=\emptyset)\to[k+\ell]=[\ell]$ and denote $\mathcal{I}_{0,\ell}=\{(\mathbf{f},\mathrm{id}_{[\ell]})\}$. Similarly, when $\ell=0$, $k\geqslant 1$, we denote $\vec{s}=\mathbf{e}$, $\mathbf{f}:[\ell]\to[k+\ell]=[k]$ and $\mathcal{I}_{k,0}=\{(\mathrm{id}_{[k]},\mathbf{f})\}$. Then the notations in Eq. (27) – (29) still make sense even if exactly one of k and ℓ is zero. More precisely, when $k=0$, $\ell\geqslant 1$, we have $h_{(\mathbf{f},\mathrm{id}_{[\ell]}),(\mathbf{e},\vec{s}),i}=s_i$, $\varepsilon_{\mathbf{f},\mathrm{id}_{[\ell]}}(i)=-1, 1\leqslant i\leqslant \ell$. Also, for any $\vec{s}$ and $\vec{t}\in\mathbb{Z}^{\ell}_{\geqslant 1}$ with $|\vec{s}|=|\vec{t}|$, we have

$$c^{\vec{t},(\mathbf{f},\mathrm{id}_{[\ell]})}_{\mathbf{e},\vec{s}}=\prod_{i=1}^{\ell}\binom{t_i-1}{s_i-1}=\prod_{i=1}^{\ell}\delta^{t_i}_{s_i}. \tag{30}$$

Similarly, if $\vec{s}=\mathbf{e}$, then for any $\vec{r},\vec{t}\in\mathbb{Z}^{k}_{\geqslant 1}$ with $|\vec{r}|=|\vec{t}|$, we have $h_{(\mathrm{id}_{[k]},\mathbf{f}),(\vec{r},\mathbf{e}),i}=r_i$, $\varepsilon_{\mathrm{id}_{[k]},\mathbf{f}}(i)=1, 1\leqslant i\leqslant k$ and

$$c^{\vec{t},(\mathrm{id}_{[k]},\mathbf{f})}_{\vec{r},\mathbf{e}}=\prod_{i=1}^{k}\delta^{t_i}_{r_i}. \tag{31}$$

We first give some conditions for the vanishing of $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}$.

Lemma 3.1. *Let $k,\ell\geqslant 1$. Let $\vec{r}\in\mathbb{Z}^{k}_{\geqslant 1}$, $\vec{s}\in\mathbb{Z}^{\ell}_{\geqslant 1}$ and $\vec{t}\in\mathbb{Z}^{k+\ell}_{\geqslant 1}$ with $|\vec{r}|+|\vec{s}|=|\vec{t}|$. Let $(\varphi,\psi)\in\mathfrak{I}_{k,\ell}$. Then $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}\neq 0$ if and only if, for $1\leqslant i\leqslant k+\ell$,*

$$\begin{cases} t_i\geqslant h_{(\varphi,\psi),i}, & \text{if } i=1 \text{ or if } i\geqslant 2 \text{ and } \varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1)=1,\\ \sum\limits_{j=1}^{i}t_j\geqslant\sum\limits_{j=1}^{i}h_{(\varphi,\psi),j}>\sum\limits_{j=1}^{i-1}t_j, & \text{if } i\geqslant 2 \text{ and } \varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1)=-1.\end{cases}$$

Proof. By definition, $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}\neq 0$ if and only if $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}(i)\neq 0$ for every $i\in[k+\ell]$. Also $\binom{a}{b}\neq 0$ if and only if $a\geqslant b\geqslant 0$. Then the lemma follows since

$$\Big(t_i-1\geqslant h_{(\varphi,\psi),i}-1\geqslant 0\Big)\Leftrightarrow\Big(t_i\geqslant h_{(\varphi,\psi),i}\geqslant 1\Big)$$

and

$$\Big(t_i-1\geqslant\sum_{j=1}^{i}t_j-\sum_{j=1}^{i}h_{(\varphi,\psi),i}\geqslant 0\Big)\Leftrightarrow\Big(-\sum_{j=1}^{i-1}t_j>-\sum_{j=1}^{i-1}t_j-1\geqslant-\sum_{j=1}^{i}h_{(\varphi,\psi),j}\geqslant-\sum_{j=1}^{i}t_j\Big).$$

□

Lemma 3.2. *Let $k,\ell,\vec{r},\vec{s},\vec{t}$ be as in Lemma 3.1.*

(a) *Let $(\varphi,\psi)\in\mathfrak{I}_{k,\ell}$. If $\varphi(1)=1$, $s_1=1$ and $t_1>r_1$ or if $\psi(1)=1$, $r_1=1$ and $t_1>s_1$, then*
$$c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}=0.$$

(b) *If $t_1<\min(r_1,s_1)$, then $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}=0$ for any $(\varphi,\psi)\in\mathfrak{I}_{k,\ell}$.*

Proof. (a). We only consider the case when $\varphi(1)=1, s_1=1$ and $t_1>r_1$. The proof of the other case is similar. Since $\varphi(1)=1$, we have $\psi(1)>1$. This means that $h_{(\varphi,\psi),i}=r_i$ for $1\leqslant i\leqslant\psi(1)-1$ and $h_{(\varphi,\psi),\psi(1)}=s_1$. Suppose $c^{\vec{t},(\varphi,\psi)}_{\vec{r},\vec{s}}\neq 0$. Then by Lemma 3.1, we have $t_i\geqslant r_i$ for $2\leqslant i\leqslant\psi(1)-1$ and $\sum\limits_{j=1}^{\psi(1)-1}r_j+s_1>\sum\limits_{j=1}^{\psi(1)-1}t_j$ by taking $i=\psi(1)$. From these two inequalities, we obtain $r_1+s_1>t_1$ and hence $r_1\geqslant t_1$ since $s_1=1$. This is a contradiction.

(b) If $t_1 < \min(r_1, s_1)$, then $t_1 < h_{(\varphi,\psi),1}$. So by Lemma 3.1, for every $(\varphi, \psi) \in \mathfrak{I}_{k,\ell}$ we have $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = 0$. ☐

We next give some relations among the numbers $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$ as the parameters vary.

Definition 3.3. *Let $\vec{e}_1$ denote $(1, 0, \cdots, 0)$ of suitable dimension. So for any vector $\vec{x} = (x_1, x_2, \cdots, x_k)$ and $a \in \mathbb{Z}$, we have*

$$\vec{x} - a\vec{e}_1 = (x_1 - a, x_2, \cdots, x_k).$$

Define

$$\vec{x}' = (x'_1, \cdots, x'_{k-1}) := (x_2, \cdots, x_k)$$

with the convention that $(x_1)' = \mathbf{e}$. For a function f on $[k]$, let $f^\sharp$ and $f^\flat$ be respectively the functions on $[k-1]$ and $[k]$ defined by

$$f^\sharp(x) = f(x+1) - 1, \quad f^\flat(x) = f(x) - 1$$

with the convention that $[0] = \emptyset$ and that, if f is a function on $[1]$, then $f^\sharp = \mathbf{f}$. Let $f^\&$ and f^* be respectively the functions on $[k+1]$ and $[k]$ defined by

$$f^\&(1) = 1,\ f^\&(x) = f(x-1) + 1, \quad f^*(y) = f(y) + 1, \quad 2 \leqslant x \leqslant r+1, 1 \leqslant y \leqslant r.$$

Also define

$$\mathfrak{I}_{k,\ell,\varphi(1)=1} = \{(\varphi, \psi) \in \mathfrak{I}_{k,\ell} \mid \varphi(1) = 1\}, \quad \mathfrak{I}_{k,\ell,\psi(1)=1} = \{(\varphi, \psi) \in \mathfrak{I}_{k,\ell} \mid \psi(1) = 1\}.$$

Lemma 3.4. *Let $k, \ell \geqslant 1$. The map*

$$(^\sharp, {}^\flat) : \mathfrak{I}_{k,\ell,\varphi(1)=1} \longrightarrow \mathfrak{I}_{k-1,\ell}, \quad (\varphi, \psi) \mapsto (\varphi^\sharp, \psi^\flat)$$

is a bijection whose inverse is given by

$$(^\&, {}^*) : \mathfrak{I}_{k-1,\ell} \longrightarrow \mathfrak{I}_{k,\ell,\varphi(1)=1}, \quad (\varphi, \psi) \mapsto (\varphi^\&, \psi^*).$$

Similarly, the map

$$(^\flat, {}^\sharp) : \mathfrak{I}_{k,\ell,\psi(1)=1} \longrightarrow \mathfrak{I}_{k,\ell-1}, \quad (\varphi, \psi) \mapsto (\varphi^\flat, \psi^\sharp)$$

is a bijection whose inverse is given by

$$(^*, {}^\&) : \mathfrak{I}_{k,\ell-1} \longrightarrow \mathfrak{I}_{k,\ell,\psi(1)=1}, \quad (\varphi, \psi) \mapsto (\varphi^*, \psi^\&).$$

Proof. From the definition we verify that

$$(^\sharp, {}^\flat)(\mathfrak{I}_{k,\ell,\varphi(1)=1}) \subseteq \mathfrak{I}_{k-1,\ell}$$

and

$$(^\&, {}^*)(\mathfrak{I}_{k-1,\ell}) \subseteq \mathfrak{I}_{k,\ell,\varphi(1)=1}.$$

Then to prove the first assertion we only need to show that $(\varphi^\sharp)^\& = \varphi$ and $(\psi^\flat)^* = \psi$ if $\varphi(1) = 1$, and that $(\varphi^\&)^\sharp = \varphi$ and $(\psi^*)^\flat = \psi$. We just check the first equation and leave the others to the interested reader. First we have $(\varphi^\sharp)^\&(1) = 1$ by definition. Since $\varphi(1) = 1$, we have $(\varphi^\sharp)^\&(i) = \varphi(i)$ when $i = 1$. If $i \geqslant 2$, then by definition we have $\varphi^\sharp(i-1) = \varphi(i) - 1$ and $(\varphi^\sharp)^\&(i) = \varphi^\sharp(i-1) + 1 = \varphi(i)$, as desired.

The proof of the second assertion in the lemma is similar. ☐

Lemma 3.5. *Let $k, \ell, \vec{r}, \vec{s}, \vec{t}$ and (φ, ψ) be as in Lemma 3.1.*

(a) *Let a and b be integers such that $a < \min(t_1, r_1)$, $b < \min(t_1, s_1)$. Then for all $i \in \{2, \cdots, k+\ell\}$, we have*

$$c_{\vec{r}-a\vec{e}_1,\vec{s}}^{\vec{t}-a\vec{e}_1,(\varphi,\psi)}(i) = c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)$$

and

$$c_{\vec{r},\vec{s}-b\vec{e}_1}^{\vec{t}-b\vec{e}_1,(\varphi,\psi)}(i) = c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i).$$

(b) *If $\varphi(1) = 1$ and $r_1 = t_1 = 1$, then*

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i+1) = c_{\vec{r}',\vec{s}}^{\vec{t}',(\varphi^\sharp,\psi^\flat)}(i), \quad 1 \leqslant i \leqslant k+\ell-1, \tag{32}$$

with the notations in Definition 3.3. Similarly, if $\psi(1) = 1$ and $s_1 = t_1 = 1$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i+1) = c_{\vec{r},\vec{s}'}^{\vec{t}',(\varphi^\flat,\psi^\sharp)}(i), \quad 1 \leqslant i \leqslant k+\ell-1. \tag{33}$$

Proof. (a) We prove the first equality. The proof for the second equality is similar. Since $a < \min(r_1, t_1)$, we have $\vec{r} - a\vec{e}_1 \in \mathbb{Z}_{\geqslant 1}^k$ and $\vec{t} - a\vec{e}_1 \in \mathbb{Z}_{\geqslant 1}^{k+\ell}$. For better distinction, we will use the full notation $h_{(\varphi,\psi),(\vec{r},\vec{s}),i}$ defined in Eq. (27) instead of its abbreviation $h_{(\varphi,\psi),i}$. Then we have

$$h_{(\varphi,\psi),(\vec{r}-a\vec{e}_1,\vec{s}),i} = \begin{cases} h_{(\varphi,\psi),(\vec{r},\vec{s}),i} & \text{if } i \neq \varphi(1), \\ h_{(\varphi,\psi),(\vec{r},\vec{s}),i} - a & \text{if } i = \varphi(1). \end{cases} \tag{34}$$

Let $i \in \{2, \cdots, k+\ell\}$. If $\varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1) = 1$, then $i \neq \varphi(1)$. Indeed, if $i = \varphi(1)$, then $i-1$ must be in $\operatorname{im}(\psi)$, implying that $\varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1) = -1$. Thus

$$c_{\vec{r}-a\vec{e}_1,\vec{s}}^{\vec{t}-a\vec{e}_1,(\varphi,\psi)}(i) = \binom{t_i - 1}{h_{(\varphi,\psi),(\vec{r}-a\vec{e}_1,\vec{s}),i-1}} = \binom{t_i - 1}{h_{(\varphi,\psi),(\vec{r},\vec{s}),i-1}} = c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i).$$

If $\varepsilon_{\varphi,\psi}(i)\varepsilon_{\varphi,\psi}(i-1) = -1$, then either $i = \varphi(j)$ or $i-1 = \varphi(j)$ for some $j \in [k]$. In either case, we have $i \geqslant \varphi(1)$ since φ keeps the order. Thus by Eq. (34), we have

$$\sum_{j=1}^{i} h_{(\varphi,\psi),(\vec{r}-a\vec{e}_1,\vec{s}),j} = \sum_{j=1}^{i} h_{(\varphi,\psi),(\vec{r},\vec{s}),j} - a.$$

So

$$c_{\vec{r}-a\vec{e}_1,\vec{s}}^{\vec{t}-a\vec{e}_1,(\varphi,\psi)}(i) = \binom{t_i - 1}{(t_1-a)+\sum\limits_{j=2}^{i} t_j - \sum\limits_{j=1}^{i} h_{(\varphi,\psi),(\vec{r}-a\vec{e}_1,\vec{s}),j}} = \binom{t_i - 1}{\sum\limits_{j=1}^{i} t_j - \sum\limits_{j=1}^{i} h_{(\varphi,\psi),(\vec{r},\vec{s}),j}} = c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i).$$

(b) Let $\varphi(1) = 1$ and $r_1 = t_1 = 1$. By Eq. (27), for $1 \leqslant i \leqslant k+\ell-1$,

$$\begin{aligned} h_{(\varphi,\psi),(\vec{r},\vec{s}),i+1} &= \begin{cases} r_j & \text{if } i+1 = \varphi(j) \\ s_j & \text{if } i+1 = \psi(j) \end{cases} = \begin{cases} r'_{j-1} & \text{if } i = \varphi(j)-1 \\ s_j & \text{if } i = \psi(j)-1 \end{cases} \\ &= \begin{cases} r'_j & \text{if } i = \varphi(j+1)-1 \\ s_j & \text{if } i = \psi(j)-1 \end{cases} = \begin{cases} r'_j & \text{if } i = \varphi^\sharp(j) \\ s_j & \text{if } i = \psi^\flat(j). \end{cases} \end{aligned}$$

Thus

$$h_{(\varphi,\psi),(\vec{r},\vec{s}),i+1} = h_{(\varphi^\sharp,\psi^\flat),(\vec{r}\,',\vec{s}),i},\ 1 \leqslant i \leqslant k+\ell-1. \tag{35}$$

Also, for $1 \leqslant i \leqslant k+\ell-1$, since $\varphi(1)=1$, we have

$$i+1 \in \mathrm{im}(\varphi) \Leftrightarrow i+1=\varphi(j), j\in\{2,\cdots,k\} \Leftrightarrow i=\varphi^\sharp(j-1), j-1\in[k-1] \Leftrightarrow i\in \mathrm{im}(\varphi^\sharp).$$

Similarly, $i+1\in\mathrm{im}(\psi) \Leftrightarrow i \in \mathrm{im}(\psi^\flat)$. Thus

$$\varepsilon_{\varphi,\psi}(i+1) = \varepsilon_{\varphi^\sharp,\psi^\flat}(i),\ 1\leqslant i\leqslant k+\ell-1. \tag{36}$$

We now verify Eq. (32) for $i=1$. Since $\varphi(1)=1$, either $2=\varphi(2)$ or $2=\psi(1)$. If $2=\varphi(2)$, then $\varepsilon_{\varphi,\psi}(2)\varepsilon_{\varphi,\psi}(1)=1$ and so

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(2) = \binom{t_2-1}{r_2-1} = \binom{t_1'-1}{r_1'-1} = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(1).$$

If $\psi(1)=2$, then $\varepsilon_{\varphi,\psi}(2)\varepsilon_{\varphi,\psi}(1)=-1$. So by the condition that $r_1=t_1=1$, we obtain

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(2) = \binom{t_2-1}{t_1+t_2-r_1-s_1} = \binom{t_2-1}{t_2-s_1} = \binom{t_2-1}{s_1-1} = \binom{t_1'-1}{s_1-1} = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(1).$$

Next consider $i\geqslant 2$. By Eq. (35) and Eq. (36), we have

$$\begin{aligned} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i+1) &= \begin{cases} \dbinom{t_{i+1}-1}{h_{(\varphi,\psi),(\vec{r},\vec{s}),i+1}-1} & \text{if } \varepsilon_{\varphi,\psi}(i+1)\varepsilon_{\varphi,\psi}(i)=1,\\ \dbinom{t_{i+1}-1}{\sum_{j=1}^{i+1} t_j - \sum_{j=1}^{i+1} h_{(\varphi,\psi),(\vec{r},\vec{s}),j}} & \text{if } \varepsilon_{\varphi,\psi}(i+1)\varepsilon_{\varphi,\psi}(i)=-1, \end{cases}\\ &= \begin{cases} \dbinom{t_i'-1}{h_{(\varphi^\sharp,\psi^\flat),(\vec{r}\,',\vec{s}),i}-1} & \text{if } \varepsilon_{\varphi^\sharp,\psi^\flat}(i)\varepsilon_{\varphi^\sharp,\psi^\flat}(i-1)=1,\\ \dbinom{t_i'-1}{\sum_{j=1}^{i} t_j' - \sum_{j=1}^{i} h_{(\varphi^\sharp,\psi^\flat),(\vec{r}\,',\vec{s}),j}} & \text{if } \varepsilon_{\varphi^\sharp,\psi^\flat}(i)\varepsilon_{\varphi^\sharp,\psi^\flat}(i-1)=-1, \end{cases} \end{aligned}$$

since $t_1=1$ and $h_{(\varphi,\psi),(\vec{r},\vec{s}),1}=r_1=1$. Therefore, we have $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i+1) = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(i)$ when $i\geqslant 2$.

The proof for Eq. (33) is similar. □

Lemma 3.6. *Let $k,\ell,\vec{r},\vec{s},\vec{t}$ and (φ,ψ) be as in Lemma 3.1.*

(a) *Suppose that $r_1\geqslant 2$ and $s_1\geqslant 2$. If $t_1\geqslant 2$, then we have*

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)} + c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)} \tag{37}$$

If $t_1=1$, then we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = 0. \tag{38}$$

(b) *Suppose that $r_1=s_1=1$. If $\varphi(1)=1$ and $t_1=1$, then we have*

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}=c_{\vec{r}',\vec{s}}^{\vec{t}',(\varphi^\sharp,\psi^\flat)} \tag{39}$$

with the notations in Definition 3.3. If $\psi(1)=1$ and $t_1=1$, then we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}=c_{\vec{r},\vec{s}'}^{\vec{t}',(\varphi^\flat,\psi^\sharp)}. \tag{40}$$

If $t_1\geqslant 2$, then we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}=0. \tag{41}$$

(c) *Suppose that $r_1=1$ and $s_1\geqslant 2$. If $\varphi(1)=1$ and $t_1=1$, then we have*

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}=c_{\vec{r}',\vec{s}}^{\vec{t}',(\varphi^\sharp,\psi^\flat)}. \tag{42}$$

If $\psi(1)=1$ and $t_1=1$, then we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}=0. \tag{43}$$

If $t_2\geqslant 2$, then we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}=c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}. \tag{44}$$

Similar statements hold when $r_1\geqslant 2$ and $s_1=1$.

Proof. (a) If $\varphi(1)=1$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1)=\binom{t_1-1}{r_1-1}=\binom{t_1-2}{r_1-2}+\binom{t_1-2}{r_1-1}=c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1)+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1).$$

Similarly, if $\psi(1)=1$, we also have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1)=c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1)+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1).$$

In either case, by Lemma 3.5.(a) we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)=c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(i)=c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(i)$$

when $i\in\{2,\cdots,k+\ell\}$. Hence

$$\begin{aligned}
c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} &= \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\\
&= (c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1)+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1))\prod_{i=2}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i)\\
&= \prod_{i=1}^{k+\ell} c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(i)+\prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(i)\\
&= c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}.
\end{aligned}$$

This proves Eq. (37). Eq (38) follows from Lemma 3.2 (b).

(b) First we assume that $t_1 = 1$. For $(\varphi, \psi) \in \mathfrak{I}_{k,\ell}$, either $\varphi(1) = 1$ or $\psi(1) = 1$. If $\varphi(1) = 1$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1) = \binom{t_1-1}{r_1-1} = \binom{0}{0} = 1$$

and by Lemma 3.5.(b) we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i+1) = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(i).$$

Hence

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \prod_{i=2}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \prod_{i=1}^{k+\ell-1} c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(i) = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}.$$

This proves Eq. (39). The proof of Eq. (40) is similar. The equality for $t_1 \geqslant 2$ follows from Lemma 3.2.(a).

(c) Suppose that $r_1 = 1$ and $s_1 \geqslant 2$.

Case 1: $t_1 = 1$**.** We consider the case of $\varphi(1) = 1$. By Lemma 3.5.(b) we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i+1) = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(i).$$

Combining this with

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1) = \binom{t_1-1}{r_1-1} = \binom{0}{0} = 1,$$

we obtain

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \prod_{i=1}^{k+\ell-1} c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}(i) = c_{\vec{r}\,',\vec{s}}^{\vec{t}\,',(\varphi^\sharp,\psi^\flat)}.$$

This proves Eq. (42). If $\psi(1) = 1$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1) = \binom{t_1-1}{s_1-1} = \binom{0}{s_1-1} = 0$$

since $s_1 - 1 \geqslant 1$ and so $c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = 0$, as needed.

Case 2: $t_1 \geqslant 2$**.** We will consider the four subcases when $\psi(1) = 1$ and $t_1 < s_1$, when $\psi(1) = 1$ and $t_1 > s_1$, when $\psi(1) = 1$ and $t_1 = s_1$, and when $\varphi(1) = 1$.

If $\psi(1) = 1$ and $t_1 < s_1$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = 0 = c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}$$

by Lemma 3.1. If $\psi(1) = 1$ and $t_1 > s_1$, then by Lemma 3.2.(a) we also have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = 0 = c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}.$$

So in these two subcases (44) holds.

Now if $\psi(1) = 1$ and $t_1 = s_1$, then

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1) = \binom{t_1-1}{s_1-1} = 1 = \binom{t_1-2}{s_1-2} = c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1).$$

If $\varphi(1) = 1$, then since $r_1 = 1$, we have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(1) = \binom{t_1-1}{r_1-1} = \binom{t_1-1}{0} = 1 = \binom{t_1-2}{0} = \binom{t_1-2}{r_1-1} = c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(1).$$

In both subcases, by Lemma 3.5.(a) we always have

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(i).$$

for $i \geqslant 2$. Therefore,

$$c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} = \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)}(i) = \prod_{i=1}^{k+\ell} c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}(i) = c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}.$$

This proves (44).

The proof for the instance of $r_1 \geqslant 2$ and $s_1 = 1$ is similar. □

4. Proof of the main theorems

We first show that, under the condition that G is an abelian group, Theorem 2.1 and Theorem 2.2 are equivalent. Then we only need to prove Theorem 2.1. This is done in Section 4.2.

4.1. **The equivalence between Theorem 2.1 and Theorem 2.2.** We start with a lemma.

Lemma 4.1. *Let G be an abelian group. With the notations in Eq. (12), (16) and (21), we have*

$$\theta(\vec{a} ш_{(\varphi,\psi)} \vec{b}) = \theta(\vec{a}) \star_{(\varphi,\psi)} \theta(\vec{b}). \tag{45}$$

Proof. Let $\vec{w} = \theta(\vec{a})$ and $\vec{z} = \theta(\vec{b})$. Then by Eq. (12), we have $w_j = 1/a_1$ when $j = 1$ and $w_j = a_{j-1}/a_j$ when $j \geqslant 2$. Similarly, $z_j = 1/b_1$ when $j = 1$ and $z_j = b_{j-1}/b_j$ when $j \geqslant 2$.

Recall Eq. (16):

$$(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i = \begin{cases} a_j & \text{if } i = \varphi(j), \\ b_j & \text{if } i = \psi(j). \end{cases}$$

If $i = 1$, we have

$$\theta(\vec{a} ш_{(\varphi,\psi)} \vec{b})_1 = (\vec{a} ш_{(\varphi,\psi)} \vec{b})_1^{-1} = \begin{cases} a_1^{-1} = w_1 & \text{if } 1 = \varphi(1) \\ b_1^{-1} = z_1 & \text{if } 1 = \psi(1) \end{cases} = (\vec{w} \star_{(\varphi,\psi)} \vec{z})_1.$$

Next let $i \geqslant 2$. Assume that $i \in \mathrm{im}(\varphi)$, say $i = \varphi(j)$ for some $j \in [k]$. If $i-1 \in \mathrm{im}(\varphi)$, then $j \geqslant 2$ and $i - 1 = \varphi(j-1)$. Thus

$$\theta(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i = \frac{(\vec{a} ш_{(\varphi,\psi)} \vec{b})_{i-1}}{(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i} = \frac{a_{j-1}}{a_j} = w_j.$$

If $i - 1 \in \mathrm{im}(\psi)$, then $i - 1 = \psi(i-j)$. Thus

$$\theta(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i = \frac{(\vec{a} ш_{(\varphi,\psi)} \vec{b})_{i-1}}{(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i} = \frac{b_{i-j}}{a_j} = \frac{w_1 \cdots w_j}{z_1 \cdots z_{i-j}}.$$

Hence by Eq. (21),

$$\theta(\vec{a} ш_{(\varphi,\psi)} \vec{b})_i = (\vec{w} \star_{(\varphi,\psi)} \vec{z})_i$$

when $i \in \mathrm{im}(\varphi)$. A similar argument shows that the above equality also holds when $i \in \mathrm{im}(\psi)$. This proves (45). □

Proposition 4.2. *When G is an abelian group, then Theorem 2.2 is equivalent to Theorem 2.1.*

Proof. By the definitions of θ, $ш_\eta$ and $ш_\rho$, we see that θ is an algebra isomorphism from $\mathcal{H}^{ш_\rho}(\widehat{G}) = (\mathcal{H}(\widehat{G}), ш_\rho)$ to $\mathcal{H}^{ш_\eta}(\widehat{G}) = (\mathcal{H}(\widehat{G}), ш_\eta)$. So for any $\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix}\right], \left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right] \in \mathcal{H}^{ш_\rho}(\widehat{G})$,

$$\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right] = \sum_{\substack{(\varphi,\psi)\in \mathcal{I}_{k,\ell} \\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}|+|\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{smallmatrix} \vec{t} \\ \vec{a}\, ш_{(\varphi,\psi)} \vec{b} \end{smallmatrix}\right]$$

$$\Leftrightarrow \theta(\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right]) = \theta(\sum_{\substack{(\varphi,\psi)\in \mathcal{I}_{k,\ell} \\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}|+|\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{smallmatrix} \vec{t} \\ \vec{a}\, ш_{(\varphi,\psi)} \vec{b} \end{smallmatrix}\right])$$

$$\Leftrightarrow \theta(\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix}\right]) ш_\eta \theta(\left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right]) = \sum_{\substack{(\varphi,\psi)\in \mathcal{I}_{k,\ell} \\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}|+|\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \theta(\left[\begin{smallmatrix} \vec{t} \\ \vec{a}\, ш_{(\varphi,\psi)} \vec{b} \end{smallmatrix}\right])$$

$$\Leftrightarrow \left[\begin{smallmatrix} \vec{r} \\ \theta(\vec{a}) \end{smallmatrix}\right] ш_\eta \left[\begin{smallmatrix} \vec{s} \\ \theta(\vec{b}) \end{smallmatrix}\right] = \sum_{\substack{(\varphi,\psi)\in \mathcal{I}_{k,\ell} \\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}|+|\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{smallmatrix} \vec{t} \\ \theta(\vec{a}\, ш_{(\varphi,\psi)} \vec{b}) \end{smallmatrix}\right]$$

$$\Leftrightarrow \left[\begin{smallmatrix} \vec{r} \\ \theta(\vec{a}) \end{smallmatrix}\right] ш_\eta \left[\begin{smallmatrix} \vec{s} \\ \theta(\vec{b}) \end{smallmatrix}\right] = \sum_{\substack{(\varphi,\psi)\in \mathcal{I}_{k,\ell} \\ \vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}|+|\vec{s}|}} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{smallmatrix} \vec{t} \\ \theta(\vec{a}) \star_{(\varphi,\psi)} \theta(\vec{b}) \end{smallmatrix}\right] \qquad \text{(by Eq. (45))}.$$

Then the proposition follows from the bijectivity of θ. □

4.2. **Proof of Theorem 2.1.** In this section we prove Theorem 2.1. We first describe recursive relations of $ш_\rho$ that we will use later in the proof.

Let $\mathcal{H}^{ш_\rho +}(\widehat{G})$ be the subring of $\mathcal{H}^{ш_\rho}(\widehat{G})$ generated by $\left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right]$ with $\vec{s} \in \mathbb{Z}_{\geqslant 1}^k, \vec{b} \in G^k, k \geqslant 1$. Then

$$\mathcal{H}^{ш_\rho}(\widehat{G}) = \mathbb{Z} \oplus \mathcal{H}^{ш_\rho +}(\widehat{G}).$$

Define the following operators

$$P : \mathcal{H}^{ш_\rho +}(\widehat{G}) \to \mathcal{H}^{ш_\rho}(\widehat{G}), \quad P(\left[\begin{smallmatrix} s_1, s_2, \cdots, s_k \\ b_1, b_2, \cdots, b_k \end{smallmatrix}\right]) = \left[\begin{smallmatrix} s_1+1, s_2, \cdots, s_k \\ b_1, b_2, \cdots, b_k \end{smallmatrix}\right],$$

$$Q_b : \mathcal{H}^{ш_\rho}(\widehat{G}) \to \mathcal{H}^{ш_\rho}(\widehat{G}), \quad Q_b(\left[\begin{smallmatrix} s_1, \cdots, s_k \\ b_1, \cdots, b_k \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 1, s_1, \cdots, s_k \\ b, b_1, \cdots, b_k \end{smallmatrix}\right], \quad Q_b(1) = \left[\begin{smallmatrix} 1 \\ b \end{smallmatrix}\right].$$

Proposition 4.3. *The multiplication* ш_ρ *on* $\mathcal{H}^{\text{ш}_\rho}(\widehat{G})$ *defined in Eq. (14) is the unique one such that*

$$
\begin{aligned}
&P(\xi_1)\,\text{ш}_\rho\, P(\xi_2) = P\big(\xi_1\,\text{ш}_\eta\, P(\xi_2)\big) + P\big(P(\xi_1)\,\text{ш}_\rho\,\xi_2\big),\ \xi_1,\xi_2 \in \mathcal{H}^{\text{ш}_\rho\,+}(\widehat{G}),\\
&Q_a(\xi_1)\,\text{ш}_\rho\, Q_b(\xi_2) = Q_a\big(\xi_1\,\text{ш}_\rho\, Q_b(\xi_2)\big) + Q_b\big(Q_a(\xi_1)\,\text{ш}_\rho\,\xi_2\big),\ \xi_1,\xi_2 \in \mathcal{H}^{\text{ш}_\rho}(\widehat{G}),\\
&P(\xi_1)\,\text{ш}_\rho\, Q_b(\xi_2) = Q_b\big(P(\xi_1)\,\text{ш}_\rho\,\xi_2\big) + P\big(\xi_1\,\text{ш}_\rho\, Q_b(\xi_2)\big),\ \xi_1 \in \mathcal{H}^{\text{ш}_\rho\,+}(\widehat{G}), \xi_2 \in \mathcal{H}^{\text{ш}_\rho}(\widehat{G}),\\
&Q_b(\xi_1)\,\text{ш}_\rho\, P(\xi_2) = Q_b\big(\xi_1\,\text{ш}_\rho\, P(\xi_2)\big) + P\big(Q_b(\xi_1)\,\text{ш}_\rho\,\xi_2\big),\ \xi_1 \in \mathcal{H}^{\text{ш}_\rho}(\widehat{G}), \xi_2 \in \mathcal{H}^{\text{ш}_\rho\,+}(\widehat{G}).
\end{aligned}
$$

with the initial condition that $1\,\text{ш}_\rho\,\xi = \xi\,\text{ш}_\rho\, 1 = \xi$ *for* $\xi \in \mathcal{H}^{\text{ш}_\rho}(\widehat{G})$.

Proof. Let $\mathcal{H}^{\text{ш}\,+}_1(\overline{G})$ be the subring of $\mathcal{H}^{\text{ш}}_1(\overline{G})$ generated by words of the form ux_b with $b \in G$. Then

$$\mathcal{H}^{\text{ш}}_1(\overline{G}) = \mathbb{Z} \oplus \mathcal{H}^{\text{ш}\,+}_1(\overline{G}).$$

Define operators

$$
\begin{aligned}
I_0 :&\mathcal{H}^{\text{ш}\,+}_1(\overline{G}) \to \mathcal{H}^{\text{ш}}_1(\overline{G}), \quad I_0(u) = x_0 u,\\
I_b :&\mathcal{H}^{\text{ш}}_1(\overline{G}) \to \mathcal{H}^{\text{ш}}_1(\overline{G}), \quad I_b(u) = \begin{cases} x_b u, & u \neq 1,\\ x_b, & u = 1,\end{cases}
\end{aligned}
$$

for $b \in G$. Then the well-known recursive formula of the shuffle product

$$(a_1\mathfrak{a})\,\text{ш}\,(b_1\mathfrak{b}) = a_1(\mathfrak{a}\,\text{ш}\,(b_1\mathfrak{b})) + b_1((a_1\mathfrak{a})\,\text{ш}\,\mathfrak{b}), a_1, b_1 \in \overline{G}, \mathfrak{a}, \mathfrak{b} \in M(\overline{G})$$

can be rewritten as the following relations of I_0 and I_a, I_b, $a, b \in G$,

$$
\begin{aligned}
&I_0(u)\,\text{ш}\, I_0(v) = I_0\big(u\,\text{ш}\, I_0(v)\big) + I_0\big(I_0(u)\,\text{ш}\, v\big), \quad u, v \in \mathcal{H}^{\text{ш}\,+}_1(\overline{G}),\\
&I_a(u)\,\text{ш}\, I_b(v) = I_a\big(u\,\text{ш}\, I_b(v)\big) + I_b\big(I_a(u)\,\text{ш}\, v\big), \quad u, v \in \mathcal{H}^{\text{ш}}_1(\overline{G}),\\
&I_0(u)\,\text{ш}\, I_b(v) = I_0\big(u\,\text{ш}\, I_b(v)\big) + I_b\big(I_0(u)\,\text{ш}\, v\big), \quad u \in \mathcal{H}^{\text{ш}\,+}_1(\overline{G}), v \in \mathcal{H}^{\text{ш}}_1(\overline{G}),\\
&I_b(u)\,\text{ш}\, I_0(v) = I_b\big(u\,\text{ш}\, I_0(v)\big) + I_0\big(I_b(u)\,\text{ш}\, v\big), \quad u \in \mathcal{H}^{\text{ш}}_1(\overline{G}), v \in \mathcal{H}^{\text{ш}\,+}_1(\overline{G}).
\end{aligned}
\tag{46}
$$

Under the bijection $\rho : \mathcal{H}^{\text{ш}}_1(\overline{G}) \to \mathcal{H}^{\text{ш}_\rho}(\widehat{G})$ in Eq. (13), I_0 and I_b, $b \in G$, are sent to P and Q_b, $b \in G$, respectively. Further the relations in Eq. (46) for I_0 and I_b ($b \in G$) take the form in Proposition 4.3. Finally, since ш is the unique multiplication on $\mathcal{H}^{\text{ш}}_1(\overline{G})$ characterized by its recursive relation Eq. (46) and the initial condition $1\,\text{ш}\, u = u\,\text{ш}\, 1 = u$, ш_ρ is also unique as characterized. □

Note that P cannot be defined by a left multiplication even though its counter part I_0 can.

For $\vec{b} \in G^k$, recall the following notation from Definition 3.3:

$$\vec{b}\,' = (b'_1, \cdots, b'_{k-1}) := (b_2, \cdots, b_k)$$

with the convention that $\vec{b}\,' = \mathbf{e}$ when $k = 1$. In the proof for Theorem 2.1 we also need the following lemma.

Lemma 4.4. *Let* $\vec{t} \in \mathbb{Z}^{k+\ell-1}_{\geqslant 1}$, $\vec{a} \in G^k$ *and* $\vec{b} \in G^\ell$.

(a) *For any* $(\varphi,\psi)\in \mathfrak{I}_{k-1,\ell}$ *we have*

$$Q_{a_1}(\left[\begin{smallmatrix} \vec{t} \\ \vec{a}'\,ш_{(\varphi,\psi)}\vec{b}\end{smallmatrix}\right]) = \left[\begin{smallmatrix} (1,\vec{t}) \\ \vec{a}\,ш_{(\varphi^{\&},\psi^*)}\vec{b}\end{smallmatrix}\right] \tag{47}$$

with the notations in Eq. (16) and Definition 3.3.

(b) *For any* $(\varphi,\psi)\in \mathfrak{I}_{k,\ell-1}$ *we have*

$$Q_{b_1}(\left[\begin{smallmatrix} \vec{t} \\ \vec{a}\,ш_{(\varphi,\psi)}\vec{b}'\end{smallmatrix}\right]) = \left[\begin{smallmatrix} (1,\vec{t}) \\ \vec{a}\,ш_{(\varphi^*,\psi^{\&})}\vec{b}\end{smallmatrix}\right]. \tag{48}$$

Proof. (a) Let $\vec{\varpi}=(\varpi_1,\cdots,\varpi_{k+\ell-1}):=\vec{a}'\,ш_{(\varphi,\psi)}\vec{b}$ and $\vec{\tau}=(\tau_1,\cdots,\tau_{k+\ell}):=\vec{a}\,ш_{(\varphi^{\&},\psi^*)}\vec{b}$. By the definition of Q_{a_1}, we only need to prove that

$$\tau_i=\begin{cases} a_1 & \text{if } i=1,\\ \varpi_{i-1} & \text{if } i\geqslant 2.\end{cases}$$

Since $\varphi^{\&}(1)=1$, we have $\tau_1=a_1$. Now let $i\geqslant 2$. We have $i\in \operatorname{im}(\varphi^{\&})$ or $i\in\operatorname{im}(\psi^*)$. If $i\in\operatorname{im}(\varphi^{\&})$, say $i=\varphi^{\&}(j)$, then $i-1=\varphi(j-1)$. Thus we have $\tau_i=a_j$ and $\varpi_{i-1}=a'_{j-1}=a_j$. This shows that $\tau_i=\varpi_{i-1}$. If $i\in\operatorname{im}(\psi^*)$, say $i=\psi^*(j)$, then $i-1=\psi(j)$. Thus $\tau_i=b_j$ and $\varpi_{i-1}=b_j$ again showing $\tau_i=\varpi_{i-1}$.

(b). The proof is similar to that for Item. (a). □

Proof of Theorem 2.1. We prove the extended form of (20) where one of k and ℓ, but not both, might be zero. We prove this by induction on $|\vec{r}|+|\vec{s}|\geqslant 1$. If $|\vec{r}|+|\vec{s}|=1$, then exactly one of k and ℓ is zero. So exactly one of $\left[\begin{smallmatrix}\vec{r}\\ \vec{a}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\vec{s}\\ \vec{b}\end{smallmatrix}\right]$ is the identity 1. Then by (30) and (31), there is nothing to prove. For any given integer $n\geqslant 2$, assume that the assertion holds for every pair $(\vec{r},\vec{s})$ with $|\vec{r}|+|\vec{s}|<n$. Now consider $\vec{r}$ and $\vec{s}$ with $|\vec{r}|+|\vec{s}|=n$. If one of k or ℓ is 0, then again by (30) and (31) there is nothing to prove. So we may assume that $k,\ell\geqslant 1$. There are four cases to consider.

Case 1. $r_1\geqslant 2$ **and** $s_1\geqslant 2$**.** Then by Proposition 4.3 and the induction hypothesis, we have

$$\begin{aligned}
\left[\begin{smallmatrix}\vec{r}\\ \vec{a}\end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix}\vec{s}\\ \vec{b}\end{smallmatrix}\right] &= P(\left[\begin{smallmatrix}\vec{r}-\vec{e}_1\\ \vec{a}\end{smallmatrix}\right]) ш_\rho P(\left[\begin{smallmatrix}\vec{s}-\vec{e}_1\\ \vec{b}\end{smallmatrix}\right])\\
&= P(\left[\begin{smallmatrix}\vec{r}-\vec{e}_1\\ \vec{a}\end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix}\vec{s}\\ \vec{b}\end{smallmatrix}\right] + \left[\begin{smallmatrix}\vec{r}\\ \vec{a}\end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix}\vec{s}-\vec{e}_1\\ \vec{b}\end{smallmatrix}\right])\\
&= P\Big(\sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}}\ \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} (c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t},(\varphi,\psi)}+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t},(\varphi,\psi)})\left[\begin{smallmatrix}\vec{t}\\ \vec{a}\,ш_{(\varphi,\psi)}\vec{b}\end{smallmatrix}\right]\Big)\\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}}\ \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} (c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t},(\varphi,\psi)}+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t},(\varphi,\psi)})\left[\begin{smallmatrix}\vec{t}+\vec{e}_1\\ \vec{a}\,ш_{(\varphi,\psi)}\vec{b}\end{smallmatrix}\right]\\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}}\ \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|,t_1\geqslant 2} (c_{\vec{r}-\vec{e}_1,\vec{s}}^{\vec{t}-\vec{e}_1,(\varphi,\psi)}+c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t}-\vec{e}_1,(\varphi,\psi)})\left[\begin{smallmatrix}\vec{t}\\ \vec{a}\,ш_{(\varphi,\psi)}\vec{b}\end{smallmatrix}\right]
\end{aligned}$$

$$= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|,t_1\geqslant 2} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \qquad \text{(by Eq. (37))}$$

$$= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \qquad \text{(by Eq. (38))}.$$

Case 2. $r_1 = s_1 = 1$**.** We will use the notations $\vec{r}\,', \vec{s}\,', \vec{a}\,'$ and $\vec{b}\,'$ in Definitions 3.3. Then

$$\begin{aligned}
&\left[\begin{array}{c} \vec{r} \\ \vec{a} \end{array} \right] \text{ш}_\rho \left[\begin{array}{c} \vec{s} \\ \vec{b} \end{array} \right] = Q_{w_1}(\left[\begin{array}{c} \vec{r}\,' \\ \vec{a}\,' \end{array} \right]) \,\text{ш}_\rho\, Q_{z_1}(\left[\begin{array}{c} \vec{s}\,' \\ \vec{b}\,' \end{array} \right]) \\
&= Q_{w_1}(\left[\begin{array}{c} \vec{r}\,' \\ \vec{a}\,' \end{array} \right] \text{ш}_\rho \left[\begin{array}{c} \vec{s} \\ \vec{b} \end{array} \right]) + Q_{z_1}(\left[\begin{array}{c} \vec{r} \\ \vec{a} \end{array} \right] \text{ш}_\rho \left[\begin{array}{c} \vec{s}\,' \\ \vec{b}\,' \end{array} \right]) \\
&= Q_{w_1}\Big(\sum_{(\varphi,\psi)\in\mathfrak{I}_{k-1,\ell}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r}\,',\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} \vec{t} \\ \vec{a}\,'\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right]\Big) \\
&\quad + Q_{z_1}\Big(\sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell-1}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}\,'}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b}\,' \end{array} \right]\Big) \\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k-1,\ell}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r}\,',\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi^{\&},\psi^{*})}\vec{b} \end{array} \right] \\
&\quad + \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell-1}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}\,'}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi^{*},\psi^{\&})}\vec{b} \end{array} \right] \quad \text{(by Eq.(47) and (48))} \\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell},\varphi(1)=1} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r}\,',\vec{s}}^{\vec{t},(\varphi^{\sharp},\psi^{\flat})} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \\
&\quad + \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell},\psi(1)=1} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}\,'}^{\vec{t},(\varphi^{\flat},\psi^{\sharp})} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \quad \text{(by Lemma 3.4)} \\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell},\varphi(1)=1} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{(1,\vec{t}),(\varphi,\psi)} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \\
&\quad + \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell},\psi(1)=1} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{(1,\vec{t}),(\varphi,\psi)} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \quad \text{(by Eq. (39) and (40))} \\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell-1},|\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{(1,\vec{t}),(\varphi,\psi)} \left[\begin{array}{c} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right] \\
&= \sum_{(\varphi,\psi)\in\mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in\mathbb{Z}_{\geqslant 1}^{k+\ell},|\vec{t}|=|\vec{r}|+|\vec{s}|,t_1=1} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \left[\begin{array}{c} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{array} \right]
\end{aligned}$$

$$= \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}\,|=|\vec{r}|+|\vec{s}|} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \qquad \text{(by Eq. (41))}.$$

Case 3. $r_1 = 1$ **and** $s_1 \geqslant 2$**.** With the notations in Definitions 3.3, we write $\vec{r} = (1, \vec{r}\,')$. Let $\vec{a}\,' = (w_2, \cdots, w_r)$. Then

$$\begin{aligned}
&\begin{bmatrix} \vec{r} \\ \vec{a} \end{bmatrix} \text{ш}_\rho \begin{bmatrix} \vec{s} \\ \vec{b} \end{bmatrix} = Q_{w_1}\Big(\begin{bmatrix} \vec{r}\,' \\ \vec{a}\,' \end{bmatrix}\Big) \text{ш}_\rho P\Big(\begin{bmatrix} \vec{s}-\vec{e}_1 \\ \vec{b} \end{bmatrix}\Big) \\
&= Q_{w_1}\Big(\begin{bmatrix} \vec{r}\,' \\ \vec{a}\,' \end{bmatrix} \text{ш}_\rho \begin{bmatrix} \vec{s} \\ \vec{b} \end{bmatrix}\Big) + P\Big(\begin{bmatrix} \vec{r} \\ \vec{a} \end{bmatrix} \text{ш}_\rho \begin{bmatrix} \vec{s}-\vec{e}_1 \\ \vec{b} \end{bmatrix}\Big) \\
&= Q_{w_1}\Big(\sum_{(\varphi,\psi)\in \mathfrak{I}_{k-1,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell-1}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r}\,',\vec{s}}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} \vec{t} \\ \vec{a}\,'\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix}\Big) \\
&\quad + P\Big(\sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix}\Big) \\
&= \sum_{(\varphi,\psi)\in \mathfrak{I}_{k-1,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell-1}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r}\,',\vec{s}}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi^{\&},\psi^{*})}\vec{b} \end{bmatrix} \\
&\quad + \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}-\vec{e}_1}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} \vec{t}+\vec{e}_1 \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \quad \text{(by Eq. (47))} \\
&= \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell},\varphi(1)=1} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell-1}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r}\,',\vec{s}}^{\vec{t},(\varphi^{\sharp},\psi^{\flat})} \begin{bmatrix} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \\
&\quad + \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}\,'}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} \vec{t}+\vec{e}_1 \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \quad \text{(by Lemma 3.4)} \\
&= \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell},\varphi(1)=1} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell-1}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{(1,\vec{t}),(\varphi,\psi)} \begin{bmatrix} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \\
&\quad + \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{\vec{t}+\vec{e}_1,(\varphi,\psi)} \begin{bmatrix} \vec{t}+\vec{e}_1 \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \quad \text{(by Eq. (42) and (44))} \\
&= \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell-1}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{(1,\vec{t}),(\varphi,\psi)} \begin{bmatrix} (1,\vec{t}) \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \\
&\quad + \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|-1} c_{\vec{r},\vec{s}}^{\vec{t}+\vec{e}_1,(\varphi,\psi)} \begin{bmatrix} \vec{t}+\vec{e}_1 \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix} \quad \text{(by Eq. (43))} \\
&= \sum_{(\varphi,\psi)\in \mathfrak{I}_{k,\ell}} \sum_{\vec{t}\in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}|=|\vec{r}|+|\vec{s}|} c_{\vec{r},\vec{s}}^{\vec{t},(\varphi,\psi)} \begin{bmatrix} \vec{t} \\ \vec{a}\,\text{ш}_{(\varphi,\psi)}\vec{b} \end{bmatrix}.
\end{aligned}$$

Case 4. $r_1 \geqslant 2$ **and** $s_1 = 1$**.** The proof for this case is similar to that for Case 3. $\square$

5. Appendix: a shuffle formulation of the Main Theorem

The main body of the paper does not depend on this Appendix. Here we give another formulation of Theorem 2.1 in terms of shuffles of permutations for those who are interested in a more precise connection between the main theorem and shuffle product.

Let integers $k, \ell \geqslant 1$ be given. Let

$$\begin{aligned} S(k,\ell) :&= \left\{\sigma \in \Sigma_{k+\ell} \mid \sigma^{-1}(1) < \cdots < \sigma^{-1}(k), \sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+\ell)\right\} \\ &= \left\{\sigma \in \Sigma_{k+\ell} \,\middle|\, \begin{array}{l} \text{if } 1 \leqslant \sigma(i) < \sigma(j) \leqslant k \\ \text{or } k+1 \leqslant \sigma(i) < \sigma(j) \leqslant k+\ell, \end{array} \ \text{then } i < j \right\}. \end{aligned} \tag{49}$$

be the set of (k,ℓ)-shuffles.

To state the shuffle form of our main theorem we need the following notations.

Define

$$\varepsilon_\sigma : [k+\ell] \to \{\pm 1\}, \quad \varepsilon_\sigma(i) = \left\{ \begin{array}{ll} 1, & 1 \leqslant \sigma(i) \leqslant k, \\ -1, & k+1 \leqslant \sigma(i) \leqslant k+\ell. \end{array} \right.$$

Let $\vec{r} = (r_1, \cdots, r_k) \in \mathbb{Z}_{\geqslant 1}^k$ and $\vec{s} = (s_1, \cdots, s_\ell) \in \mathbb{Z}_{\geqslant 1}^\ell$. Denote

$$\vec{\kappa} = (\kappa_1, \cdots, \kappa_{k+\ell}) := (r_1, \cdots, r_k, s_1, \cdots, s_\ell).$$

Let $\vec{a} \in G^k$ and $\vec{b} \in G^\ell$. Denote

$$\vec{\gamma} = (a_1, \cdots, a_k, b_1, \cdots, b_\ell).$$

For $\sigma \in S(k,\ell)$ we denote

$$\vec{a} ш_\sigma \vec{b} = (\gamma_{\sigma(1)}, \cdots \gamma_{\sigma(k+\ell)}).$$

We have the following equivalent form of Theorem 2.1.

Theorem 5.1. *Let G be a set and let $\mathcal{H}^{ш_\rho}(\widehat{G}) = (\mathcal{H}(\widehat{G}), ш_\rho)$ be as defined by Eq. (8). Then for $\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix}\right] \in \widehat{G}^k$ and $\left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right] \in \widehat{G}^\ell$ in $\mathcal{H}^{ш_\rho}(\widehat{G})$, we have*

$$\left[\begin{smallmatrix} \vec{r} \\ \vec{a} \end{smallmatrix}\right] ш_\rho \left[\begin{smallmatrix} \vec{s} \\ \vec{b} \end{smallmatrix}\right] = \sum_{\substack{\sigma \in S(k,\ell), \\ \vec{t} \in \mathbb{Z}_{\geqslant 1}^{k+\ell}, |\vec{t}| = |\vec{r}| + |\vec{s}|}} \left(\prod_{i=1}^{k+\ell} \binom{t_i - 1}{\kappa_{\sigma(i)} - 1 - \frac{1}{2}(1 - \varepsilon_\sigma(i)\varepsilon_\sigma(i-1)) \sum\limits_{j=1}^{i-1}(t_j - \kappa_{\sigma(j)})} \right) \left[\begin{smallmatrix} \vec{t} \\ \vec{a} ш_\sigma \vec{b} \end{smallmatrix}\right]$$

with the convention that $\varepsilon_\sigma(0) = \varepsilon_\sigma(1)$.

Proof. Let $\mathcal{I}_{k,\ell}$ be as defined in Eq. (15). We have the bijection between $S(k,\ell)$ and $\mathcal{I}_{k,\ell}$ given by

$$\sigma^{-1}(j) := \sigma_{\varphi,\psi}^{-1}(j) = \left\{ \begin{array}{ll} \varphi(j) & \text{if } 1 \leqslant j \leqslant k, \\ \psi(j-k) & \text{if } k+1 \leqslant j \leqslant k+\ell. \end{array} \right. \tag{50}$$

That is,

$$\sigma(i) := \sigma_{\varphi,\psi}(i) = \left\{ \begin{array}{ll} \varphi^{-1}(i) & \text{if } i \in \mathrm{im}(\varphi), \\ k + \psi^{-1}(i) & \text{if } i \in \mathrm{im}(\psi). \end{array} \right.$$

Thus we have

$$\kappa_{\sigma(i)} = \begin{cases} \kappa_{\varphi^{-1}(i)}, & i \in \mathrm{im}(\varphi) \\ \kappa_{k+\psi^{-1}(i)}, & i \in \mathrm{im}(\psi) \end{cases} = \begin{cases} r_{\varphi^{-1}(i)}, & i \in \mathrm{im}(\varphi) \\ s_{\psi^{-1}(i)}, & i \in \mathrm{im}(\psi) \end{cases} = h_{(\varphi,\psi),i} \tag{51}$$

and

$$(\vec{a} ш_\sigma \vec{b})_i = \gamma_{\sigma(i)} = \begin{cases} \gamma_{\varphi^{-1}(i)}, & i \in \mathrm{im}(\varphi) \\ \gamma_{k+\psi^{-1}(i)}, & i \in \mathrm{im}(\psi) \end{cases} = \begin{cases} a_{\varphi^{-1}(i)}, & i \in \mathrm{im}(\varphi) \\ b_{\psi^{-1}(i)}, & i \in \mathrm{im}(\psi) \end{cases} = (\vec{a} ш_{(\varphi,\psi)} \vec{b})_i. \tag{52}$$

By Eq. (52) we have

$$\vec{a} ш_\sigma \vec{b} = \vec{a} ш_{(\varphi,\psi)} \vec{b}. \tag{53}$$

Let $\varepsilon_{\varphi,\psi}$ be the function $[k+\ell] \to \{1,-1\}$ defined in Eq. (28). Then for $\sigma = \sigma_{\varphi,\psi}$,

$$\varepsilon_\sigma(i) = 1 \Leftrightarrow \sigma(i) \in [k] \Leftrightarrow i = \sigma^{-1}(j), j \in [k] \Leftrightarrow i = \varphi(j), j \in [k] \Leftrightarrow i \in \mathrm{im}(\varphi) \Leftrightarrow \varepsilon_{\varphi,\psi}(i) = 1.$$

So we have

$$\varepsilon_\sigma(i) = \varepsilon_{\varphi,\psi}(i), \qquad 1 \leqslant i \leqslant k+\ell. \tag{54}$$

Now our theorem follows from Eq. (29), (51), (53), (54) and Theorem 2.1. □

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Department of Mathematics and Computer Science, Rutgers University, Newark, NJ 07102, USA
E-mail address: `liguo@newark.rutgers.edu`

Department of Mathematics, Peking University, Beijing, 100871, China
E-mail address: `byhsie@math.pku.edu.cn`