GEOMETRY OF NEUMANN SUBGROUPS

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ABSTRACT

A Neumann subgroup of the classical modular group is by definition a complement of a maximal parabolic subgroup. Recently Neumann subgroups have been studied in a series of papers by Brenner and Lyndon, cf. [1] - [3]. There is a natural extension of the notion of a Neumann subgroup in the context of any finitely generated fuchsian group Γ acting on the hyperbolic plane $\mathcal{H}$ such that $\Gamma\backslash\mathcal{H}$ is homeomorphic to an open disk. Using a new geometric method we extend the work in [1] - [3] in this more general context.

Geometry of Neumann Subgroups

Ravi S. Kulkarni[+]

§1 Introduction (1.1) This note essentially consists of some remarks on a series of recent papers by Brenner and Lyndon concerned with the Neumann subgroups of the classical modular group, cf. [1] - [3]. We first recall their definition. Let $\Gamma = \langle x,y \mid x^2 = y^3 = e \rangle$ act as the modular group on the upper half plane $\mathcal{H}$ in the standard way. Then the subgroup $P = \langle z \overset{\text{def}}{=\!=} xy \rangle$ is a maximal parabolic subgroup of Γ and all such subgroups are conjugate. A subgroup Φ of Γ is said to be non-parabolic if it contains no parabolic element. If Φ is a complement of P in Γ, i.e. i) $P \cap \Phi = \{e\}$, and ii) $P \cdot \Phi = \Gamma$, then Φ is called a Neumann subgroup, cf. [1]. A Neumann subgroup is maximal among non-parabolic subgroups, cf. [1], (2.8).

(1.2) In connection with these subgroups, Brenner and Lyndon were led to study transitive triples (Ω,A,B), cf. [2], where Ω is a countable set, A and B are permutations of Ω of orders 2 and 3 respectively such that the group $\langle C \overset{\text{def}}{=\!=} AB \rangle$ is transitive on Ω. If (Ω,A,B) is a transitive triple then Γ, as in (1.1), acts on Ω in the obvious way so that the subgroup P is transitive on Ω. In particular $\Omega \simeq \Gamma/\Phi$ for a suitable

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subgroup Φ whose conjugacy class is well-defined. Since $P \simeq \mathbb{Z}$ it is clear that either P acts simply transitively on Ω in which case Ω is an infinite set, or else P acts ineffectively on Ω in which case Ω is a finite set. In the first case Φ is a Neumann subgroup. In the second case $(\Gamma:\Phi) < \infty$ and $\Phi\backslash\mathcal{H}$ has only one cusp. Such a subgroup was called cycloidal by Petersson, cf. [9]. Thus the study of transitive triples amounts to a simultaneous study of Neumann and cycloidal subgroups of Γ. For the earlier work on Neumann subgroups, cf. [8], [6], [13], and also [7] pp. 119 - 122.

(1.3) A principal result in [1], which extends theorem 2 of [13] is a structure-and-realization theorem for Neumann subgroups. Similar and more general results were proved by Stothers [10] - [12]. The proof in [1] is based on a correspondence between transitive triples and Eulerian paths in cuboid graphs i.e. the graphs with vertex-valences ≤ 3. For the triples associated with torsion-free Neumann-or-cycloidal subgroups the correspondence is 1-1, but in general to make the correspondence 1-1 one would need to put an extra structure on the cuboid graphs. The same method is used in [3] to produce maximal-among-non-parabolic subgroups which are not Neumann.

(1.4) In this note we extend this work to

(1.4.1) $\Gamma = \prod_{i=1}^{*n} \Gamma_i,\ \Gamma_i = \langle x_i \rangle \simeq \mathbb{Z}_{m_i},\ 2 \leq m_i,\ n < \infty.$

Except for $\Gamma \approx \mathbb{Z}_2 * \mathbb{Z}_2$, these groups can be realized as discrete subgroups of the orientation-preserving isometries of the hyperbolic plane $\mathcal{H}$ such that $\Gamma\backslash\mathcal{H}$ has finite area, and x_i acts as a rotation through angle $\frac{2\pi}{m_i}$ around its fixed point. Then the element $u = x_1 \cdot \ldots \cdot x_n$ is parabolic.

(1.5) The above remarks are meant only for motivation. In the following, hyperbolic geometry will not be used explicitly. We start with Γ as in (1.4.1). Let $u = x_1 \cdot \ldots \cdot x_n$. The conjugates of u^k, $k \neq 0$ are called parabolic elements of Γ. Let $P = \langle u \rangle$. A subgroup of Γ is called parabolic if all of its non-identity elements are parabolic. Clearly the maximal parabolic subgroups are precisely the conjugates of P. A subgroup of Γ is called non-parabolic if it contains no parabolic element. A complement Φ of P in Γ is called a Neumann subgroup. Thus, for a Neumann subgroup Φ one has i) $P \cap \Phi = \{e\}$, and ii) $P \cdot \Phi = \Gamma$. The latter implies ii)' $|P \backslash^{\Gamma} / \Phi| = 1$. Conversely if ii)' holds and $(\Gamma : \Phi) = \infty$ then Φ is a Neumann subgroup. If ii)' holds and $(\Gamma : \Phi) < \infty$ then as in [5], Φ is called a 1-cycloidal subgroup.[⊕] One sees, cf. (2.1), that a Neumann subgroup is maximal among non-parabolic subgroups.

⊕ In the correspondence between subgroups of the fuchsian groups and holomorphic maps among Riemann surfaces, the 1-cycloidal subgroups precisely correspond to meromorphic functions on closed Riemann surfaces with a single pole. These functions may be considered as generalizations of polynomial maps, cf. [5].

(1.6) Let Γ be as in (1.4.1). For $\Phi \leq \Gamma$, in [4], we attached a diagram X_Φ and its thickening $\mathbf{X}_\Phi$ with canonical projections $X_\Phi \to X_\Gamma$, $\mathbf{X}_\Phi \to \mathbf{X}_\Gamma$. Here $\mathbf{X}_\Phi$ is an orientable surface with non-empty boundary $\partial\mathbf{X}_\Phi$. One may think of $\mathbf{X}_\Gamma$ as "$\Gamma\backslash\mathcal{H}$ with the cusp cut off". This makes the "cuspidal infinity" more tangible - for example, one gets the following useful characterizations: $\Phi \leq \Gamma$ is Neumann (resp. 1-cycloidal) iff $\partial\mathbf{X}_\Phi$ is connected and non-compact (resp. connected and compact). Pinching each circle in X_Φ to a point one obtains a graph Y_Φ whose structure suggests the notion of an $(m_1,\ldots,m_n)$-<u>semiregular</u> graph, cf. (2.4). If Φ is a Neumann subgroup then the image of $\partial\mathbf{X}_\Phi$ in Y_Φ is a special type of Eulerian path which we simply call <u>admissible</u>. This provides a natural explanation of the initially intriguing Brenner-Lyndon correspondence between Neumann and 1-cycloidal subgroups of the modular group and the Eulerian paths in cuboid graphs. A natural extension of their result is: <u>the conjugacy classes of Neumann (resp. 1-cycloidal) subgroups of</u> Γ <u>are in</u> 1-1 <u>correspondence with the admissible Eulerian paths in</u> $(m_1,\ldots,m_n)$-<u>semiregular graphs</u>.

(1.7) For Γ as in (1.4.1) and $\Phi \leq \Gamma$ we have by Kurosh's theorem,

$$(1.7.1) \qquad \Phi \simeq F_r * \prod_{i=1}^{*n} \Big(\prod_{j\in J_i}^{*} \Phi_{ij} \Big)$$

where F_r denotes the free group of rank r and $\Phi_{ij} \simeq \mathbb{Z}_d$, $d|m_i$ are conjugates to subgroups of Γ_i. In (1.7.1) we assume that $\Phi_{ij} \neq \{e\}$ with the understanding that if J_i is empty then $\overset{*}{\Pi}_{j\in J_i}\Phi_{ij} = \{e\}$. Let

$$(1.7.2)\quad r_i(d) = \#\{\Phi_{ij} \mid \Phi_{ij} \simeq \mathbb{Z}_{m_i/d}\},\ d \mid m_i,\ d < m_i.$$

The numbers $r_i(d)$ may be possibly infinite. In §4 we prove a structure-and-realization theorem for Neumann subgroups. For example, <u>if at most one</u> m_i <u>is even, then</u> Φ <u>as in (1.7.1)</u> <u>is realizable as a Neumann subgroup iff either</u> 1) $r = \infty$ <u>or</u> 2) $r =$ <u>an even integer and</u> $r_i(1) = \infty$ <u>for</u> $\geq$ $n-1$ <u>values of</u> i. If there are two even m_i's there is a curious new family of Neumann subgroups, cf. (2.11), of which there is no analogue in the case of the modular group. This family makes the full structure theorem a bit complicated, but the underlying geometric idea is simple. For details cf. §4.

(1.8) Finally in §5 we give some geometric constructions of subgroups which are maximal, or maximal with respect to some additional properties such as Neumann, 1-cycloidal, non-parabolic but non-Neumann....

I wish to thank W.W. Stothers for drawing my attention to [2].

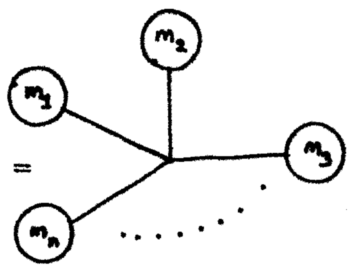
§2 Preliminaries

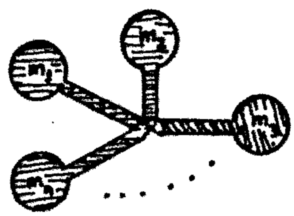
(2.0) Throughout this section let Γ, Γ_i, x_i, u be as in (1.4) and (1.5), and we use the terminology introduced there.

(2.1) Proposition: A Neumann subgroup is maximal among non-parabolic subgroups.

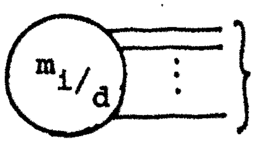
Proof: Let Φ be a Neumann subgroup of Γ, and $P = \langle u \rangle$. So P acts simply transitively on Γ/Φ. The isotropy subgroup of P at $a\Phi$ is $P \cap a\Phi a^{-1} = \{e\}$. So $a^{-1}Pa \cap \Phi = \{e\}$ i.e. Φ is a non-parabolic subgroup. If $\Phi \lneqq \Psi \leq \Gamma$ then P acts transitively but not simply transitively on Γ/Ψ. But since $P \cong \mathbb{Z}$ this means that the P-action on Γ/Ψ is ineffective and $|\Gamma/\Psi| < \infty$. Hence $P \cap \Psi \neq \{e\}$, i.e. Ψ contains parabolic elements. So Φ is maximal among non-parabolic subgroups. q.e.d.

(2.2) As in [4] let X_Γ be a diagram for Γ and $\mathbf{X}_\Gamma$ its thickening:

(2.2.1) $X_\Gamma =$ 

, $\mathbf{X}_\Gamma =$ 

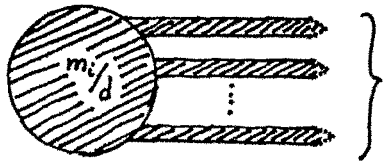
A building block of type i has the form

(2.2.2) 

d edges

and is denoted by $B_i(d)$. A diagram X_Φ is built out of such

$B_i(d)$'s and there is a canonical projection $X_\Phi \to X_\Gamma$. The thickening of $B_i(d)$ is

(2.2.3) $\mathbb{B}_i(d) =$ 

d arms.

The thickening $\mathbf{X}_\Phi$ of X_Φ is built out of $\mathbb{B}_i(d)$'s. Notice that $\mathbf{X}_\Phi$ is an orientable surface with boundary $\partial\mathbf{X}_\Phi$. There is a canonical projection[⊕] $p\colon \mathbf{X}_\Phi \to \mathbf{X}_\Gamma$ and also a "thinning" map $\mathbf{X}_\Phi \to X_\Phi$. The shape of $\mathbb{B}_i(d)$ may be described as "<u>a closed disk with</u> d <u>arms</u>". Each of the dotted edges at the end of an arm is its <u>half-outlet</u>; together they form an <u>outlet</u>. In $\mathbf{X}_\Phi$ the outlets come in groups of n. So we may use the obvious and suggestive terminology of an <u>angle formed by the half-outlets</u>. For example the interior angle formed by the half-outlets of an arm is $\frac{2\pi}{n}$. In $\mathbf{X}_\Phi$ one half-outlet of an arm of a $\mathbb{B}_i(*)$ is joined to a half-outlet of an arm of a $\mathbb{B}_{i+1}(*)$ and the other to a half-outlet of an arm of a $B_{i-1}(*)$, where the subscript i is counted mod n. In the sequel, it will be important to keep in mind that

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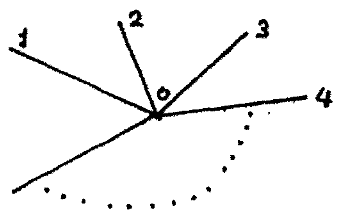
⊕ The restriction $p|_{\mathrm{int}\,\mathbf{X}_\Phi}\colon \mathrm{int}\,\mathbf{X}_\Phi \to \mathrm{int}\,\mathbf{X}_\Gamma$ is a branched covering of surfaces. If Γ is realized as an orientation-preserving, properly discontinuous group of homeomorphisms of $\mathbb{R}^2$ then $p|_{\mathrm{int}\,\mathbf{X}_\Phi}$ is equivalent to the canonical map $\Phi\backslash\mathbb{R}^2 \to \Gamma\backslash\mathbb{R}^2$.

(2.2.4) $\partial \mathbb{B}_i(d) = \{\mathbb{B}_i(d) \cap \partial \mathbf{X}_\Phi\} \cup$ {the outlets}.

(2.3) Pinching each circle in X_Φ to a point one gets a graph Y_Φ. Again one has a canonical projection denoted by $p: Y_\Phi \to Y_\Gamma$. Notice that the <u>terminal vertices</u> ⊕ of Y_Φ are precisely the images of m_i in X_Φ. The vertices adjacent to terminal vertices will be called <u>sub-terminal vertices</u>. Now

(2.3.1) $Y_\Gamma =$ 

has n+1 vertices - the image of m_i is numbered i, and the "base-vertex" is numbered 0. So the vertices of Y_Φ are divided, into n+1 disjoint subsets:

(2.3.2) $\alpha_i = \{v \mid p(v)$ has number $i\}$.

The structure of Y_Φ motivates the following

(2.4) <u>Definition</u>: Let $m_1, m_2, \ldots, m_n$ be positive integers ≥ 2 and $n \geq 2$. An $(m_1, \ldots, m_n)$-semiregular graph is a connected graph G whose vertices are divided into n+1 disjoint subsets α_i, $i = 0, 1, \ldots, n$, such that

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⊕ i.e. vertices of valence 1.

a) $v \in \alpha_i \Longrightarrow$ valence $v = n$ (resp. a divisor of m_i) if $i = 0$ (resp. ≥ 1),

b) each edge of G has one end in α_0 and the other in α_i, $i \geq 1$,

c) given $v \in \alpha_0$, and $i \geq 1$, there is a unique edge joining v to a vertex in α_i.

Clearly Y_Φ, as in (2.3), is an $(m_1,\ldots,m_n)$-semiregular graph.

If $n = 2$ (resp. some m_i is even) then the vertices in α_0 (resp. certain vertices in α_i) have valence 2, and if convenient may well be not counted as vertices. Thus for example, not counting the vertices in α_0, an (m_1,m_2)-semiregular graph is a bipartite graph. Again if G is a $(2,k)$-semiregular graph such that all vertices in α_1 resp. α_2 have valence 2 resp. k, then not counting the vertices in α_0 or α_1 one has a k-regular graph in the usual sense. Thus if $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_k$, and $\Phi \leq \Gamma$ is <u>torsionfree</u> then Y_Φ may be considered as a k-regular graph. In particular corresponding to torsion-free subgroups of the modular group one gets cubic graphs.

(2.5) <u>Remark</u>: Let G be an $(m_1,\ldots,m_n)$-semiregular graph. Then the edges at a vertex $\in \alpha_0$ come equipped with a natural cyclic order. Now suppose at each vertex $v \in \alpha_i$, $i \geq 1$ we specify some cyclic order among the edges incident with v. Then we may replace each $v \in \alpha_i$, $i \geq 1$ by a circle and attach the v-ends of the edges incident at v to the circle consistent with the prescribed cyclic order, and obtain a diagram X. This diagram has a canonical projection $p: X \to X_\Gamma$, so, cf. the proof of theorem 1 in [4], X may be considered as X_Φ for a subgroup Φ whose conjugacy classes is well-defined. Thus $G = Y_\Phi$ for some $\Phi \leq \Gamma$.

(2.6) Taking a base-point in $\partial\mathbf{X}_\Gamma$ we may represent $u = x_1 \dots x_n$ by the oriented boundary $\partial\mathbf{X}_\Gamma$. Now $p^{-1}(\partial\mathbf{X}_\Gamma) = \partial\mathbf{X}_\Phi$, so the components of $\partial\mathbf{X}_\Phi$ are in 1-1 correspondence with the double cosets $P\backslash^\Gamma/\Phi$. If C is a component of $\partial\mathbf{X}_\Phi$ and $p|_C: C \to \partial\mathbf{X}_\Gamma$ has degree d (possibly infinite), then d is the number of points in the corresponding P-orbit in Γ/Φ. In particular C is non-compact iff $d = \infty$, iff the P-action on the corresponding orbit is effective. Clearly one gets

<u>Proposition</u>: 1) Φ is a non-parabolic subgroup iff $\partial\mathbf{X}_\Phi$ has no compact component.

2) Φ is a Neumann (resp. 1-cycloidal) subgroup iff $\partial\mathbf{X}_\Phi$ is connected and non-compact (resp. connected and compact).

(2.7) We recall some elementary facts from the topology of surfaces. Let M be any connected surface possibly with non-empty boundary. A connected, compact subsurface S of M is said to be <u>tight</u> if M-int S has no compact component. Notice that if S is a compact subsurface then $M-S$ has only finitely many components. So if S is compact and connected then

$$S_1 = S \cup \{\text{compact components of } M\text{-int } S\}$$

is a tight subsurface. It is now clear that M admits an <u>exhaustion</u> by tight subsurfaces, i.e. a sequence S_i, $i = 1,2\ldots$ of tight subsurfaces such that $S_i \subseteq \text{int } S_{i+1}$, and $M = \cup_i S_i$.

Now suppose that the fundamental group of M based at $*$ is finitely generated. So there exist finitely many based loops C_i such that $\pi_1(M,*) = \langle [C_i] \rangle$ where $[C_i]$ denotes the homotopy class of C_i. One says that an arc-connected subset A of M <u>carries</u> π_1 if the canonical map $\pi_1(A) \to \pi_1(M)$ is surjective. Clearly any arc-connected subset A containing $\cup C_i$ carries π_1. Now let S be a tight subsurface which contains $\cup C_i$. In this case in fact the canonical map $\pi_1(S) \to \pi_1(M)$ is an isomorphism and it is easy to see that each component of $M-S$ is either a cylinder or a disk. If $\partial M \neq \emptyset$ these cylinders or disks may also have non-empty boundary.

(2.8) We apply the considerations in (2.7) to $\mathfrak{X}_\Phi$. Let S be a tight subsurface of $\mathfrak{X}_\Phi$. Then for each building block $\mathbb{B}_i(d)$ we see that a component of $S \cap \mathbb{B}_i(d)$ with a non-empty interior must be a closed disk, and $S \cup \mathbb{B}_i(d)$ is also tight. Let S_1 be the union of S and all $\mathbb{B}_i(d)$'s which intersect S in a subset with non-empty interior. Then S_1 has the additional property:

$$(2.8.1)\qquad \begin{cases} \partial S_1 = \{S_1 \cap \partial \mathbf{X}_\Phi\} \cup A \\ \text{where } A \text{ is the union of the half-outlets} \\ \text{on some arms of the building blocks.} \end{cases}$$

Now suppose Φ is as in (1.7.1). Then its free part F_r may be identified with $\pi_1(Y_\Phi)$ or $\pi_1(\mathbf{X}_\Phi)$, cf. the discussion in §2 of [5]. Suppose that $r < \infty$. So there exist tight subsurfaces of $\mathbf{X}_\Phi$ which carry π_1, cf. (2.7). We call a tight subsurface <u>characteristic</u> if it carries π_1 and has the additional property stated in (2.8.1).

From the above discussion it is clear that <u>if $r < \infty$, $\mathbf{X}_\Phi$ admits an exhaustion by characteristic subsurfaces</u>.

(2.9) <u>Proposition</u>: Let Φ as in (1.7.1) be a Neumann subgroup of Γ, and $r < \infty$. Let S be a characteristic subsurface of $\mathbf{X}_\Phi$. Then ∂S is connected and contains exactly one pair of half-outlets making an exterior angle $\frac{2\pi}{n}$, cf. (2.2). Moreover $\operatorname{int}(\mathbf{X}_\Phi - S)$ is homeomorphic to an open disk and $\partial(\mathbf{X}_\Phi - S)$ has two components, each homeomorphic to an open interval.

<u>Proof</u>: Since S is characteristic

$$\partial S = \{S \cap \partial \mathbf{X}_\Phi\} \cup A$$

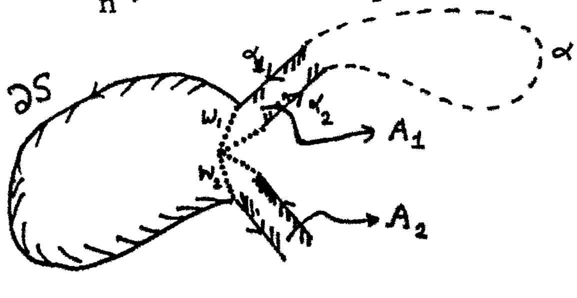
where A is a union of half-outlets. Since ∂S is compact, $\partial \mathbf{X}_\Phi$ is connected and noncompact, cf. (2.6), we see that each component of ∂S must intersect A as well as $\partial \mathbf{X}_\Phi$. Notice that the half-outlets in A come in pairs — each pair forms a connected arc, and different pairs form disjoint arcs.

First we claim that ∂S is connected. Suppose C_1, C_2 are two disjoint components of ∂S. Since C_1, C_2 contain points of $\partial \mathbf{X}_\Phi$, and $\partial \mathbf{X}_\Phi$ is connected, there is an arc $\alpha \subseteq \partial \mathbf{X}_\Phi$ joining a point p_1 in C_1 to a point p_2 in C_2. But since S is connected there is an arc $\beta \subseteq S$ joining p_1 to p_2 and passing through a base-point $*$. But then $\alpha \cup \beta$ forms a based loop whose homotopy class is clearly not contained in $\pi_1(S,*)$. This would contradict that S carries π_1. So ∂S is connected.

Next suppose if possible that there are two pairs of half-outlets, each pair forming an arc. Then $\partial S - A$ has two components which must be connected by an arc $\subseteq \partial \mathbf{X}_\Phi$, and we get a contradiction exactly as above.

Next suppose that the pair w_1, w_2 of half-outlets makes an exterior angle $> \frac{2\pi}{n}$, cf. the figure in (2.9.1).

(2.9.1)



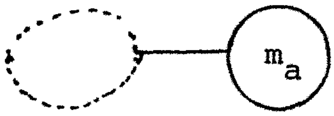
Then the arms A_1, A_2 in $\mathbf{X}_\Phi$-int S incident with w_1 and w_2 are distinct. Again since $\partial \mathbf{X}_\Phi$ is connected the components α_1, α_2 of $A_1 \cap \partial \mathbf{X}_\Phi$ are joined by an arc $\alpha \subseteq \partial \mathbf{X}_\Phi$. Clearly

$\alpha \cap S = \emptyset$. Now $\partial A_1 \cup \alpha$ forms a Jordan curve outside S. Since S carries π_1 this Jordan curve must bound a disk. But then $\mathfrak{X}_\Phi$-int S would have a compact component and S would not be tight. This contradiction shows that the exterior angle formed by w_1, w_2 must be $\frac{2\pi}{n}$, and so $w_1 \cup w_2$ is an outlet of an arm lying outside int S. This arm connects $\mathfrak{X}_\Phi$-int S to S. In particular $\mathfrak{X}_\Phi$-int S has only one component. From the remarks in (2.7) it is now clear that $\mathrm{int}(\mathfrak{X}_\Phi - S)$ is homeomorphic to a disk and $\partial(\mathfrak{X}_\Phi - S)$ has two components each sharing one endpoint of $\partial S - A$. q.e.d.

(2.10) The above proposition may be used to get an intuitive understanding of a Neumann subgroup Φ with $r < \infty$. Let $S_1 \subset S_2 \subset \ldots$ be an exhaustion of $\mathfrak{X}_\Phi$ by characteristic subsurfaces. Each S_{k+1}-int S_k is $\approx$ a closed disk. Also each S_k has exactly one pair of half-outlets with exterior angle $\frac{2\pi}{n}$. Inserting an appropriate $\mathbb{B}_i(m_i)$ in this outlet we obtain a new diagram $\tilde{S}_k \approx \mathfrak{X}_{\Phi_k}$ where Φ_k is a 1-cycloidal subgroup. Thus we get a sequence Φ_k, $k = 1,2\ldots$ of 1-cycloidal groups so that $\mathfrak{X}_{\Phi_k}$ contains some $\mathbb{B}_i(m_i)$, and $\mathfrak{X}_{\Phi_{k+1}}$ is obtained from $\mathfrak{X}_{\Phi_k}$ by removing some $\mathbb{B}_i(m_i)$, and inserting some $\mathbb{B}_i(d)$, $d < m_i$, together with some other building blocks so that the union of the newly inserted building blocks is a subset $\approx$ a closed disk. We express this by saying that Φ <u>is obtained by unfolding a sequence of 1-cycloidal subgroups</u> Φ_k.

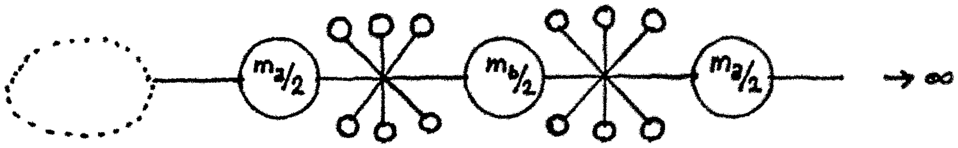
(2.11) We shall now describe a special "unfolding" of a single 1-cycloidal subgroup. It will be important in the structure theory of Neumann subgroups in §4. Suppose we have two m_i's - say m_a, m_b - even integers. Let Φ_O be a 1-cycloidal subgroup so that X_{Φ_O} contains either $B_a(m_a)$ or $B_b(m_b)$, say the first. Then we can obtain a Neumann subgroup Φ as follows, which is best described by its diagram X_Φ. Suppose

(2.11.1) X_{Φ_O} =



Let

(2.11.2) X_Φ =



Here all the unlabelled building blocks in the newly inserted portion are $B_i(m_i)$'s, $i \neq a,b$. We shall say that Φ *is a simple* (m_a,m_b)*-unfolding of a 1-cycloidal subgroup* Φ_O.

(2.12) *Remark*: Let Φ as in (2.7.1) be a Neumann subgroup with $r = \infty$. Then $\mathfrak{X}_\Phi$ contains no characteristic subsurface. But it is not difficult to see that still $\mathfrak{X}_\Phi$ admits an exhaustion S_k $k = 1,2,...$ by tight subsurfaces which satisfy the property stated in (2.8.1) and such that ∂S_k is connected. Here ∂S_k may contain several pairs of half-outlets. Filling these pairs by suitable $\mathbb{B}_i(m_i)$'s we obtain $\tilde{S}_k \simeq \mathfrak{X}_{\Phi_k}$ where Φ_k is a 1-cycloidal subgroup. In this sense Φ can still be considered as an "unfolding of a sequence of 1-cycloidal subgroups."

§3 Eulerian paths

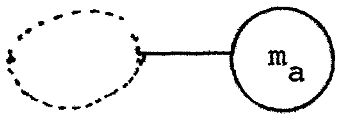
(3.1) Let G be a graph. Each edge of G can be directed in two ways and so corresponds to two directed edges, each of which is the inverse of the other. A path in G is reduced if it contains no consecutive pair of inverse edges. An Eulerian path in G is a path which contains each directed edge once and only once and which is reduced except at the terminal vertices.

(3.2) Let G be a $(m_1,\ldots,m_n)$-semiregular graph, cf. (2.4). An admissible path in G is a path in which the vertices occur in the following consecutive order:

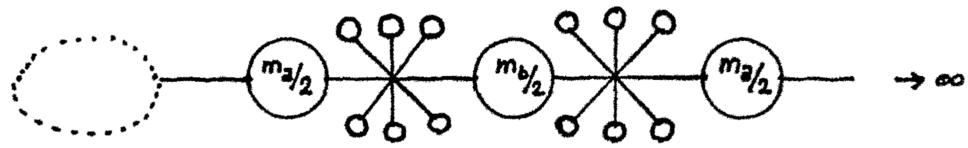
(3.2.1) $\ldots v_1 w_1 v_2 w_2 \ldots \quad v\text{'s} \in \alpha_0, \quad w_i \in \alpha_{k+i}$

where k is some fixed integer and $\alpha_{k+i} = \alpha_j$ where j is the unique positive integer, $1 \leq j \leq n$, $k+i \equiv j(n)$.

(2.11) We shall now describe a special "unfolding" of a single 1-cycloidal subgroup. It will be important in the structure theory of Neumann subgroups in §4. Suppose we have two m_i's - say m_a, m_b - even integers. Let Φ_0 be a 1-cycloidal subgroup so that X_{Φ_0} contains either $B_a(m_a)$ or $B_b(m_b)$, say the first. Then we can obtain a Neumann subgroup Φ as follows, which is best described by its diagram X_Φ. Suppose

(2.11.1) X_{Φ_0} = 

Let

(2.11.2) X_Φ = 

Here all the unlabelled building blocks in the newly inserted portion are $B_i(m_i)$'s, $i \neq a,b$. We shall say that Φ <u>is a simple</u> (m_a,m_b)-<u>unfolding of a 1-cycloidal subgroup</u> Φ_0.

(2.12) <u>Remark</u>: Let Φ as in (2.7.1) be a Neumann subgroup with $r = \infty$. Then $\mathfrak{X}_\Phi$ contains no characteristic subsurface. But it is not difficult to see that still $\mathfrak{X}_\Phi$ admits an exhaustion S_k $k = 1,2,\ldots$ by tight subsurfaces which satisfy the property stated in (2.8.1) and such that ∂S_k is connected. Here ∂S_k may contain several pairs of half-outlets. Filling these pairs by suitable $\mathbb{B}_i(m_i)$'s we obtain $\tilde{S}_k \approx \mathfrak{X}_{\Phi_k}$ where Φ_k is a 1-cycloidal subgroup. In this sense Φ can still be considered as an "unfolding of a sequence of 1-cycloidal subgroups."

§3 Eulerian paths

(3.1) Let G be a graph. Each edge of G can be directed in two ways and so corresponds to two directed edges, each of which is the inverse of the other. A path in G is reduced if it contains no consecutive pair of inverse edges. An Eulerian path in G is a path which contains each directed edge once and only once and which is reduced except at the terminal vertices.

(3.2) Let G be a $(m_1,\ldots,m_n)$-semiregular graph, cf. (2.4). An admissible path in G is a path in which the vertices occur in the following consecutive order:

(3.2.1) $\quad \ldots v_1 w_1 v_2 w_2 \ldots \quad v\text{'s} \in \alpha_0, \quad w_i \in \alpha_{k+i}$

where k is some fixed integer and $\alpha_{k+i} = \alpha_j$ where j is the unique positive integer, $1 \leq j \leq n$, $k+i \equiv j(n)$.

(3.3) <u>Theorem</u>: Let Γ be as in (1.4.1). Then the conjugacy classes of Neumann (resp. 1-cycloidal) subgroups of Γ are in 1-1 correspondence with the admissible Eulerian paths in infinite (resp. finite) $(m_1,\ldots,m_n)$-semiregular graphs.

<u>Proof</u>: Let Φ be a Neumann (resp. 1-cycloidal) subgroup of Γ. Then Y_Φ is an $(m_1,\ldots,m_n)$-semiregular graph. Since Φ is Neumann (resp. 1-cycloidal) Y_Φ is infinite (resp. finite). Now orient $\mathbf{X}_\Phi$ which also orients $\partial\mathbf{X}_\Phi$. If A is an arm of a building block of $\mathbf{X}_\Phi$ then $A \cap \partial\mathbf{X}_\Phi$ consists of two edges, which, under the canonical projection $\mathbf{X}_\Phi \to X_\Phi \to Y_\Phi$, project onto a pair of mutually inverse directed edges. It follows that the image of $\partial\mathbf{X}_\Phi$ in Y_Φ is an admissible Eulerian path.

Conversely let G be an infinite (resp. finite) $(m_1,\ldots,m_n)$-semiregular graph, and E an admissible Eulerian path in G. Let $v \in \alpha_i$, $i \geq 1$. Introduce a cyclic order among the (undirected) edges incident with v as follows: an edge f cyclically follows e iff in E the directed edge e ending in v follows the directed edge f beginning at v. By the remark in (2.5) we can construct an infinite (resp. finite) diagram X which corresponds to a conjugacy class of a subgroup Φ. But the existence of E also shows that $\partial\mathbf{X}$ is connected and is non-compact (resp. compact) so Φ is Neumann (resp. 1-cycloidal).

It is easy to see that this establishes the 1-1 correspondence asserted in the theorem. q.e.d.

§4 A structure theorem

(4.0) Throughout this section Γ is an in (1.4.1) and Φ is as in (1.7.1) and we use the notations used there. If $\Gamma \approx \mathbb{Z}_2 * \mathbb{Z}_2$ it is easy to see that the two conjugacy classes of subgroups $\approx \mathbb{Z}_2$ precisely consist of all the Neumann subgroups in Γ. Henceforth we shall assume that $\Gamma \not\approx \mathbb{Z}_2 * \mathbb{Z}_2$.

(4.1) Proposition: If $r = \infty$ then Φ is realizable as a Neumann subgroup.

Proof: The details of this proof are similar to (and simpler than) those of theorem (1.7) of [5], which deals with the case of 1-cycloidal subgroups. So we shall be brief. First of all, the diophantine condition, cf. (3.2) of [5], needed there is no longer necessary since the "difficulties can be thrown off to infinity." Recall that for $d|m_i$, $d < m_i$

$$r_i(d) = \#\{\Phi_{ij} \mid \Phi_{ij} \approx \mathbb{Z}_{m_i/d}\}$$

which may be infinite. We set $r_i(m_i) = \infty$. Choose $r_i(d)$ copies of $B_i(d)$'s, cf. (2.2.2). The objective is to construct a diagram X with these building blocks so that $\mathbf{X}$ has infinite genus and $\partial\mathbf{X}$ is connected and non-compact. Using all $\mathbb{B}_1(*)$'s and some of the $\mathbb{B}_2(*)$'s construct a complex $\mathbb{H} \approx$ the closed upper half space so that $\partial\mathbb{H}$ contains infinitely many pairs of half-outlets.[⊕] Now attach the remaining building blocks appropriately at these half-outlets so as to get $\mathfrak{X}$ with the required properties. q.e.d.

(4.2) Proposition: If Φ with $r < \infty$ is realizable as a Neumann subgroup then r is an even integer.

Proof: Indeed $F_r \cong \pi_1(\mathfrak{X}_\Phi)$. If S is a characteristic subsurface we observed in (2.7), (2.9) that $\pi_1(S) \cong \pi_1(\mathfrak{X}_\Phi)$ and S is a compact orientable surface with one boundary component. So $r = 2g$ where g is the genus of S. q.e.d.

(4.3) Proposition: Let Φ with $r = 2g < \infty$ be realizable as a Neumann subgroup. Then either

1) $r_i(1) = \infty$ for $\geq n-1$ values of i,

or 2) A) $r_i(1) = \infty$, $r_i(d) < \infty$, $d \neq 1$ for $n-2$ values of $i \neq a,b$ say,

B) $r_i(2) = \infty$, $r_i(d) < \infty$, $d \neq 2$ for $i = a,b$,

C) Φ is a simple (m_a,m_b)-unfolding of a 1-cycloidal subgroup, cf. (2.11).

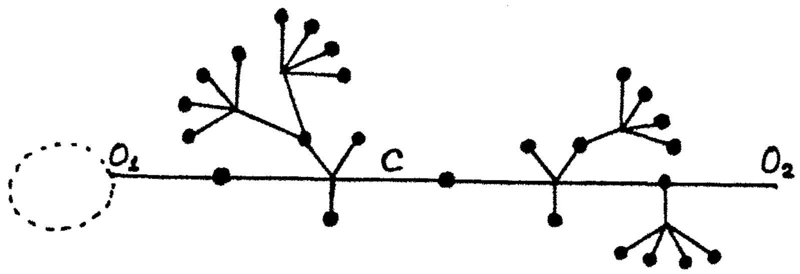
Proof: Let S_1 S_2 ... be an exhaustion of $\mathfrak{X}_\Phi$ by characteristic subsurfaces. Let $D_k = S_{k+1}-\text{int } S_k$, $k = 1,2\ldots$.

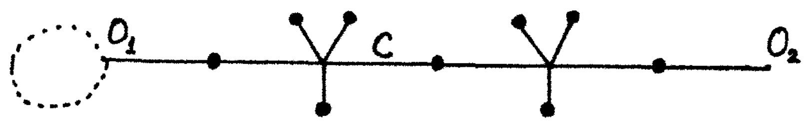
[⊕] That $\mathbb{H}$ contains infinitely many pairs of half-outlets is obvious for $n \geq 3$. For $n = 2$ this would fail exactly when $m_1 = 2 = m_2$. We have explicitly excluded this case in (4.0).

As observed in (2.10) D_k is a closed disk, and

(4.3.1) $\partial D_k = \{D_k \cap \partial \mathbf{X}_\Phi\} \cup \{$the two pairs of half-outlets in $\partial S_k \cup \partial S_{k+1}\}$

The projection of D_k in Y_Φ has the following two possible forms.

(4.3.2) 

(4.3.3) 

Here O_1, O_2 are the projections of the pairs of half-outlets in ∂D_k, and C is the shortest path joining O_1 to O_2. (Since $D_k \approx$ a closed disk, C is unique.) The large dark vertices are in $U\alpha_i$, $i \geq 1$ and the small ones are in α_0. The two forms are distinguished by the following fact. In (4.3.3) all vertices in α_0 lie on C - hence each is subterminal, cf. (2.3), and is incident with $n-2$ terminal vertices. In (4.3.2) there are some vertices in α_0 which do not lie on C, and so there are some subterminal among them which are incident with $n-1$ terminal vertices. Now each terminal vertex is an image of a $B_i(1)$ hence contributes to $r_i(1)$. So it follows that $r_i(1) = \infty$ for at least $n-2$ values of i. Suppose if possible that there actually exist two distinct values a,b of i such that $r_a(1) < \infty$, $r_b(1) < \infty$. Then the finitely many building blocks $\mathbb{B}_a(1)$'s and $\mathbb{B}_b(1)$'s are contained in some characteristic subsurface S_{k_0}.

But then for $k \geq k_0$, D_k is necessarily of the form (4.3.3) and the building blocks with two arms in D_k are necessarily $\mathbb{B}_a(2)$'s and $\mathbb{B}_b(2)$'s. Since S_{k_0} is compact it follows that $r_i(d) < \infty$ for $d \neq 1$, $i \neq a,b$, and for $d \neq 2$, $i = a,b$, as well. Finally the discussion in (2.11) shows that in this case Φ must be an (m_a,m_b)-unfolding of a suitable 1-cycloidal subgroup. q.e.d.

(4.4) <u>Proposition</u>: Let $r = 2g < \infty$, and suppose $r_i(1) = \infty$ for $\geq n-1$ values of i. Then Φ is realizable as Neumann subgroup.

<u>Proof</u>: Suppose $r_i(1) = \infty$ for $i \neq 1$. The objective is to construct a diagram X with $r_i(d)$ copies of $B_i(d)$'s, $d < m_i$, and any (possibly infinite) number of copies of $B_i(m_i)$'s so that the thickened diagram $\mathbf{X}$ is an orientable surface of genus g with $\partial\mathbf{X}$ connected and noncompact. Now using finitely many $B_i(d)$'s we can clearly construct a complex S whose

thickening $\mathbf{S}$ is a compact, orientable surface of genus g such that $\partial\mathbf{S}$ is connected.+ Now using all the remaining $B_1(d)$'s, and $B_i(f)$'s $i \geq 2$, $f \neq 1$ construct a connected complex V whose thickening $\mathbf{V}$ is an orientable surface of genus g such that $\partial\mathbf{V}$ is connected, and contains infinitely many pairs of half-outlets where the infinitely many $\mathbb{B}_i(1)$'s, $i \geq 2$ can be inserted to form $\mathbf{X}$. Clearly $\partial\mathbf{X}$ is connected, and $\mathbf{X} = \mathbf{X}_\psi$ where ψ is a Neumann subgroup $\approx \Phi$. q.e.d.

(4.5) Combining (2.11), (4.1) - (4.4), we get the following

Structure theorem. Let Γ be as in (1.4.1), $\Gamma \not\approx \mathbb{Z}_2 * \mathbb{Z}_2$, and Φ be given as an abstract group as in (1.7.1). Then Φ is realizable as a Neumann subgroup of Γ iff one of the following conditions holds.

1) $r = \infty$,

or 2) A) r = an even integer ≥ 0,

B) $r_i(1) = \infty$ for $\geq n-1$ values of i.

+ If $(m_1, m_2) \neq (2,2)$ or if $g = 0$ we can do this using only $B_i(d)$'s $i \leq 2$. Otherwise we shall need to use $B_i(d)$'s, $i \leq 3$. Here, again, we are using the assumption that $\Gamma \not\approx \mathbb{Z}_2 * \mathbb{Z}_2$.

or 3) A) r = an even integer ≥ 0,

B) $r_i(1) = \infty$, $r_i(d) < \infty$, $d \neq 1$, for $n-2$ values of $i \neq a,b$ say,

C) $r_i(2) = \infty$, $r_i(d) < \infty$, $d \neq 2$ for $i = a,b$,

D) there exists Φ_0, realizable as a 1-cycloidal subgroup such that Φ is a simple (m_a,m_b)-unfolding of Φ_0.

(4.6) <u>Remark</u>: Suppose Φ is as in (1.7.1) and 3)A) - C) are satisfied. Let ψ_0 = the finite free product of F_r and $\Phi_{ij} \simeq \mathbb{Z}_{m_i/d}$ $i \neq a,b$ and $d \neq 1$, or $i = a,b$, $d \neq 2$. If Φ is realizable as a Neumann subgroup then Φ_0 referred to in 3)D) is $\approx \psi_0 * \Theta_0$ where Θ_0 is a finite free product of groups conjugate to Γ_i $i \neq a,b$ or conjugate to the subgroups of Γ_a (resp. Γ_b) isomorphic to $\mathbb{Z}_{a/2}$ (resp. $\mathbb{Z}_{b/2}$). Moreover Φ_0 must contain at least one factor $\simeq \Gamma_a$ or Γ_b. From the way X_{Φ_0} would be constructed, cf. (2.11), it is clear that there are only <u>finitely many</u> possibilities for Θ_0 - hence, also only finitely many possibilities for Φ_0. Now theorem (1.7) of [5] gives an effective procedure for deciding whether any of these Φ_0 can be realized as a 1-cycloidal subgroup. Thus one has an effective procedure for deciding realizability of Φ as a Neumann subgroup.

(4.7) <u>Remark</u>: The condition 3)D) is <u>not</u> a consequence of

3)A) - C). For example, take $\Gamma = \mathbb{Z}_4 * \mathbb{Z}_4$ and $\Phi = \overset{*}{\Pi}\mathbb{Z}_2$ (infinite product). Write Φ as $\overset{*2}{\underset{i=1}{\Pi}} (\overset{*}{\underset{j \in J_i}{\Pi}} \Phi_{ij})$, $\Phi_{ij} \simeq \mathbb{Z}_2$ so that $|J_i| = \infty$. It is easy to see that 3)A) - C) hold, but Φ is not realizable as a Neumann subgroup.

(4.8) Remark: We should point out two possible interpretations for the phrase "Φ as in (1.7.1) is realizable as...". If m_i's are pairwise coprime then there is a unique value of i for a finite factor of Φ to be conjugate to a subgroup of Γ_i. If two or more m_i's have common factors then there may be a choice for a finite factor of Φ to be interpreted as a particular Φ_{ij}. In our statement of the structure theorem we have tacitly assumed that these choices have already been made. Thus if Φ is only given as an abstract group there may be a bit more freedom first to put it in the form (1.7.1) and then realize as a .... .

(4.9) Remark: The condition 3) C) of course requires that m_a and m_b are even integers. So if there is at most one m_i which is an even integer then the condition 3) is non-applicable.

§5 Maximal Subgroups

(5.0) In [3], [13] there are constructions of subgroups of the classical modular group which are maximal among non-parabolic subgroups, and which are different from the ones discovered by Neumann [8], or which are not Neumann subgroups in the sense of (1.1). These constructions are rather elaborate and require a very careful analysis. In terms of the diagrams X_Φ's one can give such constructions more readily, and in fact one may construct maximal, or maximal and Neumann, or maximal and 1-cycloidal, or maximal and non-parabolic but not Neumann.... subgroups.

(5.1) Let Γ be as in (1.4.1) and $\Phi \leq \Gamma$. A <u>symmetry</u> of $\mathbf{X}_\Phi$ is simply a branched-covering-transformation of $p\colon \mathbf{X}_\Phi \to \mathbf{X}_\Gamma$ i.e. a homeomorphism $\sigma\colon \mathbf{X}_\Phi \to \mathbf{X}_\Phi$ such that

(5.1.1)
$$\begin{tikzcd} \mathbf{X}_\Phi \arrow[rr, "\sigma"] \arrow[dr, "p"'] & & \mathbf{X}_\Phi \arrow[dl, "p"] \\ & \mathbf{X}_\Gamma & \end{tikzcd}$$

commutes. Then σ preserves orientation and carries building blocks into building blocks.

Notice that in an unbranched covering space a non-identity covering transformation has no fixed points. But in a branched covering it is not necessarily so.

We say that $\mathbf{X}_\Phi$ <u>has no fixed-point-free symmetry</u> if every non-identity symmetry of $\mathbf{X}_\Phi$ has a fixed point.

Notice also that a symmetry $\sigma\colon \mathbf{X}_\Phi \to \mathbf{X}_\Phi$ induces maps (again denoted by) $\sigma\colon X_\Phi \to X_\Phi$ and $\sigma\colon Y_\Phi \to Y_\Phi$, and these maps commute with the thinning map and the canonical projection $X_\Phi \to Y_\Phi$.

(5.2) Orient $\mathbf{X}_\Phi$ which also orients $\partial\mathbf{X}_\Phi$. Let C be a component of $\partial\mathbf{X}_\Phi$. <u>The pattern along</u> C is simply the finite or doubly infinite sequence of $\mathbb{B}_i(d)$'s one meets along C while walking in the "positive" direction.[⊕] The pattern is finite iff C is compact and in that case the number of terms in the pattern is a multiple of n. We say that the pattern along C is <u>not periodic</u> if either i) C is noncompact and the pattern has no finite period or ii) C is compact, the pattern contains $\alpha\cdot n$ elements, $\alpha \in \mathbb{Z}_{>0}$, and (in the cyclic order) the pattern has no period $< \alpha n$.

(5.3) Let $B = B_i(d)$ be a building block of X_Φ. The <u>neighbors</u> of B are the building blocks at the end of the paths containing two edges emanating from B. So, in all B has

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[⊕]Notice that a block $\mathbb{B}_i(d)$ with $d > 1$, cf. the picture in (2.2.3), is counted k times in the pattern along C if C contains k "circular arcs" on $\mathbb{B}_i(d)$ i.e. the components of $\partial\mathbb{B}_i(d) - \partial\{\cup \text{ arms}\}$.

$d(n-1)$ neighbors.

(5.4) <u>Theorem</u>: Let Γ be as in (1.4.1) where all m_i's are primes. Let $\Phi \leq \Gamma$ be as in (1.7.1). Assume that 1) each $B \cong B_i(1)$ in X_Φ has a $B_j(m_j)$ for each $j \neq i$ as a neighbor, 2) either A) $r = 0$ and $\mathbf{X}_\Phi$ has no fixed-point-free symmetry or B) the patterns along different components of $\partial\mathbf{X}_\Phi$ are pairwise distinct and none is periodic. Then Φ is maximal.

<u>Proof</u>: Suppose $\Phi \leq \psi \leq \Gamma$, and consider the branched covering $q: \mathbf{X}_\Phi \to \mathbf{X}_\psi$. Suppose $\mathbf{X}_\psi$ contains a branch point. Since m_i's are assumed to be primes this means that there is a building block $B \subseteq X_\Phi$ such that $B \cong B_i(1)$ and $q(B) \cong B_i(m_i)$. But then 1) implies that $q(X_\Phi) = X_\Gamma$ i.e. $\psi = \Gamma$.

Now suppose $\psi \neq \Gamma$. Hence q is <u>unbranched</u>. Under the condition 2A) $\mathbf{X}_\Phi$ is simply connected. But then q is the universal (in particular regular) covering of $\mathbf{X}_\psi$. Since we assumed that $\mathbf{X}_\Phi$ has no fixed-point-free symmetry it follows that degree $q = 1$ i.e. $\Phi = \psi$. Under the condition 2B) we see that $q|_{\partial\mathbf{X}_\Phi}$ is a homeomorphism. Also clearly $q^{-1}(\partial\mathbf{X}_\psi) = \partial\mathbf{X}_\Phi$. So again

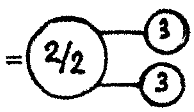
$$\text{degree } q = \text{degree } q|_{\partial\mathbf{X}_\Phi} = 1,$$

and $\Phi = \psi$. Hence Φ is maximal. q.e.d.

(5.5) <u>Remarks</u>: 1) Clearly there are many varieties of sufficient sets of conditions for maximality in terms of X_Φ's. For instance one may assume that all but finitely many building blocks of X_Φ have the property stated in 1) and then "mess up" the diagram near these finitely many blocks.

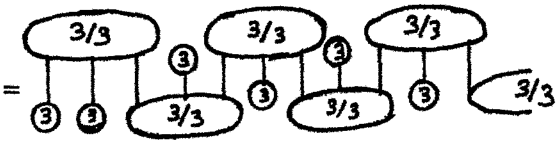
2) If $n \geq 3$ or two m_i's ≥ 3 the conditions in (5.4) are easy to ensure. For example $r = 0$ means int $\mathbf{X}_\Phi \approx \mathbb{R}^2$ and Y_Φ is a tree. The condition that $\mathbf{X}_\Phi$ has no fixed-point-symmetry is ensured if we have a compact subsurface $S \subseteq \mathbf{X}_\Phi$ satisfying the condition (2.8.1) such that $S \approx$ a closed disk and the pattern of the building blocks in S does not repeat in X_Φ - or at least the "distances" among its repetitions do not repeat. Then any symmetry of $\mathbf{X}_\Phi$ would leave S invariant and would have a fixed point by Brouwer's theorem.

3) If $n = 2$ and some $m_i = 2$ then the direct application of (5.4) produces only finitely many examples, all of finite index. But excluding the degenerate case $\Gamma \approx \mathbb{Z}_2 * \mathbb{Z}_2$ one may first pass to an appropriate 1-cycloidal subgroup in Γ and then apply the above considerations. For example let $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_3$, and let $\Phi_0 \leq \Gamma$, $\Phi_0 \approx \mathbb{Z}_3 * \mathbb{Z}_3$ whose diagram is

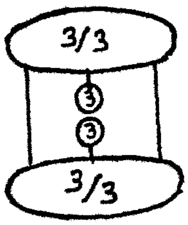
(5.5.1) $X_{\Phi_0} =$ 

, $\mathbf{X}_{\Phi_0} =$ 

Consider $\Phi \leq \Phi_0$ whose diagram is

(5.5.2) $X_\Phi =$  $\to \infty$.

Clearly Φ is a Neumann subgroup of Φ_O, and in fact maximal in Φ. Also clearly Φ is a Neumann subgroup of Γ. As a subgroup of Γ, the diagram of Φ is obtained from (5.5.2) by sticking in 2/2 on each edge. If we do this sticking and then replace one —2/2—3 by —2 we obtain a new Neumann subgroup of Γ which is clearly not contained in Φ_O. It would be also maximal in Γ. Making the pattern in (5.5.2) doubly infinite in the obvious way one obtains a subgroup $\Phi_1 \leq \Phi_O$ for which $\partial \mathbf{X}_{\Phi_1}$ contains two components both non-compact. This Φ_1 is not Neumann and it is maximal among non-parabolic subgroups in Φ_O and also in Γ but it is not maximal. For clearly $\Phi_1 \leq \psi_1 \leq \Phi_O$ where

(5.5.3) $X_{\psi_1} =$  ,

so Φ_1 is not maximal. On the other hand if $\Phi_1 \lneqq \psi \lneqq \Phi_O$ then $q\colon \mathbf{X}_{\Phi_1} \to \mathbf{X}_\psi$ must be unbranched, cf. the argument in (5.4). Now $\mathbf{X}_{\Phi_1}$ is simply connected so q must be a regular covering. One sees that the only symmetries of $\mathbf{X}_{\Phi_1}$ are the obvious "horizontal" translations, and so $\mathbf{X}_\psi$ is compact i.e. $(\Phi_O : \psi) < \infty$. So ψ contains parabolic elements. So Φ_1 is maximal among non-parabolic subgroups. On the other hand one may start with a doubly infinite version of (5.5.2) where the attachment of 3— is non-periodic. Then one would obtain a maximal-and-non-parabolic subgroup of Φ_0 which is not Neumann. By sticking in a —2 somewhere (as described above) one would obtain such subgroups also in Γ.

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