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Unitary Representations of Quantum Lorentz Group and Quantum Relativistic Toda Chain

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Abstract

The aim of this paper is to give a group theoretical interpretation of the three types of Bessel-Jackson functions. We consider a family of quantum Lorentz groups and a family of quantum Lobachevsky spaces. For three members of quantum Lobachevsky spaces the Casimir operators give rise to the two-body relativistic open Toda lattice Hamiltonians. Their eigen-functions are the modified Bessel-Jackson functions of three types. We construct the principal series of unitary irreducible representations of the quantum Lorentz groups. Special matrix elements in the irreducible spaces are the Bessel-Macdonald-Jackson functions. They are the wave functions of the two-body relativistic open Toda lattice. We obtain integral representations for these functions.

1 Introduction

There exist deep interrelations between the Harmonic Analysis on symmetric spaces and quantum integrable systems [1]. In this approach the zonal spherical functions play the role of the wave-functions of some integrable models. In particular, the Bessel functions are the zonal spherical functions related to the group symmetries of the Euclidean spaces [2]. On the other hand, they are the wave-functions of the rational Calogero-Moser model.

The modified Bessel functions and the Bessel-Macdonald functions arise in a different construction. Consider the eigen-functions of the Laplace-Beltrami operator in the horospheric coordinates on the Lobachevsky space. The modified Bessel functions and the Bessel-Macdonald functions are their Fourier transform on a horosphere [2]. From the point of view of integrable systems the Laplace-Beltrami operator in horospheric coordinates gives rise to the simplest form of the open Toda Hamiltonian. B. Kostant generalized this connection and established the similar relations between an arbitrary open Toda lattice and the Whittaker model of irreducible representations of the splitted simple group G [3] (see also reviews [4, 5]). This construction can be generalized in different ways. For example, the periodic Toda lattices arise if one replaces a simple group G by the central extended loop group [6, 7].

The zonal spherical functions for the quantum groups were investigated in numerous papers (see [8, 9]). In particular, the q-Bessel-Jackson functions were described by Vaksman and Korogodsky [10] as the zonal spherical functions related to the quantum plane.

In our paper [11] we applied the Kostant scheme to the quantum Lorentz group constructed by Podlez and Woronowicz [12]. It turns out that the corresponding integrable system is the open two-body relativistic Toda lattice [13]. Its Hamiltonian is related to the Casimir operator of the quantum Lorentz group being written in an analog of the horospheric coordinates on the corresponding quantum Lobachevsky space. Up to the q-exponents multipliers the eigen-functions are the modified q-Bessel-Jackson and the q-Bessel-Jackson-Macdonald functions. Their properties were investigated in [14, 15, 16]. For general quantum groups an analog of the Whittaker model was constructed in [17, 18]. Recently, the integral representations for the wave functions of the N-body case based on the Whittaker model of $U_q(\mathrm{sl}_N(\mathbb{R}))$ was presented in [19].

Here we come back to the two-body case and investigate it in detail. Our analysis is based on the Whittaker model for the quantum Lorentz group $U_q(\mathrm{sl}_2(\mathbb{C}))$. In this way we describe the group theoretical approach to the modified q-Bessel-Jackson and the q-Bessel-Jackson-Macdonald functions. For this purpose we construct unitary irreducible representations of a twisted family of quantum Lorentz groups. As by product we obtain some results in the Harmonic Analysis on quantum Lobachevsky spaces. The quantum Lobachevsky spaces play the role of homogeneous spaces with respect to the twisted quantum Lorentz groups. Three members from the family are distinguished. For these cases the Casimir operators realized on the quantum Lobachevsky spaces lead to second order difference operators. Up to a conjugation they coincide with the quantum relativistic two-body open Toda Hamiltonians. Their eigenfunctions are the three types of the modified Bessel-Jackson functions. We construct an analog of the principle series unitary irreducible representations. Then we consider some special matrix elements of group elements acting in an irreducible space of the class-one representations. They are the wave functions of the Toda lattice. These matrix elements have the form of the double integral. It follows from [15] that they are q-Bessel-Jackson-Macdonald functions.

The paper has the following structure. In Section 2 we recall interrelations between the Lorentz group $\mathrm{SL}(2,\mathbb{C})$ and the two-body open Toda model. In Section 3 we construct the twisted family of quantum Lorentz groups and the corresponding family of quantum Lobachevsky spaces. The principle series of unitary irreducible representations are constructed in Section 4. The eigen-functions of the Casimir operators on the quantum Lobachevsky spaces are constructed in Section 5. The integral representations of the wave functions for the open Toda model are presented in Section 6. Finally, in Section 7 we give the representation for the wave-functions as the Mellin-Barns integral. The similar representation was obtained in [19].

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2 Classical Case

1. Lorentz group. The matrix Lorentz group is the group of complex unimodular matrices of second order

$$G = \mathrm{SL}(2,\mathbb{C}) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\ (\alpha\delta - \beta\gamma = 1).$$

Its Lie algebra has three generators

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \frac{1}{2}\mathrm{diag}(1,-1),\ f = e^T$$

with commutators

$$[e,f] = 2h,\ [h,e] = e,\ [h,f] = -f.$$

They generate $\mathrm{sl}(2,\mathbb{R})$ over $\mathbb{R}$ and $\mathrm{sl}(2,\mathbb{C})$ over $\mathbb{C}$. As a real algebra $\mathrm{sl}(2,\mathbb{C})$ has six generators and one should consider the second copy of $\mathrm{sl}(2,\mathbb{R})$ generated by $(\bar{e},\bar{h},\bar{f})$.

2 Principal series [20]. The principal series $\pi_{\nu,n}$ of the unitary representations of $G = \mathrm{SL}(2,\mathbb{C})$ are defined in the following way.

Let N^- be the subgroup of the lower triangular matrices, A are the real diagonal matrices and M are the unitary diagonal matrices. The Borel subgroup B of G

$$B = N^- AM,\quad b = v\cdot \mathrm{diag}(h,h^{-1})\mathrm{diag}(e^{2\pi i\theta},e^{-2\pi i\theta}),\quad (v\in N^-)$$

has characters

$$\chi_{\nu,n}(vam) = \exp\{(i\nu-1)\log h + n\theta\}.$$

The principal series representations $\pi_{\nu,n}$ of G are induced by the characters of B in the space of smooth functions on G

$$f(bg) = \chi_{\nu,n}(b)f(g),\quad \pi_{\nu,n}(g)f(x) = f(xg),\ x\in \mathrm{SL}(2,\mathbb{C}). \tag{2.1}$$

It is amount to look on the action in the space of sections of a linear bundle over $\mathbf{P}^1$. To this end consider the Gauss decomposition of the dense subset $\{g|\alpha\neq 0\}\subset \mathrm{SL}(2,\mathbb{C})$

$$G = BN,\quad (N - \text{nilpotent subgroup of the upper triangular matrices}).$$

$$g = bn(g),\ n_{12}(g) = \frac{\beta}{\alpha} = z,\quad b_{11} = \alpha. \tag{2.2}$$

Then (2.2) means that the right action (2.1) of $\mathrm{SL}(2,\mathbb{C})$ induces the Möbius transform on $\mathbf{P}^1$

$$z\to zg = \frac{\delta z+\beta}{\gamma z+\alpha}.$$

Consider the linear bundle $\mathcal{L}_{\nu,n}$ over $\mathbf{P}^1$ with the space of sections

$$V_{\nu,n} = \Gamma(\mathcal{L}_{\nu,n}) \sim \mathcal{A}^{(-\frac{r}{2},-\frac{\tilde{r}}{2})}(\mathbf{P}^1)\ (r = i\nu+n-1,\ \tilde{r} = i\nu-n-1). \tag{2.3}$$

The sections have the form $f(z,\bar{z})(dz)^{-\frac{r}{2}}(d\bar{z})^{-\frac{\tilde{r}}{2}}$, where $f(z,\bar{z})$ are smooth functions with the asymptotic

$$f(z,\bar{z})_{z\to\infty} \sim z^r\bar{z}^{\tilde{r}}. \tag{2.4}$$

By means of (2.2) we can reduce $\pi_{\nu,n}(g)$ (2.1) to the action on $V_{\nu,n}$ (2.3)

$$\pi_{\nu,n}(g)f(z,\bar{z}) = (\gamma z+\alpha)^{i\nu+n-1}(\bar{\gamma}\bar{z}+\bar{\alpha})^{i\nu-n-1}f(zg,\bar{z}\bar{g}). \tag{2.5}$$

Since $\bar{r}+\tilde{r} = -2$ there is a Hermitian form on $\mathcal{A}^{(-\frac{r}{2},-\frac{\tilde{r}}{2})}(\mathbf{P}^1)$

$$< f_1|f_2 > = \int_{\mathbf{P}^1} f_1(z,\bar{z})\overline{f_2(z,\bar{z})}dzd\bar{z}. \tag{2.6}$$

The representation $\pi_{\nu,n}$ (2.5) is realized in $V_{\nu,n}$ by the unitary operators because

$$< \pi_{\nu,n}(g)f_1|f_2 >=< f_1|\pi_{\nu,n}(g^{-1})f_2 > .$$

The infinitesimal version of this construction takes the form:

$$e \to T^+ = \partial_z, \quad h \to T^3 = -z\partial_z + \frac{r}{2},$$

$$f \to T^- = -z^2\partial_z + rz, \quad (r = n - 1 + i\nu) \tag{2.7}$$

$$\bar{e} \to \bar{T}^+ = \partial_{\bar{z}}, \ \bar{h} \to \bar{T}^3 = -\bar{z}\partial_{\bar{z}} + \frac{\tilde{r}}{2},$$

$$\bar{f} \to \bar{T}^- = -\bar{z}^2\partial_{\bar{z}} + \tilde{r}\bar{z}, \quad (\tilde{r} = -n - 1 + i\nu). \tag{2.8}$$

This action preserves the asymptotic (2.4).

There are two Casimir operators of $\mathrm{SL}(2, \mathbb{C})$

$$\Omega = h^2 + h + fe, \quad \tilde{\Omega} = \bar{h}^2 + \bar{h} + \bar{f}\bar{e}.$$

They become the scalar operators in $V_{\nu,n}$

$$\Omega|\Psi >= (\frac{1}{4}(i\nu + n)^2 - \frac{1}{4})|\Psi >, \quad \tilde{\Omega}|\Psi >= (\frac{1}{4}(i\nu - n)^2 - \frac{1}{4})|\Psi >, \quad |\Psi >\in V_{\nu,n}. \tag{2.9}$$

If $n = 0$ then there exists a SU(2) invariant vector in $V_{\nu,0}$

$$(1 + |z|^2)^{i\nu-1}.$$

3. Lobachevsky space. A subclass of these representations $r = i\nu - 1$, $(n = 0)$ (the representations of class one) are related to the decomposition of functions on the Lobachevsky space $\mathbf{L} = \mathrm{SU}(2)\backslash\mathrm{SL}(2, \mathbb{C})$. The later can be realized as the space of second order unimodular Hermitian positive definite matrices. Any $x \in \mathbf{L}$ can be represented as

$$x = g^\dagger g = \begin{pmatrix} \bar{\alpha}\alpha + \bar{\gamma}\gamma & \bar{\alpha}\beta + \bar{\gamma}\delta \\ \bar{\beta}\alpha + \bar{\delta}\gamma & \bar{\beta}\beta + \bar{\delta}\delta \end{pmatrix}. \tag{2.10}$$

The Iwasawa decomposition

$$g = kb, \ g \in \mathrm{SL}(2, \mathbb{C}), \ k \in \mathrm{SU}(2), \ b \in AN$$

allows to introduce the horospheric coordinates on $\mathbf{L}$. If

$$b = \begin{pmatrix} h & hz \\ 0 & h^{-1} \end{pmatrix},$$

then from (2.10)

$$x = b^\dagger b = \begin{pmatrix} \bar{h}h & \bar{h}hz \\ \bar{z}\bar{h}h & \bar{z}\bar{h}hz + (\bar{h}h)^{-1} \end{pmatrix}. \tag{2.11}$$

The triple $(H = \bar{h}h, z, \bar{z})$ is uniquely determined by x. It is called the horospheric coordinates of x. It follows from (2.10) and (2.11) that

$$H = \bar{\alpha}\alpha + \bar{\gamma}\gamma,$$

$$Hz = \bar{\alpha}\beta + \bar{\gamma}\delta,$$
$$\bar{z}H = \bar{\beta}\alpha + \bar{\delta}\gamma.$$

Consider the space of smooth complex valued integrable functions $\mathcal{R}_{\mathbf{L}} = \{f(H, z, \bar{z})\}$ on $\mathbf{L}$:

$$\int |f(H, z, \bar{z})|^2 H dH dz d\bar{z} < \infty. \tag{2.12}$$

The Lie operators generated by the right shifts act on $\mathcal{R}_{\mathbf{L}}$ as

$$\begin{aligned} e &\to \mathcal{D}^+ = \partial_z, \\ h &\to \mathcal{D}^3 = \frac{1}{2}H\partial_H - z\partial_z, \\ f &\to \mathcal{D}^- = Hz\partial_H - z^2\partial_z + H^{-2}\partial_{\bar{z}}. \end{aligned} \tag{2.13}$$

The generators $\bar{e}, \bar{h}, \bar{f}$ give rise to the complex conjugate operators

$$\begin{aligned} \bar{e} &\to \bar{\mathcal{D}}^+ = \partial_{\bar{z}}, \\ \bar{h} &\to \bar{\mathcal{D}}^3 = \frac{1}{2}H\partial_H - \bar{z}\partial_{\bar{z}}, \\ \bar{f} &\to \bar{\mathcal{D}}^- = H\bar{z}\partial_H - \bar{z}^2\partial_{\bar{z}} + H^{-2}\partial_{\bar{z}}. \end{aligned} \tag{2.14}$$

Consider a family $V_\nu,\ \nu \in \mathbb{R}$ of subspaces in $\mathcal{R}_{\mathbf{L}}$

$$V_\nu = \{f(\bar{z}, aH, z) = a^{(i\nu-1)} f(\bar{z}, H, z)\}.$$

Assume that e, h, f act on the subspace of holomorphic ($\bar{z}$ independent) functions and $\bar{e}, \bar{h}, \bar{f}$ act on the subspace of antiholomorphic functions. After comparison (2.13) and (2.14) with (2.7) and (2.8) one concludes that $V_\nu = V_{\nu,0}$. The invariant integral (2.12) coincides with the hermitian form (2.6) on the irreducible subspace.

The Casimir operators in the horospheric coordinates take the form

$$\Omega = \frac{1}{4}H^2\partial_H^2 + \frac{3}{4}H\partial_H + H^{-2}\partial_{z\bar{z}}^2, \tag{2.15}$$

$$\tilde{\Omega} = \Omega.$$

According to (2.9) they are scalar operators on V_ν

$$\Omega|_{V_\nu} = -\frac{1}{4}(\nu^2 + 1)\mathrm{Id}.$$

4.Whittaker functions. Let $U_{\nu,\mu} = \{f_\nu(\bar{z}, H, z)\}$ be the space of smooth functions on $\mathbf{L}$ satisfying the following conditions:
(i) $f_\nu(\bar{z}, H, z)$ are the eigen-functions of the Casimir operator (2.15)

$$\Omega f_\nu(\bar{z}, H, z) = -\frac{1}{4}(\nu^2 + 1) f_\nu(\bar{z}, H, z).$$

(ii) $\partial_z f(\bar{z}, H, z) = i\mu f(\bar{z}, H, z), \quad \partial_{\bar{z}} f(\bar{z}, H, z) = i\mu f(\bar{z}, H, z).$

The last condition means that

$$f_\nu(\bar{z},H,z)=\exp i\mu(z+\bar{z})F_\nu(H).$$

Substituting this expression in the equation (i) one obtains

$$(\frac{1}{4}H^2\partial_H^2+\frac{3}{4}H\partial_H-H^{-2}\mu^2)F_\nu(H)=-\frac{1}{4}(\nu^2+1)F_\nu(H). \tag{2.16}$$

Two independent solutions of this equation are modified Bessel functions

$$H^{-1}I_{\pm i\nu}(2\mu H^{-1}).$$

The bounded solution is the Bessel-Macdonald function

$$H^{-1}K_{i\nu}(2\mu H^{-1})=H^{-1}\frac{\pi(I_{-i\nu}(2\mu H^{-1})-I_{i\nu}(2\mu H^{-1}))}{2i\sinh\nu}. \tag{2.17}$$

The last expression allows to find the asymptotic for $H\to 0$

$$K_{i\nu}(2\mu H^{-1})\sim\frac{\pi}{2i\sinh\nu}\left(\frac{1}{\Gamma(1-i\nu)}\left(\frac{H}{2}\right)^{-i\nu}-\frac{1}{\Gamma(1+i\nu)}\left(\frac{H}{2}\right)^{i\nu}\right).$$

The bounded solution (2.17) can be represented as a special matrix element of the operator

$$g(T^3|\varphi)=\exp 2(T^3\otimes\varphi)\in\mathcal{U}_{\rm SL}(2,\mathbb{C})\otimes\mathcal{A}_{\mathbf{L}},\quad H=e^\varphi, \tag{2.18}$$

which is a special group element defined in the tensor product of the universal enveloping algebra $\mathcal{U}_{\rm SL}(2,\mathbb{C})$ and the group algebra $\mathcal{A}_{\mathbf{L}}$. It acts on the irreducible space V_ν. Define two vectors $\psi_L,\ \ \psi_R$ such that

$$T^+\psi_R=i\mu\psi_R,\quad \bar{T}^+\psi_R=i\mu\psi_R,$$
$$(T^+-\bar{T}^-)\psi_L=(T^3-\bar{T}^3)\psi_L=0.$$

It means that ψ_R is the eigen-vector of the nilpotent subgroup N, and ψ_L is the $SU(2)$-invariant vector. The state ψ_R is called *the Whittaker vector*. Then it follows from (2.7), (2.8) that

$$\psi_R=\exp i\mu(z+\bar{z}), \tag{2.19}$$

and

$$\psi_L=\frac{1}{(1+|z|^2)^{1-i\nu}}. \tag{2.20}$$

In fact, ψ_R does not lie in V_ν, but rather it is a distribution over some subspace of V_ν. Nevertheless, the integral

$$<\psi_L|g(T^3|\varphi)|\psi_R> \tag{2.21}$$

is well defined. This construction can be generalized to an arbitrary splitted Lie group [3, 4]. The bounded solution is called *the Whittaker function*.

It will be instructive to prove that this matrix element satisfies (2.16). In fact, taking in account that $g(T^3|\varphi)T^-=e^{2\varphi}T^-g(T^3|\varphi)$ one has

$$-\frac{1}{4}(\nu^2+1)<\psi_L|g(T^3|\varphi)|\psi_R>=<\psi_L|g(T^3|\varphi)\Omega|_{V_\nu}|\psi_R>=$$

$$(\frac{1}{4}H^2\partial_H^2 + \frac{3}{4}H\partial_H) < \psi_L|g(T^3|\varphi)|\psi_R > + < \psi_L|g(T^3|\varphi)T^-T^+|\psi_R > .$$

Then the conditions (2.19), (2.20) and the commutation relation $H\partial_H g(T^3|\varphi) = g(T^3|\varphi)T^3$ give

$$< \psi_L|g(T^3|\varphi)T^-T^+|\psi_R >= -\mu^2 H^{-2} < \psi_L|g(T^3|\varphi)|\psi_R > .$$

Thus, we reproduce the left hand side of (2.16).

Substituting the explicit expressions (2.18),(2.19),(2.20) to (2.21) one finds the bounded solution of the equation (2.16)

$$< \psi_L|g(T^3|\varphi)|\psi_R >= \exp((i\nu+1)\varphi)\int\int d\bar{z}dz\frac{\exp(i\mu(z+\bar{z}))}{(1+e^{2\varphi}|z|^2)^{1+i\nu}} \sim H^{-1}K_{i\nu}(2\mu H^{-1}). \quad (2.22)$$

In this formula the angular integration in the polar coordinates allows to rewrite it as

$$H^{-1}K_{i\nu}(2\mu H^{-1}) = \Gamma(1+i\nu)H^{-i\nu-1}\int_0^\infty \frac{J_0(2\mu\rho)\rho}{(H^{-2}+\rho^2)^{1+i\nu}}d\rho. \quad (2.23)$$

Let us substitute $f_\nu(H) = e^{-\varphi}\Phi_\nu(\varphi)$ in (2.16). Then (2.16) takes the form of the quantum Liouville equation

$$\left(\frac{1}{2}\partial_\varphi^2 - 2\mu^2 e^{-2\varphi}\right)\Phi(\varphi) = -\frac{1}{2}\nu^2\Phi(\varphi). \quad (2.24)$$

The equation (2.16) has a special simple form in the momentum representation. Assume that $Hf(H) = \Phi(e^\varphi)$ and take the Fourier transform

$$\psi_\nu(p) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Phi(\varphi)e^{ip\varphi}d\varphi.$$

The equation (2.16) for $\psi_\nu(p)$ takes the form

$$\frac{1}{4}(\nu^2 - p^2)\psi_\nu(p) - \mu^2\psi_\nu(p-2i) = 0.$$

The bounded solution to this equation vanishing when $p \to \pm\infty$ is

$$\psi_\nu(p) = a(\nu)\mu^{ip}\Gamma(\frac{i}{2}(p+\nu))\Gamma(\frac{i}{2}(p-\nu)). \quad (2.25)$$

3 Quantum Lorentz group

1. General construction. Deformations of the group algebra $\mathcal{A}(\mathrm{SL}(2,\mathbb{C}))$ based on the Iwasawa and the Gauss decompositions were considered in [12] and [21]. Later three parameter deformations were found in [22]. We consider here the dual object $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$ and describe a two parameter family $\mathcal{U}_q^{(r,s)}(\mathrm{SL}(2,\mathbb{C}))$.

We start from a pair of the standard $\mathcal{U}_q(\mathrm{SL}_2)$ Hopf algebra. The first one is generated by A, B, C, D and the unit with the relations

$$AD = DA = 1,\ \ AB = qBA,\ \ BD = qDB,$$

$$AC = q^{-1}CA,\ CD = q^{-1}DC, \tag{3.1}$$
$$[B,C] = \frac{1}{q-q^{-1}}(A^2 - D^2).$$

It is a Hopf algebra where the coproduct is defined as

$$\Delta(A) = A \otimes A,\ \Delta(D) = D \otimes D,$$
$$\Delta(B) = A \otimes B + B \otimes D, \tag{3.2}$$
$$\Delta(C) = A \otimes C + C \otimes D,$$

with the counit

$$\varepsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the antipode

$$S \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix}.$$

There is a copy of this algebra $\mathcal{U}_q^*(\mathrm{SL}_2)$ generated by A^*, B^*, C^*, D^* with the relations following from (3.1). The star generators commute with A, B, C, D. The coproduct for the star generators is determined by $\Delta(X^*) = (\Delta(X))^*$ and the antipode is extracted from the rule $S \circ * \circ S \circ * = id$. The pair of two independent copies $\mathcal{U}_q(\mathrm{SL}_2)$ and $\mathcal{U}_q^*(\mathrm{SL}_2)$, which commute and cocommute, defines the untwisted version of the quantum Lorentz algebra $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$. Note that this form differs from the Iwasawa [12] and the Gauss [21] constructions.

The $\mathcal{U}_q(\mathrm{SU}(2))$ subalgebra [23] is defined as

$$\mathcal{U}_q(SU_2) = \{B^* = C, \quad C^* = B, \quad A^* = A\}. \tag{3.3}$$

There are two Casimir elements in $\mathcal{U}_q^{(r,s)}(\mathrm{SL}(2,\mathbb{C}))$ which commute with any $u \in \mathcal{U}_q^{r,s}(\mathrm{SL}(2,\mathbb{C}))$.

$$\Omega_q := \frac{(q^{-1}+q)(A^2 + A^{-2}) - 4}{2(q^{-1}-q)^2} + \frac{1}{2}(BC + CB) \tag{3.4}$$

$$\tilde{\Omega}_q := \frac{(\bar{q}^{-1}+\bar{q})(A^{*2} + A^{*-2}) - 4}{2(\bar{q}^{-1}-\bar{q})^2} + \frac{1}{2}(B^*C^* + C^*B^*) \tag{3.5}$$

We use the Casimir operator in the equivalent form

$$\Omega_q = \frac{qA^2 + q^{-1}A^{-2} - 2}{(q-q^{-1})^2} + CB. \tag{3.6}$$

To go further we following V.Drinfeld [24] introduce depending on two parameters twist in the form [25]. Let $A = q^K$, $A^* = q^{K^*}$ and

$$\mathcal{F}(r,s) = q^{rK^* \otimes K + K \otimes sK^*} \in \mathcal{U}_q(\mathrm{SL}_2) \otimes \mathcal{U}_q^*(\mathrm{SL}_2).$$

The element $\mathcal{F}(r,s)$ is the so-called left twisting because it satisfies

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F},$$

$$(\varepsilon \otimes id)\mathcal{F} = 1 = (id \otimes \varepsilon)\mathcal{F}.$$

These properties allow to generate a new coproduct from the old one:

$$\Delta^{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}.$$

In our case

$$\begin{aligned}
\Delta^{\mathcal{F}}(A) &= \Delta(A) = A \otimes A, \\
\Delta^{\mathcal{F}}(B) &= (A^*)^{-r} A \otimes B + B \otimes D(A^*)^s, \\
\Delta^{\mathcal{F}}(C) &= (A^*)^{r} A \otimes C + C \otimes D(A^*)^{-s}.
\end{aligned} \tag{3.7}$$

The antipode is changed in a consistent way with the new coproduct.

$$m(S^{\mathcal{F}} \otimes id)\Delta^{\mathcal{F}}(u) = m(id \otimes S^{\mathcal{F}})\Delta^{\mathcal{F}}(u) = \varepsilon(u),$$

where m is the multiplication. So we have

$$S^{\mathcal{F}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -q^{-1}(A^*)^{r-s}B \\ -q(A^*)^{s-r}C & A \end{pmatrix}.$$

In this way we obtain the twisted algebra $\mathcal{U}_q^{\mathcal{F}}(\mathrm{SL}_2)$. The algebra $(\mathcal{U}_q^*)^{\mathcal{F}}(\mathrm{SL}_2)$ with $\mathcal{F}$ depending on the same pair (r,s) can be constructed in the similar way.

$$\begin{aligned}
\Delta^{\mathcal{F}}(A^*) &= \Delta(A^*) = A^* \otimes A^*, \\
\Delta^{\mathcal{F}}(B^*) &= A^{-r} A^* \otimes B^* + B^* \otimes D^* A^s, \\
\Delta^{\mathcal{F}}(C^*) &= A^{r} A^* \otimes C^* + C^* \otimes D^* A^{-s}.
\end{aligned} \tag{3.8}$$

$$S^{\mathcal{F}} \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} = \begin{pmatrix} D^* & -\bar{q}A^{r-s}B^* \\ -\bar{q}^{-1}A^{s-r}C^* & A^* \end{pmatrix}.$$

The pair $\mathcal{U}_q^{\mathcal{F}}(\mathrm{SL}_2)$ and $(\mathcal{U}_q^*)^{\mathcal{F}}(\mathrm{SL}_2)$ forms the family of commutative, noncocommutative quantum Lorentz algebras $\mathcal{U}_q^{(r,s)}(\mathrm{SL}(2,\mathbb{C}))$. In particular, for $r=1,\ s=1$ its dual object is just the Gauss form of the quantum Lorentz algebra [21].

2. Quantum Lobachevsky space. The quantum Lobachevsky space $\mathbf{L}_{\kappa,q}$ is the associative $*$-algebra over $\mathbb{C}$ with a unity and three generators

$$(z^*, H, z), \quad H^* = H,\ (z)^* = z^*,\ 1^* = 1.$$

The commutation relations depend on two parameters $\kappa, q \in \mathbb{C}$

$$zH = \kappa H z, \qquad z^* H = \bar{\kappa}^{-1} H z^*, \tag{3.9}$$

$$zz^* = az^*z - bH^{-2}, \quad a = \left(\frac{\kappa}{q}\right)^2, \quad b = \bar{\kappa}^{-1}\left(\frac{\kappa}{q}\right)^2\left(1 - q^2\frac{\bar{\kappa}}{\kappa}\right) \tag{3.10}$$

The both parameters should be tune in a such a way that the coefficients a, b become real. It is happened in two cases
(i)κ and q are real,
(ii) $|q| = 1$ and $\mathrm{Arg}(q) = \mathrm{Arg}(\kappa)$.

Proposition 3.1 $\mathbf{L}_{\kappa,q}$ *is a right* $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$*-module*

Proof.
Define the right action of $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ on $\mathbf{L}_{\kappa,q}$:

$$\begin{array}{lll} z^*.A = z^*, & H.A = q^{\frac{1}{2}}H, & z.A = q^{-1}z, \\ z^*.A^* = \left(\frac{\kappa}{q}\right)^{\frac{2}{s}} z^*, & H.A^* = \left(\frac{q}{\kappa}\right)^{\frac{1}{s}} H, & z.A^* = z, \\ z^*.B = 0, & H.B = 0 & z.B = q^{-\frac{1}{2}} \\ z^*.C = q^{\frac{3}{2}}\kappa^{-1}H^{-2}, & H.C = Hz, & z.C = -q^{\frac{1}{2}}z^2, \end{array} \tag{3.11}$$

$$x.B^* = 0, \quad x.C^* = 0 \ \text{ for any } x \in \mathbf{L}_{\kappa,q}.$$

Similarly, the left action of A^*, A, B^*, C^* takes the form

$$\begin{array}{lll} A^*.z^* = \bar{q}^{-1}z^*, & A^*.H = \bar{q}^{\frac{1}{2}}H, & A^*.z = z, \\ A.z^* = z^*, & A.H = \left(\frac{\bar{q}}{\bar{\kappa}}\right)^{\frac{1}{s}} H, & A.z = \left(\frac{\bar{\kappa}}{\bar{q}}\right)^{\frac{2}{s}} z \\ B^*.z^* = \bar{q}^{-\frac{1}{2}}, & B^*.H = 0, & B^*z = 0 \\ C^*.z^* = -\bar{q}^{\frac{1}{2}}(z^*)^2, & C^*.H = z^*H, & C^*.z = \bar{q}^{\frac{3}{2}}\bar{\kappa}^{-1}H^{-2}. \end{array} \tag{3.12}$$

The coproduct in $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ (3.7),(3.8) is compatible with the commutation rules in $\mathbf{L}_{\kappa,q}$ (3.9), (3.10). Similar, the $*$-structures in $\mathbf{L}_{\kappa,q}$ and in $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$ are also compatible. It means that (3.11) and (3.12) are related as

$$(x.K)^* = K^*.x^*.$$

for $K = A, B, C$ and $x = z^*, H, z$. $\square$

To eliminate the ambiguities related to the noncommutativity of $\mathbf{L}_{\kappa,q}$ we consider only the ordered monomials putting z^* on the left side, z on the right side and keeping H in the middle of the monomials:

$$: f(z^*, H, z) := \sum_{m,k,n} c_{m,k,n} w(m,k,n), \quad w(m,k,n) = z^{*m}H^k z^n. \tag{3.13}$$

The actions (3.11),(3.12) on $w(m,k,n)$ take the form

$$\begin{gathered} w(m,k,n).A = q^{-n+\frac{k}{2}}w(m,k,n), \qquad w(m,k,n).A^* = \left(\frac{q}{\kappa}\right)^{(-2m+k)/s} w(m,k,n), \\ w(m,k,n).B = q^{-n+\frac{k+1}{2}}\frac{1-q^{2n}}{1-q^2}w(m,k,n-1), \\ w(m,k,n).C = q^{n-\frac{3(k-1)}{2}}\kappa^{k-1}\frac{1-q^{2m}}{1-q^2}w(m-1,k-2,n) - q^{-n+\frac{k+3}{2}}\frac{1-q^{2n-2k}}{1-q^2}w(m,k,n+1), \end{gathered} \tag{3.14}$$

$$\begin{gathered} A^*.w(m,k,n) = \bar{q}^{-m+\frac{k}{2}}w(m,k,n), \qquad A.w(m,k,n) = \left(\frac{\bar{q}}{\bar{\kappa}}\right)^{(-2n+k)/s} w(m,k,n), \\ B^*.w(m,k,n) = \bar{q}^{-m+\frac{k+1}{2}}\frac{1-\bar{q}^{2m}}{1-\bar{q}^2}w(m-1,k,n), \\ C^*.w(m,k,n) = \bar{q}^{m-\frac{3(k-1)}{2}}\bar{\kappa}^{k-1}\frac{1-\bar{q}^{2n}}{1-\bar{q}^2}w(m,k-2,n-1) - \bar{q}^{-m+\frac{k+3}{2}}\frac{1-\bar{q}^{2m-2k}}{1-\bar{q}^2}w(m+1,k,n). \end{gathered} \tag{3.15}$$

and

$$w(m,k,n).\Omega_q = q^{-k+1}\frac{(1-q^{k+1})^2}{(1-q^2)^2}w(m,k,n)+$$

$$+q^{-k+1}\kappa^{k-1}\frac{(1-q^{2m})(1-q^{2n})}{(1-q^2)^2}w(m-1,k-2,n-1).$$

Define the difference operators $\mathcal{D}_{z^*},\ \mathcal{D}_z$ as

$$\mathcal{D}_{z^*}f(z^*,H,z) = (z^*)^{-1}\frac{f(z^*,H,z)-f(q^2z^*,H,z)}{1-q^2},$$

$$\mathcal{D}_z f(z^*,H,z) = \frac{f(z^*,H,z)-f(z^*,H,q^2z)}{1-q^2}z^{-1},$$

and

$$\bar{\mathcal{D}}_{z^*}f(z^*,H,z) = (z^*)^{-1}\frac{f(z^*,H,z)-f(\bar{q}^2z^*,H,z)}{1-\bar{q}^2},$$

$$\bar{\mathcal{D}}_z f(z^*,H,z) = \frac{f(z^*,H,z)-f(z^*,H,\bar{q}^2z)}{1-\bar{q}^2}z^{-1}.$$

Then the action (3.14),(3.15) of generators on the ordered functions can be represent in the form

$$f(z^*,H,z).A = f(z^*,q^{\frac{1}{2}}H,q^{-1}z), \qquad f(z^*,H,z).A^* = f(\left(\frac{\kappa}{q}\right)^{\frac{2}{s}}z^*,\left(\frac{\kappa}{q}\right)^{\frac{1}{s}}H,z),$$

$$f(z^*,H,z).B = q^{\frac{1}{2}}\mathcal{D}_z f(z^*,q^{\frac{1}{2}}H,q^{-1}z), \tag{3.16}$$

$$f(z^*,H,z).C = q^{\frac{3}{2}}\kappa^{-1}\mathcal{D}_{z^*}f(z^*,q^{-\frac{3}{2}}\kappa H,q\kappa^2z)H^{-2} + \frac{q^{\frac{3}{2}}}{1-q^2}[f(z^*,q^{-\frac{3}{2}}H,qz)-f(z^*,q^{\frac{1}{2}}H,qz)]z-$$

$$-q^{\frac{3}{2}}\mathcal{D}_z f(z^*,q^{\frac{1}{2}}H,q^{-1}z)z^2,$$

and

$$A.f(z^*,H,z) = f(z^*,\left(\frac{\bar\kappa}{\bar q}\right)^{\frac{1}{s}}H,\left(\frac{\bar\kappa}{\bar q}\right)^{\frac{2}{s}}z), \qquad A^*.f(z^*,H,z) = f(\bar{q}^{-1}z^*,\bar{q}^{\frac{1}{2}}H,z),$$

$$B^*.f(z^*,H,z) = \bar{q}^{\frac{1}{2}}\bar{\mathcal{D}}_{z^*}f(\bar{q}^{-1}z^*,\bar{q}^{\frac{1}{2}}H,z), \tag{3.17}$$

$$C^*.f(z^*,H,z) = -\bar{q}^{\frac{3}{2}}(z^*)^2\bar{\mathcal{D}}_{z^*}f(\bar{q}^{-1}z^*,\bar{q}^{\frac{1}{2}}H,z) + \frac{\bar{q}^{\frac{3}{2}}}{1-\bar{q}^2}z^*[f(\bar{q}z^*,q^{-3/2}H,z)-f(\bar{q}z^*,\bar{q}^{\frac{1}{2}}H,z)]+$$

$$+\bar{q}^{\frac{3}{2}}\bar\kappa^{-1}H^{-2}\bar{\mathcal{D}}_z f(\bar{q}\kappa^2z^*,\bar{q}^{-\frac{3}{2}}\bar\kappa H,z).$$

Note, that when $q\to 1$ we come to (2.13)

$$\partial_q A \to \mathcal{D}^3, \quad B\to\mathcal{D}^+, \quad C\to\mathcal{D}^-.$$

The Haar functional $I_{\mathbf{L}_{\kappa,q}}$ on $\mathbf{L}_{\kappa,q}$ is a map $\mathbf{L}_{\kappa,q}\to\mathbb{C}$ that satisfies the following condition

$$I_{\mathbf{L}_{\kappa,q}}(f.u) = \varepsilon(u)I_{\mathbf{L}_{\kappa,q}}(f), \quad f\in\mathbf{L}_{\kappa,q}, \quad u\in\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C}))$$

Proposition 3.2 *The Jackson integral for ordered $f(z^*,H,z)$ (3.13)*

$$I_{\mathbf{L}_{\kappa,q}}(f)=\int\int\int :f(z^*,H,z)d_qz^*Hd_{q^{\frac{1}{2}}}Hd_qz:= \tag{3.18}$$
$$=(1-q)^2(1-q^{\frac{1}{2}}\sum_{s,r,t\in\mathbf{Z}}q^{s+r+t}[:f(q^s,q^{\frac{1}{2}r},q^t):+:f(-q^s,q^{\frac{1}{2}r},-q^t):$$
$$+:f(q^s,q^{\frac{1}{2}r},-q^t):+:f(-q^s,q^{\frac{1}{2}r},q^t):].$$

is the Haar functional on $\mathbf{L}_{\kappa,q}$.

Proof
The main relation one should use to check (3.16) is

$$\int f(qx)d_qx=q^{-1}\int f(x)d_qx.$$

Then the statement follows from the definitions of the actions (3.11). Moreover, from (3.16), (3.17)) we find that the integral (3.18) is invariant under the action $f\to u^*.f$. □

4 Principle series of unitary representations

The principle series representations for quantum Lorentz groups were considered in [26, 27]. Here we present the formulae for principle series operators inspired by the action of operators on the quantum Lobachevsky spaces. In what follows we forget the coalgebraic structure and for this reason use the notation $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$.

Consider the same space $V_{\nu,n}$ as in the classical case (2.3). But now we introduce another Hermitian form

$$<f|g>=q^{\frac{i\nu+n-1}{2}}\bar{q}^{\frac{i\nu-n+1}{2}}\int\int\overline{\left(f(\bar{q}^{\frac{i\nu-n+1}{2}}\bar{z},q^{\frac{i\nu+n-1}{2}}z)\right)}g(\bar{q}^{\frac{i\nu-n+1}{2}}\bar{z},q^{\frac{i\nu+n-1}{2}}z)d\bar{z}dz. \tag{4.1}$$

Define the action π_ν of $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$ on $V_{\nu,n}$. We assume that $\mathcal{U}_q(\mathrm{SL}_2)$ acts only on the holomorphic part of $V_{\nu,n}$, while $\mathcal{U}_q^*(\mathrm{SL}_2)$ acts on the antiholomorphic part. Following our notations in Section 3 we write the holomorphic action from the right hand side, while the antiholomorphic action from the left hand side. The right action takes the form

$$\begin{aligned}f(z).\pi_\nu(A)&=q^{\frac{i\nu+n-1}{2}}f(q^{-1}z),\\ f(z).\pi_\nu(B)&=q^{\frac{i\nu+n}{2}}\frac{f(q^{-1}z)-f(qz)}{1-q^2}z^{-1},\\ f(z).\pi_\nu(C)&=\frac{z}{1-q^2}[-q^{\frac{i\nu+n+2}{2}}f(q^{-1}z)+q^{-\frac{3(i\nu+n)}{2}+3}f(qz)],\end{aligned} \tag{4.2}$$

Similarly, for the left action we have

$$\begin{aligned}\pi_\nu(A^*).f(\bar{z})&=\bar{q}^{\frac{1}{2}(i\nu-n-1)}f(\bar{q}^{-1}\bar{z}),\\ \pi_\nu(B^*).f(\bar{z})&=\bar{q}^{\frac{i\nu-n}{2}}\frac{f(\bar{q}^{-1}\bar{z})-f(\bar{q}\bar{z})}{1-\bar{q}^2}\bar{z}^{-1},\\ \pi_\nu(C^*).f(\bar{z})&=\frac{\bar{z}}{1-\bar{q}^2}[-\bar{q}^{\frac{i\nu-n+2}{2}}f(\bar{q}^{-1}\bar{z})+\bar{q}^{-\frac{3(i\nu-n)}{2}+3}f(\bar{q}\bar{z})].\end{aligned} \tag{4.3}$$

Proposition 4.1 *(i) The operators π_ν (4.2),(4.3) are irreducible representations of* $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$*; (ii) for $\nu\in\mathbb{R}$ they are unitarity with respect to the form (4.1):*

$$<g|f.\pi_\nu(u)>=<\pi_\nu(S(u^*)).g|f>,$$

$$g,f\in V_{\nu,n},\quad u\in\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C})).$$

Proof. The direct calculations show that the operators $\pi_\nu(u)$ satisfy the same commutation relations as u (3.1). Moreover, the representation actions π_ν preserves the asymptotic (2.4).

Another way to see that π_ν satisfy the commutation relations for the case $n=0$ is to consider the subspace $\tilde{V}_\nu$ of $\mathbf{L}_{\kappa,q}$ of the form

$$f(z^*,cH,z)=c^{i\nu-1}f(z^*,H,z).$$

In other words, the basis functions in $\tilde{V}_\nu$ are $w(m,i\nu-1,n)=z^{m*}H^{i\nu-1}z^n$. Starting from (3.14), (3.15) we define the action of $\mathcal{U}_q^{(0,s)}(\mathrm{SL}(2,\mathbb{C})))$ on $\tilde{V}_\nu$ in a such way that the right actions of $\pi_\nu(A,B,C)$ do not touch z^*-variable ($m=0$ in (3.14)). Similarly, the left acting operators $\pi_\nu(A^*,B^*,C^*)$ do not touch z-variable ($n=0$ in (3.15)). Then $\tilde{V}_\nu$ is invariant with respect to π_ν. Since the dependence on H is fixed one can put $H=1$. Then we come to the representation (4.2) and (4.3), where V_ν is defined as $V_\nu=\tilde{V}_\nu|_{H=1}$.

Consider the action of the Casimirs (3.4),(3.5) on V_ν. They become the scalar operators

$$\pi_\nu(\Omega)=([-\frac{i\nu+n}{2}]_q)^2\mathrm{Id},\quad \pi_\nu(\Omega^*)=([-\frac{i\nu-n}{2}]_{\bar{q}})^2\mathrm{Id}. \tag{4.4}$$

In this sense the representations π_ν are irreducible.

The unitarity is proved in Appendix. □

In the classical limit we come to the principle series representations of $\mathrm{SL}(2,\mathbb{C})$ since (2.7)

$$\lim_{q\to1}\pi_\nu(B)=T^+,\quad \lim_{q\to1}\pi_\nu(C)=T^+,$$

$$\lim_{q\to1}\pi_\nu(A)=(1+T^3),$$

and the Hermitian form (4.1) takes the classical form (2.6).

It follows from (4.2) and (4.3) that ν is defined up to a constant. Two representations (ν,n) and (ν',n) are equivalent if

$$\nu'=\nu+\frac{4\pi m}{\hbar},\quad m\in\mathbb{Z},\quad \hbar=\ln q,$$

and we assume that

$$\nu\in\mathbb{R},\quad 0\leq\nu\leq\frac{4\pi}{\hbar}.$$

The case $n=0$ corresponds to the class one representations $V_{\nu,0}=V_\nu$. It was demonstrated in the Proof of Proposition 4.1 that they are related to the quantum Lobachevsky space. Moreover, as in the classical limit there exists a $\mathcal{U}_q(\mathrm{SU}(2))$ invariant state in V_ν. It will be found in Section 6.

5 The q^2-Bessel functions

We take here q and κ to be real, and put $\kappa = q^\delta$.

Consider the eigenvalue problem for the Casimir (3.4) acting on $\mathbf{L}_{\kappa,q}$

$$f^\delta_\nu(z^*,H,z).\Omega^\nu_q = ([\frac{i\nu}{2}]_q)^2 f^\delta_\nu(z^*,H,z).$$

Thereby, $f^{(\delta)}_\nu(z^*,H,z)\in V_\nu$. It follows from (3.14) that if

$$\Omega_q = \Omega_q - q^{-i\nu+2}\frac{(1-q^{i\nu})^2}{(1-q^2)^2}$$

then

$$w(m,k,n).\Omega^\nu_q = \frac{q^{-k+1}(1-q^{i\nu+k+1})(1-q^{-i\nu+k+1})}{(1-q^2)^2}w(m,k,n)+$$

$$+\frac{q^{(k-1)(\delta-1)}(1-q^{2m})(1-q^{2n})}{(1-q^2)^2}w(m-1,k-2,n-1). \qquad (5.1)$$

We seek solutions of the equation

$$f^{(\delta)}_\nu(z^*,H,z).\Omega^\nu_q = 0,$$

where $f^{(\delta)}_\nu(z^*,H,z)$ takes the form

$$f^{(\delta)}(z^*,H,z) = e_{q^2}(i\mu(1-q^2)z^*)F^{(\delta)}_\nu(H)e_{q^2}(i\mu(1-q^2)z),$$

and e_{q^2} is the q^2-exponential

$$e_{q^2}(x) = \sum_{n=0}^\infty \frac{x^n}{(q^2,q^2)_n} = \frac{1}{(x,q^2)_\infty}, \qquad |x|<1.$$

The q^2-exponentials $e_{q^2}(i\mu(1-q^2)z^*)$ and $e_{q^2}(i\mu(1-q^2)z)$ satisfy the equations

$$\mathcal{D}_{z^*}e_{q^2}(i\mu(1-q^2)z^*) = i\mu e_{q^2}(i\mu(1-q^2)z^*),\quad \mathcal{D}_z e_{q^2}(i\mu(1-q^2)z) = i\mu e_{q^2}(i\mu(1-q^2)z). \qquad (5.2)$$

On the other hand (5.1) can be rewrite in the following form

$$e_{q^2}(i\mu(1-q^2)z^*)H^k e_{q^2}(i\mu(1-q^2)z).\Omega^\nu_q =$$

$$= \frac{q^{-k+1}(1-q^{i\nu+k+1})(1-q^{-i\nu+k+1})}{(1-q^2)^2}e_{q^2}(i\mu(1-q^2)z^*)F^{(\delta)}_\nu(H)e_{q^2}(i\mu(1-q^2)z)+$$

$$+q^{(k-1)(\delta-1)}\mathcal{D}_{z^*}e_{q^2}(i\mu(1-q^2)z^*)H^{k-2}\mathcal{D}_z e_{q^2}(i\mu(1-q^2)z).$$

It follows from the last expression and (5.2) that $F^{(\delta)}_\nu(H)$ is the solution of the difference equation

$$\frac{qF^{(\delta)}_\nu(qH)-(q^{i\nu}+q^{-i\nu})F^{(\delta)}_\nu(H)+q^{-1}F^{(\delta)}_\nu(q^{-1}H)}{(1-q^2)^2} - \mu^2 q^{-\delta-1}H^{-2}F^{(\delta)}_\nu(q^{\delta-1}H) = 0. \qquad (5.3)$$

Substitution $F^{(\delta)}_\nu(H) = H^{-1}\mathcal{F}^{(\delta)}_\nu(H^{-1})$ and $H^{-1}=x$ leads to the difference equation for $\mathcal{F}^{(\delta)}_\nu(x)$

$$\mathcal{F}^{(\delta)}_\nu(q^{-1}x)-(q^{i\nu}+q^{-i\nu})\mathcal{F}^{(\delta)}_\nu(x)+\mathcal{F}^{(\delta)}_\nu(qx)-\mu^2q^{-2\delta}(1-q^2)^2x^2\mathcal{F}^{(\delta)}_\nu(q^{1-\delta}x) = 0 \qquad (5.4)$$

Define the second order difference operator

$$\Delta_q \psi(x) = \frac{\psi(qx) - 2\psi(x) + \psi(q^{-1}x)}{(q - q^{-1})^2}.$$

Then (5.4) takes the form

$$\Delta_q \mathcal{F}_\nu^{(\delta)}(x) - \mu^2 q^{-2\delta+2} x^2 \mathcal{F}_\nu^{(\delta)}(q^{1-\delta}x) = ([\frac{i\nu}{2}]_q)^2 \mathcal{F}_\nu^{(\delta)}(x).$$

It is just the discrete analog of the quantum Liouville equation (2.24) corresponding to the two-body case of the quantum open relativistic Toda lattices [13] with depending on the parameter δ Hamiltonians

$$\hat{H} = -\Delta_q + \mu^2 q^{-2\delta+2} x^2 (T_q)^{1-\delta},$$

where T_q is the shift operator $T_q \psi(x) = \psi(qx)$. The wave-functions of these systems are among the solutions of (5.4). It follows from (5.4) that the second ordered difference equations arise only for three values $\delta = 0, 1, 2$. The other cases lead to higher ordered difference equations. In what follows we consider only these three cases.

Let

$$\mathcal{F}_\nu^{(\delta)}(x) = \sum_{k=0}^{\infty} \frac{a_k (1-q^2)^{2k} (\mu x)^{i\nu+2k}}{(q^2,q^2)_k (q^{2i\nu+2},q^2)_k}$$

Then from (5.4) $a_k = (-1)^k q^{(2-\delta)k(k+i\nu)-k\delta}$, and we find that the solutions are the modified q^2-Bessel functions [14]

$$J_{i\nu}^{(j)}(2i\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) = \frac{q^{-\frac{1}{2}i\delta\nu}}{\Gamma_{q^2}(i\nu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k q^{(2-\delta)k(k+i\nu)-k\delta}(1-q^2)^{2k}(i\mu x)^{i\nu+2k}}{(q^2,q^2)_k (q^{2i\nu+2},q^2)_k}.$$

Here

$$j = \begin{cases} 1 & \text{for} \quad \delta = 2 \\ 2 & \text{for} \quad \delta = 0 \\ 3 & \text{for} \quad \delta = 1. \end{cases} \tag{5.5}$$

Three values of j correspond to the known functions [14, 15]:

$$I_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) = e^{\frac{1}{2}\nu\pi} J_{i\nu}^{(j)}(2i\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) -$$

$$-\begin{cases} \text{Jackson modified } q^2\text{−Bessel function of type 1[28]} & j=1 \\ \text{Hahn − Exton modified } q^2\text{−Bessel function [29], [30]} & j=3 \\ \text{Jackson modified } q^2\text{−Bessel function of type 2 [28]} & j=2 \end{cases}$$

Obviously $I_{-i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)$ is the solution of (5.4 also. The corresponding Wronskians are

$$W^{(j)}(x;q^2) = I_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) I_{-i\nu}^{(j)}(2\mu(1-q^2)q^{1-\frac{1}{2}\delta}x;q^2) -$$

$$-I_{-i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) I_{i\nu}^{(j)}(2\mu(1-q^2)q^{1-\frac{1}{2}\delta}x;q^2) =$$

$$= \frac{1}{\Gamma_{q^2}(i\nu+1)\Gamma_{q^2}(-i\nu+1)} \sum_{k=0}^{\infty} \frac{q^{(2-\delta)k(k-i\nu)-k\delta}(1-q^2)^{2k}(\mu x)^{2k}}{(q^2,q^2)_k} \times$$

$$\times \sum_{n=0}^{k} \begin{bmatrix} k \\ n \end{bmatrix}_{q^2} \frac{q^{-2(2-\delta)n(k-n-i\nu)}(q^{-i\nu+2k-2n} - q^{i\nu+2n})}{(q^{2i\nu+2},q^2)_n(q^{-2i\nu+2},q^2)_n}.$$

for $i\nu \neq n$, n is an integer. Here

$$\begin{bmatrix} k \\ n \end{bmatrix}_{q^2} = \frac{(q^2,q^2)_k}{(q^2,q^2)_n(q^2,q^2)_{k-n}}.$$

The interior sum $S^{(\delta)}$ is equal

$$S^{(\delta)} = \begin{cases} (-1)^k q^{-k(k+2i\nu-1)-i\nu}(1-q^{2i\nu}) & \delta = 0, \quad k \geq 0, \\ q^{-i\nu}(1-q^{2i\nu}) & \delta = 1, k = 0, \\ 0 & \delta = 1, k > 0, \\ -q^{-i\nu}(1-q^{2i\nu}) & \delta = 2, \quad k \geq 0, \end{cases}$$

So

$$W^{(j)} = \begin{cases} \frac{q^{-i\nu}(1-q^{2i\nu})}{\Gamma_{q^2}(i\nu+1)\Gamma_{q^2}(-i\nu+1)} E_{q^2}(-4\mu^2(1-q^2)q^2x^2) & \delta = 0, \\ \frac{q^{-i\nu}(1-q^{2i\nu})}{\Gamma_{q^2}(i\nu+1)\Gamma_{q^2}(-i\nu+1)} & \delta = 1, \\ \frac{q^{-i\nu}(1-q^{2i\nu})}{\Gamma_{q^2}(i\nu+1)\Gamma_{q^2}(-i\nu+1)} e_{q^2}(4\mu^2(1-q^2)q^{-2}x^2) & \delta = 2, \end{cases}$$

Thus, the modified q^2-Bessel functions $I^{(j)}_{i\nu}$, $I^{(j)}_{-i\nu}$ form the fundamental system. Though they are unbounded functions, and our goal to find their linear combination providing the bounded solution as in the classical case (2.22).

6 The q-Whittaker functions

In this Section we assume as before that $\kappa = q^\delta$ and q are real.

Consider an analog of the group element defined on $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C})) \bigotimes \mathbf{L}_{\kappa,q}$. [1]

$$g(A|H) = e^{\mathcal{K}\otimes 2\phi}, \qquad A = e^{\hbar\mathcal{K}}, \qquad H = e^{\phi}, \qquad \hbar = \ln q. \tag{6.1}$$

The action of $\pi_\nu(g(A|H))$ in V_ν (see (4.2))

$$\pi_\nu(g(A|H)) : f(\bar{z},z) \to H^{i\nu-1} f(\bar{z}, H^{-2}z). \tag{6.2}$$

We rewrite the Hermitian form (4.1) as

$$< f|g >= q^{i\nu+n-1} \int\int \overline{f(\bar{z},z)} g(q^{i\nu-n+1}\bar{z}, q^{i\nu+n-1}z) d\bar{z}dz. \tag{6.3}$$

By means of (6.3) we define the matrix element of $\pi_\nu(g(A,H))$ in V_ν

$$F^{(\delta)}_\nu(H) =< \psi_L|\psi_R.g(A|H) >=< S(g(A^*|H)).\psi_L|\psi_R >, \tag{6.4}$$

We drop here and in what follows the notion of the representation π_ν.

[1] The group elements g are defined in the tensor product $\mathcal{U}_q \otimes \mathcal{A}_q$ and satisfy the relation $\Delta g \otimes id = g_{12}g_{23}$ [31, 32].

Let $\psi_L,\ \psi_R \in V_\nu$ satisfy the following conditions

$$C^*.\psi_L(\bar{z},z) = \psi_L(\bar{z},z).B \tag{6.5}$$

$$B^*.\psi_R.B = -\bar{\mu}\mu q^{1-\delta}\psi_R.A^{2(\delta-1)}. \tag{6.6}$$

In other words, from (4.2), (4.3)

$$B^*.\psi_R(\bar{z},z).B = -\bar{\mu}\mu q^{(i\nu-2)(\delta-1)}\psi_R(\bar{z},q^{2-2\delta}z). \tag{6.7}$$

Thus ψ_R can be considered as the *quantum Whittaker vector* for $\mathcal{U}_q(\mathrm{SL}(2,\mathbb{C}))$.

Proposition 6.1 *$F_\nu^{(\delta)}(H)$ satisfies (5.3)*

Proof. Since $F_\nu^{(\delta)}(H)$ is the matrix element in the irreducible space V_ν the Casimir operator Ω_q (3.6) acts as the scalar (4.4)

$$< \psi_L|\psi_R.g(A|H)(\frac{qA^2+q^{-1}A^{-2}-2}{(q-q^{-1})^2}+CB) >= [-\frac{i\nu}{2}]_q^2 < \psi_L|\psi_R.g(A|H) > . \tag{6.8}$$

It follows from (6.1) that

$$F_\nu^{(\delta)}(qH) =< \psi_L|\psi_R.g(A|H)A^2 > . \tag{6.9}$$

Then, the A-dependent part of left hand side in this equation is boiled down to the second order difference operator

$$F_\nu^{(\delta)}(H).\left(\frac{qA^2+q^{-1}A^{-2}-2}{(q-q^{-1})^2}\right) = \tag{6.10}$$

$$< \psi_L|\psi_R.g(A|H)\left(\frac{qA^2+q^{-1}A^{-2}-2}{(q-q^{-1})^2}\right) >= \frac{qF_\nu^{(\delta)}(qH)-2F_\nu^{(\delta)}(H)+q^{-1}F_\nu^{(\delta)}(q^{-1}H)}{(q-q^{-1})^2}.$$

On the other hand (3.1) provides the following relation

$$g(A|H)C = CH^{-2}g(A|H).$$

Thereby

$$< \psi_L|\psi_R.g(A|H)CB >= H^{-2} < \psi_L|\psi_R.Cg(A|H)B > .$$

¿From (6.5) and the unitarity with respect to (6.3) we have

$$< \psi_L|\psi_R.g(A|H)CB >= H^{-2} < S(C^*).\psi_L|(\psi_R.B).g(A|H) >=$$

$$= H^{-2} < \psi_L.S(B)|(\psi_R.B).g(A|H) >= H^{-2} < \psi_L|(B^*.\psi_R.B).g(A|H) > .$$

The condition (6.7) and (6.9) allow to rewrite the last expression as

$$-\bar{\mu}\mu q^{1-\delta}H^{-2} < \psi_L|(\psi_R.A^{2(\delta-1)}).g(A|H) >= -\bar{\mu}\mu q^{1-\delta}H^{-2} < \psi_L|\psi_R.g(A|H)A^{2(\delta-1)} >=$$

$$= -\bar{\mu}\mu q^{1-\delta}H^{-2} < \psi_L|\psi_R.g(A|q^{\delta-1}H) >= -\bar{\mu}\mu q^{1-\delta}H^{-2}F_\nu(q^{\delta-1}H).$$

Thus

$$F_\nu^{(\delta)}(H).CB = -\bar{\mu}\mu q^{1-\delta}H^{-2}F_\nu(q^{\delta-1}H).$$

Substituting this equality with (6.10) in (6.8) we come to (5.3). $\square$

Corollary 6.1 *$F_\nu^{(\delta)}(H)$ is the eigenfunction of the conjugate Casimir operator Ω_q^*.*

Proof. It follows from (4.4) that

$$\Omega_q^*.F_\nu^{(\delta)}(H) = ([\frac{i\nu}{2}]_{\bar q})^2 F_\nu^{(\delta)}(H).$$

Since

$$F_\nu^{(\delta)}(H) = \left(F_\nu^{(\delta)}(H)\right)^* =< g(A^*|H).\psi_R^*|\psi_L^* >$$

one has

$$\Omega_q^*.F_\nu^{(\delta)}(H) =< (\frac{(qA^*)^2 + q^{-1}(A^*)^{-2} - 2}{(q-q^{-1})^2} + B^*C^*)g(A^*|H).\psi_R^*|\psi_L^* > .$$

Then we can use the same conditions (6.5), (6.6) because $(\psi.u)^* = u^*.\psi^*$. $\square$

Now calculate (6.4) explicitly.

Proposition 6.2 *The matrix element $F_\nu^{(\delta)}(H) =< \psi_L|g(A,H)|\psi_R >$ (6.4) has the form of the double integral*

$$F_\nu^{(\delta)}(H) = q^{i\nu-1}H^{i\nu-1}\int\int \frac{(-q^{2i\nu+4}\bar z z, q^2)_\infty}{(-q^2\bar z z, q^2)_\infty}\xi_1(q^{i\nu+1}z^*)\xi_2(q^{i\nu-1}H^{-2}z)dzdz^*, \qquad (6.11)$$

where ξ_1, ξ_2 are defined below (6.12),6.13).

Proof. Assume that

$$A^*.\psi_L(z^*,z).(A^{-1}) = \psi_L(z^*,z).$$

This condition along with (6.5) means that ψ_L is the $\mathcal{U}_q(\mathrm{SU}(2))$-invariant vector (see (3.3)) in V_ν.

It follows from (6.5) that the state ψ_L has the form

$$\psi_L(\bar z, z) = \sum_{m=0}^{\infty}\frac{b_m}{(q^2,q^2)_m}\bar z^m z^m.$$

Then, from the second condition (6.5) and from (4.2) one can calculate the coefficients $b_m = (-1)^m q^{2m}(q^{-2i\nu+2},q^2)_m$. Thereby

$$\psi_L(\bar z, z) = \sum_{m=0}^{\infty}\frac{(-1)^m q^{2m}(q^{-2i\nu+2},q^2)_m}{(q^2,q^2)_m}(\bar z z)^m = \frac{(-q^{-2i\nu+4}x,q^2)_\infty}{(-q^2 x,q^2)_\infty},$$

where $x = |z|^2$.

Now consider the Whittaker state (6.6),(6.7). Let us find it in the form of the product

$$\psi_R(\bar z, z) = \xi_1(\bar z)\xi_2(z),$$

$$\xi_1(\bar z) = \sum_{n=0}^{\infty} a_n\frac{(1-q^2)^n}{(q^2,q^2)_n}\bar\mu^n(\bar z)^n, \quad \xi_2(z) = \sum_{n=0}^{\infty} c_n\frac{(1-q^2)^n}{(q^2,q^2)_n}\mu^n z^n.$$

Then from (6.7)

$$a_n c_k = -\bar\mu\mu a_{n-1}c_{k-1}q^{i\nu\delta-2i\nu-2\delta+1+(k-1)(3-2\delta)+n} =$$
$$= a_0(i\mu)^n q^{\frac{n(n-1)}{2}+n(1-i\nu)}c_0(i\mu)^k q^{\frac{k(k-1)}{2}(3-2\delta)+k(1-i\nu)}q^{(i\nu\delta-2\delta+2)k},$$

Then assuming $a_0 = c_0 = 1$

$$\xi_1(\bar{z}) = {}_1\Phi_1(0; -q; q, -i\mu(1-q^2)q^{1-i\nu}\bar{z}), \tag{6.12}$$

where ${}_r\Phi_s$ is the basic hypergeometric series [33]. For $\xi_2(z)$ we have in the similar way

$$c_k = (i\mu)^k q^{\frac{k(k-1)}{2}(3-2\delta)+k(1-i\nu)+(i\nu\delta-2\delta+2)k}$$

Thus we obtain

$$\xi_2(z) = \begin{cases} {}_0\Phi_2(-; 0, -q; q, -i\mu(1-q^2)q^{3-i\nu}z) & \text{for} \quad \delta = 0 \\ {}_1\Phi_1(0; -q; q, -i\mu(1-q^2)qz) & \text{for} \quad \delta = 1 \\ {}_3\Phi_1(0,0,0; -q; q, -i\mu(1-q^2)q^{i\nu-1}z) & \text{for} \quad \delta = 2. \end{cases} \tag{6.13}$$

Using (6.2) and (6.3) we define the matrix element $F_\nu(H)$ in the form (6.11). □

Let us show that the (6.11) is the q-analog of the (2.22). If we rewrite the integral (6.11) in the polar coordinates $z = \rho e^{i\phi}$

$$F_\nu^{(\delta)}(H) = 2q^{i\nu-1}H^{i\nu-1}\int_0^\infty \frac{(-q^{2i\nu+4}\rho^2, q^2)_\infty}{-q^2\rho^2, q^2)_\infty}\rho d\rho \int_{-\pi}^{\pi} \xi_1(q^{i\nu+1}\rho e^{-i\phi})\xi_2(q^{i\nu-1}H^{-2}\rho e^{i\phi})d\phi,$$

and represent the functions ξ_1 and ξ_2 as series, we obtain the inner integral in the form

$$\int_{-\pi}^{\pi}\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(1-q^2)^n}{(q^2,q^2)_n}(i\mu q^2\rho)^n e^{-in\phi}\sum_{k=0}^{\infty}\frac{q^{\frac{3-2\delta}{2}k(k-1)}(1-q^2)^k}{(q^2,q^2)_k}(i\mu q^{i\nu\delta-2\delta+2}H^{-2}\rho)^k e^{ik\phi}d\phi.$$

If $\delta < \frac{3}{2}$ these series converge uniformly with respect ρ and ϕ, and we can integrate them term by term. We receive

$$2\pi\sum_{n=0}^{\infty}\frac{(-1)^n q^{(2-\delta)n^2}(1-q^2)^{2n}}{(q^2,q^2)_n^2}(\mu q^{i\nu\delta-\delta+2}H^{-1}\rho)^{2n} = 2\pi J_0^{(j)}(2\mu(1-q^2)q^{(i\nu-1)\frac{\delta}{2}+1}H^{-1}\rho; q^2).$$

The q^2-Bessel function is the holomorphic one of ρ if $\delta < \frac{3}{2}$ and it is the meromorphic function of ρ with the ordinary poles $\rho = \pm i\frac{q^{-i\nu}H}{2\mu(1-q^2)}$ if $\delta = 2$. So for $\delta = 0,\ 1,\ 2$

$$F_\nu^{(\delta)}(H) = 4\pi q^{i\nu-1}H^{i\nu-1}\int_0^\infty \frac{(-q^{2i\nu+4}\rho^2, q^2)_\infty}{(-q^2\rho^2, q^2)_\infty}J_0^{(j)}(2\mu(1-q^2)q^{(i\nu-1)\frac{\delta}{2}+1}H^{-1}\rho; q^2)\rho d\rho. \tag{6.14}$$

Let us prove now that this integral exists. Consider the first factor. Using properties of the q-binomial formula [15] we have

$$\frac{(1+\rho^2)(-q^{2i\nu+4}\rho^2, q^2)_\infty}{(-q^2\rho^2, q^2)_\infty} = \frac{(1+\rho^2)(-q^{2i\nu+2}\rho^2, q^2)_\infty}{(1+q^{2i\nu+2}\rho^2)(-q^2\rho^2, q^2)_\infty} =$$

$$= \frac{(1+\rho^2)(q^{2i\nu}, q^2)_\infty}{(1+q^{2i\nu+2}\rho^2)(q^2,q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q^{-2i\nu+2}, q^2)_n q^{2i\nu n}}{(q^2,q^2)_n(1+q^{2n+2}\rho^2)}.$$

It means that this expression is absolutely integrable function. Consider the second factor. It follows from [14] that if $\rho \neq 0$

$$|\frac{\rho}{1+\rho^2}J_0^{(1)}(2\mu(1-q^2)q^{i\nu}H^{-1}\rho; q^2)| \leq$$

$$\leq \frac{A^{(1)}\sqrt{\rho}}{1+\rho^2}|e_q(-i\mu(1-q^2)q^{i\nu}H^{-1}\rho;q^2)+ie_q(i\mu(1-q^2)q^{i\nu}H^{-1}\rho;q^2)|\leq C^{(1)},$$

$$|\frac{\rho}{1+\rho^2}J_0^{(2)}(2\mu(1-q^2)qH^{-1}\rho;q^2)|\leq$$

$$\leq \frac{A^{(2)}\sqrt{\rho}}{1+q^2}|E_q(-i\mu(1-q^2)qH^{-1}\rho;q^2)+iE_q(i\mu(1-q^2)qH^{-1}\rho;q^2)|\leq C^{(2)}.$$

For $j=3$ one has

$$J_0^{(3)}(2x,q^2)=\frac{(qx^2,q^2)_\infty}{(q^2,q^2)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k+1)}}{(q^2,q^2)_k(qx^2,q^2)_k}.$$

So

$$|\frac{\rho}{1+\rho^2}J_0^{(3)}(2\mu(1-q^2)q^{\frac{i\nu+1}{2}}H^{-1}\rho;q^2)|\leq C^{(3)},$$

and (6.14) converge absolutely.

Setting $q^{1+\frac{i\nu\delta}{2}}\rho=r$ we obtain finally

$$F_\nu^{(\delta)}(H)=4\pi q^{i\nu(\delta+1)}H^{i\nu-1}\int_0^{\infty e^{\frac{i\nu\delta}{2}\ln q}}\frac{(-q^{2i\nu+2-i\nu\delta}r^2,q^2)_\infty}{(-q^{-i\nu\delta}r^2,q^2)_\infty}J_0^{(j)}(2\mu(1-q^2)q^{-\frac{\delta}{2}}H^{-1}r;q^2)rdr=$$

$$=-\frac{8\pi q^{-\nu^2+\frac{3}{2}i\nu\delta}\ln q}{(1-q^2)\Gamma_{q^2}(i\nu+1)A_{i\nu}^{|1-\delta|}}\mu^{i\nu}H^{-1}K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{\delta}{2}}H^{-1};q^2),$$

where

$$A_{i\nu}=\sqrt{\frac{I_{i\nu}^{(2)}(2;q^2)}{I_{-i\nu}^{(2)}(2;q^2)}},\qquad(6.15)$$

and $j=1,2,3$ are connected with $\delta=2,0,1$ by relations (5.5) (see [34]). The curve of integration for $\delta=2$ ($j=1$) should satisfy the condition $\nu\ln q\neq(k+\frac{1}{2})\pi$.

7 The representation of the q-Bessel-Macdonald functions by the Mellin-Barns integral

The representation (2.25) is equivalent to the well-known Mellin transform of the classical Bessel-Macdonald function

$$\int_0^{i\infty}K_{i\nu}(2\mu H^{-1})(2\mu H^{-1})^{ip-1}dp=-i2^{-ip-2}\Gamma(\frac{i}{2}(p-\nu))\Gamma(\frac{i}{2}(p+\nu)),\quad \mathrm{Im}\,s>\mathrm{Im}\,\nu\geq 0,$$

The inversion formula has the form

$$K_{i\nu}(2\mu H^{-1}))=\frac{1}{8\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma}\Gamma(\frac{i}{2}(p-\nu))\Gamma(\frac{i}{2}(p+\nu))(2\mu H^{-1})^{-ip}dp.\qquad(7.1)$$

Here we generalized these relations for the q case (see also [19])). Consider the convergent integral

$$G(x)=\frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}g(s)x^{-s}ds,\qquad(7.2)$$

and assume that $G(x)$ satisfies (5.4). Then

$$\frac{1}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} q^{-s}(1-q^{s+i\nu})(1-q^{s-i\nu})g(s)x^{-s}ds = \frac{(1-q^2)^2}{2\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} g(s)q^{-2\delta+(\delta-1)s}x^{-(s-2)}ds,$$

and we come to the recurrence relation for $g(s)$

$$q^{-s}\frac{(1-q^{s+i\nu})(1-q^{s-i\nu})}{(1-q^2)^2}g(s) = q^{(\delta-1)s-2}g(s+2).$$

Its solution has the form

$$g(s) = q^{-\frac{\delta}{4}s^2+\frac{2+\delta}{2}s}\Gamma_{q^2}(\frac{s+i\nu}{2})\Gamma_{q^2}(\frac{s-i\nu}{2}), \tag{7.3}$$

and for $\mathrm{Re}\, s > -\mathrm{Re}\,\nu \geq 0$ the integral (7.2) does converge uniformly with respect to $x \in (1,\infty)$.

Since $I_{i\nu}^{(j)}$ and $K_{i\nu}^{(j)}$ form the fundamental system of the solutions of (5.4) [34]

$$G(x) = AI_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2) + BK_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2). \tag{7.4}$$

Let $j=1$. It was shown in [14] that $I_{i\nu}^{(1)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)$ is a meromorphic function with the ordinary poles $x = \pm\frac{q^{-r+\frac{1}{2}\delta}}{\mu(1-q^2)}$, $r=0,1,\ldots$, while $K_{i\nu}^{(1)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)$ and the left side of (7.4) are the holomorphic functions in the region $\mathrm{Re}\, x > 0$. So in this case $A=0$.

Let $j=2,3$. Then, it follows from [16], that $\lim_{x\to\infty} I_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2) = \infty$, $\lim_{x\to\infty} K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2) = 0$. Since the left hand side of (7.4) goes to zero if $x\to\infty$ we have for $j=2,3$ $A=0$.

It follows from (7.2) that $g(s)$ (7.3) is the inverse Mellin transform of $BK_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2)$

$$g(s) = B\int_0^\infty K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2)x^{s-1}dx. \tag{7.5}$$

Thus

$$K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}}x;q^2) = \frac{1}{2\pi B}\int_{\sigma-i\infty}^{\sigma+i\infty} q^{-\frac{\delta}{4}s^2+\frac{2+\delta}{2}s}\Gamma_{q^2}(\frac{s+i\nu}{2})\Gamma_{q^2}(\frac{s-i\nu}{2})x^{-s}ds$$

and we need to calculate B. The q-Bessel-Macdonald functions satisfy the following relation [16]:

$$\frac{q^{\frac{1}{2}\delta}}{\mu(1+q)x}\tilde{\mathcal{D}}_x x^{i\nu}K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) = -q^{-\frac{2-\delta}{2}(i\nu-1)}x^{i\nu-1}K_{\nu-1}^{(j)}(2(1-q^2)q^{1-\delta}x;q^2),$$

where

$$\tilde{\mathcal{D}}_x f(x) = \frac{f(x)-f(qx)}{(1-q)x}.$$

It means that

$$\int_0^\infty K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)x^{s-1}dx =$$

$$= -\frac{q^{\frac{2-\delta}{2}(s+i\nu)}}{\mu(1+q)}\int_0^\infty \tilde{\mathcal{D}}_x\left(x^{i\nu+1}K_{i\nu+1}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)\right)x^{s-i\nu-2}dx. \tag{7.6}$$

Let $f(x)$ be a smooth function and $f(x), f'(x)$ are integrable on $(0,\infty)$. Then it can be proved that

$$\int_0^\infty \tilde{\mathcal{D}}_x f(x)dx = \frac{\ln q}{1-q} f(0).$$

Assume that $s = i\nu + 2$. Remind that

$$K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) = \tag{7.7}$$

$$= \frac{1}{2} q^{\nu^2+i\nu}\Gamma_{q^2}(i\nu)\Gamma_{q^2}(1-i\nu)\left[A_{i\nu}^{|1-\delta|} I_{-i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) - A_{-i\nu}^{|1-\delta|} I_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)\right],$$

where $A_{i\nu}$ is determined by (6.15), and

$$I_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) = \frac{q^{-\frac{1}{2}i\delta\nu}}{\Gamma_{q^2}(i\nu+1)} \sum_{k=0}^\infty \frac{q^{(2-\delta)k(k+i\nu)-k\delta}(1-q^2)^{2k}(\mu x)^{i\nu+2k}}{(q^2,q^2)_k(q^{2i\nu+2},q^2)_k}. \tag{7.8}$$

Then it follows from (7.5) - (7.8)

$$q^{\frac{\delta}{4}\nu^2+i\frac{2-\delta}{2}\nu+2}\Gamma_{q^2}(i\nu+1) = -B\frac{q^{(2-\delta)(i\nu+1)}}{\mu(1+q)}\int_0^\infty \tilde{\mathcal{D}}_x\left(x^{i\nu+1}K_{i\nu+1}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2)\right)dx =$$

$$= -\frac{B\ln q}{2(1-q^2)}\mu^{-i\nu-2}\Gamma_{q^2}(i\nu+1)A_{i\nu+1}^{|1-\delta|}q^{(1+i\nu)(2-i\nu-\frac{1}{2}\delta)}.$$

Taking into account $A_{i\nu+1} = A_{i\nu}$ (see [14]), we obtain

$$B = -\frac{2(1-q^2)}{\ln q}\mu^{i\nu+2}A_{-i\nu}^{|1-\delta|}q^{-\frac{4-\delta}{4}\nu^2+\frac{1}{2}\delta}.$$

So we have proved

Proposition 7.1 *The q-Bessel-Macdonald functions can be represented by the Barns integral*

$$K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}x;q^2) =$$

$$= -\frac{\ln q\mu^{-i\nu-2}}{4\pi(1-q^2)}A_\nu^{|1-\delta|}q^{\frac{4-\delta}{4}\nu^2-\frac{1}{2}\delta}\int_{\sigma-i\infty}^{\sigma+i\infty} q^{-\frac{\delta}{4}s^2+\frac{2+\delta}{2}s}\Gamma_{q^2}(\frac{s+i\nu}{2})\Gamma_{q^2}(\frac{s-i\nu}{2})x^{-s}ds,$$

where A_ν is determined by (6.15), and $j = 1,2,3$ and $\delta = 2,0,1$ are connected by (5.5).

Assuming $x = H^{-1}$ and changing the variable of integration $s = ip$ we obtain

$$K_{i\nu}^{(j)}(2\mu(1-q^2)q^{-\frac{1}{2}\delta}H^{-1};q^2) = \tag{7.9}$$

$$= -\frac{\ln q\mu^{-i\nu-2}}{4\pi i(1-q^2)}A_\nu^{|1-\delta|}q^{\frac{4-\delta}{4}\nu^2-\frac{1}{2}\delta}\int_{-\infty-i\sigma}^{\infty-i\sigma} q^{\frac{\delta}{4}p^2+i\frac{2+\delta}{2}p}\Gamma_{q^2}(\frac{i}{2}(p+\nu))\Gamma_{q^2}(\frac{i}{2}(p-\nu))H^{ip}dp,$$

Obviously the integral representation of q^2-Bessel-Macdonald function (7.9) is the q-analog of integral representation (7.1). The same representation for $|q| = 1$ was obtained in [19].

Appendix

Proof of Theorem

Consider first the action of $\pi_\nu(A)$

$$\int [f(z^*)]^* g(q^{i\nu-1}z)).\pi_\nu(A)d_{q^2}z = \int [f(z^*)]^* q^{\frac{i\nu-1}{2}} g(q^{i\nu-1}q^{-1}z)d_{q^2}z =$$

$$= \int [q^{\frac{-i\nu+1}{2}} qf(qz)]^* g(q^{i\nu-1}z)d_{q^2}z = \int [q^{-\frac{i\nu-1}{2}} f(qz^*)]^* g(q^{i\nu-1}z)d_{q^2}z =$$

$$= \int [\pi(S^{\mathcal{F}}A^*).(f(z^*)]^* g(q^{i\nu-1}z)d_{q^2}z.$$

For $\pi_\nu(B)$ we have

$$\int [(f(z^*))^*]^* g(q^{i\nu-1}z)).\pi_\nu(B)d_{q^2}z = \int [f(z^*)]^* q^{\frac{i\nu}{2}} \mathcal{D}_z g(q^{-1}q^{i\nu-1}z)d_{q^2}z =$$

$$= -\int [q^{\frac{i\nu}{2}+1} \mathcal{D}_{z^*} f(q^{-1}z^*)]^* g(q^{i\nu-1)}z)d_{q^2}z =$$

$$= \int [\pi_\nu S^{\mathcal{F}}(B^*).f(z^*)]^* g(q^{i\nu-1)}z)d_{q^2}z.$$

Finally for $\pi_\nu(C)$

$$\int [f(z^*)]^* g(q^{i\nu-1)}z).\pi_\nu(C)d_{q^2}z =$$

$$\int [f(z^*)]^* [-q^{\frac{3}{2}i\nu} g(q^{i\nu-2)}z) + q^{-\frac{1}{2}i\nu+2} g(q^{i\nu)}z)d_{q^2}z =$$

$$= \int [-q^{-\frac{3}{2}i\nu+2} f(qz^*)z^* + q^{\frac{1}{2}i\nu} f(q^{-1}z^*)z^*]^* g(q^{i\nu-1)}z)d_{q^2}z =$$

$$= \int [\pi_\nu S^{\mathcal{F}}(C^*).f(z^*)]^* g(q^{i\nu-1)}z)d_{q^2}z.$$

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