Welschinger invariants of toric Del Pezzo surfaces with non-standard real structures

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Abstract

The Welschinger invariants of real rational algebraic surfaces are natural analogues of the Gromov-Witten invariants, and they estimate from below the number of real rational curves passing through prescribed configurations of points. We establish a tropical formula for the Welschinger invariants of four toric Del Pezzo surfaces, equipped with a non-standard real structure. Such a formula for real toric Del Pezzo surfaces with a standard real structure (i.e., naturally compatible with the toric structure) was established by Mikhalkin and the author. As a consequence we prove that, for any real ample divisor D on a surfaces Σ under consideration, through any generic configuration of $c_1(\Sigma)D-1$ generic real points there passes a real rational curve belonging to the linear system $|D|$.

Introduction

The Welschinger invariants [22, 23] play a central role in the enumerative geometry of real rational curves on real rational surfaces, providing lower bounds for the number of real rational curves passing through generic, conjugation invariant configurations of points, whereas the number of respective complex curves (Gromov-Witten invariant) serves as an upper bound. Methods of the tropical enumerative geometry, developed in [14, 15, 18], allowed one to compute and estimate the Welschinger invariants for the real toric Del Pezzo surfaces, equipped with the standard real structure [8, 10, 19]: the plane $\mathbb{P}^2$, the plane $\mathbb{P}^2_k$ with blown up $k=1$, 2, or 3 real points, and the quadric $(\mathbb{P}^1)^2$.

Along the Comessatti's classification of real rational surfaces [1, 2] (see also [12]), besides the standard real toric Del Pezzo surfaces, there are five more types, having a non-empty real points set, which we call *non-standard* and denote as $\mathbb{S}^2$, the quadric whose real point set is a sphere, $\mathbb{S}^2_{1,0},\mathbb{S}^2_{2,0},\mathbb{S}^2_{0,2}$, the sphere with blown up one or two real points, or a pair of conjugate imaginary points, respectively, and, at last,

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$(\mathbb{P}^1)^2_{0,2}$, the standard real quadric with blown up two imaginary conjugate points. In the present paper we derive the tropical formula for the Welschinger invariants of $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, and $\mathbb{S}^2_{0,2}$[1]. The surface $(\mathbb{P}^1)^2_{0,2}$ will be considered in a forthcoming paper.

As application we prove the positivity of some Welschinger invariants of the four considered surfaces, which immediately implies the existence of real rational curves belonging to given linear systems and passing through generic configurations of suitable number of real points.

We notice that the available technique of the tropical enumerative geometry applies only to toric surfaces, and among them the Welschinger invariant is well-defined only for unndodal[2] Del Pezzo surfaces, i.e., the plane, the plane with blown up $k = 1, 2$ or 3 points, and the quadric.

Welschinger invariants. For the reader's convenience, we recall the definition of Welschinger invariants. Let Σ be a real toric Del Pezzo surface with a non-empty real part, $\mathcal{L}$ a very ample real line bundle on Σ, and let non-negative integers r', r'' satisfy

$$r' + 2r'' = -c_1(\mathcal{L})K_\Sigma - 1 \ . \tag{0.1}$$

Denote by $\Omega_{r''}(\Sigma, \mathcal{L})$ the set of configurations of $-c_1(\mathcal{L})K_\Sigma - 1$ distinct points of Σ such that r' of them are real and the rest consists of r'' pairs of imaginary conjugate points. The Welschinger number $W_{r''}(\Sigma, \mathcal{L})$ is the sum of weights of all the real rational curves in the linear system $|\mathcal{L}|$, passing through a generic configuration $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma, \mathcal{L})$, where the weight of a real rational curve C is 1 if it has an even number of real solitary nodes, and is -1 otherwise. Since the complex structure of Σ determines a symplectic structure, which is generic in its deformation class, by Welschinger's theorem [22, 23], $W_{r''}(\Sigma, \mathcal{L})$ does not depend on the choice of a generic element $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma, \mathcal{L})$ (a simple proof of this fact can be found in [9]). The definition immediately implies the inequality

$$|W_{r''}(\Sigma, \mathcal{L})| \le R_{\Sigma,\mathcal{L}}(\overline{\boldsymbol{p}}) \le N_{\Sigma,\mathcal{L}} \ , \tag{0.2}$$

where $R_{\Sigma,\mathcal{L}}(\overline{\boldsymbol{p}})$ is the number of real rational curves in $|\mathcal{L}|$ passing through a generic configuration $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma, \mathcal{L})$, and $N_{\Sigma,\mathcal{L}}$ is the number of complex rational curves in $|\mathcal{L}|$, passing through generic $-c_1(\mathcal{L})K_\Sigma - 1$ points in Σ.

A tropical calculation of the Welschinger invariant. Our approach to calculating the Welschinger invariants is quite similar to that in [8, 19], and it heavily

[1] On the real toric Del Pezzo surfaces with empty real point set there are no real rational curves, and thus, Welschinger invariants vanish.

[2] Like in [8, 10] "unnodal" means the absence of $(-n)$-curves, $n \ge 2$.

relies on the enumerative tropical algebraic geometry developed in [14, 15, 18]. More precisely, we replace the complex field $\mathbb{C}$ by the field $\mathbb{K} = \bigcup_{m\ge 1} \mathbb{C}\{\{t^{1/m}\}\}$ of the complex, locally convergent Puiseux series endowed with the standard complex conjugation and with a non-Archimedean valuation

$$\mathrm{Val} : \mathbb{K}^* \to \mathbb{R}, \quad \mathrm{Val}\left(\sum_k a_k t^k\right) = -\min\{k \ : \ a_k \ne 0\}\ .$$

A rational curve over $\mathbb{K}_{\mathbb{R}}$, belonging to a linear system $|\mathcal{L}|_{\mathbb{K}}$ and passing through a generic configuration $\overline{\boldsymbol{p}} \in \Omega_{r''}(\Sigma_{\mathbb{K}}, \mathcal{L})$, is viewed as an equisingular family of real rational curves in Σ over the punctured disc. We construct an appropriate limit of the family of surfaces and embedded curves at the disc center. The central surface is usually reducible, and the adjacency of its components is encoded by a tropical curve in the real plane, which passes through the configuration $\mathrm{Val}(\overline{\boldsymbol{p}}) \subset \mathbb{R}^2$. The central curve is split into components called *limit curves*. The pair (tropical curve, limit curves) is called the]it tropical limit of the given curve $C \in |\mathcal{L}|_{\mathbb{K}}$.

We precisely describe the tropical limits of real rational curves passing through generic configurations of real points in $\Sigma_{\mathbb{K}}$, then compute the *Welschinger weights* of the respective tropical curves, i.e., the contribution to the Welschinger invariant of the real algebraic curves projecting to the given tropical curve. The result, accumulated in Theorem 1 (section 1.3), represents the Welschinger invariants $W_0(\Sigma, \mathcal{L})$ as the numbers of some combinatorial objects, forming finite discrete sets.

The proof is based on the techniques of [18, 19], both in the determining tropical limits and in the patchworking construction, which recovers algebraic curves over $\mathbb{K}$ from their tropical limits. We should like to remark that the answer rather differs from that for the standard real Del Pezzo surfaces. Namely, in the canonical case, the tropical limits are basically encoded by tropical curves, which are rational and irreducible. In the non-standard case, one obtains a relatively small number of possible tropical curves, which all split into unions of some primitive tropical curves. In contrast, the weights of the tropical curves are large and are defined in a non-trivial combinatorial way.

We also notice that the patchworking theorems from [18, 19] cover our needs in the present paper. In contrast, the determination of tropical limits requires to deal with extra difficulties, caused by the fact that the *generic* configurations of real points on the surfaces under consideration project by $\mathrm{Val} : (\mathbb{K}^*)^2 \to \mathbb{R}^2$ to *non-generic* configurations in $\mathbb{R}^2$ (cf. a similar problem in [18]).

Applications to enumerative geometry. From Theorem 1 we immediately derive the positivity of the Welschinger invariants in the considered situations, which in view of (0.2) results in Corollary 1, section 1.3, which says that , for any real very ample line bundle $\mathcal{L}$ on a non-standard real toric Del Pezzo surface $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$ and any generic configuration of $-c_1(\mathcal{L})K_\Sigma - 1$ real points on Σ there exists a real rational curve $C \in |\mathcal{L}|$ passing through the given configuration.

A detailed study of the asymptotics of the Welschinger invariants will be performed in a forthcoming paper [11].

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1 Formula for the Welschinger invariants

1.1 Lattice polygons associated with the non-standard real toric Del Pezzo surfaces

The non-standard real toric Del Pezzo surfaces $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, $\mathbb{S}^2_{0,2}$, and $(\mathbb{P}^1)^2_{0,2}$[3] can be associated with the following polygons Δ, respectively (see Figure 1):

- a square $\mathrm{Conv}\{(0,0),(d,0),(0,d),(d,d)\}$, $d \ge 1$,
- a pentagon $\mathrm{Conv}\{(0,0),(0,d),(d-d_1,d),(d,d-d_1),(d,0)\}$, $1 \le d_1 < d$,
- a hexagon $\mathrm{Conv}\{(d_2,0),(0,d_2),(0,d),(d-d_1,d),(d,d-d_1),(d,0)\}$, $1 \le d_1 \le d_2 < d$,
- a hexagon $\mathrm{Conv}\{(0,0),(0,d-d_1),(d_1,d),(d,d),(d,d_1),(d_1,0)\}$, $1 \le d_1 < d$,
- a hexagon $\mathrm{Conv}\{(0,0),(d_1-d_3,0),(d_1,d_3),(d_1,d_2),(d_3,d_2),(0,d_2-d_3)\}$, $1 \le d_3 < d_2 \le d_1$.

For the first four surfaces, the conjugation acts in the torus $(\mathbb{C}^*)^2$ by $\mathrm{Conj}(x,y) = (\overline{y},\overline{x})$, and acts in the tautological line bundle $\mathcal{L}_\Delta$, generated by monomials x^iy^j, $(i,j) \in \Delta \cap \mathbb{Z}^2$, by $\mathrm{Conj}_*(a_{ij}x^iy^j) = \overline{a}_{ij}x^jy^i$, $(i,j) \in \Delta$, resembling the reflection of Δ with respect to the bisectrix $\mathcal{B}$ of the positive quadrant. For the fifth surface, the action in $(\mathbb{C}^*)^2$ is $\mathrm{Conj}(x,y) = (\overline{x}^{-1},\overline{y}^{-1})$, and the action in $\mathcal{L}_\Delta$ is $\mathrm{Conj}_*(a_{ij}x^iy^j) = \overline{a_{i,j}}x^{d_1-i}y^{d_2-j}$, $(i,j) \in \Delta$, resembling the reflection of Δ with respect to its center.

[3] The last surface is mentioned for completeness.

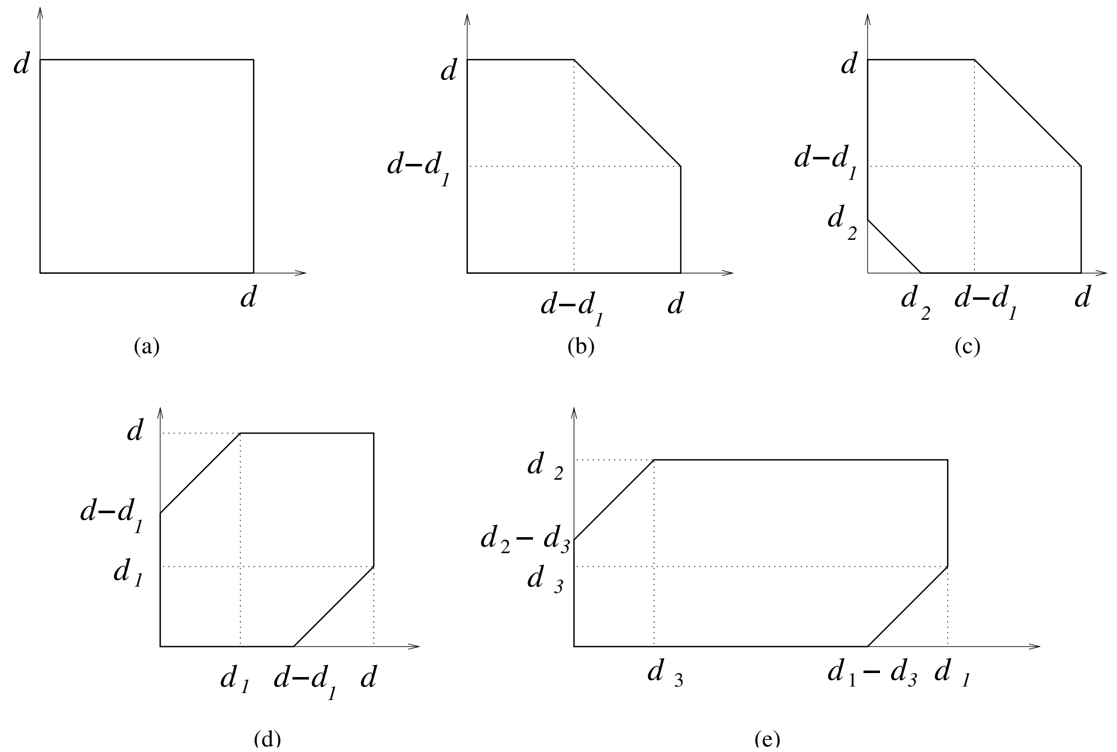


Figure 1: Polygons associated with $\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, $\mathbb{S}^2_{0,2}$, and $(\mathbb{P}^1)^2_{0,2}$

Observe that $-c_1(\mathcal{L}_\Delta)K_\Sigma - 1 = |\partial\Delta| - 1$[4].

1.2 Admissible lattice paths and graphs

Let Δ be one of the four polygons shown in Figure 1(a-d). Denote by $(\partial\Delta)_+$ the union of the sides of Δ which are not orthogonal to $\mathcal{B}$ and lie above $\mathcal{B}$. The integral points divide $(\partial\Delta)_+$ into segments s_i, $1 \le i \le m := |(\partial\Delta)_+|$. An **admissible lattice path** in Δ is a map $\gamma : [0, m] \to \Delta$ such that (see example in Figure 2(a))

- image of γ lies in the upper half-plane supported by $\mathcal{B}$,

- $\gamma(0)$ and $\gamma(m)$ are the two endpoints of $(\partial\Delta)_+$,

- composition of the functional $x + y$ with γ is a strongly increasing function,

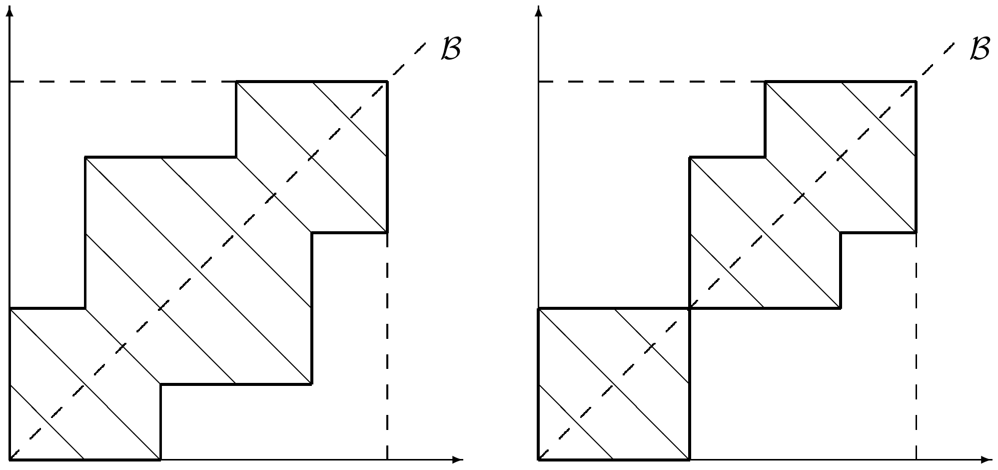
- $\gamma(i) \in \mathbb{Z}^2$, and $\gamma\big|_{[i,i+1]}$ is linear as $i \in \mathbb{Z}$,

- there is a permutation $\tau \in S_{m-1}$ such that $\gamma([i-1, i])$ is a translate of the segment $s_{\tau(i)}$, $i = 1, ..., m$,

- $\gamma([0, m]) \cap \mathcal{B} = (\partial\Delta)_+ \cap \mathcal{B}$.

An admissible lattice path γ determines a γ-admissible subdivision of Δ as follows. The part of Δ between an admissible lattice path γ and its symmetric

[4]Here and further on the symbol $|*|$ applied to lattice segments or lattice broken lines, means the lattice length.



(a) (b)

Figure 2: Lattice paths and subdivisions of Δ

image with respect to $\mathcal{B}$ is divided by the segments, joining integral points on γ with their symmetric images (see Figure 2(a)), and the remaining part of Δ is uniquely divided into parallelograms with integral vertices and Euclidean area 1. Denote the segment, joining the point $\gamma(i)$ with its symmetric image by σ_i, $i = 0, ..., m$.

A **γ-admissible graph** G is defined as follows. First, we describe some subgraph G'. Connected components of G' are lattice segments (or points) $G'_i = [(a_i, i), (b_i, i)] \subset \mathbb{R}^2$, $i = 1, ..., n := |\partial\Delta| - m$, with positive odd weights $w(G'_i)$ such that

- $0 \le a_i \le b_i \le m$ for all $i = 1, ..., n$,
- $a_i \le a_{i+1}$, and in addition $b_i \le b_{i+1}$ if $a_i = a_{i+1}$ as $i = 1, ..., n-1$,
- for all $i = 0, ..., m$,

$$\sum_{(i,j)\in G'_j} w(G'_j) = |\sigma_i| \ , \qquad (1.3)$$

- if $a_i = 0$ or $b_i = m$ then $w(G'_i) = 1$.

We then introduce new vertices $(i - \frac{1}{2}, 0)$, $i = 1, ..., m$, of the graph G and the new arcs, joining any vertex $(i - \frac{1}{2}, 0)$ with the endpoint $(i-1, j)$ of any component G'_j satisfying $b_j = i+1$, and with the endpoint (i, j) of any component G'_j satisfying $a_j = i$. Our final requirement is that the obtained graph G is a tree.

A **marking** of a γ-admissible graph G is a vector $\overline{s} = (s_1, ..., s_n) \in \mathbb{Z}^n$ such that $a_i \le s_i \le b_i$, $i = 1, ..., n$, subject to the following restriction:

$$s_i \le s_{i+1} \quad \text{as far as} \quad a_i = a_{i+1},\ b_i = b_{i+1}\ .$$

At last we define the **Welschinger number**

$$W(\gamma,G,\overline{s})=2^v\prod_{k=0}^m n_k!\left(\prod_{\substack{0\le a\le b\le m\\ c=1,3,5,...}} n_{k,a,b,c}!\right)^{-1},$$

where v is the total valency of those vertices $(i+\frac{1}{2},0)$ of G, for which $|\sigma_i|=|\sigma_{i+1}|$, and

$$n_k:=\#\{i\ :\ s_i=k,\ 0\le i\le m\},\quad n_{k,a,b,c}=\#\{i\ :\ s_i=k,\ a_i=a,\ b_i=b,\ w(G_i')=c\},$$

$$k=0,...,m,\ 0\le a\le k\le b\le m,\ c=1,3,5,...$$

1.3 Main results

Theorem 1 *In the notation of sections 1.1 and 1.2, if $\Sigma=\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, then*

$$W_0(\Sigma,\mathcal{L}(\Delta))=\sum W(\gamma,G,\overline{s})\ , \tag{1.4}$$

where the sum ranges over all admissible lattice paths γ, all γ-admissible graphs G, and all markings $\overline{s}$ of G.

It is an easy exercise to show that there always exist an admissible lattice path and a corresponding admissible graph, and hence

Corollary 1 *In the above notation, for any surface $\Sigma=\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, and any line bundle $\mathcal{L}(\Delta)$, the Welschinger invariant $W_0(\Sigma,\mathcal{L}(\Delta))$ is positive, and through any $-c_1(\mathcal{L}(\Delta))K_\Sigma-1$ generic real points on Σ there passes at least one real rational curve $D\in|\mathcal{L}(\Delta)|$.*

1.4 Examples

1.4.1 Linear systems with an elliptic general member

Let $\Sigma=\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, and let Δ be a respective associated lattice polygon as shown in Figure 1 and such that a general curve in $|\mathcal{L}(\Delta)|$ is elliptic. Then Δ is as depicted in Figure 3.

The Welschinger invariant $W_0(\Sigma,\mathcal{L}(\Delta))$ can be computed by counting rational curves in the pencil of real curves in $|\mathcal{L}(\Delta)|$ passing through $(-c_1(\mathcal{L}(\Delta))K_\Sigma-1)$ generic real points. Integrating along the pencil with respect to the Euler characteristic and noticing that the curves in the pencil have one more real base point, and

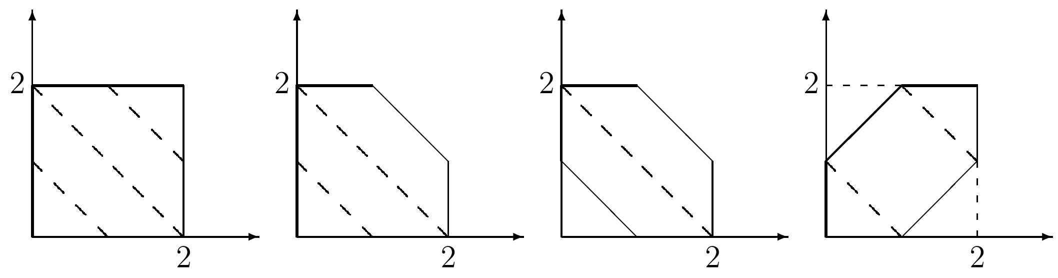


Figure 3: Linear systems with an elliptic general curve

$\chi(\mathbb{R}D)=1$ or -1 according as $D\in|\mathcal{L}(\Delta)|$ is a real rational curve with a solitary or a non-solitary node, we obtain (cf. with the case of plane cubics [10], section 3.1)

$$W_0(\Sigma,\mathcal{L}(\Delta))=-c_1(\mathcal{L}(\Delta))K_\Sigma-\chi(\mathbb{R}\Sigma)\ ,$$

which equals 6, 6, 6, or 4 as $\Sigma=\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, respectively.

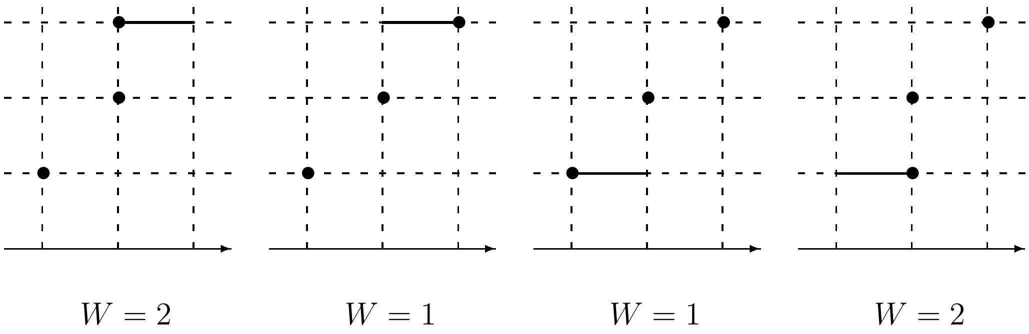
In turn in Theorem 1 we have a unique admissible path γ (fat line in Figure 3) and a unique subdivision of Δ (dashes in Figure 3). The subgraphs G' of the γ-admissible graphs, their markings and Welschinger numbers are shown in Figure 4 (the weight of any component of G' is here 1). The result, of course, coincides with the aforementioned one.

1.4.2 Linear systems of digonal curves

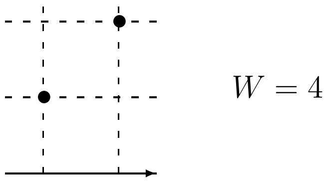
We illustrate Theorem 1 by two more examples, where one can easily write down a closed formula for the Welschinger invariant (a similar computation has been performed for digonal curves on $(\mathbb{P}^1)^2$ [10], section 3.1). Namely, we consider the surfaces $\Sigma=\mathbb{S}^2_{2,0}$ and $\mathbb{S}^2_{0,2}$ and the linear systems associated with the polygons Δ shown in Figure 5(a,b), respectively.

In the case $\Sigma=\mathbb{S}^2_{0,2}$ (see Figure 5(b)) we have a unique admissible path γ going just along $(\partial\Delta)_+$, a unique γ-admissible graph G, and a unique marking (see Figure 5(e)). Hence we obtain $W_0(\mathbb{S}^2_{0,2},\mathcal{L}(\Delta))=4^{d-1}$, d being the length of projection of Δ on a coordinate axis.

In the case $\Sigma=\mathbb{S}^2_{2,0}$, $d>2$, there are two admissible lattice paths γ_1,γ_2 (shown by fat lines in Figure 5(c,d)). The subgraph G' of an admissible graph G, and a marking $\overline{s}$ should look as shown in Figure 5(f), where we denote by k (resp., l) the number of components $[(1,j),(2,j)]$ (resp. $[(2,j),(3,j)]$), and k_1 (resp. l_1) is the number of the marked points $(2,j)$ on components $[(1,j),(2,j)]$ (resp. $[(2,j),(3,j)]$),



(a) $\Sigma = \mathbb{S}^2,\ \mathbb{S}^2_{1.0},\ \text{or}\ \mathbb{S}^2_{2,0}$



(b) $\Sigma = \mathbb{S}^2_{0,2}$

Figure 4: Admissible graphs and markings, I

and where (k, l, k_1, l_1) run over the sets $J(\gamma_1)$ and $J(\gamma_2)$ defined by

$$0 \le k_1 \le k \le d-1, \quad 0 \le l_1 \le l \le d-1, \quad d-k-l = 2a+1,\ a \ge \begin{cases} 1, & \text{if } \gamma = \gamma_1, \\ 2, & \text{if } \gamma = \gamma_2 \end{cases}$$

Here the weights of all the components of G' are equal to 1, except for the one-point component on the middle vertical line, whose weight is $2a+1$ or $2a-1$ according as $\gamma = \gamma_1$ or γ_2. Thus, we obtain

$$W_0(\mathbb{S}^2_{2,0}, \mathcal{L}(\Delta)) = \sum_{i=1}^{2} \sum_{(k,l,k_1,l_1) \in J(\gamma_i)} \frac{(d-1-k_1)!(d-1-l_1)!(k_1+l_1+1)!}{(d-1-k)!(d-1-l)!(k-k_1)!(l-l_1)!k_1!l_1!} .$$

2 Tropical limits of real rational curves: general setting

2.1 Preliminaries

Here we recall definitions and a few facts about tropical curves and tropical limits of algebraic curves over a non-Archimedean field, presented in [7, 5, 14, 15, 17, 18, 19] in more details.

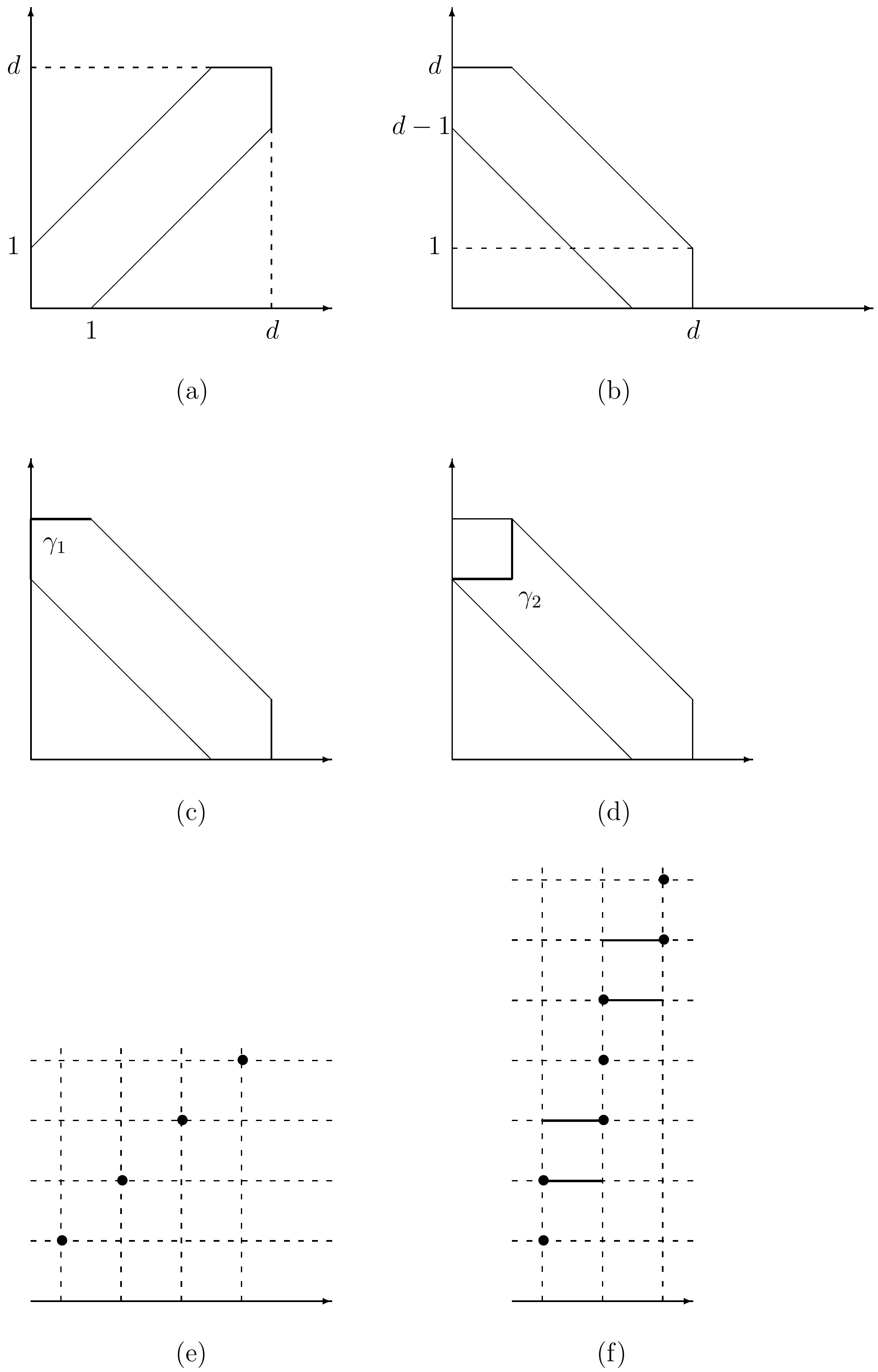


Figure 5: Admissible graphs and markings, II

By Kapranov's theorem the amoeba A_C of a curve $C \in |\mathcal{L}_\Delta|_{\mathbb{K}}$, given by an equation

$$f(x,y) := \sum_{(i,j)\in\Delta} a_{ij}x^iy^j = 0, \quad a_{ij} \in \mathbb{K},\ (i,j) \in \Delta \cap \mathbb{Z}^2, \tag{2.5}$$

with the Newton polygon Δ, is the corner locus of the convex piece-wise linear function

$$N_f(x,y) = \max_{(i,j)\in\Delta\cap\mathbb{Z}^2}(xi + yj + \mathrm{Val}(a_{ij})), \quad x, y \in \mathbb{R}\ . \tag{2.6}$$

In particular, A_C is a planar graph with all vertices of valency ≥ 3.

Take the convex polyhedron

$$\widetilde{\Delta} = \{(i,j,\gamma) \in \mathbb{R}^3\ :\ \gamma \ge -\mathrm{Val}(a_{ij}), \quad (i,j) \in \Delta \cap \mathbb{Z}^2\}$$

and define the function

$$\nu_f : \Delta \to \mathbb{R}, \quad \nu_f(x,y) = \min\{\gamma\ :\ (x,y,\gamma) \in \widetilde{\Delta}\}\ . \tag{2.7}$$

This is a convex piece-wise linear function, whose linearity domains form a subdivision S_C of Δ into convex lattice polygons $\Delta_1, ..., \Delta_N$. The function ν_f is Legendre dual to N_f, and thus, the subdivision S_C is combinatorially dual to the pair $(\mathbb{R}^2, A_C)$. Clearly, A_C and S_C do not depend on the choice of a polynomial f defining the curve C.

We define the *tropical curve*, corresponding to the algebraic curve C, as a balanced graph, supported at A_C, i.e., this is the non-Archimedean amoeba A_C, whose edges are assigned the weights equal to the lattice lengths of the dual edges of S_C. The subdivision S_C can be uniquely restored from the tropical curve A_C.

By the *tropical limit* of a curve C given by (2.5) we call a pair $(A_C, \{C_1, ..., C_N\})$, where C_k, $1 \le k \le N$, is a complex curve on the toric surface $\mathrm{Tor}(\Delta_k)$, associated with a polygon Δ_k from the subdivision S_C, and is defined by an equation

$$f_k(x,y) := \sum_{(i,j)\in\Delta_k} a^0_{ij}x^iy^j = 0\ ,$$

where $a_{ij}(t) = (a^0_{ij} + O(t^{>0})) \cdot t^{\nu_f(i,j)}$ is a coefficient from $f(x,y)$. We call $C_1, ..., C_N$ *limit curves*. Their geometrical meaning is as follows (cf. [18], section 2). By a parameter change $t \mapsto t^M$, $M >> 1$, we can make all the exponents of t in the coefficients $a_{ij} = a_{ij}(t)$ of f integral, and make the function ν_f integral-valued at integral points. The toric threefold $Y = \mathrm{Tor}(\widetilde{\Delta})$ fibers over $\mathbb{C}$ so that Y_t, $t \ne 0$, is isomorphic to $\mathrm{Tor}(\Delta)$, and Y_0 is the union of $\mathrm{Tor}(\Delta_k)$ attached to each other as the polygons of the subdivision S_C. Equation (2.5) defines an analytic surface C in Y such that the curves $C^{(t)} = C \cap Y_t$, $0 < |t| < \varepsilon$, form an equisingular family, and $C^{(0)} = C \cap Y_0 = C_1 \cup ... \cup C_N$, where $C_k = C \cap \mathrm{Tor}(\Delta_k)$.

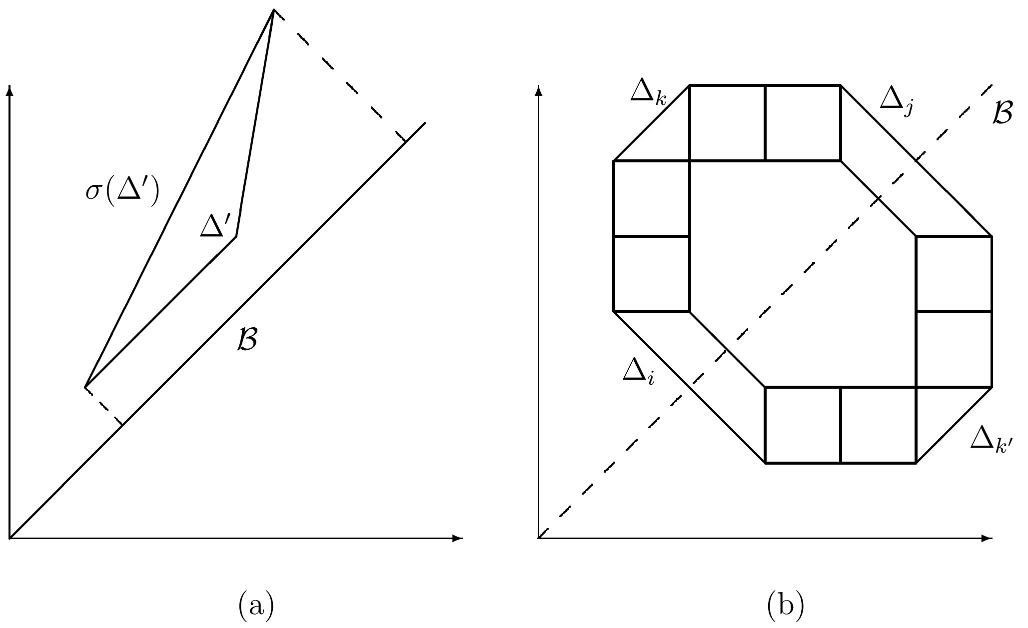


Figure 6: Bad polygons

2.2 Rank of a tropical curve

Assume that a tropical curve A_C is symmetric with respect to the bisectrix $\mathcal{B}$ of the positive quadrant of $\mathbb{R}^2$ (as well as the respective Newton polygon Δ and its dual subdivision S_C). The set of tropical curves, which are combinatorially isotopic to A_C and are symmetric with respect to $\mathcal{B}$, is parameterized by a convex polyhedron, whose dimension we denote by $\mathrm{rk}_{\mathcal{B}}(A_C)$. Put

$$\mathrm{rk}_{vir}(A_C) := \#V(S_C) - 1 - \sum_{\Delta_k \in P(S_C)} (\#V(\Delta_k) - 3) \ ,$$

where $V(\Delta_k)$ is the set of vertices of Δ_k, and $V(S_C)$ is the set of vertices of S_C.

Introduce the following auxiliary notation: $P(S_C)$ is the set of all the polygons of S_C, $P_{\mathcal{B}}(S_C) \subset P(S_C)$ is the set of the polygons, whose interior crosses $\mathcal{B}$, $N_m(\mathcal{B})$ is the number of m-gons in $P_{\mathcal{B}}(S_C)$, $N'_{2m}(\mathcal{B})$ is the number of $2m$-gons having no sides orthogonal to $\mathcal{B}$, N_{2m}^{par} the number of $2m$-gons in $P(S_C)\backslash P_{\mathcal{B}}(S_C)$, whose opposite sides are parallel (further on we call them *parallelogons*).

A polygon $\Delta' \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ is called *bad*, if it has no sides, orthogonal to $\mathcal{B}$, and there is a side $\sigma(\Delta') \subset \Delta'$ such that Δ' is contained in the quadrangle (or triangle) spanned by σ and its orthogonal projection $\mathrm{pr}_{\mathcal{B}}\sigma$ on $\mathcal{B}$ (see Figure 6(a)).

Lemma 2.1 *In the above notation, it holds*

$$\mathrm{rk}_{\mathcal{B}}(A_C) = \mathrm{rk}_{vir}(A_C) + d_{\mathcal{B}}(A_C) \ , \tag{2.8}$$

where

$$2d_{\mathcal{B}}(A_C) \le \sum_{m\ge 2}((2m-3)N_{2m} - N_{2m}^{par} - N_{2m}(\mathcal{B}))$$
$$+\sum_{m\ge 1}((2m-2)N_{2m+1} - N_{2m+1}(\mathcal{B})) \ . \qquad (2.9)$$

Furthermore, if

- *either* $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ *is empty or contains only triangles and parallelogons, and* $P_{\mathcal{B}}(S_C)$ *contains a polygon having at least three sides, which are not orthogonal to* $\mathcal{B}$*,*
- *or* $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ *contains only triangles and parallelogons, among them a polygon with* ≥ 6 *sides,*
- *or* $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ *contains a polygon, which is neither a parallelogon, nor a bad polygon,*

then

$$2d_{\mathcal{B}}(A_C) \le \sum_{m\ge 2}((2m-3)N_{2m} - N_{2m}^{par} - N_{2m}(\mathcal{B}))$$
$$+\sum_{m\ge 1}((2m-2)N_{2m+1} - N_{2m+1}(\mathcal{B})) - 1 \ . \qquad (2.10)$$

Proof. We follow the lines of the proof of Lemma 2.2 [18], where similar bounds have been established for arbitrary plane tropical curves.

Step 1. Iso-combinatorial deformations of A_C are parameterized by the values $\mathrm{Val}(a_{ij})$, corresponding to the vertices of S_C. The dependence between these parameters is caused by vertices of valency > 3, which means that m planes forming the graph of ν intersect at one point. An extra dependence comes from the symmetry condition, for example, the vertices lying on $\mathcal{B}$ must remain on $\mathcal{B}$ along the deformation. To estimate the number of independent parameters, we introduce a linear order of the polygons $\Delta_1, ..., \Delta_N \in P(S_C)$, and then, for any $k = 1, ..., N$, count how many new independent conditions are imposed by the set of vertices $V(\Delta_k)\backslash\bigcup_{i<k}\Delta_i$.

Step 2. Take a vector $\overline{a} = (-1, 1+\varepsilon) \in \mathbb{R}^2$ with a small nonzero ε. It defines a partial order on $P(S_C)$ as follows: for $\Delta_i, \Delta_j \in P(S_C)$ with a common edge σ, put $\Delta_j \succ \Delta_i$ if $\overline{a}$ enters Δ_i crossing σ. We then extend this partial order up to a linear one, assuming that the polygons $\Delta_i \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ which lie below $\mathcal{B}$ precede the polygons $\Delta_j \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ which lie above $\mathcal{B}$.

If Δ_i is a $2m$-gon which either belongs to $P_{\mathcal{B}}(S_C)$, or is a parallelogon, then it imposes at least $m-1$ new independent conditions. If $\Delta_i \in P_{\mathcal{B}}(S_C)$ is an $(2m+1)$-gon, and $\overline{a}$ enters Δ_i through the edge orthogonal to $\mathcal{B}$, then Δ_i imposes at least $m-1$ new independent conditions. If $\Delta_i \in P_{\mathcal{B}}(S_C)$ is an $(2m+1)$-gon, and $\overline{a}$ emanates from Δ_i through the edge orthogonal to $\mathcal{B}$, then Δ_i imposes at least m new independent conditions: when $m \ge 2$ the conditions mean that the m planar faces of the graph of ν_f pass through a point determined by the preceding polygons, and when $m=1$ we impose the condition that the vertex of A_C dual to Δ_i lies on $\mathcal{B}$.

Next, if Δ_i, $1 \le i \le N$, is not as above, denote by $e_{\overline{a}}(\Delta_i)$ the number of edges of Δ_i, through which $\overline{a}$ emanates from Δ_i. Then Δ_i imposes at least $\min\{e_{\overline{a}}(\Delta_i)-1,\ \#V(\Delta_i)-3\}$ new independent conditions.

Doing a similar count for the vector $-\overline{a}$ and summing up all the conditions, and observing that

$$\min\{e_{\overline{a}}(\Delta_i)-1,\ \#V(\Delta_i)-3\}+\min\{e_{-\overline{a}}(\Delta_i)-1,\ \#V(\Delta_i)-3\} \ge \#V(\Delta_i)-3\ ,$$

we obtain (2.9).

Step 3. Assume that $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ is empty or consists of only triangles and parallelogons, and there is $\Delta_i \in P_{\mathcal{B}}(S_C)$ with at least three sides, which are not orthogonal to $\mathcal{B}$.

Using the above vector $\overline{a}$, we define a partial order on $P(S_C)$ as follows: for $\Delta_i, \Delta_j \in P(S_C)$ with a common edge σ, put $\Delta_j \succ \Delta_i$ if (i) σ crosses $\mathcal{B}$ or lies in the closure of the upper component of $\mathbb{R}^2\backslash\mathcal{B}$, and $\overline{a}$ crosses σ entering Δ_j, or (ii) σ lies in the lower component of $\mathbb{R}^2\backslash\mathcal{B}$, and $\overline{a}$ crosses σ entering Δ_i. We then extend this partial order up to a linear one, assuming that the polygons from $P_{\mathcal{B}}(S_C)$ precede the polygons from $P(S_C)\backslash P_{\mathcal{B}}(S_C)$.

As in Step 2, we obtain that any $2m$-gon $\Delta_j \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ imposes at least $m-1$ new independent conditions. Among the polygons of $P_{\mathcal{B}}(S_C)$ there are at least $\frac{1}{2}\sum N_{2m+1}(\mathcal{B}) + N^{even}(\mathcal{B}$ of those, which have no predecessors in the partial order defined, where $N^{even}(\mathcal{B})$ denotes the number of even-gons in $P_{\mathcal{B}}(S_C)$, which have no sides orthogonal to $\mathcal{B}$. Each of these polygons Δ_k imposes $\#V(\Delta)-2$ independent conditions, and any other polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$ imposes at least $\#V(\Delta_k)-3$ new independent conditions. Altogether this yields that

$$2d_{\mathcal{B}}(A_C) \le \sum_{m\ge 2}(2m-4)N^{par}_{2m} - \sum_{m\ge 1}(2m-3)N_{2m+1}(\mathcal{B}) - 2N^{even}(\mathcal{B})\ . \qquad (2.11)$$

By the hypotheses of Step 3, there is $\Delta_i \in P_{\mathcal{B}}(S_C)$, which is either an odd-gon with at least 5 sides, or an even-gon with at least 6 sides, or an even-gon without sides orthogonal to $\mathcal{B}$. In each case (2.11) immediately implies (2.10).

Step 4. Assume that $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ consists only of triangles and parallelogons, and there is $\Delta_i \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ such that $\#V(\Delta_i) \ge 6$. Then we again order the set $P(S_C) := \{\Delta_1, ..., \Delta_N\}$ as described in Step 3, assuming without loss of generality that the polygons in $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ lying below $\mathcal{B}$ precede the polygons from $P_{\mathcal{B}}(S_C)$, and the latter polygons precede the polygons in $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ lying above $\mathcal{B}$. Let Δ_i be the first polygon in $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ with ≥ 6 sides. We then change the order in $P(S_C)$ in the following way:

$$\Delta_1 \succ \Delta_2 \succ ... \succ \Delta_{i-1} \succ \Delta_N \succ \Delta_{N-1} \succ ... \succ \Delta_i \ .$$

Notice that, for any parallelogram Δ_k, $1 \le k < i$, again the number of sides which are not contained in its predecessors is equal to 2, and thus, it again contributes one new independent condition. On the other hand, since Δ_i has now no predecessors, it imposes $\#V(\Delta_i) - 3 > \frac{1}{2}(\#V(\Delta_i) - 2)$ independent conditions, which improves the upper bound (2.9) up to (2.10).

Step 5. Assume that $\Delta_i \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ is neither a parallelogon, nor a bad polygon. Comparing with the computation of Step 1, we see that to gain an extra 1 in the right-hand side of the upper bound to $2d_{\mathcal{B}}(S_C)$, the number of independent conditions imposed by Δ_i and its symmetric copy $\Delta_{i'}$ with respect to $\mathcal{B}$ should be at least $\#V(\Delta_i) - 2$. This is the case for any non-bad polygon: either (choosing a suitable sign of ε in the definition of $\overline{a}$, if necessary) we have $e_{\pm\overline{a}}(\Delta_i) \ge 2$, and then

$$\min\{e_{\overline{a}}(\Delta_i) - 1,\ \#V(\Delta_i) - 3\} + \min\{e_{-\overline{a}}(\Delta_i) - 1,\ \#V(\Delta_i) - 3\}$$

$$= e_{\overline{a}}(\Delta_i) + e_{-\overline{a}}(\Delta_i) - 2 = \#V(\Delta_i) - 2 \ ,$$

or we have $e_{\overline{a}}(\Delta_i) = 1$ with Δ_i lying below $\mathcal{B}$, respectively, $e_{-\overline{a}}(\Delta_i) = 1$ with Δ_i lying above $\mathcal{B}$, and then $\Delta_{i'}$ imposes $\#V(\Delta_{i'}) - 2 = \#V(\Delta_i) - 2$ new independent conditions (including the requirement that the vertices of A_C dual to Δ_i and $\Delta_{i'}$ are symmetric with respect to $\mathcal{B}$). $\square$

3 Tropical limits of real rational curves on non-standard real toric Del Pezzo surfaces

Let $\Sigma = \mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, $\mathbb{S}^2_{2,0}$, or $\mathbb{S}^2_{0,2}$, and $C \in |\mathcal{L}_\Delta|_{\mathbb{K}}$ a real curve, passing through a configuration $\overline{\boldsymbol{p}} = \{\boldsymbol{p}_i \ : \ i = 1, ..., -c_1(\mathcal{L}_\Delta)K_\Sigma - 1\}$ of real generic points in Σ.

We can define C by an equation (2.5) with $a_{ji}=\overline{a}_{ij}$, $(i,j)\in\Delta$, Δ being the suitable polygon shown in Figure 1(a,b,c,d). Since the anti-holomorphic involution acts on $(\mathbb{K}^*)^2\subset\Sigma_{\mathbb{K}}$ by $\mathrm{Conj}(\xi,\eta)=(\overline{\eta},\overline{\xi})$, the configuration $\overline{\boldsymbol{p}}$ should satisfy $\boldsymbol{p}_i=(\xi_i(t),\overline{\xi}_i(t))$, $i=1,...,-c_1(\mathcal{L}_\Delta)K_\Sigma-1$, in particular, the configuration $\overline{\boldsymbol{x}}=\mathrm{Val}(\overline{\boldsymbol{p}})\subset\mathbb{Q}^2$ must lie on the line $\{x=y\}$. We choose $\overline{\boldsymbol{p}}$ to be generic in $\Omega_0(\Sigma_{\mathbb{K}},\mathcal{L}_{\mathbb{K}})$. Observe also that the tropical curve $A_C\subset\mathbb{R}^2$ is symmetric with respect to $\mathcal{B}$, and so is the subdivision S_C of Δ.

3.1 Tropical curves and dual subdivisions

Proposition 3.1 *In the above notation, let $\overline{\boldsymbol{p}}\in\Omega_0(\Sigma_{\mathbb{K}},\mathcal{L}_{\mathbb{K}})$ be a generic configuration, $C\in|\mathcal{L}(\Delta)|_{\mathbb{K}}$ a real rational curve passing through $\overline{\boldsymbol{p}}$. Then*

(1) A_C has precisely $|(\partial\Delta)_+|$ vertices on $\mathcal{B}$, and they all belong to $\overline{\boldsymbol{x}}$; the set $A_C\backslash\mathcal{B}$ is a union of rays emanating from the vertices of A_C on $\mathcal{B}$, two from each vertex;

*(2) the subdivision S_C is as follows: there is an admissible lattice path γ in Δ *in the sense of section 1.2) such that the part of Δ between γ and its symmetric with respect to $\mathcal{B}$ image is divided by the segments, joining each integral point on γ with its symmetric image, and the remaining part of Δ is divided into lattice parallelograms with Euclidean area 1.*

In particular, Proposition 3.1 yields that

- $V(S_C)\cap\partial\Delta$ consists exactly of all the integral points on $\partial\Delta$, which are not interior points of the sides orthogonal to $\mathcal{B}$;

- $P_{\mathcal{B}}(S_C)$ consists of $|(\partial\Delta)_+|$ polygons, which are triangles, or trapezes with a pair of edges orthogonal to $\mathcal{B}$.

To describe limit curves, we introduce a few notations.

For a polygon $\Delta_k\in P(S_C)$, we denote by $\mathrm{Tor}(\partial\Delta_k)$ the union of the divisors in $\mathrm{Tor}(\Delta_k)$, associated with the edges of Δ_k.

If $\Delta_k\in P_{\mathcal{B}}(S_C)$, denote by $(\partial\Delta_k)_\perp$ the union of its edges orthogonal to $\mathcal{B}$. The edges $\sigma\subset(\partial\Delta_k)_\perp$, $\Delta_k\in P_{\mathcal{B}}(S_C)$, are dual to the edges of A_C lying on $\mathcal{B}$. Let $\boldsymbol{x}_i=(\alpha_i,\alpha_i)$, $i\in I\subset\{1,...,|\partial\Delta|-1\}$, be all the points in $\overline{\boldsymbol{x}}$, which are interior points of the edges of A_C lying on $\mathcal{B}$. Introduce the set

$$\Phi=\left\{(\xi_i^{(0)},\overline{\xi_i^{(0)}})\in\mathbb{P}^1\ :\ \boldsymbol{p}_i=(\xi_i^{(0)},\overline{\xi_i^{(0)}})t^{-\alpha_i}+O(t^{-\alpha_i+1}),\quad i\in I\right\}\ . \tag{3.12}$$

On the divisors $\mathrm{Tor}(\sigma)$, where $\sigma\subset(\partial\Delta_k)_\perp$, $\Delta_k\in P_{\mathcal{B}}(S_C)$, we have the naturally defined coordinate systems, and hence, for any such divisor, we obtain the set Φ_σ of the points with the coordinates $(\xi_i^{(0)},\overline{\xi_i^{(0)}})\in\Phi$.

Proposition 3.2 *Under the hypotheses of Proposition 3.1, it holds that*

(i) any curve C_k, corresponding to a parallelogram $\Delta_k \notin P_{\mathcal{B}}(S_C)$, is a union of curves defined by binomials,

(ii) any curve C_k, corresponding to a polygon $\Delta_k \in P(S_C)$, is a union of a real rational nodal curve C'_k and some real curves defined by binomials; the curve C'_k crosses each divisor $\mathrm{Tor}(\sigma)$, $\sigma \subset \partial\Delta_k \backslash (\partial\Delta_k)_\perp$, precisely at one point, and crosses each divisor $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, at some points of Φ_σ, furthermore, C'_k is non-singular along $\partial\Delta_k$; the binomial components of C_k cross each divisor $\mathrm{Tor}(\sigma)$, $\sigma \subset (\partial\Delta_k)_\perp$, at some points of $\Phi_\sigma \backslash C'_k$,

(iii) if $\Delta_k \in P_{\mathcal{B}}(S)C)$ is a triangle or a trapeze with two non-parallel sides, then C'_k is non-singular; if $\Delta_k \in P_{\mathcal{B}}(S_C)$ is a rectangle, then C'_k has no solitary nodes.

The main ingredients of the proof are similar to that used in [18], section 3.3, and [19], section 2.2. For the reader convenience, we divide our reasoning in few steps.

3.2 Proof of Propositions 3.1 and 3.2: preliminary estimates

Step 1. The points $\boldsymbol{x}_i$, $i \in \{1, ..., |\partial\Delta| - 1\} \backslash I$, are either vertices of A_C, or lie on edges of A_C orthogonal to $\mathcal{B}$ (and which are dual to edges of S_C, lying on $\mathcal{B}$). Due to the general position of the configuration $\overline{\boldsymbol{x}}$ on $\mathcal{B}$,

$$\mathrm{rk}_{\mathcal{B}} A_C \ge |\partial\Delta| - 1 - \#I \ . \tag{3.13}$$

In view of (2.9), we obtain

$$2\#V(S_C) - 2 - 2\sum_{k=1}^{N}(\#V(\Delta_k) - 3) + 2d_{\mathcal{B}}(A_C) \ge 2|\partial\Delta| - 2 - 2\#I \ ,$$

and then, using the Euler formula,

$$2\#E(S_C) - 2\sum_{k=1}^{N}(\#V(\Delta_k) - 2) + 2d_{\mathcal{B}}(A_C) \ge 2|\partial\Delta| - 2 - 2\#I \ ,$$

where $E(S_C)$ is the set of edges of the subdivision S_C, and we end up with

$$\#E(S_C, \partial\Delta) + N_3 - \sum_{m\ge 5}(m-4)N_m + 2d_{\mathcal{B}}(A_C) \ge 2|\partial\Delta| - 2 - 2\#I \ , \tag{3.14}$$

where $E(S_C, \partial\Delta)$ is the set of edges of S_C lying on $\partial\Delta$.

Step 2. Consider now the limit curves $C_1, ..., C_N$. Let C_{ij}, $1 \le j \le n_i$, be all the components of C_i, $1 \le i \le N$, counting multiplicities, and let s_{ij} be the number of local branches of C_{ij} centered along $\mathrm{Tor}(\partial\Delta_i)$. The curves C_{ij} are rational (see, for instance, Step 1 in the proof of Proposition 2.1 [19]). Denote by $\mathcal{C}_b$ the set of components C_{ij}, defined by irreducible binomials, and by $\mathcal{C}_{nb}$ the set of remaining components C_{ij}. Then

$$2 \le \sum_{k=1}^{N} \sum_{C_{kj} \in \mathcal{C}_{nb}} (2 - s_{kj}) + s(\partial\Delta) \ , \tag{3.15}$$

where $s(\partial\Delta)$ stands for the number of local branches (counting multiplicities) of the curves $C_1, ..., C_N$ centered at $\mathrm{Tor}(\sigma)$ with σ running over all the edges of Δ. Notice that the equality in (3.15) can be attained only when in the deformation $C^{(t)}$, $t \ge 0$,

(**E1**) the intersection points of distinct components C_{ij}, C_{il}, which belong to $(\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_i)$, persist, and

(**E2**) in a regular small neighborhood U of any point $z \in C_{ij} \cap C_{kl} \cap \mathrm{Tor}(\sigma)$, $\sigma = \Delta_i \cap \Delta_k$, in the threefold $Y = \mathrm{Tor}(\widetilde{\Delta})$, the Euler characteristic of the normalization of $C^{(t)} \cap U$, $t \ne 0$, is zero.

Step 3. Next we have

$$\sum_{\Delta_k \notin P_{\mathcal{B}}(S_C)} \sum_{C_{kj} \in \mathcal{C}_{nb}} (s_{kj} - 2) \ge \sum_{m \ge 1} (N_{2m+1} - N_{2m+1}(\mathcal{B}))$$

$$+ \sum_{m \ge 2} (N_{2m} - N_{2m}(\mathcal{B}) - N_{2m}^{par}) \ , \tag{3.16}$$

where the equality is attained only when

(**E3**) for each parallelogon $\Delta_k \notin P_{\mathcal{B}}(S_C)$, the limit curve C_k splits into binomial components, for any other polygon $\Delta_k \notin P_{\mathcal{B}}(S_C)$, precisely one of the components C_{kj} of C_k satisfies $s_{kj} = 3$, it is non-multiple, and all the other components are binomial.

To estimate

$$\sum_{\Delta_k \in P_{\mathcal{B}}(S_C)} \sum_{C_{kj} \in \mathcal{C}_{nb}} (s_{kj} - 2) \ , \tag{3.17}$$

we introduce an auxiliary graph $\widetilde{G}$. Write $\Phi = \{z_i \ : \ i = 1, ..., \#I\}$, where numbering reflects the natural order of the points $\boldsymbol{x}_j$, $j \in I$, on the line $\mathcal{B}$. Let $\sigma_1, ..., \sigma_m$ be all the naturally ordered edges from $\bigcup_{\Delta_k \in P_{\mathcal{B}}(S_C)} (\partial\Delta_k)_\perp$. A point (i, j), where $1 \le i \le m$, $1 \le j \le \#I$, is chosen as a vertex of $\widetilde{G}$ if $z_j \in \mathrm{Tor}(\sigma_i) \cap C_k$, where $\sigma_i \subset (\partial\Delta_k)_\perp$. We join two points (i, j), $(i+1, j)$ of $\widetilde{G}$ by a segment if σ_i and σ_{i+1} are edges of the same polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$, and the corresponding points in

$\mathrm{Tor}(\sigma_i)$ and $\mathrm{Tor}(\sigma_{i+1})$ are joined by a binomial component of C_k. The obtained graph we denote by $\widetilde{G}'$. Next, we introduce new vertices w_i, $i = 1, ..., n$, of $\widetilde{G}$, which are in 1-to-1 correspondence with the components $C_{kl} \in \mathcal{C}_{nb}$ of all the curves C_k, $\Delta_k \in P_{\mathcal{B}}(S_C)$. We join a vertex w_i with a vertex (j_1, j_2), $j_2 > 0$ by an arc, if the corresponding to (j_1, j_2) point of $\mathrm{Tor}(\sigma_j)$ belongs to the component $C_{kl} \in \mathcal{C}_{nb}$ of the curve C_k, corresponding to w_i, as $\sigma_j \subset (\partial\Delta_k)_\perp$.

Observe that the expression (3.17) is at least the total valency of the vertices w_i of $\widetilde{G}$. The subgraph $\widetilde{G}' \subset \widetilde{G}$ splits into components. Denote by J the set of the univalent vertices of $\widetilde{G}$, which belong to the subgraph $\widetilde{G}'$. Then

$$\sum_{\Delta_k \in P_{\mathcal{B}}(S_C)} \sum_{C_{kj} \in \mathcal{C}_{nb}} (s_{kj} - 2) \ge 2\#I - \#J \ , \tag{3.18}$$

where the equality is attained only when

(**E4**) $C_k \cap \mathrm{Tor}(\sigma) \subset \Phi_\sigma$ for any $\sigma \subset (\partial\Delta_k)_\perp$, $\Delta_k \in P_{\mathcal{B}}(S_C)$;

(**E5**) the vertices of $\widetilde{G}'$, located on one horizontal line, belong to the same component of $\widetilde{G}'$;

(**E6**) if $\Delta_k \in P_{\mathcal{B}}(S_C)$, then the components of C_k, belonging to $\mathcal{C}_{nb}$, are not multiple, they intersect each other only in $(\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_k)$, they are unibranch at the points of intersection with $\mathrm{Tor}(\partial\Delta_k)$, and, furthermore, any such component crosses $\bigcup_{\sigma \subset \partial\Delta_k \backslash (\partial\Delta_k)_\perp} \mathrm{Tor}(\sigma)$ precisely at two points;

(**E7**) if $z \in \Phi_\sigma$, where $\sigma = \Delta_k \cap \Delta_i$, $\Delta_k, \Delta_i \in P_{\mathcal{B}}(S_C)$, then z belongs precisely to one component of $(C_k)_{\mathrm{red}}$ and to one component of $(C_i)_{\mathrm{red}}$;

(**E8**) the total number of branches of the non-binomial components C'_k of the curves C_k for all $\Delta_k \in P_{\mathcal{B}}(S_C)$, centered on the divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset \bigcup_{\Delta_k \in P_{\mathcal{B}}(S_C)} (\partial\Delta_k \backslash (\partial\Delta_k)_\perp$, is equal to $2\#I - \#J$.

Step 4. Inequalities (3.14), (3.15), (3.16) together with (3.18) give

$$2d_{\mathcal{B}}(S_C) \ge \sum_{m \ge 1} ((2m-2)N_{2m+1} - N_{2m+1}(\mathcal{B})) + \sum_{m \ge 2} ((2m-3)N_{2m} - N_{2m}(\mathcal{B}) - N_{2m}^{par})$$

$$+2|\partial\Delta| - \#E(\partial\Delta, S_C) - s(\partial\Delta) - \#J \ . \tag{3.19}$$

Notice that

$$\#E(\partial\Delta, S_C) + \#J \le |\partial\Delta| + \delta \ , \tag{3.20}$$

where δ is the number of the sides of Δ orthogonal to $\mathcal{B}$ (i.e., is 0 for $\mathbb{S}^2$, is 1 for $\mathbb{S}^2_{1,0}$ and 2 for $\mathbb{S}^2_{2,0}$). The equality here happens only when

(**E9**) for each side of Δ, orthogonal to $\mathcal{B}$, there is a polygon $\Delta_k \in P_{\mathcal{B}}(S_C)$ with an edge $\sigma = (\partial\Delta_k)_\perp \cap \partial\Delta$; the corresponding limit curve C_k satisfies $C_k \cap \mathrm{Tor}(\sigma) \subset \Phi_\sigma$, and $(C_k \cdot \mathrm{Tor}(\sigma))_z = 1$ for any point $z \in C_k \cap \mathrm{Tor}(\sigma)$;

(**E10**) all the edges of S_C on $\partial\Delta$, which are not contained in $\bigcup_{\Delta_k\in P_{\mathcal{B}}(S_C)}(\partial\Delta_k)_\perp$, have lattice length 1.

Thus, (2.9) and (3.19) yield

$$M-\delta\le 2d_{\mathcal{B}}(S_C)\le M\ ,$$

where

$$M:=\sum_{m\ge 1}((2m-2)N_{2m+1}-N_{2m+1}(\mathcal{B}))+\sum_{m\ge 2}((2m-3)N_{2m}-N_{2m}(\mathcal{B})-N_{2m}^{par})\ .$$

3.3 Proof of Propositions 3.1 and 3.2 for $\Sigma=\mathbb{S}^2$, $\mathbb{S}^2_{1,0}$, and $\mathbb{S}^2_{2,0}$

3.3.1 The case $2d_{\mathcal{B}}(S_C)=M-\delta$

Under the assumption made, relations (3.13), (3.14), (3.15), (3.16) (3.18), (3.19), and (3.20) turn into equalities, and hence conditions (E1-E10) hold true.

Step 1. We have

$$\begin{aligned}|\partial\Delta|-1-\#I&\le\#P_{\mathcal{B}}(S_C)+\#\{\sigma\in E(S_C),\ \sigma\subset\mathcal{B}\}\}\\ &\le\#P_{\mathcal{B}}(S_C)+2\#\{\sigma\in E(S_C),\ \sigma\subset\mathcal{B}\}\}\\ \frac{1}{2}\sum_{\substack{\sigma\subset\partial\Delta_k\backslash(\partial\Delta_k)_\perp\\ \Delta_k\in P_{\mathcal{B}}(S_C)}}|\sigma|&+2\sum_{\substack{\sigma\in E(S_C)\\ \sigma\subset\mathcal{B}}}|\sigma|\le|(\partial\Delta)_+|\end{aligned}\tag{3.21}$$

(the latter inequality coming from the projection to $\partial\Delta$ in the direction orthogonal to $\mathcal{B}$). This yields, in particular,

$$\#I\ge|\partial\Delta|-|(\partial\Delta)_+|-1\ .$$

In view of

$$\#J\le|\partial\Delta|-2|(\partial\Delta)_+|$$

we derive that the total number of branches of the non-binomial components C'_k of the curves C_k for all $\Delta_k\in P_{\mathcal{B}}(S_C)$, centered on the divisors $\mathrm{Tor}(\sigma)$, $\sigma\subset\bigcup_{\Delta_k\in P_{\mathcal{B}}(S_C)}(\partial\Delta_k\backslash(\partial\Delta_k)_\perp$, is at least

$$2\#I-\#J\ge\#I+|(\partial\Delta)_+|-1\ .$$

Since the graph $\widetilde{G}$ has no cycles (otherwise, in the deformation $C^{(t)}$, $t\ge 0$, the curves C_k for all $\Delta_k\in P_{\mathcal{B}}(S_C)$ will glue up into a curve of a positive genus), we obtain that the number of the non-binomial components of the curves C_k for all

$\Delta_k \in P_{\mathcal{B}}(S_C)$ plus $\#I$ is at least $\#I + |(\partial\Delta)_+|$, and hence the number of the non-binomial components of the curves C_k for all $\Delta_k \in P_{\mathcal{B}}(S_C)$ is at least $|(\partial\Delta)_+|$, which implies

$$\frac{1}{2} \sum_{\substack{\sigma \subset \partial\Delta_k \backslash (\partial\Delta_k)_\perp \\ \Delta_k \in P_{\mathcal{B}}(S_C)}} |\sigma| \ge |(\partial\Delta)_+| \ .$$

Together with (3.21) this turns all the above relations into equalities, whose meaning is as follows:

(**E11**) $\#I = |\partial\Delta| - |(\partial\Delta)_+| - 1$, $\#J = |\partial\Delta| - 2|(\partial\Delta)_+|$, $\#P_{\mathcal{B}}(S_C) = |(\partial\Delta)_+|$;

(**E12**) S_C has no edges lying on $\mathcal{B}$; the edges of S_C on $\partial\Delta$ are the sides of Δ, orthogonal to $\mathcal{B}$, and the remaining ones are vertical and horizontal unit segments; all the vertices of A_C on $\mathcal{B}$ belong to $\overline{\boldsymbol{x}}$; the polygons in $P_{\mathcal{B}}(S_C)$ are triangles or trapezes with a pair of edges orthogonal to $\mathcal{B}$, and the projection of any such polygon to $\partial\Delta$ in the direction orthogonal to $\mathcal{B}$ is a vertical or horizontal unit segment.

Step 2. Arguing on the contrary, we shall show that the polygons in $P(S_C) \backslash P_{\mathcal{B}}(S_C)$ are only parallelogons.

Assume that there is $\Delta_k \in P(S_C) \backslash P_{\mathcal{B}}(S_C)$, which is not a parallelogon, and lies above $\mathcal{B}$. The dual to it vertex v_k of A_C lies above $\mathcal{B}$ as well. Due to (E3), v_k is odd-valent. Let a straight line $\langle \overline{a}, (x, y) \rangle = c_k$ pass though v_k, where $\overline{a}$ is the vector introduced in Step 2 of the proof of Lemma 2.1.

Suppose that, for any odd-valent vertex v_k of A_C, lying above $\mathcal{B}$, the number of edges of A_C, emanating from v_k and crossing the line $\langle \overline{a}, (x, y) \rangle = c_k + t$ in a neighborhood of v_k, is always greater for $t > 0$ than for $t < 0$. However, this contradicts the fact, that, for a suitable $c < \min\{c_k\}$ and sufficiently small ε, the line $\langle \overline{a}, (x, y) \rangle = c$ crosses all the edges of A_C, emanating from the vertices of A_C, which lie on $\mathcal{B}$, to the upper half-plane of $\mathbb{R}^2 \backslash \mathcal{B}$, and, along (E11, E12), the number of these edges is equal to $|(\partial\Delta)_+|$, and on the other hand, the line $\langle \overline{a}, (x, y) \rangle = c$, $c > \max\{c_k\}$ crosses precisely $|(\partial\Delta)_+|$ (unbounded) edges of A_C, not lying on $\mathcal{B}$.

Hence there is an odd-valent vertex v_k of A_C, lying above $\mathcal{B}$ and such that the line $\langle \overline{a}, (x, y) \rangle = c_k + t$ crosses fewer edges of A_C, emanating from v_k, as $t > 0$ than for $t < 0$. That is, the polygon Δ_k has two sides, σ', σ'', through which the vector $\overline{a}$ enters Δ_k, and such that a non-binomial component C'_k of the limit curve C_k crosses $\mathrm{Tor}(\sigma')$ and $\mathrm{Tor}(\sigma'')$. In turn, σ' is a side of some $\Delta_l \in P(S_C)$, preceding Δ_k in the order determined by the vector $\overline{a}$. If $\Delta_l \notin P_{\mathcal{B}}(S_C)$, then for any component C'_l of C_l, which contains the point $C'_k \cap \mathrm{Tor}(\sigma')$, there is an edge σ'_1 of Δ_l, through which $\overline{a}$ enters Δ_k and such that $C' + l \cap \mathrm{Tor}(\sigma'_1) \ne \emptyset$. Proceeding in this manner, we shall necessarily obtain a polygon $\Delta_i \in P_{\mathcal{B}}(S_C)$ with a marked edge, which is not orthogonal to $\mathcal{B}$. Similarly, starting with the edge σ'' we build a sequence of polygons ending with some $\Delta_j \in P_{\mathcal{B}}(S_C)$. Of course, the polygon $\Delta_{k'}$ symmetric to Δ_k with respect to $\mathcal{B}$ is joined with the same $\Delta_i, \Delta_j \in P_{\mathcal{B}}(S_C)$ via mirror sequences of polygons (see, for example, Figure 6(b)). In the deformation $C^{(t)}$, $t \ge 0$, The non-binomial components C'_k and $C'_{k'}$ glue up with the respective components of the limit curves, corresponding to the polygons in the constructed sequences, and, in particular, with the non-binomial components C'_i, C'_j of the limit curves C_i, C_j. But then we obtain a curve with a positive genus, which contradicts our initial assumptions, and thus, completes the proof of the assertion of this Step.

Step 3. We finally show that the subdivision S_C and the limit curves $C_1, ..., C_N$ are as stated in Propositions 3.1 and 3.2.

Since the polygons of $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ are parallelogons, $A_C\backslash\mathcal{B}$ is a union of rays emanating from the vertices of A_C lying on $\mathcal{B}$. Since these vertices are elements of the generic (on $\mathcal{B}$) configuration $\overline{\boldsymbol{x}}$ (property (E12)), the rays have only double intersection points. Furthermore, the rays in the upper half-plane must be orthogonal to the sides of $(\partial\Delta)_+$, and their number is equal to $|(\partial\Delta)_+|$. hence $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ consists of unit squares. Notice also that

$$\partial\left(\bigcup_{\Delta_k \in P_{\mathcal{B}}(S_C)} \Delta_k\right) \cap \mathcal{B} = \partial\Delta \cap \mathcal{B} \ ,$$

that is the situation like that shown in Figure 2(b) is not possible, since otherwise one would get a family of reducible curves $C^{(t)}$, $t > 0$. By the same reason, the graph $\widetilde{G}$, constructed in Step 2 of section 3.2, is a connected tree.

Thus, all the requirements of Propositions 3.1 and 3.2 are fulfilled. In particular, the case of $\Sigma = \mathbb{S}^2$ is completely settled.

3.3.2 The case $2d_{\mathcal{B}}(S_C) = M - \delta + 1$, and $\Sigma = \mathbb{S}^2_{1,0}$ or $\mathbb{S}^2_{2,0}$

Observe that in this case, the number of the odd-gons in $P_{\mathcal{B}}(S_C)$ is of the opposite parity with δ. That means, there is a side σ of Δ, which is orthogonal to $\mathcal{B}$, has even length ≥ 2, and does not contain edges of polygons from $P_{\mathcal{B}}(S_C)$. Then, as in Step 1 of section 3.3.1, we derive that

- the number of the non-binomial components of the limit curves C_k for all $\Delta_k \in P_{\mathcal{B}}(S_C)$ does not exceed $|(\partial\Delta)_+|$;
- $\#J \le |\partial\Delta| - 2|(\partial\Delta)_+| - |\sigma|$;
- the total number of the arcs of the graph $\widetilde{G}$, joining the vertices in $\widetilde{G}\backslash\widetilde{G}'$ with the components of $\widetilde{G}'$ is at least

$$2\#\pi_0(\widetilde{G}') - \#J \ge \#\pi_0(\widetilde{G}') + \#I - \#J \ge \#\pi_0(\widetilde{G}') + |(\partial\Delta)_+| - 1 + |\sigma| \ .$$

The latter expression is greater than $\#\pi_0(\widetilde{G}') + |(\partial\Delta)_+|$, which bounds from above the number of the vertices of the graph $\widetilde{G}$ after contracting the components of $\widetilde{G}'$ to points. Hence $\widetilde{G}$ has cycles, and thus, in the deformation $C^{(t)}$, $t \ge 0$, we necessarily obtain a curve of a positive genus.

The case asserted in the title does not occur. In particular, the consideration of $\Sigma = \mathbb{S}^2_{1,0}$ is completed.

3.3.3 The case $2d_{\mathcal{B}}(S_C) = M$ and $\Sigma = \mathbb{S}^2_{2,0}$

In the considered situation, the number of odd-gons in $P_{\mathcal{B}}(S_C)$ is even. That is either no side of Δ contains an edge of a polygon from $P_{\mathcal{B}}(S_C)$, or both sides of Δ, orthogonal to $\mathcal{B}$ contains edges of polygons from $P_{\mathcal{B}}(S_C)$.

The case when no side of Δ contains an edge of a polygon from $P_{\mathcal{B}}(S_C)$ can be prohibited by means of the argument of section 3.3.2. In the case when each of the two sides of Δ, orthogonal to $\mathcal{B}$, contains an edge of a polygon from $P_{\mathcal{B}}(S_C)$, we repeat the computations of Step 2 in section 3.3.1 and derive the properties (E11,E12).

By Lemma 2.1, $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ should consist of parallelogons and bad polygons. However, bad polygons cannot occur, what one can establish as in Step 2 of section 3.3.1 by reducing the existence of bad polygons to the positivity of the genus of the curves $C^{(t)}$, $t \ne 0$.

Furthermore, as in Step 3 of section 3.3.1, the property (E12) yields that $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ consists of only unit squares. Next we notice that a limit curve C_k, corresponding to a square $\Delta_k \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ splits into binomial components, since otherwise, in (3.16), the right-hand side should be raised by 4 (counting the contributions of Δ_k and its symmetric image with respect to $\mathcal{B}$), which together with other estimates would lead to an impossible inequality $2d_{\mathcal{B}}(S_C) \ge M-\delta+4 = M+2$.

Thereby we end up with all the requirements of Propositions 3.1 and 3.2, which moreover, imply $2d_{\mathcal{B}}(S_C) = M - \delta$.

The case $\Sigma = \mathbb{S}^2_{2,0}$ is completed.

3.4 Proof of Propositions 3.1 and 3.2 for $\Sigma = \mathbb{S}^2_{0,2}$

The results of section 3.2 say that $2d_{\mathcal{B}}(S_C) = M$, and the conditions (E1-E10) hold true.

By Lemma 2.1, $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ consists of parallelogons and bad polygons.

Let $\Delta_k \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ be a bad polygon, lying above $\mathcal{B}$. Clearly, the projection of its side $\sigma(\Delta_k)$ (see Figure 6(a)) on $\mathcal{B}$ has Euclidean length $\ge \sqrt{2}$.

If $\sigma(\Delta_k) \subset \partial\Delta$, then $\sigma(\Delta_k)$ is just the unit length segment parallel to $\mathcal{B}$. Assume that $\sigma(\Delta_k)$ is an edge of another polygon $\Delta_{k+1} \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$. If Δ_{k+1} is a parallelogon, we take its edge σ_1 opposite and parallel to $\sigma(\Delta_k)$, and which, in fact is a translate of $\sigma(\Delta_k)$ due to (E3). If Δ_{k+1} is a bad polygon, we take $\sigma_1 = \sigma(\Delta_{k+1}$. Checking whether $\sigma_1 \subset \partial\Delta$, and taking, if not, an adjacent polygon Δ_{k+2}, we continue the procedure, building a sequence $\sigma_1, \sigma_2, ... \in E(S_C)$, which ends up with some $\sigma_n \subset \partial\Delta$. Observe that the Euclidean length of the projections on $\mathcal{B}$ does not decrease in the sequence $\sigma_1, \sigma_2, ...$, and even jumps when the corresponding polygon Δ_{k+j} is bad. On the other hand, σ_n can only be a unit lattice length segment parallel to $\mathcal{B}$.

So, we deduce that Δ_k is a triangle, the edge $\sigma(\Delta_k)$ is a unit lattice length segment, parallel to $\mathcal{B}$, which is joined with its translate $\sigma'(\Delta_k)$ on $\partial\Delta$ via a sequence of parallel opposite sides of parallelogons. Moreover, $\sigma'(\Delta_k) \neq \sigma'(\Delta_l)$ for distinct bad triangles Δ_k, Δ_l lying above $\mathcal{B}$.

Applying the procedure of Step 2 in section 3.3.1, we can join the two other sides of Δ_k by sequences of parallelogons with edges σ', σ'' of some polygons $\Delta_i, \Delta_j \in P_{\mathcal{B}}(S_C)$ (see Figure 6(b)). Notice again that the edges σ', σ'' are different from the corresponding edges, which can be obtained from any other bad triangle.

If the points $C_i \cap \mathrm{Tor}(\sigma')$ and $C_j \cap \mathrm{Tor}(\sigma'')$[5] both belong to non-binomial components of the limit curves C_i, C_j, then we obtain a contradiction as it was in Step 2 of section 3.3.1. Hence, say, the point $C_i \cap \mathrm{Tor}(\sigma')$ belongs to a binomial component of C_i. Since σ' is not parallel to $\mathcal{B}$, we deduce that, for another edge σ of Δ_i, lying above $\mathcal{B}$, at least one point in $C_i \cap \mathrm{Tor}(\sigma)$ belongs to another binomial component of C_i.

Let $n \le d_1$ be the number of bad triangles, lying above $\mathcal{B}$ (d_1 being the parameter of Δ indicated in Figure 1(d)). Counting the intersections of the non-binomial components of the limit curves C_i for all $\Delta_i \in P_{\mathcal{B}}(S_C)$, with the divisors $\mathrm{Tor}(\sigma)$, $\sigma \subset \partial\Delta_i \backslash (\partial\Delta_i)_\perp$, and taking into account that any such edge σ is joined via a sequence parallelogons with its translate in a bad triangle or on $\partial\Delta$, we obtain that the number of such non-binomial components does not exceed $2d - d_1 - n$, whereas the number of polygons in $P_{\mathcal{B}}(S_C)$ does not exceed $2d - d_1$ (d being the parameter shown in Figure 1(d)). As in Step 1 of section 3.3.1, we successively get that

$$\#I \ge |\partial\Delta| - |(\partial\Delta)_+| - 1 = 2d - d_1 - 1 \ ,$$

the number of arcs of the graph $\widetilde{G}$, joining the vertices in $\widetilde{G}\backslash\widetilde{G}'$ with the components

[5] Clearly the lattice length of the segments σ', σ'' is 1, and thus, the intersections are one-point sets.

of $\widetilde{G}'$, is at least (here $J=\emptyset$, because Δ has no sides orthogonal to $\mathcal{B}$)

$$2\#\pi_0(\widetilde{G}')\ge\#\pi_0(\widetilde{G}')+\#I\ge\#\pi_0(\widetilde{G}')+2d-d_1-1\ .$$

If $n>0$, the latter expression is greater or equal than $\#\pi_0(\widetilde{G}')+2d-d_1-n$, which bounds from above the number of the vertices of the graph $\widetilde{G}$ after contracting the components of $\widetilde{G}'$ to points, and hence $\widetilde{G}$ has cycles, which in turn results in the existence of a positive genus curves in the deformation $C^{(t)}$, $t\ge 0$ (cf. section 3.3.2).

By Lemma 2.1, the absence of bad polygons says that the remaining polygons in $P(S_C)\backslash P_{\mathcal{B}}(S_C)$ are parallelograms. Then the properties (E1-E10) complete the proof of all the statements of Propositions 3.1 and 3.2 for $\Sigma=\mathbb{S}^2_{02}$.

4 Proof of Theorem 1

4.1 Restoring the subdivision S and the tropical curve A out of the configuration of points

We start with an admissible lattice path γ as defined in section 1.2, and then construct the uniquely determined γ-admissible subdivision S of Δ again as described in section 1.2.

To restore the tropical curve A, we pick $|(\partial\Delta)_+|$ points in the configuration $\overline{\boldsymbol{x}}$, appoint them as vertices of A, dual to the polygons in $P_{\mathcal{B}}(S)$. Then we take rays, not lying on $\mathcal{B}$, emanating from the chosen vertices and orthogonal to the respective sides of the polygons of $P_{\mathcal{B}}(S)$. Then we take the segment joining the extreme vertices and, in case Δ has sides orthogonal to $\mathcal{B}$, append rays, lying on $\mathcal{B}$ and emanating from the extreme vertices.

The tropical curve A and the subdivision S determine the convex piece-wise linear function $\nu:\Delta\to\mathbb{R}$ uniquely up to multiplication by a constant and addition of a linear affine function.

Now we explain how to choose the vertices of A once we know a marking $\overline{s}$. The vertices of G, located on the horizontal coordinate axis (see section 1.2) divide the axis into $|(\partial\Delta)_+|-1$ segments and two rays. Then we pick $|(\partial\Delta)_+|$ points in $\overline{\boldsymbol{x}}$ so that the distribution of the remaining points $\boldsymbol{x}_i\in\overline{\boldsymbol{x}}$, $i\in I\subset\{1,...,|\partial\Delta|-1\}$, in the complement of the chosen vertices in $\mathcal{B}$ will coincide with the distribution of the coordinates of $\overline{s}$ in the segments and rays on the horizontal axis.

4.2 Restoring limit curves

The points $\boldsymbol{p}_i$, $i\in I$, determine a finite set Φ in each of the divisors $\mathrm{Tor}(\sigma)$, $\sigma\subset\bigcup_{\Delta_k\in P_{\mathcal{B}}(S)}(\partial\Delta_k)_\perp$, as defined by (3.12). We define a correspondence between the components of the subgraph G' and the points of the set Φ as follows. Divide the plane by certain $|(\partial\Delta)_+|$ horizontal lines into $|(\partial\Delta)_+|-1$ strips and two half-planes so that the distribution of the components of G', ordered by increasing height, among these pieces will coincide with the distribution of the naturally ordered points $\boldsymbol{x}_i$, $i\in I$, in the complement of the vertices of A in $\mathcal{B}$. Then we establish the correspondence between the components of G' and the points of Φ in each element of the distribution in an arbitrary way. The integral points in the components of G' correspond to the respectively ordered edges $\sigma\subset\bigcup_{\Delta_k\in P_{\mathcal{B}}(S)}(\partial\Delta_k)_\perp$. For a trapeze or a rectangle $\Delta_k\in P_{\mathcal{B}}(S)$, we include into the limit curve C_k a binomial component, passing through some point $\tau\in\Phi$ if and only if the corresponding component of G' contains the segment, joining the integral points associated with the sides of Δ_k orthogonal to $\mathcal{B}$. The weight of a component of G' determines the multiplicity of respective binomial curves in the limit curves C_k, $\Delta_k\in P_{\mathcal{B}}(S)$. The non-binomial components C'_k of the limit curves C_k, $\Delta_k\in P_{\mathcal{B}}(S)$, correspond to the vertices of G lying on the horizontal coordinate axis, and the intersections of such components with the divisors $\mathrm{Tor}(\sigma)$, $\sigma\subset(\partial\Delta_k)_\perp$, are defined by the arcs of G emanating from the given vertex, where the intersection multiplicities are the weights of the corresponding components of G'.

If Δ_k is a triangle or a trapeze, then C'_k is determined uniquely. Indeed, an equation for C'_k can be written as

$$\alpha x^ay^b\prod_i(x\overline{\xi_i^{(0)}}-y\xi_i^{(0)})^{m_i}+\beta x^cy^d\prod_j(x\overline{\xi_j^{(0)}}-y\xi_j^{(0)})^{m_j}=0,\quad \alpha,\beta\in\mathbb{R}\ ,$$

where $(\xi_i^{(0)},\overline{\xi_i^{(0)}})$ runs over the set $C'_k\cap\mathrm{Tor}(\sigma_{k-1})$, and $(\xi_j^{(0)},\overline{\xi_j^{(0)}})$ runs over the set $C'_k\cap\mathrm{Tor}(\sigma_k)$. Substituting $(x,y)=(\xi_l^{(0)},\overline{\xi_l^{(0)}})$, where $\boldsymbol{x}_l$ is the vertex of A dual to Δ_k, into the above equation, we determine α/β.

If Δ_k is a rectangle, then the number of real curves C'_k, satisfying the given requirements can be derived from

Lemma 4.1 *Let Δ_0 be the rectangle with vertices $(m,0)$, $(m+1,1)$, $(0,m)$, and $(1,m+1)$, the real structure of the surface $\mathrm{Tor}(\Delta_0)$ is given by $\mathrm{Conj}(x,y)=(\overline{y},\overline{x})$, $(x,y)\in(\mathbb{C}^*)^2$, and we are given*

- *generic distinct real points $(\alpha_i,\overline{\alpha}_i)\in\mathrm{Tor}(\sigma')$, $1\le i\le p$, $\sigma'=[(m,0),(0,m)]$,*
- *generic distinct real points $(\beta_j,\overline{\beta}_j)\in\mathrm{Tor}(\sigma';)$, $1\le j\le q$, $\sigma;'=[(m+,1),(1,m+1)]$,*
- *positive integers $m_1,...,m_p$ and $n_1,...,n_q$ such that $m_1+...+m_p=n_1+...+n_q=m$,*

- *a generic real point* $(\xi_0, \overline{\xi}_0) \in (\mathbb{C}^*)^2 \subset \mathrm{Tor}(\Delta_0)$.

Then there are precisely 2^{p+q} *distinct real rational curves* $D \in |\mathcal{L}(\Delta_0)|$*, which pass through* $(\xi_0, \overline{\xi}_0)$ *and satisfy*

$$(D \cdot \mathrm{Tor}(\sigma'))_{(\alpha_i, \overline{\alpha}_i)} = m_i,\ 1 \le i \le p, \quad (D \cdot \mathrm{Tor}(\sigma''))_{(\beta_j, \overline{\beta}_j)} = n_j,\ 1 \le j \le q\ .$$

All these curves are nodal and non-singular along $\mathrm{Tor}(\sigma') \cup \mathrm{Tor}(\sigma'')$.

Proof. A curve D in the assertion can be parameterized as follows. Assuming that D crosses $\mathrm{Tor}(\sigma')$ for the distinct parameter values $t = 0, \lambda_1, ..., \lambda_{p-1} \in \mathbb{R}$, crosses $\mathrm{Tor}(\sigma'')$ for the distinct parameter values $t = \mu_1, ..., \mu_{q-1}, \infty$, crosses the other two toric divisors of $\mathrm{Tor}(\Delta_0)$ at $t = \tau, \overline{\tau}$, $\tau \in \mathbb{C}\backslash\mathbb{R}$, and passes through $(\xi_0, \overline{\xi}_0)$ at $t = 1$, we obtain

$$x = a\beta_q \frac{t-\tau}{t-\overline{\tau}} \cdot t^{m_p} \prod_{i=1}^{p-1} (t-\lambda_i)^{m_i} \left(\prod_{j=1}^{q-1} (t-\mu_j)^{n_j} \right)^{-1} ,$$

$$y = a\overline{\beta}_q \frac{t-\overline{\tau}}{t-\tau} \cdot t^{m_p} \prod_{i=1}^{p-1} (t-\lambda_i)^{m_i} \left(\prod_{j=1}^{q-1} (t-\mu_j)^{n_j} \right)^{-1}$$

with certain $a \in \mathbb{R}^*$. The conditions imposed on D read as equations to the unknowns $a, \lambda_1, ..., \lambda_{p-1}, \mu_1, ..., \mu_{q-1}, \tau$:

$$\left(\frac{\tau}{\overline{\tau}}\right)^2 = \frac{\alpha_p \overline{\beta}_q}{\overline{\alpha}_p \beta_q} \quad \text{at } t=0, \quad \frac{1-\tau}{1-\overline{\tau}} = \frac{\xi_0}{a\beta_q} \prod_{i=1}^{p-1} (1-\lambda_i) \left(\prod_{j=1}^{q-1} (1-\mu_j) \right)^{-1} \quad \text{at } t=1\ ,$$

$$\left(\frac{\lambda_i - \tau}{\lambda_i - \overline{\tau}}\right)^2 = \frac{\alpha_i \overline{\beta}_p}{\overline{\alpha}_i \beta_p} \quad \text{at } t = \lambda_i, \quad \left(\frac{\mu_j - \tau}{\mu_j - \overline{\tau}}\right)^2 = \frac{\beta_j \overline{\beta}_p}{bet_j \beta_p} \quad \text{at } t = \mu_j\ .$$

Due to the generic choice of ξ_0, α_i, β_j, the first two equations determine four distinct values of τ, and, for each of them, the remaining equations give two independent values for any of $\lambda_1, ..., \lambda_{p-1}, \mu_1, ..., \mu_{q-1}$. $\square$

At last we observe that, given limit curves C_k for $\Delta_k \in P_{\mathcal{B}}(S_C)$, the curves C_l for $\Delta_l \in P(S_C)\backslash P_{\mathcal{B}}(S_C)$ are determined uniquely.

4.3 Computation of Welschinger invariants

Observe that the Welschinger number $W(\gamma, G, \overline{s})$ is just the number of ways to restore the subdivision S, the tropical curve A and the limit curves C_k, $\Delta_k \in P(S)$ along the above procedure. We claim that the restored data produce precisely one real rational curve $C \in |\mathcal{L}(\Delta)|_{\mathbb{K}}$ passing through the configuration $\overline{\boldsymbol{p}}$, and its Welschinger sign is $+1$.

For, we apply the patchworking theory, presented in [18], section 5. First, we should complete the given data by deformation patterns associated with the components of G', and which similarly to [18, 19] are represented by real rational curves with Newton triangles like $\text{Conv}\{(0,0),(0,2),(m,1)\}$, m being the weight of the corresponding component of G'. The completed data satisfy the hypotheses of Theorem 5 from [18], and give rise to families of real rational curves $C^{(t)} \in |\mathcal{L}(\Delta)|$, $t > 0$, which in turn smoothly depend on $|\partial\Delta| - 1$ extra real parameters, which can be fixed from the condition to pass through the configuration $\overline{\boldsymbol{p}}$ (considered as a real configuration depending on the parameter t). Recall that a family $C^{(t)}$, $t \neq 0$, can be interpreted as a real rational curve $C \in |\mathcal{L}(\Delta)|_{\mathbb{K}}$.

We notice here that whenever an even weight m is assigned to a component of G', either there are no suitable deformation patterns, or there are two deformation patterns having distinct parity of the number of solitary real nodes (see [18], proof of Proposition 6.1), and hence the produced real curves in $|\mathcal{L}(\Delta)|_{\mathbb{K}}$ contribute zero to the Welschinger invariant. At last, if m is odd then there exists precisely one suitable real deformation pattern, and it has an even number of solitary nodes (see again [18], proof of Proposition 6.1). The conditions to pass through the points of $\overline{\boldsymbol{p}}$ have unique real solution (all the solutions, as presented in formula (5.4.26) of [18], contain m-th root of unity among which only one is real), and hence (1.4) follows.

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