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MODULI OF CUBIC SURFACES AND THEIR ANTICANONICAL DIVISORS

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ABSTRACT. We consider the moduli space of log smooth pairs formed by a cubic surface and an anticanonical divisor. We describe all compactifications of this moduli space which are constructed using Geometric Invariant Theory and the anticanonical polarization. The construction depends on a weight on the divisor. For smaller weights the stable pairs consist of mildly singular surfaces and very singular divisors. Conversely, a larger weight allows more singular surfaces, but it restricts the singularities on the divisor.

1. INTRODUCTION

The moduli space of (marked) cubic surfaces is a classic space in algebraic geometry. Indeed, its GIT compactification was first described by Hilbert in 1893 [13], and several alternative compactifications have followed it (see [14, 16, 12]). In this article, we enrich this moduli problem by parametrizing pairs (S, D) where $S \subset \mathbb{P}^3$ is a cubic surface, and $D \in |-K_S|$ is an anticanonical divisor. There are several motivations for our construction. Firstly, it was recently established that the GIT compactification of cubic surfaces corresponds to the moduli space of K-stable del Pezzo surfaces of degree three [17]. The concept of K-stability has a natural generalization to log-K-stability for pairs, and our GIT quotients are the natural candidates for compactifications of log K-stable pairs of cubic surfaces and their anticanonical divisors. Therefore, our description is a first step toward a generalization of [17]. Secondly, a precise description of the GIT of cubic surfaces is important for describing the complex hyperbolic geometry of the moduli of cubic surfaces, and constructing new examples of ball quotients (see [1]). We expect similar applications for our GIT quotients.

The GIT quotients considered depend on a choice of a linearization $\mathcal{L}_t$ of the parameter space $\mathcal{H}$ of cubic forms and linear forms in $\mathbb{P}^3$. We have that $\mathcal{H} \cong \mathbb{P}^{19} \times \mathbb{P}^3$. Although $\mathrm{Pic}(\mathbb{P}^{19} \times \mathbb{P}^3) \cong \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle$, it can be shown that the different GIT quotients arising by picking different polarizations of $\mathcal{H}$ are controlled by the parameter $t = \frac{b}{a} \in \mathbb{Q}_{>0}$ (see Section 2 for a thorough treatment). For each value of t, there is a GIT compactification $\overline{M(t)}$ of the moduli space of pairs (S, D) where S is a cubic surface and $D \in |-K_S|$ is an anticanonical divisor. It follows from the general theory of variations of GIT (see [19, 5], c.f. [10, Theorem 1.1]) that $0 \leqslant t \leqslant 1$ and that there are only finitely many different GIT quotients associated to t. Indeed, there is a *set of chambers* (t_i, t_{i+1}) where the GIT quotients $\overline{M(t)}$ are isomorphic for all $t \in (t_i, t_{i+1})$, and there are finitely many *GIT walls* $t_1, \ldots, t_k$ where the GIT quotient is a birational modification of $\overline{M(t)}$ where $0 < |t - t_i| < \epsilon \ll 1$. Additionally there are an initial and end walls $t_0 = 0$ and $t_{k+1} = 1$.

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Lemma 1.1. *The GIT walls are*

$$t_0 = 0,\ t_1 = \frac{1}{5},\ t_2 = \frac{1}{3},\ t_3 = \frac{3}{7},\ t_4 = \frac{5}{9},\ t_5 = \frac{9}{13},\ t_6 = 1.$$

Given $t \in \mathbb{Q}_{>0}$, a pair (S, D) is *t-stable* (respectively *t-semistable*) if it is t-stable (respectively t-semistable) under the $\mathrm{SL}(4, \mathbb{C})$-action. A pair is *strictly t-semistable* if it is t-semistable but not t-stable. The space $M(t)$ parametrizes t-stable pairs and $\overline{M(t)}$ parametrizes closed strictly t-semistable orbits.

The GIT walls can be interpreted geometrically as follows. Let T be one of the possible isolated singularities in a cubic surface (see Proposition 3.1), let $w(T)$ be the sum of its associated weights (see Definition 3.4). For example, the set of weights for the $\boldsymbol{A}_n$ singularity is $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n+1}\right)$ and $w(\boldsymbol{A}_n) = \frac{n+2}{n+1}$. We define $\mathrm{Wall}(T) \coloneqq \frac{4}{w(T)} - 3$.

Theorem 1.2. *There are 13 non-isomorphic GIT quotients $\overline{M(t)}$. Seven of these quotients correspond to the walls t_i in Lemma 1.1 and they can be recovered as $t_i = \mathrm{Wall}(T)$ for some isolated ADE singularity T in some irreducible cubic surface:*

$$t_0 = \mathrm{Wall}(\boldsymbol{A}_2) = 0, \qquad t_1 = \mathrm{Wall}(\boldsymbol{A}_3) = \frac{1}{5}, \quad t_2 = \mathrm{Wall}(\boldsymbol{A}_4) = \frac{1}{3}$$
$$t_3 = \mathrm{Wall}(\boldsymbol{A}_5) = \mathrm{Wall}(\boldsymbol{D}_4) = \frac{3}{7}, \quad t_4 = \mathrm{Wall}(\boldsymbol{D}_5) = \frac{5}{9}$$
$$t_5 = \mathrm{Wall}(\boldsymbol{E}_6) = \frac{9}{13}, \qquad t_6 = \mathrm{Wall}(\widetilde{\boldsymbol{E}}_6) = 1.$$

the other six GIT quotients $\overline{M(t)}$ corresponding to linearizations $t \in (t_i, t_{i+1})$, $i = 1, \dots, 6$. All the points in $\overline{M(t_0)}$ and $\overline{M(t_6)}$ correspond to strictly semi-stable pairs, while all other $\overline{M(t)}$ with $t \in (0, 1)$ have stable points. The GIT quotient is empty for any $t \notin [0, 1]$.

The quotient $\overline{M(0)}$ is isomorphic to the GIT of cubic surfaces and the quotient $\overline{M(1)}$ is the GIT of plane cubic curves (see [10, Lemma 4.1]). These spaces are classical and have been thoroughly studied (see [15]). Henceforth focus on the case $t \in (0, 1)$.

The next theorem gives a full of classification of t-stable pairs (S, D) appearing in $M(t)$. A nice feature of $M(t)$ is that for each $t \in (0, 1)$ and each t-stable pair (S, D), the surface S has isolated ADE singularities. Table 1 gives a summary of the t-stable pairs (S, D) for each t in terms of their *worst* singularities and the intersection of the components of D. See Definition 3.2 and Figure 2 for the notion of *worst singularity.* See Table 2 to reinterpret D in the language of ADE singularities.

Theorem 1.3. *Consider a pair (S, D) formed by a cubic surface S and a hyperplane section $D \in |-K_S|$.*

- *(i) Let $t \in (0, \frac{1}{5})$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_2$ and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.*
- *(ii) Let $t = \frac{1}{5}$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_2$, D is reduced, and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.*

t	$(0, \frac{1}{5})$	$\frac{1}{5}$	$(\frac{1}{5}, \frac{1}{3})$	$\frac{1}{3}$
Sing(S)	$\boldsymbol{A}_2$	$\boldsymbol{A}_2$	$\boldsymbol{A}_3$	$\boldsymbol{A}_3$
Sing(D)	on smooth or $\boldsymbol{A}_1 \in S$	isolated on smooth or $\boldsymbol{A}_1 \in S$	isolated on smooth or $\boldsymbol{A}_1 \in S$	isolated or cuspidal at $\boldsymbol{A}_1 \in S$
t	$(\frac{1}{3}, \frac{3}{7})$	$\frac{3}{7}$	$(\frac{3}{7}, \frac{5}{9})$	$\frac{5}{9}$
Sing(S)	$\boldsymbol{A}_4$	$\boldsymbol{A}_4$	$\boldsymbol{A}_5$, $\boldsymbol{D}_4$	$\boldsymbol{A}_5$, $\boldsymbol{D}_4$
Sing(D)	isolated or cuspidal at $\boldsymbol{A}_1 \in S$	tacnodal or normal crossings at $\boldsymbol{A}_1 \in S$	tacnodal or normal crossings at $\boldsymbol{A}_1 \in S$	cuspidal or normal crossings at $\boldsymbol{A}_1 \in S$
t	$(\frac{5}{9}, \frac{9}{13})$	$\frac{9}{13}$	$(\frac{9}{13}, 1)$	
Sing(S)	$\boldsymbol{A}_5$, $\boldsymbol{D}_5$	$\boldsymbol{A}_5$, $\boldsymbol{D}_5$	$\boldsymbol{E}_6$	
Sing(D)	cuspidal or normal crossings at $\boldsymbol{A}_1 \in S$	normal crossings on smooth or $\boldsymbol{A}_1 \in S$	normal crossings on smooth or $\boldsymbol{A}_1 \in S$	

TABLE 1. Worst singularities possible in a t-stable pair (S, D) for each $t \in (0, 1)$.

(iii) *Let $t \in (\frac{1}{5}, \frac{1}{3})$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_3$, D is reduced and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.*

(iv) *Let $t = \frac{1}{3}$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_3$, D is reduced and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S and D has at worst a cuspidal singularity at P.*

(v) *Let $t \in (\frac{1}{3}, \frac{3}{7})$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_4$, D is reduced and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S and D has at worst a normal crossing singularity at P.*

(vi) *Let $t = \frac{3}{7}$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_4$, D has at worst a tacnodal singularity and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S and D has at worst a normal crossing singularity at P.*

(vii) *Let $t \in (\frac{3}{7}, \frac{5}{9})$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_5$ or $\boldsymbol{D}_4$, D has at worst a tacnodal singularity and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S and D has at worst a normal crossing singularity at P.*

(viii) *Let $t = \frac{5}{9}$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_5$ or $\boldsymbol{D}_4$, D has at worst an $\boldsymbol{A}_2$ singularity and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S and D has at worst a normal crossing singularity at P.*

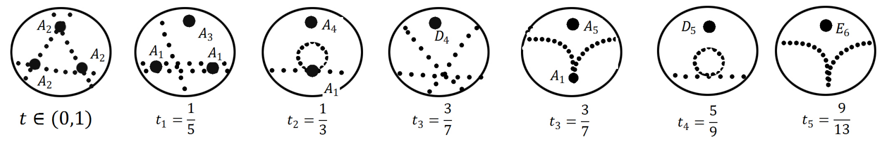

FIGURE 1. Pairs in $\overline{M(t)} \setminus M(t)$ for each $t \in (0,1)$. The dotted lines represent the divisor D. The bold points are singularities of the surface.

(ix) *Let $t \in (\frac{5}{9}, \frac{9}{13})$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_5$ or $\boldsymbol{D}_5$, D has at worst a cuspidal singularity and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S and D has at worst a normal crossing singularity at P.*

(x) *Let $t = \frac{9}{13}$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_5$ or $\boldsymbol{D}_5$, D has at worst normal crossing singularities and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.*

(xi) *Let $t \in (\frac{9}{13}, 1)$. The pair (S, D) is t-stable if and only if S has finitely many ADE singularities, D has at worst normal crossing singularities and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.*

Our last theorem gives a full of classification of the pairs (S, D) associated to each of the unique closed orbits in $\overline{M(t)} \setminus M(t)$ for each $t \in (0, 1)$. Figure 1 gives sketches of each of these pairs. Recall that an *Eckardt point* of a cubic surface S is a point where three coplanar lines of S intersect.

Theorem 1.4. *Let $t \in (0,1)$. If $t \neq t_i$, then $\overline{M}(t)$ is the compactification of the stable loci $M(t)$ by the closed $\mathrm{SL}(4,\mathbb{C})$-orbit in $\overline{M}(t) \setminus M(t)$ represented by the pair (S_0, D_0), where S_0 is the unique $\mathbb{C}^*$-invariant cubic surface with three $\boldsymbol{A}_2$ singularities and D_0 is the union of the unique three lines in S_0, each of them passing through two of those singularities.*

If $t = t_i$, $i = 1, 2, 4, 5$, then $\overline{M}(t_i)$ is the compactification of the stable loci $M(t_i)$ by the two closed $\mathrm{SL}(4,\mathbb{C})$-orbits in $\overline{M}(t_i) \setminus M(t_i)$ represented by the uniquely defined pair (S_0, D_0) described above and the $\mathbb{C}^*$-invariant pair (S_i, D_i) uniquely defined as follows:

(i) *the cubic surface S_1 with an $\boldsymbol{A}_3$ singularity and two $\boldsymbol{A}_1$ singularities and the divisor $D_1 = 2L + L' \in |-K_S|$ where L and L' are lines such that L is the line containing both $\boldsymbol{A}_1$ singularities and L' is the only line in S not containing any singularities;*

(ii) *the cubic surface S_2 with an $\boldsymbol{A}_4$ singularity and an $\boldsymbol{A}_1$ singularity and the divisor $D_2 \in |-K_S|$ which is a tacnodal curve singular at the $\boldsymbol{A}_1$ singularity of S;*

(iii) *the cubic surface S_4 with a $\boldsymbol{D}_5$ singularity and the divisor $D_4 \in |-K_S|$ which is a tacnodal curve whose support does not contain the surface singularity;*

(iv) *the cubic surface S_5 with an $\boldsymbol{E}_6$ singularity and the cuspidal rational curve $D_5 \in |-K_S|$ whose support does not contain the surface singularity.*

The space $\overline{M}(t_3)$ *is the compactification of the stable loci* $M(t_3)$ *by the three closed* $\mathrm{SL}(4,\mathbb{C})$*-orbits in* $\overline{M}(t_3) \setminus M(t_3)$ *represented by the* $\mathbb{C}^*$*-invariant pairs uniquely defined as follows:*

(i) the pair (S_0, D_0) *described above;*
(ii) the pair (S_3, D_3) *where* S_3 *is the cubic surface with a* $\boldsymbol{D}_4$ *singularity and and Eckardt point and* D_3 *consists of the unique three coplanar lines intersecting at the Eckardt point;*
(iii) the pair (S_3', D_3') *where* S_3' *is the cubic surface with an* $\boldsymbol{A}_5$ *and an* $\boldsymbol{A}_1$ *singularity and the divisor* D_3' *which is an irreducible curve with a cuspidal point at the* $\boldsymbol{A}_1$ *singularity of* S_3'*.*

Notation used and structure of the article. Throughout the article a pair (S, D) consists of a cubic surface $S \subset \mathbb{P}^3_{\mathbb{C}}$ and an anticanonical section $D \in |-K_S| \cong \mathbb{P}(H^0(S, \mathcal{O}_S(1)))$ Hence, $D = S \cap H$ in the case for some hyperplane $H = \{l(x_0, \dots, x_3) = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$. Whenever we consider a parameter $t \in (t_i, t_{i+1})$ we implicitly mean $t \in (t_i, t_{i+1}) \cap \mathbb{Q}$.

In Section 2 we describe in detail the GIT setting we consider. We introduce the required singularity theory in Section 3. GIT-stability depends on a finite list of geometric configurations characterized in Section 4. We prove Theorem 1.3 in Section 5. We prove Theorems 1.2 and 1.4 in Section 5.

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Our results use some computations done via software. The computational results are summarized in Appendix A. The computations, together with full source code written in Python can be found in [11]. The code is based on the theory developed in our previous article [10] and a rough idea of the algorithm can be found there. More detailed algorithms will appear in [9]. The source code and data, but not the text of this article, are released under a Creative Commons CC BY-SA 4.0 license. See [11] for details. If you make use of the source code and/or data in an academic or commercial context, you should acknowledge this by including a reference or citation to [10] —in the case of the code— or to this article —in the case of the data.

2. GIT set up and Computational Methods

In this section, we briefly describe the GIT setting for constructing our compact moduli spaces. We refer the reader to [10] where the problem is thoroughly discussed and solved for pairs formed by a hyperplane and a hypersurface of $\mathbb{P}^{n+1}$ of a fixed degree. Our GIT quotients are given by

$$\overline{M}\left(\frac{b}{a}\right) := \left(\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))) \times \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)))\right)^{ss} /\!\!/_{\mathcal{O}(a,b)} \mathrm{SL}(4, \mathbb{C})$$

and they depend only of one parameter $t := \frac{b}{a} \in \mathbb{Q}_{\geqslant 0}$. The use of GIT requires three initial combinatorial steps which are computed with the algorithm [9] implemented in [11]. The first step is to find a set of *candidate GIT walls* which includes all GIT walls (see [10, Theorem 1.1]). Some of these walls may be redundant and they are removed by comparing if there is any geometric change to the t-(semi)stable pairs (S, D) for $t = t_i \pm \epsilon$ for $0 < \epsilon \ll 1$. The set of candidate GIT walls is precisely the one in Lemma 1.1 and once Theorem 1.4 is proven this proves Lemma 1.1.

The second step (see [10, Lemma 3.2]) is to find the finite set $S_{2,3}$ of one-parameter subgroups that determine the t-stability of all pairs (S, D) for all t. For convenience, given a one-parameter subgroup $\lambda = \mathrm{Diag}(r_0, \dots, r_3)$, we define its dual one as $\overline{\lambda} = \mathrm{Diag}(-r_3, \dots, -r_0)$.

Lemma 2.1. *The elements $S_{2,3}$ are λ_k and $\overline{\lambda}_k$ where λ_k is one of the following:*

$$\begin{array}{rrr}
\lambda_1 = \mathrm{Diag}(1,0,0,-1) & \lambda_2 = \mathrm{Diag}(2,0,-1,-1) & \lambda_3 = \mathrm{Diag}(5,1,-3,-3) \\
\lambda_4 = \mathrm{Diag}(13,1,-3,-11) & \lambda_5 = \mathrm{Diag}(3,1,-1,-3) & \lambda_6 = \mathrm{Diag}(9,1,-3,-7) \\
\lambda_7 = \mathrm{Diag}(5,5,-3,-7) & \lambda_8 = \mathrm{Diag}(1,1,1,-3) & \lambda_9 = \mathrm{Diag}(5,1,1,-7) \\
\lambda_{10} = \mathrm{Diag}(1,1,-1,-1) & &
\end{array}$$

Let Ξ_k be the set of all monomials in four variables of degree k. Let $g \in \mathrm{SL}(4, \mathbb{C})$. Suppose $g \cdot S$ is given by the vanishing locus of a homogeneous polynomial F of degree 3 and $g \cdot D$ is given by the vanishing locus of F and a homogeneous polynomial l of degree 1. We say that F and l are *associated* to the pair (S, D). Let $\lambda = \mathrm{Diag}(r_0, \dots, r_3)$. Denote by $\mathcal{S} \subseteq \Xi_3$ and $\mathcal{D} \subseteq \Xi_1$ the monomials with non-zero coefficients in F and l, respectively. There is a natural pairing pairing $\langle v, \lambda \rangle \in \mathbb{Z}$ for any $v \in \Xi_k$. We define

$$\mu_t(g \cdot S, g \cdot D, \lambda) \coloneqq \min_{v \in \mathcal{S}} \langle v, \lambda \rangle + t \min_{x_i \in \mathcal{D}} \langle x_i, \lambda \rangle.$$

Lemma 2.2 (Hilbert-Mumford Criterion, see [10, Lemma 3.2])**.** *A pair (S, D), where $D = S \cap H$ is not t-stable if and only if there is $g \in \mathrm{SL}_n$ satisfying*

$$\mu_t(S, D) = \max_{\lambda \in S_{2,3}} \{\mu_t(g \cdot S, g \cdot D, \lambda)\} \geqslant 0.$$

Given $t \in (0, 1)$, and $\lambda \in S_{2,3}$ and $i \in \{0, \dots, 3\}$, the next step is to find the pairs of sets $N_t^{\oplus}(\lambda, x_i) \coloneqq \left(V_t^{\oplus}(\lambda, x_i), B^{\oplus}(x_i)\right)$ defined as:

$$\begin{aligned}
V_t^{\oplus}(\lambda, x_i) &= \{v \in \Xi_d \mid \langle v, \lambda \rangle + t \langle x_i, \lambda \rangle > 0\}, \\
B^{\oplus}(x_i) &= \{x_k \in \Xi_1 \mid k \leqslant i\}
\end{aligned} \tag{2.1}$$

which are maximal with respect to the containment order. For convenience, we list them in the Appendix (see Lemma A.1).

Theorem 2.3 ([10, Theorem 1.4])**.** *Let $t \in (0, 1)$. A pair $(S, S \cap H)$ is not t-stable if and only if there exists $g \in \mathrm{SL}(4, \mathbb{C})$ such that the set of monomials associated to $(g \cdot S, g \cdot H)$ is contained in a pair of sets $N_t^{\oplus}(\lambda, x_i)$ as given in Lemma A.1.*

Given $N^{\oplus}_t(\lambda, x)$, define $N^0_t(\lambda, x_i) := \big(V^0_t(\lambda, x_i), B^0(x_i)\big)$ (see [10, Prop. 5.3]) where $V^0_t(\lambda, x_i) \times B^0(x_i)$ is equal to

$$\{(v, m) \in V^{\oplus}_t(\lambda, x_i) \times B^{\oplus}(x_i) \mid \langle v, \lambda\rangle + t\langle m, \lambda\rangle = 0\}. \tag{2.2}$$

Theorem 2.4 ([10, Theorem 1.6]). *Let $t \in (0, 1)$. If a pair $(S, S \cap H)$ belongs to a closed strictly t-semistable orbit, then there exist $g \in \mathrm{SL}(4, \mathbb{C})$, $\lambda \in S_{2,3}$ and x_i such that the set of monomials associated to $(g \cdot S, g \cdot D)$ corresponds to those in a pair of sets $N^0_t(\lambda, x_i)$ as given in Lemma A.2.*

3. Preliminaries in singularity theory

We recall the admissible singularities in normal cubic surfaces (see also [6, Section 9.2.2]).

Propositon 3.1 ([4]). *Let X be an irreducible and reduced cubic surface and $p \in X$ be an isolated singular point. Then, the singularity at p is either a singularity of type $\boldsymbol{A}_k$, $\boldsymbol{D}_k$ with $k \leq 5$, $\boldsymbol{E}_6$, or a simple elliptic singularity of type $\widetilde{\boldsymbol{E}}_6$.*

Definition 3.2 ([3, p.88]). A class of singularities T_2 is adjacent to a class T_1, and one writes $T_1 \leftarrow T_2$ if every germ of $f \in T_2$ can be locally deformed into a germ in T_1 by an arbitrary small deformation. We say that the singularity T_2 is *worse* than T_1; or that T_2 is a *degeneration* of T_1.

The degenerations of the isolated singularities that appear in a cubic surface (or in their anticanonical divisors, which are plane cubic curves) are described in Figure 2 (for details see [3, p. 88] and [2, §13]). The above theory considers only local

$$\begin{tikzcd}
\boldsymbol{A}_1 & \boldsymbol{A}_2 \arrow[l] & \boldsymbol{A}_3 \arrow[l] & \boldsymbol{A}_4 \arrow[l] & \boldsymbol{A}_5 \arrow[l] & \\
 & & \boldsymbol{D}_4 \arrow[u] & \boldsymbol{D}_5 \arrow[l] \arrow[u] & \boldsymbol{E}_6 \arrow[l] \arrow[u] & \widetilde{\boldsymbol{E}}_6 \arrow[l]
\end{tikzcd}$$

Figure 2. Degeneration of germs of isolated singularities appearing in cubic surfaces.

deformations of singularities. When we study degenerations in the GIT quotient we are interested in global deformations.

Lemma 3.3 ([18, Theorem 1]). *Let $V(T_1, \dots T_r)$ be the set of cubic hypersurfaces in $\mathbb{P}^n$ for $n \leqslant 3$ with r isolated singular points of types $T_1, \dots T_r$. The germ of the linear system $|\mathcal{O}_{\mathbb{P}^3}(3)|$ at any $X \in V(T_1, \dots T_r)$ is a joint versal deformation of all singular points of X if $\sum_{i=1}^r \mu(T_i) \leq 9$ where $\mu(T_i)$ is the Milnor number of T_i.*

Recall that $\mu(\boldsymbol{A}_k) = k$, $\mu(\boldsymbol{D}_k) = k$ and $\mu(\boldsymbol{E}_6) = 6$. By checking carefully how these singularities appear together in each cubic surface (see [4, p. 255]) we conclude that $\sum_{i=1}^r \mu(T_i) \leqslant 6$ for all cubic surfaces with ADE singularities. Furthermore, by looking at Table 2, we see that $\sum_{i=1}^r \mu(T_i) \leqslant 4$ for any plane cubic curve with isolated singularities . Hence, Lemma 3.3 implies that for cubic plane curves and cubic surfaces, any local deformation of isolated singularities is induced by a global deformation.

Definition 3.4 ([4]). A polynomial F in $n+1$ variables is *semi-quasi-homogeneous (SQH)* with respect to the weights $(w_1, w_2, \dots, w_n)$ if all the monomials of F have weight larger or equal than 1 and those monomials of weight 1 define a function with an isolated singularity. In particular, the weights associated to the ADE singularities $\boldsymbol{A}_k$, $\boldsymbol{D}_k$ and $\boldsymbol{E}_6$ are

$$\left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{k+1}\right), \qquad \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{(k-2)}{2(k-1)}, \frac{1}{k-1}\right), \qquad \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right),$$

respectively. Furthermore, the weight of $\widetilde{\boldsymbol{E}}_6$ is $\left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. These weights are uniquely associated to their respective singularity.

Non-singular	-
Nodal cubic	$\boldsymbol{A}_1$
Cuspidal cubic	$\boldsymbol{A}_2$
Line and conic intersecting at a tacnode	$\boldsymbol{A}_3$
Line and conic intersecting in two points	$2\boldsymbol{A}_1$
Three lines intersecting in three points	$3\boldsymbol{A}_1$
Three lines intersecting at a point	$\boldsymbol{D}_4$

Table 2. Plane cubic curves and their singularities

Lemma 3.5 ([4, p. 246]). *If $F(x_0, x_1, x_2)$ is SQH with respect to one of the sets of weights in Definition 3.4 we can, by a locally analytic change of coordinates, reduce the terms of weight 1 to the normal forms for $\boldsymbol{A}_k$, $\boldsymbol{D}_k$, $\boldsymbol{E}_6$, which are locally analytically isomorphic to the following surface singularities:*

$$\boldsymbol{A}_k\colon\ x_1^{k+1} + x_2^2 + x_3^2 \ (k \geqslant 1), \qquad \boldsymbol{D}_k\colon\ x_1^{k-1} + x_1x_2^2 + x_3^2 \ (k \geqslant 4),$$

$$\boldsymbol{E}_6\colon\ x_1^3 + x_2^4 + x_3^2, \qquad \widetilde{\boldsymbol{E}}_6\colon\ x_1^3 + x_2^3 + x_3^3 + 3\lambda x_1x_2x_3,\ \lambda^3 \neq -1.$$

and the resulting function will remain SQH.

Reduced plane cubic curves are completely characterized according to the number and type of their ADE singularities (see Table 2).

4. Geometric characterization of pairs

In this section we relate the classifications of pairs in terms of singularity theory and the equations defining them. We have divided our lemmas in four groups: classification of singular cubic surfaces, classification of pairs (S, D) with singular boundary D, classification of pairs (S, D) where S is singular at a point $P \in D$ and classification of pairs (S, D) invariant under a $\mathbb{C}^*$-action. We will denote homogenous polynomials of degree d in $n+1$ variables as $f_d(x_0, \dots, x_n), g_d$, etc.

Singular cubic surfaces.

Lemma 4.1 ([4, Lemma 3]). *Let $F = x_0x_1x_3 + f_3(x_0, x_1, x_2)$, $P = (0,0,0,1)$, $Q = (0,0,1,0)$, $H = \{x_3 = 0\} \cong \mathbb{P}^2_{(x_0,x_1,x_2)}$ and $H_i = \{x_i = x_3 = 0\} \subset H$ for $i = 0, 1$.*

(1) The singularities of $\{F=0\}$ other than that at P correspond to the intersection of $C=\{x_0x_1=0\}\subset H$ and $C'=\{f_3=0\}$ at points R other than Q. Indeed, if $\operatorname{mult}_R(C\cdot C')=k$, then R is an $\boldsymbol{A}_{k-1}$ singularity.

(2) If $f_3(0,0,1)\neq 0$, then P is an $\boldsymbol{A}_2$ singularity. Let $k_i=\operatorname{mult}_Q(H_i\cdot C')$. If both k_0 and k_1 are both at least 2, then $\{F=0\}$ has non-isolated singularities. Otherwise P is an $\boldsymbol{A}_{k_0+k_1+1}$ singularity for $\{k_0,k_1\}=\{1,1\},\{1,2\},\{1,3\}$.

Lemma 4.2. *A pair (S,D) has an $\boldsymbol{A}_2$ singularity at a point $P\in D$ or a degeneration of one if and only if P is conjugate to $(0,0,0,1)$ and simultaneously (S,D) is conjugate by $\operatorname{Aut}(\mathbb{P}^3)$ to the pair defined by equations*

$$x_3f_2(x_0,x_1)+f_3(x_0,x_1,x_2)=0,\qquad l_1(x_0,x_1,x_2)=0.$$

Proof. Without loss of generality, we may assume $P=(0,0,0,1)$. By Lemma 4.1, S has (a degeneration of) an $\boldsymbol{A}_2$ singularity at P if and only if it is given by the equation $x_0x_1x_3+f_3(x_0,x_1,x_2)=0$. Any quadric $f_2(x_0,x_1)$ can be transformed to x_0x_1 or to a degeneration of x_0x_1 (e.g. x_0^2) by a change of coordinates preserving x_2 and x_3. The lemma follows because a hyperplane section D contains P if and only if D is given by a linear form $l_1(x_0,x_1,x_2)$. □

Lemma 4.3. *A surface S has an $\boldsymbol{A}_3$ singularity or a degeneration of one if and only if it is conjugate by $\operatorname{Aut}(\mathbb{P}^3)$ to:*

$$\{x_3f_2(x_0,x_1)+x_2^2f_1(x_0,x_1)+x_2g_2(x_0,x_1)+g_3(x_0,x_1)=0\}.$$

Proof. By Lemma 4.1, we may assume $S=\{x_0x_1x_2+f_3(x_0,x_1,x_2)=0\}$ and $P=(0,0,0,1)$. Moreover, the singularity is of type $\boldsymbol{A}_k$ with $k\geqslant 3$ if and only if $f_3(0,0,1)=0$. Therefore $f_3(x_0,x_1,x_2)=x_2^2f_1(x_0,x_1)+x_2g_2(x_0,x_1)+g_3(x_0,x_1)$. □

Lemma 4.4. *A surface S has an $\boldsymbol{A}_4$ singularity or a degeneration of one if and only if it is conjugate by $\operatorname{Aut}(\mathbb{P}^3)$ to*

$$\{x_3x_0l_1(x_0,x_1)+x_0x_2^2+x_2g_2(x_0,x_1)+g_3(x_0,x_1)=0\}.$$

Proof. By Lemma 4.1, the surface S is defined by the equation $x_0x_1x_3+f_3(x_0,x_1,x_2)=0$ where $f_3(x_0x_1x_2)=x_2^2f_1(x_0,x_1)+x_2g_2(x_0,x_1)+g_3(x_0,x_1)$ and $k_0=\operatorname{mult}_Q(H_0\cdot C')\geqslant 2$ and $k_1=\operatorname{mult}_Q(H_1\cdot C')\geqslant 1$ if and only if P is (a degeneration of) an $\boldsymbol{A}_4$ singularity. Notice that

$$k_i=\operatorname{mult}_Q(H_i\cdot C')=\dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x_0,x_1]}{\langle x_i,f_1+g_2+g_3\rangle}\right).$$

Therefore $k_0\geqslant 2$ if and only if $f_1(0,1)=0$. Hence, $f_1=x_0$. The lemma follows from noticing that $x_0x_1x_3$ is conjugate to $x_0x_3l_1(x_0,x_1)$ by an element of $\operatorname{Aut}(\mathbb{P}^3)$ fixing x_0,x_2,x_3. □

Lemma 4.5. *A surface S has an $\boldsymbol{A}_5$ singularity or a degeneration of one if and only if it is conjugate by $\operatorname{Aut}(\mathbb{P}^3)$ to*

$$\{x_3x_0l_1(x_0,x_1)+x_0x_2f_1(x_0,x_1,x_2)+f_3(x_0,x_1)=0\}.$$

Proof. As for Lemma 4.4, we may use Lemma 4.1 to assume that S is defined by $x_0x_1x_3+x_2^2f_1(x_0,x_1)+x_2g_2(x_0,x_1)+g_3(x_0,x_1)=0$ and $P=(0,0,0,1)$ is an $\boldsymbol{A}_5$ singularity if and only if

$$k_0=\operatorname{mult}_Q(H_i\cdot C')=\dim\left(\frac{\mathbb{C}[x_0,x_1]}{\langle x_0,f_1+g_2g_3\rangle}\right)\geqslant 3,$$

or equivalently $f_1(0,1)=0$ and $g_2(0,1)=0$. Likewise, $f_1 = ax_0$ and $g_2 = b_0x_0^2 + b_1x_0x_1$ and regrouping terms the proof follows. $\square$

In Figure 2 we see that the only non-trivial degenerations of a $\boldsymbol{D}_4$ singularity in a cubic surface which are not a $\tilde{\boldsymbol{E}}_6$ singularity are $\boldsymbol{D}_5$ and $\boldsymbol{E}_6$ singularities. Hence the next lemma follows at once from [4, Case C].

Lemma 4.6. *A surface S has a $\boldsymbol{D}_4$ singularity or a degeneration of one if and only if it is conjugate by* $\mathrm{Aut}(\mathbb{P}^3)$ *to* $\{x_3x_0^2 + f_3(x_0,x_1,x_2)=0\}$.

Lemma 4.7. *A surface S has a $\boldsymbol{D}_5$ singularity or a degeneration of one if and only if it is conjugate by* $\mathrm{Aut}(\mathbb{P}^3)$ *to*

$$\{f_3(x_0,x_1) + x_2g_2(x_0,x_1) + x_0x_2^2 + x_0^2x_3 = 0\}.$$

Proof. By Lemma 4.6 and Figure 2, we may assume that S is given by $x_3x_0^2 + f_3(x_0,x_1,x_2)$ since $\boldsymbol{D}_5$ is a degeneration of $\boldsymbol{D}_4$. Let $H=\{x_3=0\}$ and $C=\{x_3=f_3(x_0,x_1,x_2)=0\}\subset H$ and $C'=\{x_3=x_0=0\}\subset H$. We can rewrite $f_3 = x_2^2g_1(x_0,x_1) + x_2g_2(x_0,x_1) + g_3(x_0,x_1)$. By [4, Lemma 4], the point $P=(0,0,0,1)$ is (a degeneration of) a $\boldsymbol{D}_5$ singularity if and only if $C\cap C'$ consist of at most two points. The equation of $S\cap H\subset H$ localized at $Q=(0,0,1,0)$ is $g_1(x_0,x_1)+g_2(x_0,x_1)+g_3(x_0,x_1)=0$, and $C\cap C'$ intersects in at most two points if and only if

$$\dim_Q\left(\frac{\mathbb{C}[x_0,x_1]}{\langle x_0, g_1+g_2+g_3\rangle}\right)\geqslant 2.$$

The latter is equivalent to take $g_1 = ax_0$, which by rescaling x_2 gives the result. $\square$

Lemma 4.8. *A surface S has a $\boldsymbol{E}_6$ singularity or a degeneration of one if and only if it is conjugate by* $\mathrm{Aut}(\mathbb{P}^3)$ *to*

$$\{x_3x_0^2 + x_0x_2l_1(x_0,x_1,x_2) + f_3(x_0,x_1)=0\}.$$

Proof. Using the same notation as in Lemma 4.7 and following [4, Lemma 4], S is defined by $x_3x_0^2 + x_2^2g_1(x_0,x_1) + x_2g_2(x_0,x_1) + g_3(x_0,x_1) = 0$, and has (a degeneration of) an $\boldsymbol{E}_6$ singularity if and only if

$$\dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x_0,x_1]}{\langle x_0, g_1+g_2+g_3\rangle}\right)\geqslant 3.$$

The latter is equivalent to take $g_1 = x_0$ and $g_2 = x_0l_1(x_0,x_1)$. $\square$

Lemma 4.9 (see [4, Case E])**.** *A surface S has an isolated $\widetilde{\boldsymbol{E}}_6$ singularity if and only if S is the cone over a smooth plane cubic curve given by $f_3(x_0,x_1,x_2)=0$.*

By Serre's criterion, any hypersurface of dimension 2 is non-normal if and only if it has non-isolated singularities, which are classified in [4, Case E]:

Lemma 4.10. *Any irreducible non-normal cubic surface is conjugated by* $\mathrm{Aut}(\mathbb{P}^3)$ *to*

$$\{x_3f_2(x_0,x_1) + f_3(x_0,x_1) + x_2g_2(x_0,x_1) = 0\}.$$

Lemma 4.11. *A surface S is reducible if and only if it is conjugate by* $\mathrm{Aut}(\mathbb{P}^3)$ *to*

$$\{x_0f_2(x_0,x_1,x_2,x_3)=0\}.$$

Pairs with singular boundary. Consider a pair (S,D) and a point $P \in D \subset S$. By choosing coordinates appropriately we can suppose that $P = (0,0,0,1)$ and $(S,D) = ((F=0),(F=H=0))$ for F and H given as

$$\begin{aligned} F &= x_0 f_2(x_0,x_1,x_2,x_3) + x_3^2 f_1(x_1,x_2) + x_3 g_2(x_1,x_2) + f_3(x_1,x_2), \\ H &= x_0 \end{aligned} \tag{4.1}$$

Lemma 4.12. *A pair (S,D) has D with an $\boldsymbol{A}_2$ singularity at a point P or a degeneration of one if and only if (S,D) is conjugate by $\mathrm{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_0 f_2(x_0,x_1,x_2,x_3) + x_3 x_1^2 + f_3(x_1,x_2) = 0, \qquad x_0 = 0. \tag{4.2}$$

Proof. Without loss of generality we can suppose (S,D) given by (4.1). The equation of (a degeneration of) a plane cubic curve in $(x_0 = 0)$ with an $\boldsymbol{A}_2$ singularity at P is given by $x_1^2 x_3 + f_3(x_1,x_2) = 0$, where the curve has an $\boldsymbol{A}_2$ singularity at P if and only if x_2^3 has a non-zero coefficient in f_3. Therefore D is as in the statement if and only if in (4.1) we take $f_1 = 0$ and $g_2 = x_1^2$. □

Lemma 4.13. *A pair (S,D) has D with an $\boldsymbol{A}_3$ singularity at P or a degeneration of one if and only if (S,D) is conjugate by $\mathrm{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_0 f_2(x_0,x_1,x_2,x_3) + x_1(x_2^2 + x_1 l_1(x_1,x_2,x_3)) = 0, \qquad x_0 = 0.$$

Proof. We may assume that the equations of (S,D) are as in (4.1) and $P = (0,0,0,1)$. By restricting to $\{x_0 = 0\} \cong \mathbb{P}^2$ and localizing at P, the equation for D is $f_1(x_1,x_2) + g_2(x_1,x_2) + f_3(x_1,x_2)$ and by choosing coordinates appropriately we may assume that $L = \{x_1 = 0\}$ and $C = \{x_2^2 + x_1 l_1(x_1,x_2) = 0\}$ are a line and a conic intersecting at P, where l is a polynomial of degree 1, not necessarily homogeneous. Therefore $D|_{x_0=0}$ has equation $x_1(x_2^2 + x_1 l_1(x_1,x_2,x_3))$ so $f_1 \equiv 0$, $g_2 \equiv a x_1^2$, $f_3 = x_1 x_2^2 + x_1 l_1(x_1,x_2,0)$ and the result follows. □

Lemma 4.14. *A pair (S,D) has D with a $\boldsymbol{D}_4$ singularity at P or a degeneration of one if and only if (S,D) is conjugate by $\mathrm{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_0 f_2(x_0,x_1,x_2,x_3) + f_3(x_1,x_2) = 0, \qquad x_0 = 0.$$

Proof. We may assume that the equations of (S,D) are given as in (4.1). A plane cubic curve as in the statement is given by a homogeneous polynomial $f_3(x_1,x_2)$ in $\mathbb{P}^2_{(x_1,x_2,x_3)} \cong \{x_0 = 0\}$. By comparing with (4.1), this is equivalent to have $f_1 = g_2 \equiv 0$. □

Lemma 4.15. *A pair (S,D) has D non-reduced if and only if it is conjugate by $\mathrm{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_0 f_2(x_0,x_1,x_2,x_3) + x_1^2 f_1(x_1,x_2,x_3) = 0, \qquad x_0 = 0.$$

Proof. The result follows from (4.1) by noting that any two distinct lines in $\mathbb{P}^2$ are projectively equivalent to any other two lines. □

Lemma 4.16. *A pair (S,D) has $D = L + C$ where L is a line and C is a conic such that $3L \in |-K_S|$ if and only if it is conjugate by $\mathrm{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_0 f_2(x_0,x_1,x_2,x_3) + a x_1^3 = 0, \qquad l_1(x_0,x_1) = 0.$$

where L and $3L$ are conjugated to $\{x_0 = x_1 = 0\}$ and $= \{x_0 = 0\}|_S$, respectively. This surface has a point $Q \in L \subset \operatorname{Supp}(D)$ such that S has a singularity at Q which is not of type $\boldsymbol{A}_1$.

Proof. Suppose (S, D) as in the statement. Without loss of generality, we may suppose that the equation of S is as in (4.1), $D = \{x_0 + bx_1 = 0\}$ and let $D' := \{x_0 = 0\}$. Clearly $L \subset \operatorname{Supp}(D') \cap \operatorname{Supp}(D)$ and $D = D'$ if and only if $b = 0$. In this case, the equation of $D = D'$ in $\{x_0 = 0\} \cong \mathbb{P}^2$ is given by $x_3^2 f_1(x_1, x_2) + x_3 g_2(x_1, x_2) + f_3(x_1, x_2) = 0$ and $3L \in |-K_S|$ if and only if $f_1 = g_2 \equiv 0$ and $f_3 = ax_1^3$. If $b \neq 0$, then $x_1 = -\frac{x_0}{b}$. Take $x_0 = 0$ in (4.1). The equation of $D' = \{x_0 = 0\}|_S$ is $x_3^2 f_1 + x_3 g_2 + f_3 = 0$ and $D' \equiv 3L$ if and only if $f_1 = g_2 = 0$ and $f_3 = x_1^3$. But then, the equation of D in $\{x_0 + bx_1 = 0\}$ is $x_1(bf_2 + x_1^2)$ and $C = \{bf_2 + x_1^2 = x_0 + bx_1 = 0\}$. It is a well known fact that the line L contains a point Q at which S is singular and Q is not of type $\boldsymbol{A}_1$ (see [15, p. 227]). □

Pairs (S, D) where S is singular at a point $P \in D$.

Lemma 4.17 (see [4, Section 2, pp. 247–252])**.** *Let S be a surface with a singularity at $P = (0, 0, 0, 1)$. Then, the equation of S can be written as*

$$F = x_3 f_2(x_0, x_1, x_2) + f_3(x_0, x_1, x_2).$$

Lemma 4.18. *Given a pair (S, D), S is singular at a point $P \in D$ and D is an $\boldsymbol{A}_2$ singularity at P or a degeneration of one if and only if (S, D) is conjugate by $\operatorname{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_3 x_0 l_1(x_0, x_1, x_2) + x_3 x_1^2 + f_3(x_1, x_2) + x_0 f_2(x_0, x_1, x_2) = 0, \qquad x_0 = 0. \tag{4.3}$$

Proof. Without loss of generality we can assume $P = (0, 0, 0, 1)$. From Lemma 4.17, the equation of S is

$$\begin{aligned} & x_3 h_2(x_0, x_1, x_2) + h_3(x_0, x_1, x_2) = \\ & = a_0 x_3 x_1^2 + x_0 f_2(x_0, x_1, x_2) + f_3(x_1, x_2) + x_1 x_3 g_1(x_0, x_2) + x_3 g_2(x_0, x_2). \end{aligned}$$

By comparing with the equation in Lemma 4.12, D has (a degeneration of) an $\boldsymbol{A}_2$ singularity at P if and only if $g_1(x_0, x_2) = ax_0$ and $g_2(x_0, x_2) = bx_0^2 + cx_0 x_2$. The lemma follows. □

Lemma 4.19. *Given a pair (S, D), S is singular at a point $P \in D$ and D has an $\boldsymbol{A}_3$ singularity at P or a degeneration of one if and only if (S, D) is conjugate by $\operatorname{Aut}(\mathbb{P}^3)$ to the pair defined by equations:*

$$x_0^2 l_1(x_0, x_1, x_2, x_3) + x_0 f_2(x_1, x_2) + x_0 x_3 g_1(x_1, x_2) + x_1^2 h_1(x_1, x_2, x_3) + x_1 x_2^2 = 0,$$

$$x_0 = 0.$$

Proof. Without loss of generality we can assume $P = (0, 0, 0, 1)$. From Lemma 4.17, the equation of S is

$$\begin{aligned} & x_3 h_2(x_0, x_1, x_2) + h_3(x_0, x_1, x_2) = \\ & = a_0 x_3 x_0^2 + x_0 x_3 g_1(x_1, x_2) + x_3 g_2(x_1, x_2) + q_3(x_1, x_2) + x_0 q_2(x_0, x_1, x_2). \end{aligned}$$

By comparing with the equation in Lemma 4.13, D is (a degeneration of) an $\boldsymbol{A}_3$ singularity at P if and only if $g_2(x_1, x_2) = a_0 x_1^2$ and $q_3(x_1, x_2) = a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1^2 x_2$. Hence, after rescaling x_1, the equation of S is

$$a_0 x_3 x_0^2 + x_0 x_3 g_1(x_1, x_2) + a_0 x_1^2 x_3 + a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + x_0 q_2(x_0, x_1, x_2)$$

which is equal to

$$x_0^2 l_1(x_0, x_1, x_2, x_3) + x_0 f_2(x_1, x_2) + x_0 x_3 g_1(x_1, x_2) + x_1^2 q_1(x_1, x_2, x_3) + x_1 x_2^2.$$

□

Pairs (S, D) invariant under a $\mathbb{C}^*$-action.

Lemma 4.20. *Let (S, D) be a pair which is invariant under a non-trivial $\mathbb{C}^*$-action. Suppose the singularities of S and D are given as in the first and second columns of Table 3, respectively. Then (S, D) is conjugate by $\mathrm{Aut}(\mathbb{P}^3)$ to $(\{F = 0\}, \{F = H = 0\})$ for F and H as in the third and fourth columnns in Table 3, respectively. In particular, any such pair (S, D) is unique. Conversely, if (S, D) is given by equations as in the third and fourth columns of Table 3, then (S, D) has singularities as in the first and second columns of Table 3 and (S, D) is $\mathbb{C}^*$-invariant. Furthermore the element $\lambda \in \mathrm{SL}(4, \mathbb{C}^*)$, as defined in Lemma 2.1, given in the fifth column of Table 3 is a generator of the $\mathbb{C}^*$-action.*

$\mathrm{Sing}(S)$	$\mathrm{Sing}(D)$	F	H	λ
$P_i = \boldsymbol{A}_2$, $i = 1, 2, 3$	$\boldsymbol{A}_1$ at each P_i	$x_0x_1x_3 + x_2^3$	x_2	$\overline{\lambda}_2$
$P = \boldsymbol{A}_3$, $Q_1 = \boldsymbol{A}_1$, $Q_2 = \boldsymbol{A}_1$	$D = 2L + L'$, $Q_1, Q_2 \in L$, $\mathrm{Sing}(S) \cap L' = \emptyset$	$x_0x_1x_3 + x_1x_2^2 + x_0x_2^2$	x_3	$\overline{\lambda}_3$
$P = \boldsymbol{A}_4$, $Q = \boldsymbol{A}_1$	$\boldsymbol{A}_3$ at Q	$x_0x_1x_3 + x_0x_2^2 + x_1^2x_2$	x_3	λ_5
$P = \boldsymbol{A}_5$, $Q = \boldsymbol{A}_1$	$\boldsymbol{A}_2$ at Q	$x_0x_2^2 + x_0x_1x_3 + x_1^3$	x_3	λ_6
$P = \boldsymbol{D}_4$	$\boldsymbol{D}_4$ not at P	$x_0^2x_3 + x_1^3 + x_2^3$	x_3	λ_9
$P = \boldsymbol{D}_5$	$\boldsymbol{A}_3$ not at P	$x_0^2x_3 + x_0x_2^2 + x_1^2x_2$	x_3	$\overline{\lambda}_6$
$P = \boldsymbol{E}_6$	$\boldsymbol{A}_2$ not at P	$x_0^2x_3 + x_0x_2^2 + x_1^3$	x_3	$\overline{\lambda}_4$

TABLE 3. Some pairs (S, D) invariant under a $\mathbb{C}^*$-action.

Proof. There is a unique surface S with three $\boldsymbol{A}_2$ singularities [4, p. 255] which corresponds to the equation in Table 3. When a surface S has singularities $\boldsymbol{A}_4+\boldsymbol{A}_1$, $\boldsymbol{A}_5+\boldsymbol{A}_1$, $\boldsymbol{D}_4$, $\boldsymbol{D}_5$ or $\boldsymbol{E}_6$, and a $\mathbb{C}^*$-action, the equation for F follows from [8, Table 3]. If S has singularities $\boldsymbol{A}_3 + 2\boldsymbol{A}_1$, then [8, Table 3] gives that S has equation $x_3 f_2(x_0, x_1) + x_2^2 l_1(x_0, x_1) = 0$, where x_0x_1 has a non-zero coefficient in f_2, since otherwise S is singular along a line. Hence, after a change of coordinates involving only variables x_0 and x_1 and rescaling x_3, we obtain the desired result. It is trivial to check that each one-parameter subgroup λ in the table leaves S invariant, and therefore λ is a generator of the $\mathbb{C}^*$-action.

Given H, denote $D_H = \{F = H = 0\} \subset S$. We need to show that for (S, D) with prescribed singularities, $D_H = D$ if and only if H is as stated in Table 3. Verifying that for F and H as in the table, the pair (S, D) has the exepected singularities is stright forward and we omit it. We verify the converse.

Suppose that S has three $\boldsymbol{A}_2$ singularities. Then we may assume that $F = x_0x_1x_3 + x_2^3$ and the singularities correspond to $P_1 = (1,0,0,0)$, $P_2 = (0,1,0,0)$ and $P_3 = (1,0,0,0)$. There are only three lines L_1, L_2, L_3 in S [4, p. 255], which correspond to $\{x_2 = x_i = 0\}$ for $i = 0,1,3$, respectively. Clearly any two of these intersect at each of the points P_j. Moreover $D_H = D = \sum L_i$ and D has an $\boldsymbol{A}_1$ singularity at each P_i, as stated in Table 3.

Suppose that S has an $\boldsymbol{E}_6$ singularity at a point P and D has an $\boldsymbol{A}_2$ singularity at a point $Q \neq P$ and (S,D) is $\mathbb{C}^*$-invariant. Without loss of generality, we can now assume that $F = x_0^2x_3 + x_0x_2^2 + x_1^3$, $H = \sum a_ix_i$ for some parameters a_i and $P = (0,0,0,1)$. Since $\overline{\lambda}_4$ is a generator of the $\mathbb{C}^*$-action, then $\overline{\lambda}_4(t) \cdot H = a_0t^{11}x_0 + a_1t^3x_1 + a_2t^{-1}x_2 + a_3t^{-13}x_3$. Therefore D_H is $\mathbb{C}^*$-invariant if and only if $H = x_i$ for some $i = 0,\dots,3$. Notice that this happens every time the entries of λ are distinct. If $H = x_0$, then D_H is a triple line. If $H = x_1$, then D_H is the union of a conic and a line, and therefore D_H does not have an $\boldsymbol{A}_2$ singularity. If $H = x_2$, then D_H has an $\boldsymbol{A}_2$ singularity at P. If $H = x_3$, then D_H has an $\boldsymbol{A}_2$ singularity at $Q = (1,0,0,0) \neq P$ and $D_H = D$.

Suppose S has a $\boldsymbol{D}_5$ singularity at a point P, D has an $\boldsymbol{A}_3$ singularity at a point $Q \neq P$ and (S,D) is $\mathbb{C}^*$-invariant. Reasoning as in the previous case, we may assume $\overline{\lambda}_6$ generates the $\mathbb{C}^*$-action, $F = x_0^2x_3 + x_0x_2^2 + x_1^2x_2$, $H = x_i$ for some $i = 0,\dots,3$ and $P = (0,0,0,1)$. If $H = x_0$ or $H = x_2$, then the support of D_H contains a double line. If $H = x_2$, then D_H has an $\boldsymbol{A}_3$ singularity at P. If $H = x_3$, then D_H has an $\boldsymbol{A}_3$ singularity at $Q = (1,0,0,0) \neq P$ and $D_H = D$.

Suppose S has an $\boldsymbol{A}_5$ singularity at a point P and an $\boldsymbol{A}_1$ singularity at a point Q, D has an $\boldsymbol{A}_2$ singularity at Q and (S,D) is $\mathbb{C}^*$-invariant. We may assume λ_6 generates the $\mathbb{C}^*$-action, $F = x_0x_2^2 + x_0x_1x_3 + x_1^3$, $H = x_i$ for some $i = 0,\dots,3$, $P = (0,0,0,1)$ and $Q = (1,0,0,0)$. If $H = x_0$ then D_H is a triple line. If $H = x_1$, then D_H has a double line in its support. If $H = x_2$, then D_H has two $\boldsymbol{A}_1$ singularities. If $H = x_3$, then D_H has an $\boldsymbol{A}_2$ singularity at $Q = (1,0,0,0) \neq P$ and $D_H = D$.

Suppose S has an $\boldsymbol{A}_4$ singularity at a point P and an $\boldsymbol{A}_1$ singularity at a point Q, D has an $\boldsymbol{A}_3$ singularity at Q and (S,D) is $\mathbb{C}^*$-invariant. We may assume λ_5 generates the $\mathbb{C}^*$-action, $F = x_0x_1x_3 + x_0x_2^2 + x_1^2x_2$, $H = x_i$ for some $i = 0,\dots,3$, $P = (0,0,0,1)$ and $Q = (1,0,0,0)$. If $H = x_0$ or $H = x_1$ then D_H contains a double line in its support. If $H = x_2$, then D_H has three $\boldsymbol{A}_2$ singularities and if $H = x_3$, then D_H has an $\boldsymbol{A}_2$ singularity at Q and $D_H = D$.

Suppose S has a $\boldsymbol{D}_4$ singularity at a point P, D has a $\boldsymbol{D}_4$ singularity at a point $Q \neq P$ and (S,D) is $\mathbb{C}^*$-invariant. We may assume the generator of the $\mathbb{C}^*$-action is λ_9, $F = x_0^2x_3 + x_1^3 + x_2^3$ and $P = (0,0,0,1)$. If D_H is λ_9-invariant, either $H = x_i$ for some $i = 0,\dots,3$ or $H = x_1 - ax_2$ for $a \neq 0$. If $H = x_0$, then D_H has a $\boldsymbol{D}_4$ singularity at P. If $H = x_1$ or $H = x_2$, then D_H has an $\boldsymbol{A}_2$ singularity. If $H = x_1 - ax_2$ with $a \neq 0$, then $D_H = \{x_0^2x_3 + \left(1 + \frac{1}{a}\right)x_1^3 = 0, x_2 = \frac{x_1}{a}\}$ has an $\boldsymbol{A}_2$ singularity. If $H = x_3$, then D_H has a $\boldsymbol{D}_4$ singularity at $Q = (1,0,0,0) \neq P$ and $D_H = D$.

Suppose S has an $\boldsymbol{A}_3$ singularity at a point P, two $\boldsymbol{A}_1$ singularities at points Q_1 and Q_2, $D = 2L + L'$ where L is a line containing Q_1 and Q_2 and L' is a line such that $P, Q_1, Q_2 \notin L'$. Furthermore, suppose (S,D) is $\mathbb{C}^*$-invariant. We may assume that $\overline{\lambda}_3$ is the generator of the $\mathbb{C}^*$-action, $F = x_0x_1x_3 + x_1x_2^2 + x_0x_2^2$, $P = (0,0,0,1)$,

$Q_1 = (1,0,0,0)$, $Q_2 = (0,1,0,0)$ and $L = \{x_2 = x_3 = 0\}$. Moreover, if D_H is $\overline{\lambda}_3$-invariant, either $H = x_i$ for some $i = 0,\dots,3$ or $H = x_0 - ax_1$ for $a \neq 0$. If $H = x_0$ or $H = x_1$, then D_H does not contain L in its support. If $H = x_2$ or $H = x_0 - ax_1$, then D_H is reduced. If $H = x_3$, then $D_H = 2L + L'$, where $L' = \{x1 + x_0 = x_3 = 0\}$. Since $P, Q_1, Q_2 \notin L$, then $D_H = D$.

□

5. Proof of main theorems

We present the proofs of theorems 1.3 and 1.4. First, we reduce the amount of pairs we need to consider to those with isolated singularities:

Lemma 5.1. *Let (S, D) be a pair.*

(1) If S is reducible or not normal, then (S, D) is t-unstable for $t \in [0, 1)$.
(2) If D is not reduced, then, (S, D) is t-unstable for $t \in (1/5, 1]$.

Proof. The case where S is reducible follows from [10, Theorem 1.3]. If S is not normal we may assume S is as in Lemma 4.10. Then $\mu_t(S, D, \lambda_{10}) \geqslant 1 - t > 0$. If D is not reduced, we may assume (S, D) is as in Lemma 4.15. Then $\mu_t(S, D, \lambda_3) = -1 + 5t > 0$, if $t > \frac{1}{5}$. □

Proof of Theorem 1.3. Let (S, D) be a pair defined by equations F and H. Notice that Lemma 5.1 tells us that S being normal is a necessary condition for (S, D) to be t-stable for any $t \in (0, 1)$. In particular S has a finite number of singularities, since it is a surface. By Theorem 2.3, the pair (S, D) is t-stable if and only if for any $g \in \mathrm{SL}(4, \mathbb{C})$ the monomials with non-zero coefficients of $(g \cdot F, g \cdot H)$ are not contained in $N_t^{\oplus}(\lambda, x_i)$ for any of the pairs of sets in Table 4 —characterized geometrically in Section 3— which are maximal for every given t, as stated in Lemma A.1 and Theorem 2.3. This is equivalent to the conditions in the statement. We verify the conditions for each $t \in (0, 1)$. We will refer to the singularities of D in terms of the ADE classification as in sections 3 and 4. These will be equivalent to the global description used in the statement of Theorem 1.3 by Table 2.

Suppose $t \in (0, \frac{1}{5})$ and $(\lambda, x_i) = (\overline{\lambda}_3, x_3)$. Then S cannot have an $\boldsymbol{A}_3$ singularity or a degeneration of one. When $(\lambda, x_i) = (\lambda_9, x_3)$, we deduce that S cannot have a $\boldsymbol{D}_4$ singularity or a degeneration of one (this condition is redundant since $\boldsymbol{D}_4$ is a degeneration of $\boldsymbol{A}_3$). From $(\lambda, x_i) = (\lambda_1, x_2)$ or $(\lambda, x_i) = (\overline{\lambda}_2, x_2)$ we deduce that if $P \in D$ then P is a singular point of S of type at worst $\boldsymbol{A}_1$. We obtain the same condition if $(\lambda, x_i) = (\lambda_2, x_1)$. This completes the proof when $t \in (0, \frac{1}{5})$.

When $t = \frac{1}{5}$, the maximal sets $N_t^{\oplus}(\lambda, x_i)$ are the same as for $t \in \left(0, \frac{1}{5}\right)$ with the addition of $N_t^{\oplus}(\lambda_3, x_0)$, which represents the monomials of the equations of any pair (S', D') such that D' is not reduced. Therefore (S, D) is $\frac{1}{5}$-stable if and only if in addition to the conditions for t-stability when $t \in (0, \frac{1}{5})$, D is not reduced. Hence (ii) follows.

Let $t \in \left(\frac{1}{5}, \frac{1}{3}\right)$. The maximal t-non-stable sets $N_t^{\oplus}(\lambda, x_i)$ are the same as for $t = \frac{1}{5}$ but replacing the set $N_t^{\oplus}(\overline{\lambda}_3, x_3)$ with both $N_t^{\oplus}(\lambda_7, x_3)$ and $N_t^{\oplus}(\lambda_5, x_3)$. A pair (S', D') whose defining equations have coefficients in $N_t^{\oplus}(\overline{\lambda}_3, x_3)$, $N_t^{\oplus}(\lambda_7, x_3)$ and $N_t^{\oplus}(\lambda_5, x_3)$ require that S' has (a degeneration of) an $\boldsymbol{A}_3$ singularity, S' is not normal or S' has (a degeneration of) an $\boldsymbol{A}_4$ singularity, respectively. The second condition is redundant by Lemma 5.1. Hence a t-stable pair (S, D) may now have $\boldsymbol{A}_3$ singularities but not $\boldsymbol{A}_4$ singularities. However, the coefficients of the equations of (S,D) cannot be in $N_t^{\oplus}(\lambda_9, x_3)$ and hence S cannot have (degenerations of) $\boldsymbol{D}_4$ singularities. Therefore (S,D) is t-stable if and only if S has at worst $\boldsymbol{A}_3$ singularities, D is reduced and if D supports a surface singularity P, then P must be an $\boldsymbol{A}_1$-singularity and (iii) follows.

Let $t = \frac{1}{3}$. The maximal sets $N_t^{\oplus}(\lambda, x_i)$ are the same as for $t \in \left(\frac{1}{5}, \frac{1}{3}\right)$ with the addition of $N_t^{\oplus}(\lambda_5, x_0)$, which represents the monomials of the equations of any pair (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_3$ singularity at a singular point P of S. Hence (S,D) is $\frac{1}{3}$-stable if and only if it is t-stable for $t \in \left(\frac{1}{5}, \frac{1}{3}\right)$ but D does not have (a degeneration of) an $\boldsymbol{A}_3$ singularity at a singular point of P. Hence (iv) follows.

Let $t \in \left(\frac{1}{3}, \frac{3}{7}\right)$. The maximal sets are $N_t^{\oplus}(\lambda, x_i)$ the same as for $t = \frac{1}{3}$ but replacing the set $N_t^{\oplus}(\lambda_5, x_3)$ —parametrizing pairs (S', D') where S' has (a degeneration of) an $\boldsymbol{A}_4$ singularity— with the set $N_t^{\oplus}(\lambda_6, x_3)$ —parametrizing pairs (S', D') where S' has (a degeneration of) an $\boldsymbol{A}_5$ singularity. Hence a t-stable pair (S,D) may now have $\boldsymbol{A}_4$ singularities but not $\boldsymbol{A}_5$ ones. However, the coefficients of the equations of (S,D) cannot be in $N_t^{\oplus}(\lambda_9, x_3)$ and hence S cannot have (degenerations of) $\boldsymbol{D}_4$ singularities. Furthermore the restrictions for $t = \frac{1}{3}$ regarding D still apply. Therefore a pair (S,D) is t-stable if and only if satisfies the conditions in (v).

Let $t = \frac{3}{7}$. The maximal sets $N_t^{\oplus}(\lambda, x_i)$ are the same as for $t \in \left(\frac{1}{3}, \frac{3}{7}\right)$ but replacing the set $N_t^{\oplus}(\lambda_5, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_3$ singularity at a surface singularity of S'—, for both the set $N_t^{\oplus}(\overline{\lambda}_6, x_0)$—parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_2$ singularity at a surface singularity of S'— and the set $N_t^{\oplus}(\overline{\lambda}_9, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_4$ singularity. Hence (vi) follows.

Let $t \in \left(\frac{3}{7}, \frac{5}{9}\right]$. The difference between the maximal sets for $N_t^{\oplus}(\lambda, x_i)$ and for $N_{\frac{3}{7}}^{\oplus}(\lambda, x_i)$ consists of three new sets ($N_t^{\oplus}(\overline{\lambda}_6, x_3)$, $N_t^{\oplus}(\lambda_8, x_3)$ and $N_t^{\oplus}(\lambda_{10}, x_3)$) and three sets that do not appear for t anymore ($N_t^{\oplus}(\lambda_9, x_3)$, $N_t^{\oplus}(\lambda_6, x_3)$, $N_t^{\oplus}(\lambda_7, x_3)$). The three new sets parametrize pairs (S', D') such that S' has at least either (a degeneration of) one $\boldsymbol{D}_5$ singularity , a degeneration of one $\tilde{\boldsymbol{E}}_6$ singularity or one line of singularities, respectively. The three sets that are not maximal non-stable sets for t parametrize pairs (S', D') such that S' has (a degeneration of) a $\boldsymbol{D}_4$, an $\boldsymbol{A}_5$ and a line of singularities, respectively. Hence, the only difference with respect to $t = \frac{3}{7}$ is that we include pairs (S,D) such that S has at worst $\boldsymbol{A}_5$ or $\boldsymbol{D}_4$ singularities and (vii) follows.

Let $t = \frac{5}{9}$. The difference between the maximal sets for $N_t^{\oplus}(\lambda, x_i)$ for $t \in \left(\frac{3}{7}, \frac{5}{9}\right)$ and for $N_{\frac{5}{9}}^{\oplus}(\lambda, x_i)$ consists of replacing the set $N_t^{\oplus}(\lambda_3, x_0)$ —parametrizing pairs (S', D') such that D' is non-reduced— for the set $N_t^{\oplus}(\lambda_6, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_3$ singularity. Hence a $\frac{5}{9}$-stable pair (S,D) is a t-stable pair for $t \in \left(\frac{3}{7}, \frac{5}{9}\right)$ such that D has at worst an $\boldsymbol{A}_2$ singularity. Notice that D is still reduced by Lemma 5.1. Hence (viii) follows.

Let $t \in \left(\frac{5}{9}, \frac{9}{13}\right)$. The difference between the maximal sets for $N_t^{\oplus}(\lambda, x_i)$ for $t \in \left(\frac{5}{9}, \frac{9}{13}\right)$ and for $N_{\frac{5}{9}}^{\oplus}(\lambda, x_i)$ consists of replacing the set $N_t^{\oplus}(\overline{\lambda}_6, x_3)$ —parametrizing pairs (S', D') such that S' has (a degeneration of) a $\boldsymbol{D}_5$ singularity— for the set

$N_t^{\oplus}(\overline{\lambda}_4, x_3)$ —parametrizing pairs (S', D') such that S' has (a degeneration of) an $\boldsymbol{E}_6$ singularity. Hence (ix) follows.

Let $t = \frac{9}{13}$. The difference between the maximal sets for $N_t^{\oplus}(\lambda, x_i)$ for $t \in \left(\frac{5}{9}, \frac{9}{13}\right)$ and for $N_{\frac{9}{13}}^{\oplus}(\lambda, x_i)$ consists of replacing the set $N_t^{\oplus}(\overline{\lambda}_6, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_2$ singularity at a singular point of S'—, the set $N_t^{\oplus}(\overline{\lambda}_9, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) a $\boldsymbol{D}_4$ singularity— and the set $N_t^{\oplus}(\lambda_6, x_0)$— parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_3$ singularity— for the set $N_t^{\oplus}(\lambda_4, x_0)$ —parametrizing pairs (S', D') such that D' has (a degeneration of) an $\boldsymbol{A}_2$ singularity. Hence (x) follows.

Let $t \in \left(\frac{9}{13}, 1\right)$. The maximal sets $N_t^{\oplus}(\lambda, x_i)$ are the same as for $N_{\frac{9}{13}}^{\oplus}(\lambda, x_i)$ but removing the set $N_t^{\oplus}(\overline{\lambda}_4, x_3)$, which parametrizes pairs (S', D') where S' has an $\boldsymbol{E}_6$ singularities. Hence such surfaces are now t-stable providing they do not violate any other conditions. This concludes the proof of the theorem. □

Proof of Theorem 1.4. Suppose (S, D) —defined by polynomials F and H— belongs to a closed strictly t-semistable orbit. By Lemma 4.20 and Lemma A.2, we may assume that the monomials with non-zero coefficients of F and H correspond to the fourth and fifth columns in Table 5. Notice that for each pair (λ, x_i), there is a change o f coordinates that gives a natural bijection between $N^0(\lambda, x_i)$ and $N^0(\overline{\lambda}, x_{3-i})$. Therefore about half of the values are redundant and we have two possible choices for each F and H if $t \neq t_1, \dots, t_5$ three choices if $t = t_1, t_2, t_4, t_5$ and four if $t = t_3$.

Notice that the pair $(\overline{S}, \overline{D})$ corresponding to $\overline{F} = x_0x_3x_1 + x_2^3$, $\overline{H} = x_2$ is strictly t-semistable by Lemma A.2. Suppose that $(\lambda, x_i) = (\lambda_1, x_2)$. Then $F = x_0x_3f_1(x_1, x_2) + f_3(x_1, x_2)$ and $H = g_1(x_1, x_2)$. After a change of variables involving only x_1 and x_2, we may assume that $F = x_0x_3x_1 + f_3(x_1, x_2)$. We will show that the closure of (S, D) contains $(\overline{S}, \overline{D})$. Let $\gamma = \mathrm{Diag}(1, 1, 0, -2)$ be a one-parameter subgroup. Then $\lim_{t\to 0} \gamma(t) \cdot F = x_0x_1x_3 + bx_2^3$ and $\lim_{t\to 0} \gamma(t) \cdot H = x_2$. If $b = 0$, then $\lim_{t\to 0} \gamma(t) \cdot S$ is reducible, which is impossible as it is not t-stable for any value of $t \in (0, 1)$ by Lemma 5.1. Therefore $b \neq 0$ and by rescaling we see that $\lim_{t\to 0} \gamma(t) \cdot (S, D) = (\overline{S}, \overline{D})$. Hence, the closure of the orbit of (S, D) contains $(\overline{S}, \overline{D})$, which we tackle next.

Suppose that $(\lambda, x_i) = (\lambda_2, x_1)$. Then $F = x_1^3 + x_0f_2(x_2, x_3)$ and $H = x_1$. After a change of variables involving only x_2 and x_3 we may assume that $F = x_1^3 + x_0x_2x_3$. We can do similar changes of variables in the rest of the cases and end up with F and H not depending on any parameters. Observe that since (S, D) is strictly t-semistable, the stabilizer subgroup of (S, D) $G_{(S,D)} \subset \mathrm{SL}(4, \mathbb{C})$ is infinite (see [7, Remark 8.1 (5)]). In particular there is a $\mathbb{C}^*$-action on (S, D). Lemma 4.20 classifies the singularities of (S, D) uniquely according to their equations. For each $t \in (0, 1)$, the proof of Theorem 1.4 follows once we recall the classification of plane cubic curves according to their isolated singularities (see Table 2). □

Appendix A. Maximal sets of non-stable pairs

Table 4 has all pairs of sets $N_t^{\oplus}(\lambda, x_i) = \left(V_t^{\oplus}(\lambda, x_i), B^{\oplus}(x_i)\right)$ which are maximal under the containtment order, for each $t \in (0, 1)$ and all $\lambda \in S_{2,3}$ and $x_i \in \Xi_1$. Consider t, λ_i and x_i for one of the rows in Table 4. Suppose that a pair of polynomials

λ	x_i	t	$V_t^{\oplus}(\lambda, x_i)$	$B^{\oplus}(x_i)$	Sing(S)	Sing(D)	Lem.
λ_1	x_2	$(0,1)$	$x_3x_0\{x_0,x_1,x_2\}$, $\{x_0,x_1,x_2\}^3$	x_0,x_1,x_2	$\boldsymbol{A}_2$	$P \in D$	4.2
$\overline{\lambda}_2$	x_2	$(0,1)$	$x_3\{x_0,x_1\}^2, \{x_0,x_1,x_2\}^3$	x_0,x_1,x_2	$\boldsymbol{A}_2$	$P \in D$	4.2
λ_2	x_1	$(0,1)$	$x_1^3, x_0\{x_0,x_1,x_2,x_3\}^2$	x_0,x_1	$\boldsymbol{A}_2$	$P \in D$	4.16
$\overline{\lambda}_3$	x_3	$\left(0,\frac{1}{5}\right]$	$x_3\{x_0,x_1\}^2$,$x_2^2\{x_0,x_1\}$, $x_2\{x_0,x_1\}^2$, $\{x_0,x_1\}^3$	x_0,x_1,x_2,x_3	$\boldsymbol{A}_3$		4.3
$\overline{\lambda}_8$	x_3	$(0,1)$	$x_0\{x_0,x_1,x_2,x_3\}^2$	x_0,x_1,x_2,x_3	reducible		4.11
λ_9	x_3	$\left(0,\frac{3}{7}\right]$	$x_3x_0^2, \{x_0,x_1,x_2\}^3$	x_0,x_1,x_2,x_3	$\boldsymbol{D}_4$		4.6
λ_3	x_0	$\left[\frac{1}{5},\frac{5}{9}\right)$	$x_0\{x_0,x_1,x_2,x_3\}^2$, $x_1^2\{x_1,x_2,x_3\}$	x_0		non-reduced	4.15
λ_5	x_3	$\left(\frac{1}{5},\frac{1}{3}\right]$	$\{x_0,x_1\}^3$, $x_2\{x_0,x_1\}^2$, $x_0x_2^2$, $x_0x_3\{x_0,x_1\}$	x_0,x_1,x_2,x_3	$\boldsymbol{A}_4$		4.4
λ_7	x_3	$\left(\frac{1}{5},\frac{3}{7}\right]$	$x_3\{x_0,x_1\}^2$, $\{x_0,x_1\}^3$, $x_2\{x_0,x_1\}^2$	x_0,x_1,x_2,x_3	$\boldsymbol{A}_\infty$		4.10
λ_5	x_0	$\left[\frac{1}{3},\frac{3}{7}\right]$	$x_0^2\{x_0,x_1,x_2,x_3\}$, $x_0\{x_1,x_2\}^2$, $x_0x_3\{x_1,x_2\}$, $x_1^2\{x_1,x_2,x_3\}$, $x_1x_2^2$	x_0,	P	D is $\boldsymbol{A}_3$ at P	4.19
λ_6	x_3	$\left(\frac{1}{3},\frac{3}{7}\right]$	$x_0x_3\{x_0,x_1\}$, $x_0x_2\{x_0,x_1,x_2\}$, $\{x_0,x_1\}^3$	x_0,x_1,x_2,x_3	$\boldsymbol{A}_5$		4.5
$\overline{\lambda}_6$	x_0	$\left[\frac{3}{7},\frac{9}{13}\right)$	$x_0x_3\{x_0,x_1,x_2\}$, $\{x_1,x_2\}^3$, $x_0\{x_0,x_1,x_2\}^2$, $x_1^2x_3$	x_0	P	D is $\boldsymbol{A}_2$ at P	4.18
$\overline{\lambda}_9$	x_0	$\left[\frac{3}{7},\frac{9}{13}\right)$	$x_0\{x_0,x_1,x_2,x_3\}$, $\{x_1,x_2\}^3$	x_0		$\boldsymbol{D}_4$	4.14
λ_{10}	x_3	$\left(\frac{3}{7},1\right)$	$x_3\{x_0,x_1\}^2$, $\{x_0,x_1\}^3$, $x_2\{x_0,x_1\}^2$	x_0,x_1,x_2,x_3	$\boldsymbol{A}_\infty$		4.10
λ_8	x_3	$\left(\frac{3}{7},1\right)$	$\{x_0,x_1,x_2\}^3$	x_0,x_1,x_2,x_3	$\tilde{\boldsymbol{E}}_6$		4.9
$\overline{\lambda}_6$	x_3	$\left(\frac{3}{7},\frac{5}{9}\right]$	$\{x_0,x_1\}^3$, $x_2\{x_0,x_1\}^2$, $x_0x_2^2$, $x_0^2x_3$	x_0,x_1,x_2,x_3	$\boldsymbol{D}_5$		4.7
λ_6	x_0	$\left[\frac{5}{9},\frac{9}{13}\right)$	$x_0\{x_0,x_1,x_2,x_3\}^2$, $x_1x_2^2$, $x_1^2\{x_1,x_2,x_3\}$	x_0		$\boldsymbol{A}_3$	4.13
$\overline{\lambda}_4$	x_3	$\left(\frac{5}{9},\frac{9}{13}\right)$	$x_0^2x_3$, $x_0x_2\{x_0,x_1,x_2\}$, $\{x_0,x_1\}^3$	x_0,x_1,x_2,x_3	$\boldsymbol{E}_6$		4.8
λ_4	x_0	$\left[\frac{9}{13},1\right)$	$x_0\{x_0,x_1,x_2,x_3\}^2$, $x_1^2x_3$, $\{x_1,x_2\}^3$	x_0		$\boldsymbol{A}_2$	4.12

TABLE 4. Maximal non-stable sets for $t \in (0,1)$.

F and H has monomials with non-zero coefficients only for monomials in $V_t^{\oplus}(\lambda, x_i)$ and $B^{\oplus}(x_i)$, respectively. Let (S,D) be a pair defined by F and H as in Section 2. Then the singularities of (S,D) correspond to a (possibly trivial) degeneration

λ	x_i	t	$V_t^0(\lambda, x_i)$	$B^0(x_i)$
λ_1	x_2	$(0,1)$	$x_0\{x_1,x_2\}x_3, \{x_1,x_2\}^3$	x_1, x_2
λ_2	x_1	$(0,1)$	$x_1^3, x_0\{x_2,x_3\}^2$	x_1
$\overline{\lambda_2}$	x_2	$(0,1)$	$x_2^3, \{x_0,x_1\}^2x_3$	x_2
λ_3	x_0	$t_1 = \frac{1}{5}$	$x_1^2\{x_2,x_3\}, x_0\{x_2,x_3\}^2$	x_0
$\overline{\lambda_3}$	x_3	$t_1 = \frac{1}{5}$	$x_2^2\{x_0,x_1\}, x_3\{x_0,x_1\}^2$	x_3
λ_5	x_0	$t_2 = \frac{1}{3}$	$x_1x_2^2, x_0x_2x_3, x_1^2x_3$	x_0
λ_5	x_3	$t_2 = \frac{1}{3}$	$x_1^2x_2, x_0x_1x_3, x_0x_2^2$	x_3
$\overline{\lambda_6}$	x_0	$t_3 = \frac{3}{7}$	$x_1^2x_3, x_0x_2x_3, x_2^3$	x_0
λ_9	x_3	$t_3 = \frac{3}{7}$	$\{x_1,x_2\}^3, x_0^2x_3$	x_3
λ_6	x_3	$t_3 = \frac{3}{7}$	$x_0x_2^2, x_0x_1x_3, x_1^3$	x_3
$\overline{\lambda_9}$	x_0	$t_3 = \frac{3}{7}$	$\{x_1,x_2\}^3, x_0x_3^2$	x_0
$\overline{\lambda_6}$	x_3	$t_4 = \frac{5}{9}$	$x_1^2x_2, x_0x_2^2, x_0^2x_3$	x_3
λ_6	x_0	$t_4 = \frac{5}{9}$	$x_1x_2^2, x_0x_3^2, x_1^2x_3$	x_0
λ_4	x_0	$t_5 = \frac{9}{13}$	$x_1^2x_3, x_0x_3^2, x_2^3$	x_0
$\overline{\lambda_4}$	x_3	$t_5 = \frac{9}{13}$	$x_1^3, x_0x_2^2, x_0^2x_3$	x_3

TABLE 5. Maximal non-stable sets for $t \in (0, \frac{1}{5})$.

of the singularities appearing in the sixth and seventh entries of the corresponding row. This is proven in the Lemma referred to in the eigth column of the table. The notation $\boldsymbol{A}_\infty$ denotes that S contains a line of singularities where the general point is an $\boldsymbol{A}_1$ surface singularity.

Table 5 contains all pairs of sets $N_t^0(\lambda, x_i)$ for each $N_t^\oplus(\lambda, x_i)$ appearing in Table 4 such that the associated pair (S, D) is t-semistable.

Lemma A.1. *Let $t \in (0,1)$. Consider each $\lambda \in S_{2,3}$ and each $i = 0, \dots, 3$. The pairs of sets $N_t^\oplus(\lambda, x_i) = (V_t^\oplus(\lambda, x_i), B^\oplus(x_i))$ defined in* (2.1) *which are maximal with respect to the containment order of sets are given in Table 4.*

Proof. Since $S_{2,3}$ is a finite set, there is a finite number of pairs of sets $N_t^\oplus(\lambda, x_i)$. Finding the maximal ones among them is a straight forward computation which can be carried out by software (see [9] for a detailed algorithm and [11] for the code). $\square$

Lemma A.2. *Let $t \in (0,1)$. Consider each of the pairs of sets $N_t^0(\lambda, x_i) = (V^0(\lambda, x_i), B^0(x_i))$ defined in* (2.2)*, for each (λ, x_i) such that $N_t^\oplus(\lambda, x_i)$ is maximal with respect to the containment order of sets, as described in Lemma A.1. Let (S, D) be a pair defined by polynomials F and H such that its monomials with non-zero coefficients corresponds to those in $V^0(\lambda, x_i)$ and $B^0(x_i)$, respectively. If the pair (S, D) is strictly t-semistable, then $N_t^\oplus(\lambda, x_i)$ correspond to those sets in Table 5.*

Proof. Computing $N_t^0(\lambda, x_i)$ is immediate from the definition. Deciding if each pair (S, D) is strictly t-semistable is a combinatorial application of the Centroid Criterion [10, Lemma 1.5]. Both operations can be carried by software [11]. $\square$

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