Classification of finite dimensional representations of one noncommutative quadratic algebra

Natalia K. Iyudu[1,2]

[1]Dept. of Mathematics and Mechanics, Moscow State University, Vorobjovy Gory, Moscow, 119899, Russia
[2]Max-Planck-Institut für Mathematik, Germany, 53117 Bonn, Vivatsgasse 7
e-mails: `ioudu@mail.ru, iyudu@mpim-bonn.mpg.de`

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1 Introduction

We consider here a quadratic RIT algebra given by the following presentation: $R = k\langle x, y\rangle/(xy - yx - y^2)$. This algebra appeared in different areas of mathematics and physics. First of all it is a kind of a quantum plane: one of the two Auslander regular algebras of global dimension two in the Artin–Shelter classification [7]. There were considered and studied, deformations of $GL(2)$ analogues to $GL_q(2)$ related to this algebra in Manin's school [8], [9], where this algebra appeared under the name "Jordan algebra". This algebra is also a RIT algebra of type $(1,1)$. The class of relativistic internal time (RIT) algebras has a physical origin, it appeared and was investigated in papers [10], [11], [12]. It could be seen from there that the the class of RIT algebras arises from deformations of the Poincare algebra of the Lorenz group SO(3,1) by means of introducing the relativistic internal time.

We are going to describe here all finite dimensional representations of algebra R, to prove some structural results on the images of these representations in the endomorphism rings and on this basis to start a classification of reps considering the quiver equivalence of their images.

The intermediate results state that our algebra is residually finite dimensional. At the same time we show that in any fixed dimension it has an infinite representation type.

We suggest to consider some interesting families of representations defined by the type of the partition of n, for example a union by n of subvarieties $\{(X,Y) \in mod(R,n) \,|\, rkY = n-1\}$. While R is wild, we occur to be able to show that this families can be completely classified by parameters.

We also study the properties of affine varieties of R-module structures on $\mathbf{k}^n$. For example, we could state that the subvariety of n-dimensional representations which form our family of representations after taking union by n is an irreducible component of $\mathrm{mod}(R,n)$.

Thus we get results in three different approaches to the classification of representations of this infinite dimensional algebra: vertical classification, horizontal classification for some interesting families of representations and classification by quivers appearing from the images of representations in the endomorphism ring.

Let us mention that our RIT algebra is a subalgebra of the first Weyl algebra A_1.The latter has no finite dimensional representations, but the RIT algebra occurs to have quite a rich structure of them.

Category of finite dimensional modules over RIT mod R contains for example as a full subcategory mod $GP(n,2)$, where $GP(n,2)$ is a Gelfand–Ponomarev algebra [3] with the nilpotency degrees of variables x and y, n and 2 respectively.

We are interested here in complex representations, hence we fix once and for all a field $\mathbf{k} = \mathbb{C}$. Let Mod R be the category of all R-modules, mod R — category of finite dimensional R-modules and $\rho_n \in \mathrm{mod}R$ — n-dimensional representation of R.

In §2 we derive some structural properties of the images A_n of ρ_n. For any n there is a finite number of equivalence classes of images defined by a finite number of quivers. These classes contain for example all isomorphic algebras: quiver is an isomorphism invariant of an algebra. We are interested in the question which quivers could be realized by these images. Since we intend to consider here, first of all, the family of representations defined by the special type of partition related to the Jordan form of Y, we clarify the question above for this family. It turns out that only one quiver appears from this family (independently of the dimension n of representation).

These could be shown on the basis of structural results obtained in the first part of §2. We prove that the semisimple parts of all images A_n are direct products of fields, that is all images ρ_n are in fact basic algebras. Hence there exists a complete system of orthogonal idempotents in A_n/J consisting of m elements, where m is a number of different eigenvalues of $\rho_n(x)$. We are able to describe them explicitly in terms of polynomials. It is also possible to lift them to A_n and to construct a quiver with vertices corresponding to the idempotents e_i and $m_{ij} = \dim_k e_i(J/J^2)e_j$ arrows from e_i to e_j, where J is a Jacobson radical of A_n.

In §3 we describe the set of all n-dimensional reps of R and in §4 we shall construct the sequence ε_n of finite dimensional representations with zero intersection of their kernels. This shows that R is residually finite dimensional.

In §5 we give some series of examples of simple infinite dimensional modules over RIT algebra. As a consequence of the Cohen–Macaulay property possessed by RIT algebra, which we derive from [6] in [13] we can state that there are no infinite dimensional holonomic modules.

In §6 we prove an analog of the Gerstenhaber theorem on the dimensions of images of representations of two-generated commutative algebra. It turns out that we are able just to list all possible values of dimensions.

In §7 we consider a variety of R-module stuctures on $\mathbf{k}^n$. We solve the classification problem (horizontal, i.e. by parameters) in the family of representations appearing as a union by n of subvarieties of $mod(R,n)$ defined as $\{(X,Y) \in mod(R,n) | \operatorname{rk} Y = n-1\}$.

In §8 we mention that our investigation of the finite-dimensional representations of RIT algebras have lead to the discovery of surprising connection between this theory and problems around the Milnor's conjecture of 70th about the existence of left-invariant affine structures on solvable Lie groups.

2 Structural properties of the images of representations and quivers

Let us consider now the algebras A_n which are the images of all above described representations in dimension n. We derive here some structural properties of these algebras and then will be able to classify them. In this section we fix the dimension n and sometimes omit it in the notation (will write A instead of A_n, Y instead of Y_n etc.).

Let $J(A) = J$ be the Jacobson radical of the algebra $A = \rho(R)$. We describe now the semisimple part of A and show that A is *basic* (the quotient by radical $A/J(A)$ is a direct product of division rings). Since we consider as a basic field the algebraically closed field $\mathbb{C}$ over which there are no other finite dimensional division ring besides itself, we can call by basic those rings for which quotient by radical is a product of fields.

Hence we show that our algebra R is a basic object from the categorical point of view. Namely, due to the Wedderburn-Artin theorem (and equivalency of categories M_R and M_{R^n}) any finite dimensional semisimple algebra is Morita equivalent to some basic semisimple one, that is their categories of modules are equivalent.

THEOREM **2.1.** *Let $A = \rho_n(R)$ be the image of $R = \mathbf{k}\langle x, y\rangle/[x, y] = y^2$, with respect to an n-dimensional representation ρ_n, generated by $X = \rho_n(x)$ and $Y = \rho_n(y)$. Then A/J is a commutative one-generated ring $\mathbf{k}[x]/q(x)$, where $q(x) = (x - \lambda_1) \dots (x - \lambda_k)$ and $\lambda_1, \dots, \lambda_k$ are all different eigenvalues of the matrix X.*

LEMMA **2.1.** *Let $Y = \rho_n(y)$. Then the matrix Y is nilpotent.*

Proof. Suppose that matrix Y is not nilpotent and hence has a nonzero eigenvalue. We take a projector P on the subspace corresponding to this eigenvalue. It is obviously commute with any matrix, particularly with Y : $PY = YP$, and is an idempotent operator: $P^2 = P$. Hence multiplying our relation $XY - YX = Y^2$ from the right and from the left side by P and using above two notices we can observe that operators $X' = PXP$ and $Y' = PYP$ also satisfy the same relation: $X'Y' - Y'X' = Y'^2$. Taking into account that Y' has a form ofone or more Jordan blocks with the same nonzero eigenvalue λ, we get that traces of right and left parts of the relation can not coincide. This contradiction complete the proof. ∎

LEMMA **2.2.** *Let $X = \rho_n(x)$. Then the matrix $Q = (X - \lambda_1 I) \dots (X - \lambda_k I)$ is nilpotent.*

Proof. Note that $\operatorname{Spec} p(X) = p(\operatorname{Spec} X)$ for any polynomial p. $\operatorname{Spec} X$ in our case is $\{\lambda_1, \dots, \lambda_k\}$ and hence $\operatorname{Spec} Q = \{0\}$. Therefore the matrix Q is nilpotent. ∎

LEMMA **2.3.** *Any nilpotent element of the algebra $A = \rho(R)$ belongs to the radical $J(A)$.*

Proof. Let $Q \in \mathbb{C}[x]$ and $Q = Q(X) \in A$ be a nilpotent element with the degree of nilpotency N: $Q^N = 0$. We show first that $Q \in J(A)$. We have to show that for any $a \in \mathbb{C}\langle x, y\rangle$, $1 - a(X, Y)Q(X)$ is invertible. It suffices to verify that $a(X,Y)Q(X)$ is nilpotent. By Lemma 1 Y is nilpotent. Denote by m the degree of nilpotency of Y: $Y^m = 0$. Let us verify that $(a(X,Y)Q(X))^{mN} = 0$. Present $a(X,Y)$ as $u(X) + Yb(X,Y)$. If then we consider a word of length not less then mN of letters $\alpha = u(X)Q(X)$ and $\beta = Yb(X,Y)Q(X)$ then we can see that it is equal to zero. Indeed, if there are at least m letters β then using the relation $XY - YX = Y^2$ one can represent our word as a sum of words having a subword Y^m. Otherwise our word has the subword $\alpha^N = u(X)^N Q(X)^N = 0$. Thus, $Q(X) \in J(A)$.

Note now that if we have an arbitrary nilpotent polynomial $G(X,Y)$, we can separate the terms containing Y: $G(X,Y) = Q(X) + YH(X,Y)$. To obtain nilpotency of any element $a(X,Y)G(X,Y)$ it suffices to verify nilpotency of $a(X,Y)Q(X)$, which was already proved because the relation $[X,Y] = Y^2$ allows us to commute with Y, preserving the degree of it. ∎

COROLLARY **2.1.** *The Jacobson radical of $A = \rho(R)$ consists precisely of all nilpotent elements.*

COROLLARY **2.2.** *Let $Y = \rho(y)$. Then $Y \in J(A)$.*

From Corollary 2 we can see that A/J is an algebra of one variable x: $A/J \equiv \mathbf{k}[x]/I$ and we have to find now the generator of the ideal I.

First of all by Lemmas 2 and 3 $Q \in J(A)$. Let us show now that Q divides any element of $J(A)$. If some polynomial $P \in \mathbb{C}[x]$ does not vanish in some eigenvalue λ of X then $P(X) \notin J(A)$. Indeed the matrix $P(X)$ has a non-zero eigenvalue $P(\lambda)$ and hence $E - \frac{1}{p(\lambda)}P(X)$ is non-invertible. Therefore $P(X) \notin J(A)$. Thus, Q is the generator of I. This finishes the proof of Theorem 1.

THEOREM **2.2.** *The system $e_i = P_i(X)/P_i(\lambda_i)$, where $P_i(X) = (X - \lambda_1)...\widehat{(X - \lambda_i)}...(X - \lambda_k)$ and λ_i is a different eigenvalues of $X = \rho(x)$ is a complete system of orthogonal idempotents of A/J.*

P r o o f. Orthogonality of e_i is clear from the presentation of A/J as $\mathbf{k}[x]/(q)$ proven in theorem 1. ∎

THEOREM **2.3.** *The semisimple part of A is a product of a finite number of copies of the field $\mathbf{k}$ and radical splits:*

$$A = J \oplus \prod_{i=1}^{k} \mathbf{k}_i,$$

where k is the number of different eigenvalues of the matrix $X = \rho(x)$.

P r o o f. We shall construct an isomorphism of A/J and $\prod_{i=1}^{k} \mathbf{k}_i$ using the system e_i, $i = 1, \ldots, k$ of idempotents constructed in Theorem 2. Clearly e_i

form a linear basis in A/J considered as a linear space over $\mathbf{k}$. From the presentation of A/J as a quotient $\mathbf{k}[x]/q$ given in Theorem 1 it is clear that the dimension of A/J is equal to the degree of q, which coincides with the number of different eigenvalues of the matrix $X = \rho(x)$. Since the idempotents e_i are orthogonal they are linearly independent and therefore form a basis of A/J. The multiplication of two arbitrary elements $a, b \in A/J$, $a = a_1e_1 + \ldots + a_ke_k$, $b = b_1e_1 + \ldots + b_ke_k$ is given by the formula $ab = a_1b_1e_1 + \ldots + a_kb_ke_k$ due to orthogonality of the idempotents e_i. Hence the map $a \mapsto (a_1, \ldots, a_k)$ is the desired isomorphism of A/J and $\prod_{i=1}^{k} \mathbf{k}_i$.

In this case the radical splits. Theorem is proved. ∎

Now we can construct for the basic algebra $A/J = \prod_{i=1}^{k} \mathbf{k}_i$, an associated quiver by a standard way (see for example [1]). The vertices will correspond to the idempotents e_i or by theorem 2 equivalently, to the different eigenvalues of matrix X. The number of arrows from vertex e_i to the vertex e_j is the $\dim_k e_i(J/J^2)e_j$. There are a finite number of such quivers in fixed dimension n (the number of vertices bounded by n, the number of arrows between any two vertices roughly by n^2). Now we can define an equivalence of representations using their images (which are basic algebras).

Definition. *Two representations ρ_1 and ρ_2 of the algebra R are quiver-equivalent if the quivers associated to algebras $\rho_1(R)$ and $\rho_2(R)$ coincide.*

Let us clarify the question on how many quiver-equivalence classes appear in the family of representations

$$\mathcal{M} = \bigcup_n \{(X, Y) \in mod(R, n) | \operatorname{rk} Y = n - 1\}$$

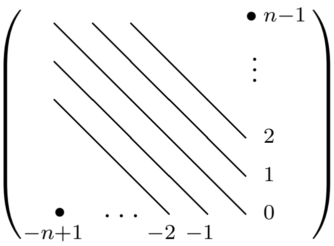
and which quivers are realised.

Theorem **2.4.** *The whole family of representations $\mathcal{M}$ belongs to one quiver-equivalence class. Corresponding quiver consits of one vertex and two loops.*

3 Description of all finite dimensional representations.

Let us construct first a sequence $\rho_n : R \to M_n(\mathbb{C})$ of representations of R for any $n \in \mathbb{N}$. We can assume that the image of one of the generators $Y = \rho_n(y)$ is in normal Jordan form. Let us first find all possible matrices

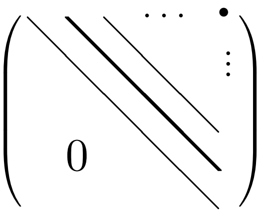
X'th ($X = \rho_n(x)$) in the case when Y is just one Jordan block: $Y = J_n$. We have to find matrices $X = (a_{ij})$ satisfying the relation $[X, Y] = Y^2$. Let $B = (b_{ij})$ be $[X, Y] = [X, J_n] = B$, then $b_{ij} = a_{i+1,j} - a_{i,j-1}$. From the condition $B = Y^2$ it follows that $b_{ij} = 0$ if $i \neq j - 2$ and $b_{ij} = 1$ if $i = j - 2$. For convenience we use the following numeration of diagonals: main diagonal has number 0, upper diagonals have positive numbers $1, 2, \ldots, n-1$ and lower diagonals have negative numbers $-1, -2, \ldots, -n+1$:



Note that b_{ij} is the difference between neighboring elements in the diagonal under the diagonal where b_{ij} appears. Hence, the first condition above means that in the matrix X elements of any diagonal with number k for $k \neq 1$ coincide and are zero for $k < 0$. From the second condition it follows that the elements of the first upper diagonal form an arithmetic progression with difference 1: $a + 1, \ldots, a + n - 1$.

Therefore we have the following sequence of representations:

$$Y_n = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{pmatrix}, \quad X_n = \tag{1}$$



Here and below we will draw a diagonal as a continuous line if all its elements coincide and as a thick line if its elements form am arithmetic progression with difference one.

Note that this family of representations can also be obtained by the following way. One can mention that the matrix

$$X^0 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & \mathbf{0} & \\ & & 0 & 3 & & \\ & & & \ddots & \ddots & \\ & \mathbf{0} & & & 0 & n-1 \\ & & & & & 0 \end{pmatrix} \tag{2}$$

satisfies the relation $[X^0Y] = Y^2$ for $Y = J_n$. On the other hand a matrix $X = X^0 + M$ satisfies the relation $[X, Y] = Y^2$ if and only if M commutes with $Y = J_n$. Any matrix having only one non-zero diagonal with equal elements on it commutes with $Y = J_n$. Hence we obtain the same family of representations.

Consider now the general case when the Jordan normal form of Y contains several Jordan blocks:

$$Y = \begin{pmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_m} \end{pmatrix}$$

Let us cut an arbitrary matrix X into the square and rectangular blocks of corresponding size:

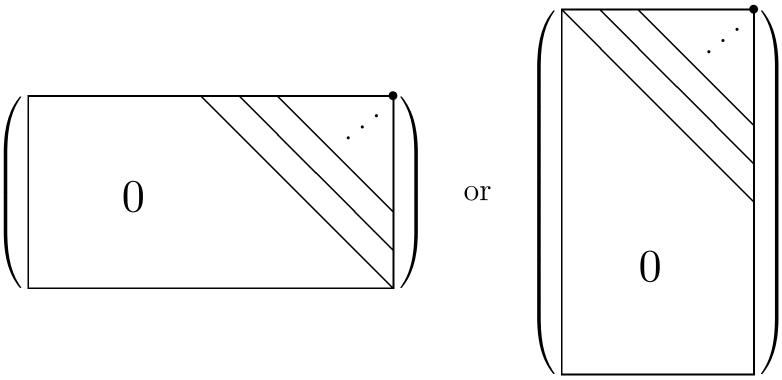
$$X = \left(\begin{array}{|c|c|c|c|} \hline \boxed{A_{11}} & A_{12} & & A_{1m} \\ \hline & \boxed{A_{22}} & & \\ \hline & & \ddots & \\ \hline A_{m1} & A_{m2} & & \boxed{A_{mm}} \\ \hline \end{array}\right)$$

Then we can describe the structure of the matrix $B = [X, Y] = XY - YX$ by the following way:

$$B = \left(\begin{array}{|c|c|c|c|} \hline \boxed{[A_{11}]} & [A_{12}] & & [A_{1m}] \\ \hline & \boxed{[A_{22}]} & & \\ \hline & & \ddots & \\ \hline [A_{m1}] & [A_{m2}] & & \boxed{[A_{mm}]} \\ \hline \end{array}\right), \quad \text{where } [A_{ij}] = A_{ij}J_i - J_jA_{ij}.$$

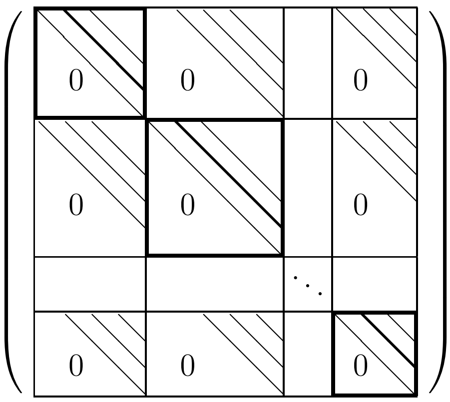
From the condition $B = Y^2$ we have that $[A_i, J_i] = J_i^2$ and hence A_i is the same as in the previous case when Y was just a Jordan block and $A_{ij}J_i -$

$J_j A_{ij} = 0$ for $i \neq j$. The latter condition means that A_{ij} has the following structure.



The elements of any diagonal here are equal and they are equal to zero below the upper diagonal of maximal length (the matrix is non-square in general). As a result we have the following family of representations:

$$Y_n = \begin{pmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_m} \end{pmatrix}, \quad X_n =$$



$$. \tag{3}$$

We mean that the elements of each diagonal marked as a strip are equal to each other while the elements of different diagonals can be different.

We have proved the following theorem

THEOREM **3.1.** *There exist a finite dimensional representations of RIT of any dimension. Description of the complete set of them (subject to the Jordan form of Y) are given by (3).*

From this description it immediately follows:

LEMMA **3.1.** *All irreducible representations of R are one-dimensional and* *have form $Y = 0$, $X = a \in \mathbb{C}$; all completely reducible representations are*

$$Y_n = 0, \quad X_n = \begin{pmatrix} a_1 & & & \\ & a_2 & & 0 \\ 0 & & \ddots & \\ & & & a_n \end{pmatrix}.$$

4 Residually finite dimensionality of R

Let us consider now the main sequence of representation constructed in §2:

$$Y_n = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \quad X_n^0 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & 0 & \\ & & 0 & 3 & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & n-1 \\ & & & & & 0 \end{pmatrix}.$$

Denote it by ε_n. As was actually shown in §2, all representations (1) corresponding to Yth with one Jordan block could be obtained from ε_n by the following automorphism of R, $\varphi : R \longrightarrow R : x \mapsto x + a, y \mapsto y$ where $a \in R$ such that $[a, y] = 0$.

In addition to the usual equivalence relation on the representations given by simultaneous conjugation of matrices from which one obtains the isomorphism classes of modules, we introduce here one more equivalence relation.

Definition 4.1. We say that two representation of the algebra R ρ' and ρ'' *are equivalent* more precisely, let us introduce the following equivalence relations on the representations of R $\rho' \sim \rho'' \iff \exists \varphi \in \mathrm{Aut(R,R)} : \rho'\varphi = \rho''$.

Then we can state that any representation of the type (1) corresponding to Yth with one Jordan block is equivalent to ε_n for some n.

We will prove now that the sequence of representations ε_n asymptotically is faithful.

THEOREM **4.1.** *Let ε_n be the sequence of representations of R as above. Then $\bigcap_{n=0}^{\infty} \ker \varepsilon_n = 0$.*

Let us divert first and describe the computation of a linear basis of R.

LEMMA **4.1.** *The system of monomials $y^n x^m$ form a basis of algebra R as a linear space over $\mathbf{k}$.*

P r o o f. We shall use here the method of construction of the linear basis of an algebra, given by relations based on the construction of a Gröbner (standard) basis of an ideal.

Let $A = \mathbf{k}\langle X\rangle/I$. The first essential point is to fix an ordering on the semigroup ß $= \langle X\rangle$. We fix some linear ordering in the set X. Then we have to extend it to an *admissible* ordering on ß, i.e. it has to satisfy the conditions:

1) if $u, v, w \in$ ß and $u < v$ then $uw < vw$ and $wu < wv$

2) the descending chain condition (d.c.c.): there is no infinite properly descending chain of elements of ß.

We shall use the *degree-lexicographical* ordering in the semigroup ß, namely

$$\forall u = x_{i_1} \dots x_{i_n}, v = x_{j_1} \dots x_{j_k} \in \text{ß} \quad \text{we say} \quad u > v, \quad \text{when} \quad deg\, u > deg$$

$$\text{or} \quad deg\, u = deg\, v \quad \text{and} \quad \exists l, \quad \text{for which} \quad x_{i_m} = x_{j_m}\, \forall m < l \quad \text{and} \quad x_{i_l} > x_{j_l}.$$

This ordering is admissible.

Denote by $\bar{f}$ the highest term of polynomial $f \in \mathbf{k}\langle\mathbf{X}\rangle$ with respect to introduced above order.

Definition 4.2. Let say that subset $G \in I, I \lhd \mathbf{k}\langle\mathbf{X}\rangle$ is a *Groebner basis* of ideal if the set of highest terms of elements of G generates an ideal of highest terms of $I : id\{\bar{G}\} = \bar{I}$.

Definition 4.3. We will say that monomial $u \in \langle X\rangle$ is *normal* if it does not contain as a subword any highest term of element of the ideal I.

From these two definitions it is clear that normal monomial is a monomial which does not contain any highest term of element of Groebner basis of ideal I.

It is easy, but useful fact that $\mathbf{k}\langle\mathbf{X}\rangle$ is isomorphic to the direct sum $I \oplus \langle N\rangle_{\mathbf{k}}$, as a linear space over $\mathbf{k}$, where $\langle N\rangle_{\mathbf{k}}$ is the linear span of the set of normal monomials from $\langle X\rangle$ with respect to the ideal I. We claim here also that the set of normal words form a linear basis. Hence given a Gröbner basis G of an ideal I, we can construct a linear basis of $A = \mathbf{k}\langle\mathbf{X}\rangle/\mathbf{I}$ as a set of all irreducible (with respect to G) monomials.

We say that algebra is *standard finitely presented* if it is presented by its Gröbner basis.

P r o o f. (of the theorem 4.1.)

We are going to show that $\varepsilon_n(f) \neq 0$ for $n \geqslant 2 \deg f$. Suppose that n is sufficiently large and $\varepsilon_n(f)$ is zero and get a contradiction. Denote $\deg f$ by l, and let $f = f_1 + ... + f_l$ be a decomposition of f on the homogeneous components of degrees $i = 1, ..., l$ respectively. Let us compute first the matrix which is the image of an arbitrary monomial $y^k x^m$. Image of the monomial x^m, $\varepsilon(x^m)$ is a matrix with vector $[1{\cdot}2{\cdot}...{\cdot}m, 2{\cdot}3{\cdot}...{\cdot}(m{+}1), ...]$ on the upper diagonal number m at the above numeration and zeros elsewhere, $\varepsilon(y^k)$ acts on matrix by the moving up all rows on k steps. We can see from here that matrix corresponding to the polynomial $y^k x^m$ can have only one nonzero diagonal, in above numeration it is the upper diagonal number $m+k$, and vector in this diagonal is following: $[(k+1)...(m+k), (k+2)...(m+k+1), ...]$.

Hence for

$$f = f_l = \sum_{k+m=l} a_{k,m} y^k x^m = \sum_{r=0}^{l} a_r y^{l-r} x^r$$

we shall have the sum of matrices $\sum_{r=0}^{l} a_r M_r$, where M_r has the vector $(P(0), ..., P(j)$:

$$\left(\prod_{i=1}^{r} (l-r+i), \prod_{i=1}^{r} (l-r+i+1), \dots, \prod_{i=1}^{r} (l-r+i+j), \dots \right)$$

on the diagonal number l (all other diagonals are zero). The number standing on the j-th place of this diagonal is the value in j of a polynomial

$$P(j) = (l-r+j) \cdot ... \cdot (l+j-1)$$

of degree exactly r. Therefore the sum $\sum_{r=0}^{l} a_r M_r$ has a polynomial of j of degree $N = \max\{r : a_r \neq 0\}$ on the diagonal number l. Since any polynomial of degree N has at most N zeros we arrive to a contradiction in the case when lth diagonal has length more than l. Hence for any $n \geqslant 2 \deg f$, $\varepsilon_n(f) \neq 0$.
■

Let recall that an algebra R *residually has some property* $\mathcal{P}$ means that there exists a system of equivalence relations τ_i on R with trivial intersection, and such that in the quotient of R by any τ_i property $\mathcal{P}$ holds.

From theorem 1 we get the following corollary considering equivalence relations modulo ideals $\ker \varepsilon_n$.

Corollary **4.1.** *Algebra R is residually finite dimensional.*

5 Infinite dimensional simple modules

Using the above results on finite dimensional modules over R we give here a description of infinite dimensional simple modules. It happens that infinite dimensional simple modules of finite length over R do exist. The most important step to investigate them is a description of simple modules.

First we will give just an example of infinite dimensional simple module.

PROPOSITION **5.1.** *(Left) ideal generated by polynomial $1 + y$ in R is maximal and has an infinite codimension.*

Proof. If one suppose that the left ideal $I = id_L(1 + y)$ generated by $1+y$ has a finite codimension, than by lemma 1 y^n have to be contained in I for some degree n. Indeed, for any finite dimensional representation ρ_n, $\rho_n(y)$ have to be nilpotent, for the representation on the R/I it means that we have to had $y^nR \subset I$. Since R contains the unit, $y^n \in I$. But $1+y$ and y^n together generate all R (there exist $u, v \in \mathbf{k}[y]$ such that $u(1+y)+vy^n = 1$) and we get contradiction with maximality of I. ■

Now we will give some kind of description of all infinite dimensional simple modules.

THEOREM **5.1.** *An ideal I is maximal (left) ideal in R of infinite codimension if and only if it contains some element of the form $1 + ry$, $r \in R$.*

Proof. We divide the proof of the theorem for two lemmas which correspond to 'if' and 'only if' parts. Only nuance is that in lemma 1 we need not maximality of an ideal, we use only infinite codimension condition.

LEMMA **5.1.** *Let I be a left ideal in R and $I \ni 1+ry$ for some $r \in R$. Then I has infinite codimension.*

Proof. We intend to show that the row $(1 + ry, y^n)$ is left unimodular in R for any $n \in \mathbb{N}$ and $r \in R$ or equivalently ${}_R\langle 1 + ry, y^n\rangle = R$. Denote $J = {}_R\langle 1 + ry, y^n\rangle$. Let us prove that $y^{n-1} \in J$. Indeed since $1 + ry \in J$ we have that $y^{n-1}(1 + ry) = y^{n-1} + y^{n-1}ry \in J$. Since $y^{n-1}ry$ has form sy^n for some $s \in R$ and $y^n \in J$, we have that $y^{n-1}ry \in J$ and therefore $y^{n-1} = y^{n-1}(1 + ry) - y^{n-1}ry \in J$. Repeating these arguments for ${}_R\langle 1 + ry, y^{n-1}\rangle$, ${}_R\langle 1+ry, y^{n-2}\rangle$ etc. we consequently obtain that $y^{n-1}, y^{n-2}, \ldots, 1 \in J$. Hence $J = R$.

Therefore $y^n \notin I$ for any $n = 0, 1, \ldots$ since $I \neq R$. As we have shown above finite codimension of a left ideal K in R implies the inclusion $y^n \in K$ for some $n \in \mathbb{N}$. Thus the codimension of I is infinite. ■

LEMMA **5.2.** *Let I be a maximal left ideal in R of infinite codimension. Then I contains an element of the form $1 + ry$ for some $r \in R$.*

Proof. Any element $u \in R$ admits a unique representation of the form $u = p_u + v_uy$, where $p_u \in \mathbf{k}\langle x\rangle$ and $v_u \in R$. Consider the set of polynomials p_u for $u \in I$. A priori two cases are possible:

Case 1: $p = \mathrm{GCD}\{p_u : u \in I\} \neq 1$;

Case 2: $p = \mathrm{GCD}\{p_u : u \in I\} = 1$.

Let us show that only Case 2 is possible. In Case 1 there exists a polynomial $p \in \mathbf{k}\langle x \rangle$ of degree $\geqslant 1$ such that $p \big| p_u$ for any $u \in I$. Then any $u \in I$ can be rewritten in the form $wp + vy$ for some $w, v \in R$. Therefore $I \subset J$, where $J = {}_R\langle p, y \rangle$. One can easily verify that codimension of J is equal to the degree of p and therefore is finite. Hence I is contained in a maximal left ideal of finite codimension (e.g. in any maximal left ideal containing J). Since I is itself a maximal left ideal of R, we obtain that I has infinite codimension, which is a contradiction.

In Case 2 there exist $u_1, \dots, u_n \in I$ and $\alpha_1, \dots, \alpha_n \in \mathbf{k}\langle x \rangle$ such that $\alpha_1 p_{u_1} + \ldots + \alpha_n p_{u_n} = 1$. Consider the corresponding linear combination of u_j: $u = \alpha_1 u_1 + \ldots + \alpha_n u_n \in I$. If we substitute $u_j = p_{u_j} + v_{u_j} y$, we get $u = 1 + (\alpha_1 v_{u_1} + \ldots + \alpha_n v_{u_n})y$. Hence $u \in I$ has the form $1 + ry$ for $r \in R$. ∎

Lemmas 1 and 2 immediately imply Theorem 1.

∎

6 Analogue of the Gerstenhaber theorem for commuting matrices

In this section we intend to prove an analog of the Gerstenhaber theorem about the dimensions of images of representations of two-generated algebra of commutative polynomials. Theorem of Gerstenhaber says that any algebra generated by two matrices $A, B \in M_n(k)$ of size n which commute $AB = BA$ has dimension not exceeding n. We consider instead of commutativity the relation $XY - YX = Y^2$ and prove the following

Theorem. *Let $X, Y \in M_n(k)$ be matrices of the size n over the field k, satisfying the relation $XY - YX = Y^2$ and Y has in Jordan normal form one full block. Denote $\mathcal{A}$ the algebra generated by X and Y. Then for odd $n = 2m+1$, $\dim \mathcal{A} = \frac{(n+1)^2}{4}$ and for even $n = 2m$, $\dim \mathcal{A} = \frac{n(n+2)}{4}$.*

Proof. In the proof of Theorem 1 we already computed the matrices, which are images of monomials $y^k x^l$, in case when Y has one block in the Jordan form, $\varepsilon(y^k x^l)$ in the above notation. Now we should calculate the dimension of the linear span of matrices $\varepsilon(y^k x^l)$. Let us recall how they look like. The matrix $\varepsilon(y^{l-r} x^r)$ on the l-th upper diagonal has a vector $(P(0), P(1), \dots, P(j), \dots)$, where

$$P(j) = (l - r + j) \dots (l + j - 1) = \prod_{i=1}^{r} (l + j - r + i)$$

and zeros elsewhere. In the j-th place of the l-th diagonal we have a value of a polynomial of degree exactly r. Those diagonals which have number less then then the number of elements in it give the impact to the dimension equal to the dimension of the space of polynomials of corresponding degree. When the diagonals become shorter (the number of elements less then the number of the diagonal) then the impact to the dimension of this diagonal equals to the number of the elements in it. Thus, if $n = 2m + 1$, $\dim \mathcal{A} = 1 + \cdots + m + (m + 1) + m + \cdots + 1 = (m + 1)^2 = \frac{(n+1)^2}{4}$. When $n = 2m$, we have $\dim \mathcal{A} = 1 + \cdots + m + m + \cdots + 1 = m(m + 1) = \frac{n(n+1)}{4}$. ■

7 Tameness results for families of representations

Let us consider the variety of R-module structures on k^n and denote it by $mod(R, n)$. Each such structure corresponds to a k-algebra homomorphism $R \to M_n(k)$, or equivalently to a pair of matrices (X, Y), $X, Y \in M_n(k)$, satisfying the relation $XY - YX = Y^2$. The group $GL_n(k)$ acts on $mod(R, n)$ by simultaneous conjugation and orbits of this action are exactly the isomorphism classes of n-dimensional A-modules. Denote this orbit of a module M or a pair of matrices (X, Y) as $O(M)$ or $O(X, Y)$ respectively. Consider also the following strata. Let $U(Y, P)$ be the set of all pairs (X, Y) satisfying the relation, where Y has a fixed Jordan form. Here P is a partition of n, which defines the Jordan form. We will write $U(Y, n)$ for the stratum corresponding to Y with one Jordan block.

Clearly

$$U(Y, P) = \cup_{X;\, Y \text{with fixed Jordan form}} O(X, Y).$$

Our goal now is to show that for any fixed dimension n there are infinitely many orbits $O(X, Y)$ in the stratum $U(Y, n)$. This will mean that for any dimension n there are infinitely many non-isomorphic indecomposable representations, i.e. in any dimension R has infinite representation type.

Moreover we will be able to state that the representation type is *tame* for the family of reps corresponding to Y with full Jordan block. Namely we will prove that in the stratum $U(Y, n)$ isoclasses of indecomposables could be parameterized by two parameters (for any fixed n). Since number of parameters does not depend of n, this family of representations corresponding to the stratum $\cup_{n \in N} U(Y, n)$ is *tame*, i.e. could be parameterized by the finite number of parameters.

Concerning any stratum $U(Y, P)$ for arbitrary partition P of n we could say that it should be *tame* but the number of parameters growth with n and hence complete family of reps is *wild*.

We will consider here the case of the simplest partition $P = (n)$, that is the case of one full block in the Jordan form of Y.

We can restrict ourself by consideration of the action of GL_n on the section of stratum $U(Y, n)$ consisting of pairs (X, Y), where Y is fixed (and has as a Jordan form the full block). Denote this subset of $U(Y, n)$ by W_Y. Corresponding section of the orbit $O(X, Y)$ denote by $O_Y(X) = O(X, Y) \cap W_Y$.

Consider the induced action on W_Y of the subgroup $G = SL_n \cap Z(Y)$, where $Z(Y)$ is a centralizer of Y: $Z(Y) = \{C \in GL_n | CY = YC\}$. For the section of the orbit of X $O_Y(X)$ we will also write $O_Y(X, G)$. Group G can be presented as

$$G = \{E + \alpha_1 Y + \alpha_2 Y^2 + ... + \alpha_{n-1} Y^{n-1}\},$$

due to our description of the centralizer of Y in §3. This group acts on the affine space of the dimension n:

$$W_Y = \{\lambda E + X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1} Y^{n-1}\}$$

here λ is the eigenvalue of X and X^0 is the matrix defined in §3:

$$X^0 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 2 & & 0 & \\ & & 0 & 3 & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & n-1 \\ & & & & & 0 \end{pmatrix}$$

Let fix first the eigenvalue λ ($\lambda = 0$), we get then the space of dimension $n - 1$:

$$W'_Y = \{X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1} Y^{n-1}\}.$$

We intend to calculate now the dimension of the orbit $O_Y(X, G)$ of X with fixed eigenvalue $\lambda = 0$ under G action.

Let us consider the map $\varphi : G \longrightarrow W'_Y$ defined by this action: $\varphi(C) = CXC^{-1}$, then $\mathrm{Im}\varphi = O_Y(X, G)$. We will calculate now the rank of Jacobian of this map and show that it is constant on G and equal to $n - 2$.

7.1 Calculation of the rank of Jacobian

THEOREM **7.1.** *Let G be an intersection of SL_n with centralizer of Y. Consider the action of this group on the affine space $W'_Y = \{X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1} Y^{n-1}\}$ by conjugation. Then the rank of the Jacobian of the map $\varphi : G \longrightarrow W'_Y$ is equal to $n - 2$ in any point $C \in G$.*

P r o o f. Consider $d\varphi(C)(\Delta) = (C+\Delta)^{-1}X(C+\Delta) - C^{-1}XC$, where

$$C = E + \alpha_1 Y + \alpha_2 Y^2 + ... + \alpha_{n-1}Y^{n-1},$$

$$X = X^0 + c_1 Y + c_2 Y^2 + ... + c_{n-1}Y^{n-1},$$

$$\Delta = \beta_1 Y + \beta_2 Y^2 + ... + \beta_{n-1}Y^{n-1}.$$

Let us present $(C+\Delta)^{-1}$ by the following way:

$$(C+\Delta)^{-1} = (E + \Delta C^{-1})^{-1}C^{-1} =$$

$$(E - \Delta C^{-1} + \text{lower order terms of}\,\Delta)C^{-1}.$$

Then

$$(C+\Delta)^{-1}X(C+\Delta) - C^{-1}XC =$$

$$(E - \Delta C^{-1} + \text{lower order terms of}\,\Delta)C^{-1}X(C+\Delta) - C^{-1}XC =$$

$$-\Delta C^{-2}XC + C^{-1}X\Delta + \text{lower order terms of}\,\Delta =$$

$$(-\Delta C^{-1}\cdot C^{-1}X + C^{-1}X\cdot \Delta C^{-1})C + \text{lower order terms of}\,\Delta.$$

Denote $\tilde{\Delta} := \Delta C^{-1}$ and $\tilde{X} := C^{-1}X$. Obviously multiplication by C preserves the rank and rank of linear map $d\varphi(C)(\Delta)$ is equal to the rank of the map $T(\tilde{\Delta}) = [\tilde{\Delta}, \tilde{X}]$.

Here again $\tilde{\Delta}$ has a form

$$\tilde{\Delta} = \gamma_1 Y + \gamma_2 Y^2 + ... + \gamma_{n-1}Y^{n_1}.$$

Let us compute commutator of $\tilde{X}$ with Y^k, taking in account that C^{-1} is a polynomial on Y, hence commute with Y^k and also the relation in the RIT algebra:

$$XY^k - Y^kX = kY^{k+1}.$$

We get $\tilde{X}Y - Y\tilde{X} = C^{-1}XY^k - Y^kC^{-1}X = C^{-1}(XY^k - Y^kX) = C^{-1}kY^{k+1}$. Hence

$$\tilde{X}p(Y) - p(Y)\tilde{X} = C^{-1}Y^2p'(Y)$$

for arbitrary polynomial p. Applying this for the polynomial $\tilde{\Delta}$ we get

$$T(\tilde{\Delta}) = [\tilde{\Delta}, \tilde{X}] = \sum_{k=1}^{n-2} \gamma_k k C^{-1} Y^{k+1},$$

hence this linear map has rank $n-2$. ∎

Now by the theorem on locally flat map ([5]) we have that locally Imφ has rank $n-2$, that is for any point of G there exists a neighborhood σ of this point such that $\varphi(\sigma)$ has dimension $n-2$.

Since the affine space G is separable, we can choose countable covering from all $\varphi(\sigma)$, and hence cover the orbit by countably many spaces of dimension $n-2$.

The space $W'_Y = \cup_{X,\lambda=0} O_Y(X,G)$ could not be covered by countably many spaces of dimension $n-2$, hence there are uncountably many orbits $O_Y(X,G)$ and we get

Corollary **7.1.** *Let Y be the matrix with fixed Jordan structure consisting of one full block. Then there are uncountably many isomorphism classes of indecomposable modules in the stratum $U(Y,n)$. Hence for any fixed n algebra R has an infinite representation type.*

From the theorem 1 we could also deduce the following statement concerning parameterization of isoclasses of indecomposable modules.

Corollary **7.2.** *Let $U(Y,n)$ be the stratum as above. Then the set of isomorphism classes of indecomposable modules from $U(Y,n)$ could be parameterized by two parameters.*

Particularly number of parameters does not depends of n in this case.

Proposition **7.1.** *Parameters μ amd λ are invariant under the action of G on the set of matrices* $\left\{ \begin{pmatrix} \lambda & \mu+1 & & & * & \\ & \lambda & \mu+2 & & & \\ & & \lambda & \mu+3 & & \\ & & & \ddots & \ddots & \\ & \mathit{0} & & & \lambda & \mu+n-1 \\ & & & & & \lambda \end{pmatrix} \right\}$.

Proof. Direct calculation of ZMZ^{-1} for $Z \in G$ shows that elements in first two diagonals of M will be preserved. ∎

Hence from the corollary 2 and proposition 1 we have the following classification result for the family of reps with one full Jordan block for Y, or equivalently with the condition $n - rkY = 1$.

Theorem **7.2.** *Let $P_{\lambda,\mu}$ denotes the pair $(X_{\lambda,\mu}, Y)$, where*

$$X_{\lambda,\mu} = \begin{pmatrix} \lambda & \mu+1 & & & & 0 \\ & \lambda & \mu+2 & & & \\ & & \lambda & \mu+3 & & \\ & & & \ddots & \ddots & \\ & 0 & & & \lambda & \mu+n-1 \\ & & & & & \lambda \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & & & & 0 \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}.$$

Every pair X, Y is conjugate to $P_{\lambda,\mu}$ for some λ, μ. No two pairs $P_{\lambda,\mu}$ with different (λ, μ) are conjugate.

These two parameters λ, μ for reps of any dimension gives the following tameness result.

THEOREM **7.3.** *The subset of all finite dimensional representations corresponding to Y with full Jordan block, or equivalently defined by the condition $n - rkY = 1$ is tame (dimension of representations supposed to be not fixed here).*

We conjecture here that there exist tameness results also for the families of representations with other fixed type of block structure. It could be proved when for fixed 'type of partition' the number of parameters does not growth with n.

8 Consequences for the Milnor conjecture

The above investigation of the finite-dimensional representations of RIT algebras lead to the discovery of surprising connection between this theory and Milnor's conjecture of 70th about the existence of left-invariant affine structures on solvable Lie groups. Knowledge on representations of RIT algebras presented here gives a solution of series of question related to Milnor's problem. More precisely it will be discussed in a separate paper.

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